11.2

α) $\hat{T}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}-c_2^*\\c_1^*\end{pmatrix}$

Έστω $(d_1^* \;\; d_2^*)\,\hat{T}\begin{pmatrix}c_1\\c_2\end{pmatrix} = (d_1^* \;\; d_2^*)\begin{pmatrix}-c_2^*\\c_1^*\end{pmatrix} = -d_1^*c_2^* + d_2^*c_1^*$

$= (c_1^* \;\; c_2^*)\begin{pmatrix}d_2^*\\-d_1^*\end{pmatrix}$. Η εξ 11.4 για $|\psi\rangle = \begin{pmatrix}c_1\\c_2\end{pmatrix}$, $|\phi\rangle = \begin{pmatrix}d_1\\d_2\end{pmatrix}$ δίνει

$\hat{T}^\dagger\begin{pmatrix}d_1\\d_2\end{pmatrix} = \begin{pmatrix}d_2^*\\-d_1^*\end{pmatrix} = -\hat{T}\begin{pmatrix}d_1\\d_2\end{pmatrix}$, άρα $\hat{T}^\dagger = -\hat{T}$

Αφού $\hat{T}^\dagger\hat{T} = \hat{I} \rightarrow \hat{T}^2 = -\hat{I}$

β) Για $|\psi\rangle = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \longrightarrow \vec{r}_\psi = \left(c_1 c_2^* + c_2^* c_1,\ i(c_2^* c_1 - c_1^* c_2),\ |c_1|^2 - |c_2|^2\right)$

$|\psi'\rangle = \hat{T}|\psi\rangle = \begin{pmatrix} -c_2^* \\ c_1^* \end{pmatrix}$, $\vec{r}_{\psi'} = \left(-c_1^* c_2 - c_2^* c_1,\ -i(c_2^* c_1 - c_1^* c_2,\ |c_2|^2 - |c_1|^2\right)$

γ) Βρίσκουμε $\hat{\sigma}_1 \hat{T}|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -c_2^* \\ c_1^* \end{pmatrix} = \begin{pmatrix} c_1^* \\ -c_2^* \end{pmatrix}$ $= (-r_1, r_2, r_3)$

$\hat{T}\hat{\sigma}_1|\psi\rangle = \hat{T}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \hat{T}\begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} -c_1^* \\ c_2^* \end{pmatrix} = -\hat{\sigma}_1 \hat{T}|\psi\rangle$

Άρα $\hat{\sigma}_1 \hat{T} = -\hat{T}\hat{\sigma}_1 \rightarrow \hat{T}^{-1}\hat{\sigma}_1 \hat{T} = -\hat{\sigma}_1 \longrightarrow \hat{T}^\dagger \hat{\sigma}_1 \hat{T} = -\hat{\sigma}_1$

ομοίως για $\hat{\sigma}_2, \hat{\sigma}_3$

δ) $i\frac{\partial}{\partial t}|\psi\rangle = n\cdot\vec{\sigma}|\psi\rangle \longrightarrow -i\frac{\partial}{\partial t}\hat{T}|\psi\rangle = \hat{T}\vec{n}\cdot\vec{\sigma}|\psi\rangle = \sum_i n_i \hat{T}\hat{\sigma}_i|\psi\rangle$

$= -\sum_i n_i \vec{\sigma}\,\hat{T}|\psi\rangle \longrightarrow i\frac{\partial}{\partial t}\hat{T}|\psi\rangle = \vec{n}\cdot\vec{\sigma}\ \hat{T}|\psi\rangle$

η εξ. Σρέντινγκερ παραμένει αναλλοίωτη

$\boxed{3}$ για $\underline{n} = \left(\frac{3}{5}, 0, -\frac{4}{5}\right)$, $\lambda_n = \begin{pmatrix} 0 & -\frac{4}{5} & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \\ 0 & -\frac{3}{5} & 0 \end{pmatrix}$

ο πίνακας $i\lambda_n$ έχει ιδιοτιμές, με ιδιοδιάνυσμα, φασματικό προβολέα

$0 \qquad \begin{pmatrix} 3/5 \\ 0 \\ -4/5 \end{pmatrix} \qquad \begin{pmatrix} 9/25 & 0 & -\frac{12}{25} \\ 0 & 0 & 0 \\ -\frac{12}{25} & 0 & \frac{16}{25} \end{pmatrix}$

