## Κεφ. 7

① $\hat{\rho} = \frac{1}{4}\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ -i & 1 & 2 \end{pmatrix}$ , $\hat{A} = \varepsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \varepsilon \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{P_\varepsilon} + 3\varepsilon \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{P_{3\varepsilon}}$

Τα δυνατά αποτελέσματα της μέτρησης είναι $\varepsilon$, $3\varepsilon$ με πιθανότητες

$$Prob(\varepsilon) = Tr(\hat{\rho}\hat{P}_\varepsilon) = \frac{1}{4} Tr \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 1 \\ -i & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{4} Tr \begin{pmatrix} 1 & 0 & i \\ 0 & 0 & 1 \\ -i & 0 & 2 \end{pmatrix} = \frac{3}{4}$$

και

$$Prob(3\varepsilon) = 1 - Prob(\varepsilon) = \frac{1}{4}$$

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② $Tr\hat{\rho} = 1 \rightarrow \frac{1}{7}(2\alpha + 2\beta + 1) = 1 \rightarrow \alpha + \beta = 3$ (I), $\alpha, \beta \geq 0$

Για να δούμε αν είναι καθαρή, αρκεί $Tr\,\rho^2 = 1$

$$\frac{1}{49} Tr \begin{pmatrix} \alpha & -i & \beta & i \\ i & 1 & 2i & -1 \\ \beta & -2i & 2\beta & 2i \\ -i & -1 & -2i & \alpha \end{pmatrix}\begin{pmatrix} \alpha & -i & \beta & i \\ i & 1 & 2i & -1 \\ \beta & -2i & 2\beta & 2i \\ -i & -1 & -2i & \alpha \end{pmatrix}$$

(για το ίχνος αρκεί ο υπολογισμός των διαγωνίων στοιχείων του $\rho^2$)

$$= \frac{1}{49} Tr \begin{pmatrix} \alpha^2 + 2 + \beta^2 & & & \\ & 7 & & \\ & & 5\beta^2 + 8 & \\ & & & \alpha^2 + 6 \end{pmatrix} = \frac{1}{49}(2\alpha^2 + 6\beta^2 + 23)$$

$$Tr(\hat{\rho}^2)=1 \longrightarrow 2\alpha^2+6\beta^2+23=49 \rightarrow \alpha^2+3\beta^2=13 \xrightarrow{(I)}$$

$$(3-\beta)^2+3\beta^2=13 \rightarrow 9+\beta^2-6\beta+3\beta^2=13 \longrightarrow 4\beta^2-6\beta-4=0$$

$$\rightarrow 2\beta^2-3\beta-2=0 \rightarrow \beta=\frac{3\pm\sqrt{25}}{4} = \begin{matrix} 2 \\ -\frac{1}{2} \leftarrow \text{μη αποδεκτό} \end{matrix}$$

$$\beta=2 \rightarrow \alpha=1$$

Άρα καθαρή για $\alpha=1$, $\beta=2$

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(4) Η γενική μήτρα πυκνότητας στο $\mathbb{C}^3$ είναι

$$\hat{\rho}=\begin{pmatrix} x & \alpha & \beta \\ \alpha^* & y & \gamma \\ \beta^* & \gamma^* & z \end{pmatrix} \quad \text{με } x,y,z\geq 0 \text{ και } x+y+z=1$$

η συνθήκη $\langle \lambda_1\rangle=0 \rightarrow Tr(\hat{\rho}\hat{\lambda}_1)=0 \longrightarrow Tr\begin{pmatrix} x & \alpha & \beta \\ \alpha^* & y & \gamma \\ \beta^* & \gamma^* & z \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}=0$

$$\longrightarrow Tr\begin{pmatrix} \alpha & & \\ & \alpha^*+\gamma & \\ & & \gamma^* \end{pmatrix}=0 \rightarrow \alpha+\alpha^*+\gamma+\gamma^*=0 \rightarrow Re(\alpha+\gamma)=0$$

