NOTES ON LIE GROUPS

1. Lecture 1

1.1. **Lie Groups.**

Definition 1.1.1. A *Lie group* G is a C^∞ manifold with a group structure so that the group operations are smooth. More precisely, the maps

$$m : G \times G \to G \qquad \text{(multiplication)}$$
$$\text{inv} : G \to G \qquad \text{(inversion)}$$

are C^∞ maps of manifolds.

Example 1.1.2. Take $G = \mathbb{R}$ with $m(a, b) = a + b$, $\text{inv}(a) = -a$ for all $a, b \in \mathbb{R}$. Then G is an abelian Lie group.

Example 1.1.3. Let V be a finite dimensional vector space over $\mathbb{R}$. Then, V has a canonical manifold structure, and is a group under vector addition. It can be shown that vector addition and negation are smooth, so V is a Lie group.

Example 1.1.4. Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices over $\mathbb{R}$. Define

$$GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det A \neq 0\}$$

Then, $GL(b, \mathbb{R})$ is a group unde the operations $m(A, B) = AB$ and $\text{inv}(A) = A^{-1} = \frac{\text{adj}\, A}{\det A}$ where $\text{adj}\, A$ denotes the adjugate of A. As these operations are smooth on $GL(n, \mathbb{R})$ considered as a submanifold of $\mathbb{R}^{n^2}$, $GL(n, \mathbb{R})$ is a Lie group called the *real general linear group*. Completely analogously, we have the Lie group

$$GL(n, \mathbb{C}) = \{A = M_n(\mathbb{C}) \mid \det A \neq 0\}$$

the *complex general linear group*.

Example 1.1.5. The *orthogonal group* $O(n) = \left\{A \in M_n(\mathbb{R}) \mid AA^T = I\right\}$ is a Lie group as a subgroup and submanifold of $GL(n, \mathbb{R})$.

We shall not prove the following theorem for now, but rather leave it as an advertisement of coming attractions.

Theorem 1.1.6. *(Closed Subgroup Theorem) Let G be a Lie group and $H < G$ a closed subgroup of G. Then, H is a Lie group in the induce topology as an embedded submanifold of G.*

As a direct consequence we have

Corollary 1.1.7. *If G and G' are Lie groups and $\phi : G \to G'$ is a continuous homomorphism, then ϕ is smooth.*

From the Closed Subgroup Theorem we can generate quite a few more examples of Lie groups.

Example 1.1.8. The following groups are Lie groups:
- The *real special linear group* $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) \mid \det A = 1\}$.
- The *complex special linear group* $SL(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) \mid \det A = 1\}$.
- The *special orthogonal group* $SO(n, \mathbb{R}) = SL(n, \mathbb{R}) \cap O(n)$.

- The *unitary group* $U(n) = \{A \in GL(n, \mathbb{C}) \mid AA^* = I\}$ (where A^* denotes the Hermitian transpose of A).
- The *special unitary group* $SU(n) = U(n) \cap SL(n, \mathbb{C})$

Exercise 1.1.9. Prove that each of the groups in Example 1.8 are Lie groups (assuming the Closed Subgroup Theorem).

Example 1.1.10. We now define the Euclidean group of rigid motions, $Euc(n)$. Let $End(V, W)$ denote the vector space of all linear endomorphisms from a vector space V to itself. As a set, we have

$$Euc(n) = \{T \in End(\mathbb{R}^n) \mid \|Tx - Ty\| = \|x - y\| \forall x, y \in R^n\}$$

where $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$. Now, one can check that if $T \in Euc(n)$ and $T(0) = 0$, then $T \in O(n)$. Then, we can write $x \mapsto Tx - T(0) \in O(n)$ and so $T(x) = (T(x) - T(0)) + T(0)$. This shows that $T \in \mathbb{R}^n \times O(n)$.

We can think of $Euc(n)$ as a slightly different set. Write

$$Euc(n) = \left\{ \begin{bmatrix} A & v \\ 0 & 1 \end{bmatrix} \,\middle|\, A \in O(n), v \in \mathbb{R}^n \right\}$$

If we identify $\mathbb{R}^n$ with the set of all vectors of the form $\begin{bmatrix} w \\ 1 \end{bmatrix}$ with $w \in \mathbb{R}^n$, then we have

$$\begin{bmatrix} A & v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ 1 \end{bmatrix} = \begin{bmatrix} Aw + v \\ 1 \end{bmatrix}$$

Exercise 1.1.11. Is $Euc(n) \cong \mathbb{R}^n \times O(n)$ as groups?