$1 \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} -4i/5 \\ 1 \\ -3i/5 \end{pmatrix} \qquad \frac{1}{50}\begin{pmatrix} 16 & -20i & 12 \\ 20i & 25 & 15i \\ 12 & 15i & 9 \end{pmatrix}$

$-1 \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 4i/5 \\ 1 \\ 3i/5 \end{pmatrix} \qquad \frac{1}{50}\begin{pmatrix} 16 & 20i & 12 \\ -20i & 15 & -15i \\ 12 & 15i & 9 \end{pmatrix}$

Άρα $e^{\theta\lambda_n} = 1\cdot\begin{pmatrix} 9/25 & 0 & -12/25 \\ 0 & 0 & 0 \\ -\frac{12}{25} & 0 & \frac{16}{25} \end{pmatrix} + \frac{e^{-i\theta}}{50}\begin{pmatrix} 16 & -20i & 12 \\ 20i & 25 & 15i \\ 12 & -15i & 9 \end{pmatrix} + \frac{e^{i\theta}}{50}\begin{pmatrix} 16 & 20i & 12 \\ -20i & 15 & -15i \\ 12 & 15i & 9 \end{pmatrix}$

4) Έστω $[\hat{A}, J_1]=0$, $[\hat{A}, \hat{J}_2]=0$

$$[A, J_3] = -i\,[A,[J_1, J_2]] = -i\,[J_2,\underbrace{[A,J_1]}^{0}] - i\,[J_1,\underbrace{[J_2,A]}^{0}] = 0$$

ταυτότητα Jacobi

5

Εφόσον $Tr\, U J_i U^{\dagger} = Tr\, J_i$, μπορούμε να διαλέξουμε $\hat{U}$ που κάνει περιστροφή ώστε να πάρουμε $U J_i U^{\dagger} = \hat{J}_3$. Άρα

$$Tr\,\hat{J}_1 = Tr\,\hat{J}_2 = Tr\,\hat{J}_3 = \sum_{m=-j}^{j} \langle j,m|\,J_3\,|j,m\rangle = \sum_{m=-j}^{j} m = 0$$

Για το $T_r(J_i J_j)$ πάλι με το ίδιο επιχείρημα

$$Tr(J_1 J_2) = Tr(J_2 J_3) = Tr(J_3 J_1)$$

και

$$Tr\,(J_1^2) = Tr(J_2^2) \pm Tr(J_3^2)$$

υπολογίζουμε $Tr(\hat{J}_1 \hat{J}_3) = \sum_{m=-j}^{j} \langle j,m|\,J_1 J_3\,|j,m\rangle = \sum_{m=-j}^{j} m \underbrace{\sqrt{j(j+1)-m(m+1)}\,\langle j,m|j,m+1\rangle}_{0} = 0$

$$Tr(J_3^2) = \frac{1}{3}\, Tr\,\vec{J}^2 = \frac{1}{3}\sum_{m=-j}^{j} \langle j,m|\vec{J}^2|j,m\rangle$$

$$= \frac{1}{3}\, j(j+1) \sum_{m=-j}^{j} 1 = \frac{1}{3}\, j(j+1)(2j+1)$$

άρα $Tr(\hat{J}_i \hat{J}_j) = \frac{1}{3}\, j(j+1)(2j+1)$

---

6 $\hat{T}|j,m\rangle = (-1)^{j-m}|j,-m\rangle$

Για δεδομένο j $\quad |\psi\rangle = \sum_{m=-j}^{j} c_m |j,m\rangle$

$$T|\psi\rangle = \lambda|\psi\rangle \longrightarrow \sum_{m=-j}^{j} c_m^* (-1)^{j-m} |j,-m\rangle = \lambda \sum_{m=-j}^{j} c_m |j,m\rangle = \lambda \sum_{m=-j}^{j} c_{-m} |j,-m\rangle$$

Άρα $\quad c_m^* (-1)^{j-m} = \lambda c_{-m} \quad (1)$

Έχουμε δεί ότι $\hat{T}^2 = (-1)^{2j} \rightarrow$

Για ακέραιο j, $\lambda^2 = 1 \rightarrow \lambda = \pm 1$, η (1) γίνεται $c_m^* (-1)^{j-m} = \pm c_{-m}$.

