Σύνθεση 2 καθέτων α.α.τ. ίσης συχνότητας

Έστω:
$$x = \alpha \sin(\omega t + \varphi) \quad (1)$$
$$y = \beta \sin(\omega t + \psi) \quad (2)$$

Θέτω: $X \equiv \frac{x}{\alpha}$, $Y = \frac{y}{\beta}$, $\gamma \equiv \omega t + \varphi$, $\delta = \psi - \varphi$ (3)

$$\left.\begin{matrix} (1) \\ (2) \end{matrix}\right\} \overset{(3)}{\Longrightarrow} \begin{cases} X = \sin\gamma & (4) \\ Y = \sin(\gamma+\delta) = \operatorname{Im}\left[e^{i(\gamma+\delta)}\right] = \operatorname{Im}\left[e^{i\gamma}e^{i\delta}\right] = \end{cases}$$

$$= \operatorname{Im}\left[(\cos\gamma + i\sin\gamma)\cdot(\cos\delta + i\sin\delta)\right] = \cdots =$$

$$= \operatorname{Im}\left[(\cdots) + i(\sin\gamma\cos\delta + \sin\delta\cos\gamma)\right] =$$

$$= \sin\gamma\cos\delta + \sin\delta\cos\gamma \quad (5)$$

$$(4),(5) \Rightarrow X^2 + Y^2 - 2XY\cos\delta = \sin^2\gamma + \sin^2\gamma\cos^2\delta + \sin^2\delta\cos^2\gamma$$

$$+ 2\sin\gamma\cos\gamma\sin\delta\cos\delta - 2\sin\gamma\cos\delta\sin\gamma\cos\delta -$$

$$- 2\sin\gamma\cos\delta\sin\delta\cos\gamma =$$

$$= \sin^2\gamma + \sin^2\gamma\cos^2\delta + \sin^2\delta\cos^2\gamma - 2\sin^2\gamma\cos^2\delta =$$

$$= \sin^2\gamma\cdot 1 - \sin^2\gamma\cos^2\delta + \sin^2\delta\cos^2\gamma =$$

$$= \sin^2\gamma(1 - \cos^2\delta) + \sin^2\delta\cos^2\gamma =$$

$$= \sin^2\gamma\sin^2\delta + \cos^2\gamma\sin^2\delta =$$

$$= \sin^2\delta\,(\underbrace{\sin^2\gamma + \cos^2\gamma}_{1}) = \sin^2\delta \overset{(3)}{\Longrightarrow}$$

$$\Rightarrow \boxed{\left(\frac{x}{\alpha}\right)^2 + \left(\frac{y}{\beta}\right)^2 - 2\frac{xy}{\alpha\beta}\cos(\varphi-\psi) = \sin^2(\varphi-\psi)},$$

Που είναι έλλειψη καθώς είναι της γενικής μορφής:

$$Ax^2 + Bxy + \Gamma y^2 + \Delta x + Ey + Z = 1, \quad B^2 - 4A\Gamma < 0.$$