1.2. **Lie Algebras.**

Definition 1.2.1. A (real) *Lie algebra* L is a vector space over $\mathbb{R}$ with a bilinear map (called the *Lie bracket*)

$$\begin{aligned} [\cdot, \cdot] : L \times L &\to L \\ (X, Y) &\mapsto [X, Y] \end{aligned}$$

such that for all $X, Y, Z \in L$,

(1) $[X, Y] = -[Y, X]$
(2) $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$

Remark 1.2.2. If we write $\mathrm{ad}(X)Y = [X, Y]$, then 2) reads: $\mathrm{ad}(X)$ is a derivation of $(L, [,])$.

Example 1.2.3. Let $L = M_n(\mathbb{R})$. Then, L is a Lie algebra with the *commutator*; i.e. $[X, Y] = XY - YX$.

Exercise 1.2.4. Prove that $M_n(\mathbb{R})$ is a Lie algebra with the commutator bracket.

2. Lecture 2

2.1. Left Invariant Vector Fields.

Definition 2.1.1. Let G be a Lie group and M a smooth manifold. An *action* of G on M is a smooth map

$$G \times M \to M$$

satisfying

(1) $1_G \cdot x = x$ for each $x \in M$
(2) $g \cdot (g' \cdot x) = (gg') \cdot x$ for each $g, g' \in G, x \in M$.

Example 2.1.2. Any Lie group G acts on itself by left multiplication. If $a \in G$ is fixed, we denote this action by

$$L_a(g) = ag \qquad \forall g \in G$$

G also acts on itself by right multiplication (we denote this by R_a).

Remark 2.1.3. L_a (resp. R_a) is a diffeomorphism for each $a \in G$ since we have a smooth inverse given by

$$L_a^{-1}(g) = a^{-1}g = L_{a^{-1}}(g)$$

for any $g \in G$.

Remark 2.1.4. If G is a Lie group acting on a manifold M and we write $g_M : m \mapsto g \cdot m$, then we have a map

$$\rho : G \to Diff(M)$$
$$g \mapsto g_M$$

for each $g \in G$. Now, $\rho(1_G) = \mathrm{id}_M$ and $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$ so that ρ is a group homomorphism from G to the group of diffeomorphisms of M.

Example 2.1.5. Define a map $L : G \to Diff(G); g \mapsto L_g$ by $L_g(g') = gg'$. Then L_g is a homomorphism for each fixed $g \in G$ and represents the usual left action of G on itself.

Definition 2.1.6. A vector field X on a Lie group G is *left invariant* if

$$(dL_g)(X(x)) = X(L_g(x)) = X(gx)$$

for each $x, g \in G$.

We will now use left invariant vector fields to show that the tangent space of G at the identity, denoted T_1G, is a Lie algebra.

Proposition 2.1.7. *Let G be a Lie group. Then, the vector space of all left invariant vector fields on G is ismomorphic (as a vector space) to T_1G.*

Proof. Since X is left invariant the following diagram commutes

$$\begin{tikzcd} TG \arrow[r, "dL_a"] & TG \\ G \arrow[u, "X"] \arrow[r, "L_a"] & G \arrow[u, "X"'] \end{tikzcd}$$

so that $X(a) = (dL_a)_1(X(1))$ for all $a \in G$. Denote by $\Gamma(TG)^G$ the set of all left invariant vector fields on G. Define a map $\phi : \Gamma(TG)^G \to T_1G$ by $\phi : X \mapsto X(1)$. Then, ϕ is linear and injective since if $X, Y \in \Gamma(TG)^G$ and $\phi(X) = \phi(Y)$

$$X(g) = dL_g(X(1)) = dL_g(Y(1)) = Y(g)$$

for each $g \in G$.

Now, ϕ is also surjective. For $v \in T_1G$, define $X_v \in \Gamma(TG)^G$ by $X_v(a) = (dL_a)_1(v)$ for $a \in G$. We claim that X_v is a left invariant vector field. Now, $X_v : G \to TG$ is a C^∞ map of manifolds since if $f \in C^\infty G$, then for $a \in G$

$$\begin{aligned}(X_v(f))(a) &= (dL_a(v))f \\ &= v(f \circ L_a)\end{aligned}$$

Now, if $x \in G$ we have

$$(f \circ L_a)(x) = (f \circ m)(a, x)$$

which is a smooth map of a, x (here, m is the multiplication map on G). Thus, $v(f \circ L_a)$ is smooth and hence so is X_v.