Αν πάρουμε οποιαδήποτε c_m για $m>0$, αμέσως βρίσκουμε τα c_{-m}. Αν $m=0$, $c_0^* (-1)^j = \pm c_0$

αν $(-1)^j \lambda = 1$ $\quad c_0^* = c_0 \rightarrow c_0$ πραγματικό

αν $(-1)^j \lambda = -1$ $\quad c_0^* = -c_0 \rightarrow c_0$ φανταστικό

Για ημιακέραιο j, $\lambda^2 = -1 \rightarrow \lambda = \pm i$, η (1) γίνεται $c_m^* (-1)^{j-m} = \pm i c_{-m}$

Για οποιαδήποτε c_m για $m>0$ βρίσκουμε τα c_{-m}. Αφού j ημιακέραιο δεν υπάρχει $m=0$.

---

7 α) $[\hat{x}_i, \hat{\ell}_j] = \sum_{k,m} [\hat{x}_i, \varepsilon_{jkm} x_k p_m] = \sum_{k,m} \varepsilon_{jkm} x_k [x_i, p_m] =$

$= \sum_{k,m} \varepsilon_{jkm} x_k (i\delta_{im}) = i \sum_k \varepsilon_{jki} \hat{x}_k = i \sum_k \varepsilon_{ijk} \hat{x}_k$

ομοίως για $[\hat{p}_i, \hat{\ell}_j]$

β) $\hat{\vec{r}} \cdot \hat{\vec{\ell}} = \sum_i \hat{x}_i \hat{\ell}_i = \sum_i \hat{x}_i \sum_{j,k} \varepsilon_{ijk} \hat{x}_j \hat{p}_k = \sum_{i,j,k} \varepsilon_{ijk} \underbrace{\hat{x}_i \hat{x}_j}_{\text{συμμετρικό}} \hat{p}_k = 0$

$\vec{\ell} \cdot \vec{r} = \sum_{i,j,k} \varepsilon_{ijk} \hat{x}_j \hat{p}_k \hat{x}_i = \underbrace{\sum_{ijk} \varepsilon_{ijk} (x_j x_i p_k - i x_j \delta_{ik})}_{=0}$ αντισυμμετρικό

8

$\ell=0$: $Y_{00}(\theta,\phi)=c$. $\Big|\int \sin\theta d\theta d\phi\, |Y_{00}|^2=1 \rightarrow c^2\cdot 4\pi=1\rightarrow c=\frac{1}{\sqrt{4\pi}}$

$Y_{00}(\theta,\phi)=\frac{1}{2\sqrt{\pi}}$

$\ell=1$: $Y_{11}(\theta,\phi)=C\sin\theta e^{i\phi}$ $\Big| \int \sin\theta d\theta d\phi\, |Y_{11}|^2=1 \rightarrow C^2\cdot 2\pi\int_0^\pi \sin^3\theta d\theta=1$

$\xrightarrow[\xi=\cos\theta]{} 2\pi C^2\int_{-1}^{1} d\xi\,(1-\xi^2)=1\rightarrow 2\pi C^2\cdot\frac{4}{3}=1\rightarrow C^2=\frac{3}{8\pi}\rightarrow C=\sqrt{\frac{3}{8\pi}}$

$Y_{11}(\theta,\phi)=\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$

$Y_{10}(\theta,\phi)\sim \hat{\ell}_- Y_{11}(\theta,\phi)\sim e^{-i\phi}\left(-\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)(\sin\theta e^{i\phi})=-2\cos\theta$

Άρα $Y_{10}(\theta,\phi)=C\cos\theta$ $\Big| \int \sin\theta d\theta d\phi\, |Y_{10}|^2=1\rightarrow c^2\cdot 2\pi\int \sin\theta d\theta \cos^2\theta=1$

$\rightarrow 2\pi c^2\int_{-1}^{1} d\xi\, \xi^2=1\rightarrow 2\pi c^2\,\frac{2}{3}=1\rightarrow c=\sqrt{\frac{3}{4\pi}}$

$Y_{10}(\theta,\phi)=\sqrt{\frac{3}{4\pi}}\cos\theta$

Από συμμετρία: $Y_{1,-1}(\theta,\phi)=\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$

$\ell=2$: $Y_{22}(\theta,\phi)=C\sin^2\theta\, e^{2i\phi}$ $\Big| \int \sin\theta d\theta d\phi\, |Y_{22}|^2=1\rightarrow$

$\rightarrow c^2\cdot 2\pi\cdot\int_0^\pi \sin^5\theta d\theta=1\rightarrow c^2\cdot 2\pi\int_{-1}^{1} d\xi\,(1-\xi^2)^2=1\rightarrow c^2\, 2\pi\cdot\frac{16}{15}=1$