η συνθήκη $\langle \lambda_2\rangle=0 \rightarrow Tr(\hat{\rho}\hat{\lambda}_2)=0 \rightarrow Tr\begin{pmatrix} x & \alpha & \beta \\ \alpha^* & y & \gamma \\ \beta^* & \gamma^* & z \end{pmatrix}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}=0$

$$\rightarrow Tr\begin{pmatrix} i\alpha & & \\ & -i\alpha^*+i\gamma & \\ & & -i\gamma^* \end{pmatrix}=0 \rightarrow i\left(\alpha-\alpha^*+\gamma-\gamma^*\right)=0 \rightarrow Im(\alpha+\gamma)=0$$

Άρα $\alpha+\gamma=0 \longrightarrow \hat{\rho}=\begin{pmatrix} x & \alpha & \beta \\ \alpha^* & y & -\alpha \\ \beta^* & -\alpha^* & z \end{pmatrix}$

(α)

$$\langle \lambda_3\rangle=1 \longrightarrow Tr\begin{pmatrix} x & \alpha & \beta \\ \alpha^* & y & -\alpha \\ \beta^* & -\alpha^* & z \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}=0 \longrightarrow Tr\begin{pmatrix} x & & \\ & 0 & \\ & & -z \end{pmatrix}=1$$

$x = z + 1$ , αλλά αφού $x+y+z=1$ και $x,y,z \geq 0$ αυτό είναι εφικτό μόνο αν $z=0$ , οπότε $x=1$ και $y=0$

Άρα $\hat{\rho} = \begin{pmatrix} 1 & \alpha & \beta \\ \alpha^* & 0 & -\alpha \\ \beta^* & -\alpha^* & 0 \end{pmatrix}$ . Αλλά μια μήτρα πυκνότητας ικανοποιεί $|\rho_{mn}| \leq \sqrt{\rho_{nn}\,\rho_{mm}}$

$\downarrow$

$\left.\begin{matrix} |\alpha| \leq 0 \\ |\beta| \leq 0 \end{matrix}\right\} \longrightarrow \alpha = \beta = 0$

$\hat{\rho} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ καθαρή αφού $\hat{\rho}^2 = \hat{\rho}$

γ) $\langle 1_z \rangle = 0 \longrightarrow x - z = 0 \longrightarrow y = 1 - 2x$

$\hat{\rho} = \begin{pmatrix} x & \alpha & \beta \\ \alpha^* & 1-2x & -\alpha \\ \beta^* & -\alpha^* & x \end{pmatrix}$ η συνθήκη $|\rho_{mn}| \leq \sqrt{\rho_{nn}\,\rho_{mm}}$

$|\alpha| \leq \sqrt{x(1-2x)}$

$|\beta| \leq x$ (II)

για να είναι η $\hat{\rho}$ καθαρή $Tr\hat{\rho}^2 = 1$

Βρίσκουμε $Tr\hat{\rho}^2 = Tr \begin{pmatrix} x^2+|\alpha|^2+|\beta|^2 & & \\ & |\alpha|^2+(1-2x)^2+|\alpha|^2 & \\ & & |\beta|^2+|\alpha|^2+x^2 \end{pmatrix} = 1$

$\longrightarrow 2x^2 + 4|\alpha|^2 + 2|\beta|^2 + (1-2x)^2 = 1 \xrightarrow{(II)}$

$6x^2 - 4x + 4|\alpha|^2 + 2|\beta|^2 = 0 \longrightarrow 3x^2 - 4x + 2|\alpha|^2 + |\beta|^2 = 0$

$\longrightarrow 3x^2 - 4x + 2x(1-2x) + x^2 \geq 0 \longrightarrow -2x \geq 0$ . Αφού $x \geq 0$ αυτό είναι δυνατό μόνο για $x = 0$ άρα $y = 1$

