Κεφ. 9.

① Έστω $\hat{\rho}' = \hat{U}\hat{\rho}\hat{U}^\dagger \to Tr\rho'^2 = Tr(\hat{U}\hat{\rho}\hat{U}^\dagger\hat{U}\hat{\rho}\hat{U}^\dagger) = Tr(\hat{U}^\dagger\hat{U}\hat{\rho}^2) = Tr(\hat{\rho}^2)$

Αντίθετα αν $\hat{\rho}' = \sum_n \hat{P}_n\hat{\rho}\hat{P}_n \to Tr\tilde{\rho}^2 = \sum_{n,m} Tr(\hat{P}_n\hat{\rho}\underbrace{\hat{P}_n\hat{P}_m}_{\delta_{nm}\hat{P}_n}\hat{\rho}\hat{P}_m) = \sum_n Tr(\hat{P}_n\hat{\rho}\hat{P}_n\hat{\rho})$

Αν $\hat{\rho} = |\psi\rangle\langle\psi|$, $Tr\hat{\rho}'^2 = \sum_n Tr(P_n|\psi\rangle\langle\psi|P_n|\psi\rangle\langle\psi|) =$

$= \sum_n |\langle\psi|\hat{P}_n|\psi\rangle|^2 = \sum_n p_n^2 < \sum_n p_n = 1 \to$ δεν ισχύει η ισότητα

$p_n = Prob(a_n) = \langle\psi|\hat{P}_n|\psi\rangle$

---

② $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$

Φασματικοί προβολείς του $\hat{\sigma}_1$: $\hat{P}_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$

Φασματικοί προβολείς του $\hat{\sigma}_2$: $Q_\pm = \frac{1}{2}\begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix}$

<u>πρώτη μέτρηση $\hat{\sigma}_1$, δεύτερη $\hat{\sigma}_2$:</u>

$Prob(+1,+1) = \langle\psi|P_+Q_+P_+|\psi\rangle = \frac{1}{16}(-i\ 1)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} =$

$= \frac{1}{16}(-i+1\quad -i+1)\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1+i \\ 1+i \end{pmatrix} = \frac{1}{16}(1-i\quad 1-i)\begin{pmatrix} 2 \\ 2i \end{pmatrix} = \frac{1}{4}$

$Prob(+1,-1) = \langle\psi|\hat{P}_+\hat{Q}_-\hat{P}_+|\psi\rangle = \frac{1}{16}(-i\ 1)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} =$

$= \frac{1}{16}(1-i\quad 1-i)\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1+i \\ 1+i \end{pmatrix} = \frac{1}{16}(1-i\quad 1-i)\begin{pmatrix} 2i \\ 2 \end{pmatrix} = \frac{1}{4}$

$Prob(-1,+1) = \langle\psi|\hat{P}_-\hat{Q}_+\hat{P}_-|\psi\rangle = \frac{1}{16}(-i\ 1)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \dots = \frac{1}{4}$

$Prob(-1,-1) = \langle\psi|\hat{P}_-\hat{Q}_-\hat{P}_-|\psi\rangle = \frac{1}{16}(-i\ 1)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \dots = \frac{1}{4}$

πρώτη μέτρηση $\hat{\sigma}_2$, δεύτερη $\hat{\sigma}_1$

Παρατηρούμε ότι $\hat{Q}_+|\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = 0$, $\hat{Q}_-|\psi\rangle = \dots = |\psi\rangle$

Άρα $Prob(+1,+1) = \langle\psi|Q_+P_+Q_+|\psi\rangle = 0$

$$\text{Prob}(+1,-1) = \langle\varphi|Q_+ P_- Q_+|\varphi\rangle = 0$$

$$\text{Prob}(-1,+1) = \langle\varphi|Q_- P_+ Q_-|\varphi\rangle = \langle\varphi|P_+|\varphi\rangle = \frac{1}{4}(-i\ 1)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}i\\ 1\end{pmatrix} = \frac{1}{2}$$

$$\text{Prob}(-1,-1) = \langle\varphi|Q_- P_- Q_-|\varphi\rangle = \langle\varphi|P_-|\varphi\rangle = \frac{1}{4}(-i\ 1)\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}i\\ 1\end{pmatrix} = \frac{1}{2}$$

---

③ ο προβολικός του σ_1 με ιδιοτιμή $+1$ είναι $P_+ = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$

" σ_2 " $+1$ " $Q_+ = \frac{1}{2}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}$

" σ_3 " $+1$ " $R_+ = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$

$$\text{Prob}(+,+,+) = \langle 0|P_+ Q_+ P_+ Q_+ P_+|0\rangle = \frac{1}{16}(0\ 1)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$= \frac{1}{16}(1\ 1)\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & -i\\ i & 1\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \frac{1}{16}(1+i\ \ 1-i)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1-i\\ 1+i\end{pmatrix} =$$

$$= \frac{1}{16}(1+i\ \ 1-i)\begin{pmatrix}1-i\\ 0\end{pmatrix} = \frac{1}{8}$$

---

④ $\gamma' = \sum_{n,m} Tr(P_n\rho P_n P_m \rho P_m) = \sum_n Tr(P_n \rho P_n \rho)$

Η Cauchy-Schwarz για τελεστές $|Tr(AB)| \leq \sqrt{Tr(A^\dagger A)\,Tr(B^\dagger B)}$

για $A = B = P_n\rho$ παίρνουμε $|Tr(P_n\rho P_n\rho)| \leq Tr(P_n\rho\rho P_n) = Tr(P_n\rho^2)$

άρα $\gamma' = \sum_n Tr(P_n\rho P_n\rho) \leq \sum_n |Tr(P_n\rho P_n\rho)| \leq \sum_n Tr(P_n\rho^2) = Tr\,\rho^2$ (αφού $\sum_n P_n = I$)

Η ισότητα ισχύει όταν $A = A^\dagger \rightarrow \hat{P}_n\rho = \rho\hat{P}_n$ ✓

(c) Με βάρος $\lambda_1 = 0,4$ έχουμε ένα στατιστικό υποσύνολο στο οποίο δε γίνεται πρώτη μέτρηση, άρα η κατάσταση στη δεύτερη μέτρηση είναι η αρχική.

Θα έχουμε $\langle \sigma_3 \rangle_1 = \langle 1|\sigma_3|1\rangle = (1 \ \ 0)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$

Με βάρος $\lambda_2 = 0,6$ έχουμε ένα στατιστικό υποσύνολο, στο οποίο γίνεται πρώτα μέτρηση του $\hat{\sigma}_1$. Άρα η κατάσταση μετά την πρώτη μέτρηση θα είναι $\rho' = \hat{P}_+ \hat{\rho}_0 \hat{P}_+ + \hat{P}_- \hat{\rho}_0 \hat{P}_-$, όπου $P_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}$, $\rho_0 = |1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Άρα $\rho' = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} =$

$= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Άρα $\langle \sigma_3 \rangle_2 = Tr(\rho' \sigma_3) = \frac{1}{2} Tr \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2} Tr \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0$

Παίρνοντας την μίξη των δυο υποσυνόλων

$$\langle \sigma_3 \rangle = \lambda_1 \langle \sigma_3 \rangle_1 + \lambda_2 \langle \sigma_3 \rangle_2 = 0,4 \cdot 1 + 0,6 \cdot 0 = 0,4$$