$\rightarrow c=\sqrt{\frac{15}{32\pi}}$ $\qquad Y_{22}(\theta,\phi)=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$

$Y_{21}(\theta,\phi)\sim\hat{\ell}_- Y_{22}(\theta,\phi)=e^{-i\phi}\left(-\frac{\partial}{\partial\theta}+i\cot\theta\frac{\partial}{\partial\phi}\right)(\sin^2\theta\, e^{i\phi})=-4\sin\theta\cos\theta e^{i\phi}$

$Y_{21}(\theta,\phi)=c\sin\theta\cos\theta\, e^{i\phi}\longrightarrow \int \sin\theta d\theta d\phi |Y_{21}|^2=1\rightarrow c^2\cdot 2\pi\cdot\int_0^\pi \sin^3\theta\cos^2\theta d\theta$

$=c^2\cdot 2\pi\cdot\int d\xi\,\xi^2(1-\xi^2)=1\rightarrow c=\sqrt{\frac{15}{8\pi}}\longrightarrow Y_{21}(\theta,\phi)=\frac{1}{2}\sqrt{\frac{15}{2\pi}}\sin\theta\cos\theta e^{i\phi}$

$$Y_{2,0}(\theta,\phi) \sim L_- Y_{2,1}(\theta,\phi) \sim e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right)\left(\sin\theta\cos\theta e^{i\phi}\right) = -3\cos^2\theta + 1$$

$$Y_{20}(\theta,\phi) = c(3\cos^2\theta - 1) \quad \Big| \quad \int \sin\theta\, d\theta\, |Y_{2,0}|^2 = 1 \rightarrow c^2 \cdot 2\pi \int \sin\theta\, d\theta\, (3\cos^2\theta - 1)^2 = 1$$

$$\rightarrow 2\pi c^2 \int_{-1}^{1} d\xi\, (3\xi^2 - 1)^2 = 1 \rightarrow c = \sqrt{\frac{5}{16\pi}},$$

Από συμμετρία

$$Y_{20}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}(3\cos^2\theta - 1)$$

$$Y_{2,-1}(\theta,\phi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}} \sin\theta\cos\theta\, e^{-i\phi}$$

$$Y_{2,-2}(\theta,\phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta\, e^{-2i\phi}$$

---

9 Παρατηρούμε ότι $\psi(\theta,\phi) = \frac{C}{2i}\sin\theta\cos\theta\, e^{i\phi} - \frac{C}{2i}\sin\theta\cos\theta\, e^{-i\phi} =$

$$= \frac{1}{\sqrt{2}i}\left(Y_{2,1}(\theta,\phi) - Y_{2,-1}(\theta,\phi)\right) \leftarrow \text{βλ. άσκηση 9}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}i}\left(|2,1\rangle - |2,-1\rangle\right) \rightarrow \begin{matrix} Prob(l=2) = 1 \\ Prob(l \neq 2) = 0 \end{matrix} \Bigg\} \begin{matrix} Prob(m=1) = \frac{1}{2} \\ Prob(m=-1) = \frac{1}{2} \end{matrix}$$

10

$$\hat{H} = \frac{1}{2I_1}(\hat{\ell}_1^2 + \hat{\ell}_2^2) + \frac{1}{2I_3}\hat{\ell}_3^2 = \frac{1}{2I_1}(\hat{\ell}^2 - \hat{\ell}_3^2) + \frac{1}{2I_3}\hat{\ell}_3^2$$

$$= \frac{1}{2I_1}\vec{\ell}^2 + \frac{1}{2}\left(\frac{1}{I_3} - \frac{1}{I_1}\right)\hat{\ell}_3^2$$

Άρα ιδιοδιανύσματα τα $|l,m\rangle$, με ιδιοτιμές

$$E_{lm} = \frac{1}{2I_1} l(l+1) + \frac{1}{2}\left(\frac{1}{I_3} - \frac{1}{I_1}\right) m^2$$

11

1) α) $j_1=j_2=j_3=\frac{1}{2}$, $0\le j_{12}\le 1 \to j_{12}=0,1$

$\left.\begin{matrix} j_{12}=0 \\ j_3=\frac{1}{2}\end{matrix}\right\}$ $\frac{1}{2}\le j_{o\lambda}\le\frac{1}{2} \to j_{o\lambda}=\frac{1}{2}$