$\text{II} \longrightarrow \left.\begin{matrix} |\alpha| \leq 0 \\ |\beta| \leq 0 \end{matrix}\right\} \longrightarrow \alpha = \beta = 0 \longrightarrow \hat{\rho} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(6) $\hat{\rho}=\sum_i \lambda_i \hat{\rho}_i \rightarrow \hat{\rho}^2=\sum_{i,j} \lambda_i \lambda_j \hat{\rho}_i \hat{\rho}_j \rightarrow \gamma = Tr\hat{\rho}^2 = \sum_{i,j} \lambda_i \lambda_j Tr(\hat{\rho}_i \hat{\rho}_j) \leq$

$\sum_{i,j} \lambda_i \lambda_j |Tr(\rho_i \rho_j)| \leq \sum_{i,j} \lambda_i \lambda_j \sqrt{Tr\rho_i^2}\sqrt{Tr\rho_j^2} = \sum_{i,j} \lambda_i \lambda_j \sqrt{\gamma_i}\sqrt{\gamma_j} = \left(\sum_i \lambda_i \sqrt{\gamma_i}\right)^2$

$\leq \sum_i \lambda_i (\sqrt{\gamma_i})^2 = \sum_i \lambda_i \gamma_i$

Cauchy schwarz

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(9) στ αρμονικό ταλαντωτή: $\hat{x} = \frac{1}{\sqrt{2m\omega}}(\hat{a}+\hat{a}^\dagger)$

$\hat{p} = i\sqrt{\frac{m\omega}{2}}(\hat{a}^\dagger - \hat{a})$

$\langle \hat{x} \rangle = \frac{1}{\sqrt{2m\omega}} \langle n|\hat{a}+\hat{a}^\dagger|n\rangle = 0$ , $\langle p \rangle = i\sqrt{\frac{m\omega}{2}} \langle n|a^\dagger - a|n\rangle = 0$

$\langle \hat{x}^2 \rangle = \frac{1}{2m\omega}\langle n|(\hat{a}+\hat{a}^\dagger)^2|n\rangle = \frac{1}{2m\omega}\langle n|a^2 + a^\dagger a + a a^\dagger + a^{\dagger 2}\rangle =$

$= \frac{1}{2m\omega}\langle n|2a^\dagger a + I|n\rangle = \frac{1}{2m\omega}(2n+1)$

$\langle \hat{p}^2 \rangle = \frac{m\omega}{2}\langle n|-(a^\dagger - a)^2|n\rangle = \frac{m\omega}{2}\langle n|-a^{\dagger 2} - a^2 - a a^\dagger - a^\dagger a|n\rangle$

$= \frac{m\omega}{2}\langle n|2a^\dagger a + I|n\rangle = \frac{m\omega}{2}(2n+1)$

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(11) a) $\psi_0(x) = \sqrt{\gamma}\, e^{-\gamma|x|}$ $\quad \langle x \rangle = \gamma\int_{-\infty}^{\infty} x e^{-2\gamma|x|}dx = 0$

$\langle x^2 \rangle = \gamma \int_{-\infty}^{\infty} x^2 e^{-2\gamma|x|}dx = 2\gamma\int_0^\infty x^2 e^{-2\gamma x}dx = 2\gamma \frac{1}{4\gamma^3} = \frac{1}{2\gamma^2}$

βολικό να υπολογίσουμε $\tilde{\psi}_0(p) = \frac{\sqrt{\gamma}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} e^{-\gamma|x|}dx = \frac{\sqrt{\gamma}}{\sqrt{2\pi}}\, 2\int_0^\infty \cos px\, e^{-\gamma x}dx$

$= \frac{\sqrt{\gamma}}{\sqrt{2\pi}}\frac{2\gamma}{\gamma^2+p^2} = \sqrt{\frac{2}{\pi}}\gamma^{3/2}\frac{1}{\gamma^2+p^2}$

$\langle p \rangle = \frac{2}{\pi}\gamma^3 \int_{-\infty}^{\infty}\frac{dp\, p}{(\gamma^2+p^2)^2} = 0$

$\langle p^2 \rangle = \frac{2}{\pi}\gamma^3\int_{-\infty}^{\infty}\frac{dp\, p^2}{(\gamma^2+p^2)^2} = \frac{2\gamma^3}{\pi}\frac{\pi}{2\gamma} = \gamma^2$