$\left.\begin{matrix} j_{12}=1 \\ j_3=\frac{1}{2}\end{matrix}\right\}$ $\frac{1}{2}\le j_{o\lambda}\le\frac{3}{2} \to j_{o\lambda}=\frac{1}{2},\frac{3}{2}$

$j_{o\lambda}=\frac{1}{2},\frac{1}{2},\frac{3}{2}$

$\downarrow$

$C^2\otimes C^2\otimes C^2 = C^2\oplus C^2\oplus C^4$

$\underline{2}\otimes\underline{2}\otimes\underline{2} = 2\oplus 2\oplus 4$

β) $j_1=j_2=\frac{1}{2}$, $j_3=1$, $0\le j_{12}\le 1\to j_{12}=0,1$

$\left.\begin{matrix} j_{12}=0 \\ j_3=1\end{matrix}\right\}$ $j_{o\lambda}=1$

$\left.\begin{matrix} j_{12}=1 \\ j_3=1\end{matrix}\right\}\to$ $0\le j_{o\lambda}\le 2\to j_{o\lambda}=0,1,2$

$j_{o\lambda}=0,1,1,2$

$C^2\otimes C^2\otimes C^3 = C\oplus C^3\oplus C^3\oplus C^5$

$\underline{2}\otimes\underline{2}\otimes\underline{3} = \underline{1}\oplus\underline{3}\oplus\underline{3}\oplus\underline{5}$

γ) $j_1=j_2=j_3=j_4=\frac{1}{2} \to j_{12}=0,1$

$j_{34}=0,1$

$\left.\begin{matrix} j_{12}=0 \\ j_{34}=0\end{matrix}\right\}$ $j_{o\lambda}=0$

$\left.\begin{matrix} j_{12}=0 \\ j_{34}=1\end{matrix}\right\}$ $j_{o\lambda}=1$

$\left.\begin{matrix} j_{12}=1 \\ j_{34}=0\end{matrix}\right\}$ $j_{o\lambda}=1$

$\left.\begin{matrix} j_{12}=1 \\ j_{34}=1\end{matrix}\right\}$ $0\le j_{o\lambda}\le 2\to j_{o\lambda}=0,1,2$

$j_{o\lambda}=0,0,1,1,1,2$

$C^2\otimes C^2\otimes C^2\otimes C^2 = C\oplus C\oplus C^3\oplus C^3\oplus C^3\oplus C^5$

$\underline{2}\otimes\underline{2}\otimes\underline{2}\otimes\underline{2} = \underline{1}\oplus\underline{1}\oplus 3\oplus\underline{3}\oplus\underline{3}\oplus\underline{5}$

12

Οι δυνατές τιμές του J είναι $0, 1, 2$.

Ξεκινάμε από τη $|2,-2\rangle = |1,-1\rangle_1 \otimes |1,-1\rangle_2$

$\hat{J}_+ |2,-2\rangle = 2\,|2,-1\rangle$

$\hat{J}_+ |2,-2\rangle = {}^1\hat{J}_+ |1,-1\rangle_1 \otimes |1,-1\rangle_2 + |1,-1\rangle_1 \otimes \hat{J}_+ |1,-1\rangle_2$

$= \sqrt{2}\,|1,0\rangle_1 \otimes |1,-1\rangle_2 + |1,-1\rangle_1 \otimes \sqrt{2}\,|1,0\rangle_2 \otimes \longrightarrow$

$$\underline{|2,-1\rangle = \frac{1}{\sqrt{2}}\,|1,0\rangle_1 \otimes |1,-1\rangle_2 + \frac{1}{\sqrt{2}}\,|1,-1\rangle \otimes |1,0\rangle_2}$$

$\hat{J}_+ |2,-1\rangle = \sqrt{6}\,|2,0\rangle$

$\hat{J}_+ |2,-1\rangle = \frac{1}{\sqrt{2}}\ {}^1\hat{J}_+ |1,0\rangle \otimes |1,-1\rangle_2 + \frac{1}{\sqrt{2}}\,|1,0\rangle_1 \otimes {}^2\hat{J}_+ |1,-1\rangle_2$

$+ \frac{1}{\sqrt{2}}\ {}^1\hat{J}_+ |1,-1\rangle \otimes |1,0\rangle_2 + \frac{1}{\sqrt{2}}\,|1,-1\rangle_1 \otimes {}^2J_+ |1,0\rangle_2 =$

$= \frac{1}{\sqrt{2}}\sqrt{2}\ |1,1\rangle_1 \otimes |1,-1\rangle_2 + \frac{1}{\sqrt{2}}\,|1,0\rangle_1 \otimes \sqrt{2}\,|1,0\rangle_2$

$+ \frac{1}{\sqrt{2}}\sqrt{2}\ |1,0\rangle_1 \otimes |1,0\rangle_2 + \frac{1}{\sqrt{2}}\,|1,-1\rangle_1 \otimes \sqrt{2}\,|1,1\rangle_2 =$

$= |1,1\rangle_1 \otimes |1,-1\rangle_2 + |1,-1\rangle_1 \otimes |1,1\rangle_2 + 2\,|1,0\rangle_1 \otimes |1,0\rangle_2$

Άρα

$$\underline{|2,0\rangle = \frac{1}{\sqrt{6}}\left(|1,1\rangle_1 \otimes |1,-1\rangle_2 + |1,-1\rangle_1 \otimes |1,1\rangle_2 + 2|1,0\rangle_1 \otimes |1,0\rangle_2\right)}$$

Τα $|2,1\rangle, |2,2\rangle$ βγαίνουν από συμμετρία.

Το $|1,-1\rangle$ είναι $|1,-1\rangle = a\,|1,0\rangle_1 \otimes |1,-1\rangle_2 + b\,|1,-1\rangle_1 \otimes |1,0\rangle_2$

η συνθήκη $\langle 2,-1|1,-1\rangle = 0 \rightarrow a = -b = \frac{1}{\sqrt{2}}$

άρα $\underline{|1,-1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle_1 \otimes |1,-1\rangle_2 - |1,-1\rangle_1 \otimes |1,0\rangle_2\right)}$

$\hat{J}_+ |1,-1\rangle = \sqrt{2}\,|1,0\rangle$

$$\hat{J}_+|1,-1\rangle = \frac{1}{\sqrt{2}}\ {}^1\hat{J}_+|1,0\rangle_1 \otimes |1,-1\rangle_2 - \frac{1}{\sqrt{2}}|1,-1\rangle_1 \otimes {}^2\hat{J}_+|1,0\rangle_2$$

$$+\frac{1}{\sqrt{2}}|1,0\rangle_1 \otimes {}^2J_+|1,-1\rangle_2 - \frac{1}{\sqrt{2}}\ {}^1J_+|1,-1\rangle_1 \otimes |1,0\rangle_2$$

$$= \frac{1}{\sqrt{2}}\sqrt{2}\,|1,1\rangle_1 \otimes |1,-1\rangle_2 - \frac{1}{\sqrt{2}}|1,-1\rangle_1 \otimes \sqrt{2}\,|1,1\rangle_2$$

$$+\frac{1}{\sqrt{2}}|1,0\rangle \otimes \sqrt{2}|1,0\rangle_2 - \frac{1}{\sqrt{2}}\sqrt{2}\,|1,0\rangle_1 \otimes |1,0\rangle_2 =$$

$$= |1,1\rangle_1 \otimes |1,-1\rangle_2 - |1,-1\rangle_1 \otimes |1,1\rangle_2$$

Άρα $\underline{|1,0\rangle = \frac{1}{\sqrt{2}}\left(|1,1\rangle_1 \otimes |1,-1\rangle_2 - |1,-1\rangle_1 \otimes |1,1\rangle_2\right)}$

Το $|1,1\rangle$ προκύπτει από συμμετρία.

Τέλος το $|0,0\rangle$ είναι της μορφής

$$\alpha|1,1\rangle_1 \otimes |1,-1\rangle_2 + b|1,-1\rangle_1 \otimes |1,1\rangle_2 + c|1,0\rangle_1 \otimes |1,0\rangle_2$$

Ικανοποιεί $\langle 0,0|1,0\rangle = \langle 0,0|2,0\rangle = 0$,

$$\left.\begin{array}{l} \alpha+b+2c=0 \\ a-b=0 \end{array}\right\} \quad a=b=-c \qquad \text{άρα}$$

$$\underline{|0,0\rangle = \frac{1}{\sqrt{3}}\left(|1,1\rangle_1 \otimes |1,-1\rangle_2 + |1,-1\rangle_1 \otimes |1,1\rangle_2 - |1,0\rangle_1 \otimes |1,0\rangle_2\right)}$$