$(\Delta x)^2(\Delta p)^2 = \frac{1}{2\gamma^2}\gamma^2 = \frac{1}{2} \rightarrow (\Delta x)(\Delta p) = \frac{1}{\sqrt{2}}$

β) εκτός ύλης

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$\psi(x) = 2\lambda^{3/2} x e^{-\lambda x}$ (προσοχή, διόρθωση στην κανονικοποίηση)

(α) $\langle x\rangle = 4\lambda^3 \int_0^\infty dx\, x^3 e^{-2\lambda x} = 4\lambda^3 \frac{3!}{(2\lambda)^4} = \frac{3}{2\lambda}$

$\langle x^2\rangle = 4\lambda^3 \int_0^\infty dx\, x^4 e^{-2\lambda x} = 4\lambda^3 \frac{4!}{(2\lambda)^5} = \frac{3}{\lambda^2}$

$(\Delta x)^2 = \frac{3}{\lambda^2} - \frac{9}{4\lambda^2} = \frac{5}{4\lambda^2}$

(β) Τα γενικευμένα ιδιοδιανύσματα του $|\hat{p}|$ είναι $f_k(x) = \sqrt{\frac{2}{\pi}} \sin kx \equiv |k\rangle$

Άρα η πυκνότητα πιθανότητας είναι $|\langle k|\psi\rangle|^2 = p(k)$

$\langle k|\psi\rangle = 2\lambda^{3/2} \sqrt{\frac{2}{\pi}} \int_0^\infty dx\, \sin kx\, x\, e^{-2\lambda x} = \sqrt{\frac{2}{\pi}}\, 2\lambda^{3/2} \frac{4\lambda k}{(k^2+4\lambda^2)^2}$

↑ πίνακας ολοκληρωμάτων

$= 8\sqrt{\frac{2}{\pi}} \frac{\lambda^{5/2} k}{(k^2+4\lambda^2)^2}$

Άρα $p(k) = \frac{128\lambda^5}{\pi} \frac{k^2}{(k^2+4\lambda^2)^4}$

(γ) $\langle|\hat{p}|\rangle = \int_0^\infty k\,dk\, p(k) = \frac{128\lambda^5}{\pi} \int_0^\infty dk \frac{k^3}{(k^2+4\lambda^2)^4} = \frac{128\lambda^5}{\pi} \frac{1}{192\lambda^4} = \frac{2\lambda}{3\pi}$

$\langle p^2\rangle = \frac{128\lambda^5}{\pi} \int_0^\infty dk \frac{k^4}{(k^2+4\lambda^2)^4} = \frac{128\lambda^5}{\pi} \frac{\pi}{256\lambda^3} = \frac{\lambda^2}{2}$

$(\Delta|p|)^2 = \lambda^2\left(\frac{1}{2} - \frac{4}{9\pi^2}\right)$

ολοκληρώματα $\int_0^\infty \frac{x^3 dx}{(x^2+1)^4} = \frac{1}{12}$

$\int_0^\infty \frac{x^4 dx}{(x^2+1)^4} = \frac{\pi}{32}$

$\Delta x\, \Delta|p| = \sqrt{\frac{5}{4}} \frac{1}{\lambda} \cdot \lambda \sqrt{\frac{1}{2} - \frac{4}{9\pi^2}}$

$\approx 0{,}75$

(δ) ο τελεστής $\hat{Q}$ έχει ιδιοδιανύσματα $f_q(x) = \frac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}+iq}$

$|q\rangle$

Υπολογίζουμε $\langle q|\psi\rangle = 2\lambda^{3/2} \frac{1}{\sqrt{2\pi}} \int_0^\infty dx \, x^{\frac{1}{2}+iq} e^{-2\lambda x} =$

$$= \sqrt{\frac{2}{\pi}} \lambda^{3/2} \frac{1}{(2\lambda)^{\frac{3}{2}+iq}} \Gamma\left(\frac{3}{2}+iq\right) = \frac{1}{2}\sqrt{\frac{1}{\pi}} (2\lambda)^{-iq} \Gamma\left(\frac{3}{2}+iq\right)$$