13 Οι δυνατές τιμές του J είναι 2 και 1

$$\underline{|2,-2\rangle = |\tfrac{3}{2},-\tfrac{3}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2}$$

$$\hat{J}_+|2,-2\rangle = 2|2,-1\rangle$$

$$\hat{J}_+|2,-2\rangle = {}^1J_+|\tfrac{3}{2},-\tfrac{3}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2 + |\tfrac{3}{2},-\tfrac{3}{2}\rangle_1 \otimes {}^2J_+|\tfrac{1}{2},-\tfrac{1}{2}\rangle_2$$

$$= \sqrt{3}\,|\tfrac{3}{2},-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2 + |\tfrac{3}{2},-\tfrac{3}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2$$

Άρα $|2,-1\rangle = \frac{\sqrt{3}}{2}|\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},-\frac{1}{2}\rangle_2 + \frac{1}{2}|\frac{3}{2},-\frac{3}{2}\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2$

$\hat{J}_+|2,-1\rangle = \sqrt{6}\,|2,0\rangle$

$\hat{J}_+|2,-1\rangle = \frac{\sqrt{3}}{2}\,\hat{J}_+|\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},-\frac{1}{2}\rangle_2 + \frac{\sqrt{3}}{2}|\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes \hat{J}_+|\frac{1}{2},-\frac{1}{2}\rangle_2$

$+ \frac{1}{2}\hat{J}_+|\frac{3}{2},-\frac{3}{2}\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2 + \frac{1}{2}|\frac{3}{2},-\frac{3}{2}\rangle_1 \otimes \hat{J}_+|\frac{1}{2},\frac{1}{2}\rangle_2 =$

$= \frac{\sqrt{3}}{2}\,2\,|\frac{3}{2},\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},-\frac{1}{2}\rangle_2 + \frac{\sqrt{3}}{2}|\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2 + \frac{1}{2}\sqrt{3}\,|\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2 + 0$

$= \sqrt{3}\,|\frac{3}{2},\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},-\frac{1}{2}\rangle_2 + \sqrt{3}\,|\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2$

Άρα $|2,0\rangle = \frac{1}{\sqrt{2}}\left(|\frac{3}{2},\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},-\frac{1}{2}\rangle_2 + |\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2\right)$

Τα $|2,1\rangle$, $|2,2\rangle$ βγαίνουν από συμμετρία

$|1,-1\rangle = a|\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},-\frac{1}{2}\rangle_2 + b|\frac{3}{2},-\frac{3}{2}\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2$

$\langle 2,-1|1,-1\rangle = 0 \rightarrow \sqrt{3}a + b = 0 \rightarrow b = -\sqrt{3}a$

$|1,-1\rangle = \frac{1}{2}|\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},-\frac{1}{2}\rangle_2 - \frac{\sqrt{3}}{2}|\frac{3}{2},-\frac{3}{2}\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2$

$\hat{J}_+|1,-1\rangle = \sqrt{2}\,|1,0\rangle$

$\hat{J}_+|1,-1\rangle = \frac{1}{2}\hat{J}_+|\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},-\frac{1}{2}\rangle_2 + \frac{1}{2}|\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes \hat{J}_+|\frac{1}{2},-\frac{1}{2}\rangle_2$

$- \frac{\sqrt{3}}{2}\hat{J}_+|\frac{3}{2},-\frac{3}{2}\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2 - \frac{\sqrt{3}}{2}|\frac{3}{2},-\frac{3}{2}\rangle_1 \otimes \hat{J}_+|\frac{1}{2},\frac{1}{2}\rangle_2 =$

$= \frac{1}{2}\cdot 2\,|\frac{3}{2},\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},-\frac{1}{2}\rangle_2 + \frac{1}{2}|\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2 - \frac{\sqrt{3}\sqrt{3}}{2}|\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2 =$

$= |\frac{3}{2},\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},-\frac{1}{2}\rangle_2 - |\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2$

$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|\frac{3}{2},\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},-\frac{1}{2}\rangle_2 - |\frac{3}{2},-\frac{1}{2}\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2\right)$

14 Η $|7,-7\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle = |\psi\rangle$ είναι συμβατή με τη $|\frac{15}{2},-\frac{13}{2}\rangle$ και με τη $|\frac{13}{2},-\frac{13}{2}\rangle$ του σύνθετου συστήματος.

Βρίσκουμε την $|\frac{15}{2},\frac{13}{2}\rangle$ με το γνωστό αλγόριθμο

$$|\tfrac{15}{2},-\tfrac{15}{2}\rangle = |7,-7\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2$$

$$\hat{J}_+|\tfrac{15}{2},-\tfrac{15}{2}\rangle = \sqrt{15}\,|\tfrac{15}{2},-\tfrac{13}{2}\rangle$$

Αλλά $\hat{J}_+|\frac{15}{2},-\frac{15}{2}\rangle = \hat{J}_+|7,-7\rangle_1 \otimes |\frac{1}{2},-\frac{1}{2}\rangle + |7,-7\rangle_1 \otimes \hat{J}_+|\frac{1}{2},-\frac{1}{2}\rangle_2 =$

$$= \sqrt{14}\,|7,-6\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2 + |7,-7\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2$$

Άρα $|\frac{15}{2},-\frac{13}{2}\rangle = \sqrt{\frac{14}{15}}\,|7,-6\rangle_1 \otimes |\frac{1}{2},-\frac{1}{2}\rangle_2 + \frac{1}{\sqrt{15}}|7,-7\rangle_1 \otimes |\frac{1}{2},\frac{1}{2}\rangle_2$

$$\text{Prob}(j=\tfrac{15}{2}) = \left|\langle\psi|\tfrac{15}{2},-\tfrac{13}{2}\rangle\right|^2 = \frac{1}{15}$$

$$\text{Prob}(j=\tfrac{13}{2}) = 1-\text{Prob}(j=\tfrac{15}{2}) = \frac{14}{15}$$

---

15 Καθώς $\hat{J}_3(|j,m\rangle \otimes |j,-m\rangle) = (m-m)\,|j,m\rangle \otimes |j,-m\rangle = 0$

$$\hat{J}_3|\psi\rangle = 0$$

Υπολογίζω $\hat{J}_+|\psi\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} (-1)^{j-m}\left[{}^{1}\hat{J}_+|j,m\rangle \otimes |j,-m\rangle + |j,m\rangle \otimes {}^{2}\hat{J}_+|j,-m\rangle\right]$

$$= \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j-1} (-1)^{j-m}\left(\sqrt{j(j+1)-m(m+1)}\;|j,m+1\rangle \otimes |j,-m\rangle\right)$$

$$+\frac{1}{\sqrt{2j+1}} \sum_{m=-j+1}^{j} (-1)^{j-m}\left(\sqrt{j(j+1)-m(m-1)}\;|j,m\rangle \otimes |j,-m+1\rangle\right)$$

στο δεύτερο όρο θέτω όπου $m \to m+1$, οπότε γίνεται ο πρώτος με συντελεστή $(-1)^{j-m-1} = -(-1)^{j-m}$

Άρα ακυρώνονται και $\hat{J}_+|\psi\rangle = 0$

~~Όμοιως $J_-|\psi\rangle = 0$.~~

Άρα έχει τη μέγιστη τιμή του M για δεδομένο J. Αφού $M=0$, άρα και $J=0$

$\boxed{17}$ Το $\hat{J}_1 \cdot \hat{J}_2$ έχει ιδιοτιμές $\frac{1}{2}\left(J_{o\lambda}(J_{o\lambda}+1) - J_1(J_1+1) - J_2(J_2+1)\right)$

για $J_1 = 1$, $J_2 = \frac{3}{2}$, $\frac{1}{2} \leq J_{o\lambda} \leq \frac{5}{2} \rightarrow J_{o\lambda} = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

Για $J_{o\lambda} = \frac{1}{2}$ έχουμε ιδιοτιμή ενέργειας $\frac{g}{2}\left(\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{3}{2}\left(\frac{3}{2}+1\right) - 1(1+1)\right) = -\frac{5g}{2}$
και εκφυλισμό $(2J_{o\lambda}+1) = 2$

Για $J_{o\lambda} = \frac{3}{2}$ " ιδιοτιμή $\frac{g}{2}\left(\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{3}{2}\left(\frac{3}{2}+1\right) - 1(1+1)\right) = -g$
και εκφυλισμό $(2J_{o\lambda}+1) = 4$

Για $J_{o\lambda} = \frac{5}{2}$ " ιδιοτιμή $\frac{g}{2}\left(\frac{5}{2}\left(\frac{5}{2}+1\right) - \frac{3}{2}\left(\frac{3}{2}+1\right) - 1(1+1)\right) = \frac{3g}{2}$
και εκφυλισμό $(2J_{o\lambda}+1) = 6$