Άρα $p(q) = |\langle q|\psi\rangle|^2 = \frac{1}{4\pi} \left|\Gamma\left(\frac{3}{2}+iq\right)\right|^2$

(13) $\psi(x) = \sqrt{\frac{30}{L^5}}\, x(L-x)$. Τα ιδιοδιανύσματα του $\widehat{|p|}$ είναι οι συναρτήσεις

$$f_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi}{L}x$$

$$\langle n|\psi\rangle = \int_0^L dx\, \psi(x) f_n^*(x) = \sqrt{\frac{2}{L}}\sqrt{\frac{30}{L^5}} \int_0^L dx\, x(L-x) \sin\frac{n\pi}{L}x$$

$\frac{x}{L}$ →

$$= \sqrt{\frac{60}{L^6}}\, L^3 \int_0^1 dy\, y(1-y) \sin n\pi y = \sqrt{60}\, \frac{2-2\cos(n\pi) - n\pi\sin(n\pi)}{(n\pi)^3}$$

$$= \sqrt{60}\, \frac{2-(-1)^n}{n^3\pi^3}$$

$$P_n = |\langle n|\psi\rangle|^2 = 60\, \frac{5-4(-1)^n}{n^6\pi^6} = \frac{60}{\pi^6 n^6}\left(5-4(-1)^n\right)$$

$$\langle x\rangle = \frac{30}{L^5}\int_0^L dx\, x\, x^2(L-x)^2 = \frac{30}{L^5} L^6 \int_0^1 dy\, y^3(1-y)^2 = 30L\,\frac{1}{60} = \frac{L}{2}$$

$$\langle x^2\rangle = \frac{30}{L^5}\int_0^L dx\, x^2 x^2(L-x)^2 = \frac{30}{L^5} L^7 \int_0^1 dy\, y^4(1-y)^2 = 30L^2\,\frac{1}{105} = \frac{2L^2}{7}$$

$$(\Delta x)^2 = \frac{2L^2}{7} - \frac{L^2}{4} = L^2\,\frac{8-7}{28} = \frac{L^2}{28}$$

$$\langle|\hat{p}|\rangle = \sum_{n=1}^{\infty} P_n \cdot \frac{n\pi}{L} = \frac{60}{L\pi^6}\left(\sum_{n=1}^{\infty}\frac{5}{n^5} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5}\right)$$

πίνακες σειρών

$$= \frac{60}{L\pi^6}\left(5\zeta(5) - 4\left(-\frac{15}{16}\zeta(5)\right)\right) = \frac{60}{L\pi^6}\cdot\frac{35}{4}\zeta(5) = \frac{525\zeta(5)}{L\pi^6}$$

$$\langle|p|^2\rangle = \sum_{n=1}^{\infty} P_n \frac{n^2\pi^2}{L^2} = \frac{60}{L^2\pi^5}\left(5\sum_{n=1}^{\infty}\frac{1}{n^4} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\right)$$

$$= \frac{60}{L^2\pi^5}\left[5\zeta(4) - 4\left(-\frac{7}{8}\zeta(4)\right)\right] = \frac{60}{L^2\pi^5}\cdot\frac{17}{2}\zeta(4) = \frac{30\cdot 17}{L^2\pi^5}\,\frac{\pi^4}{90} = \frac{17}{3\pi L^2}$$

$$\Delta|p|^2 = \frac{17}{3\pi L^2} - \frac{525\zeta(5)}{\pi^6}\,\frac{1}{L^2} = \frac{1}{\pi L^2}\left(\frac{17}{3} - \frac{525\zeta(5)}{\pi^5}\right) \approx \frac{1{,}24}{L^2}$$

$$(\Delta x)^2 (\Delta|p|)^2 = \frac{L^2}{28}\cdot\frac{1{,}24}{L^2} = 0{,}044$$