We now show X_v is left invariant. For $a, g \in G$, we have

$$\begin{aligned}(dL_g)(X_v(a)) &= dL_g((dL_a)_1(v)) \\ &= d(L_g \circ L_a)(v) \\ &= d(L_{ga})(v) \\ &= X_v(ga) \\ &= X_v(L_g(a))\end{aligned}$$

so that X_v is left invariant. Therefore ϕ is onto and $\Gamma(TG)^G \cong T_1G$. □

We now give T_1G a Lie algebra structure by idetifying it with $\Gamma(TG)^G$ with the Lie bracket of vector fields. But, we need to show that $[,]$ is in fact a binary operation on $\Gamma(TG)^G$. Recall if $f : M \to N$ is a smooth map of manifolds and X, Y are vector fields on M and N respectively, we say that X, Y are *f-related* if $df(X(x)) = Y(f(x))$ for every $x \in M$. It is a fact from manifold theory that is X, Y and X', Y' are *f-related*, then so are $[X, Y]$ and $[X', Y']$. But, left invariant vector fields are L_a related for all $a \in G$ by definition. This justifies

Proposition 2.1.8. *The Lie bracket of two left vector fields is a left invariant vector field.*

Thus, we can regard T_1G as a Lie algebra and make the following definition.

Definition 2.1.9. Let G be a Lie group. The *Lie algebra* $\mathfrak{g}$ of G is T_1G with the Lie bracket induced by its identification with $\Gamma(TG)^G$.

Example 2.1.10. Let $G = (\mathbb{R}^n, +)$. What is $\mathfrak{g}$? Notice that for this group, $L_a(x) = a + x$, so that $(dL_a)_0 = id_{T_0\mathbb{R}^n}$. So, $(dL_a)_0(v) = v$ for all $v \in T_0R^n$ and thus $\mathfrak{g} = T_0\mathbb{R}^n \cong \mathbb{R}^n$. So the Lie algebra contains all constant vector fields, and the Lie bracket is identically 0.

Example 2.1.11. Consider the Lie group $GL(n, \mathbb{R})$. We have $T_IGL(n, \mathbb{R}) = M_n(\mathbb{R})$, the set of all $n \times n$ real matrices. For any $A, B \in M_n(\mathbb{R})$, the Lie bracket is the commutator; that is,

$$[A, B] = AB - BA$$

To prove this, we compute X_A, the left invariant vector field associated with the matrix $A \in T_1 GL(n, \mathbb{R})$. Now, on $M_n(\mathbb{R})$, we have global coordinate maps given by $x_{ij}(A) = A_{ij}$, the ijth entry of the matrix B. So, for $g \in GL(n, \mathbb{R})$, $(X_A(x_{ij}))(g) = X_A(I)(x_{ij} \circ L_g)$. Also, if $h \in GL(n, \mathbb{R})$,

$$\begin{aligned} (x_{ij} \circ L_g)(h) &= x_{ij}(gh) \\ &= \sum_k g_{ik} h_{kj} \\ &= \sum_k g_{ik} x_{kj}(h) \end{aligned}$$

which implies that $x_{ij} \circ L_g = \sum_k g_{ik} x_{kj}$.

Now, if $f \in C^\infty(GL(n, \mathbb{R}))$, $X_A(I)f = \frac{d}{dt}\big|_{t=0} f(I + tA)$ so that

$$\begin{aligned} X_A(I)x_{ij} &= \frac{d}{dt}\bigg|_{t=0} x_{ij}(I + tA) \\ &= A_{ij} \end{aligned}$$

Putting these remarks together, we see that $X_A(x_{ij} \circ L_g) = \sum_k g_{ik} A_{kj} = \sum_k x_{ik}(g) A_{kj}$.

We are now in a position to calculate the Lie bracket of the left invariant vector fields associated with elements of $M_n(\mathbb{R})$:

$$\begin{aligned} ([X_A, X_B](I))_{ij} &= [X_A, X_B](I)x_{ij} \\ &= X_A X_B(x_{ij}) - X_B X_A(x_{ij}) \\ &= \Big(X_A\Big(\sum_k B_{kj} x_{ik}\Big) - X_B\Big(\sum_k A_{kj} x_{ik}\Big)\Big)(I) \\ &= \Big(\sum_{k,l} B_{kj} x_{il} A_{lk} - A_{kj} x_{il} B_{lk}\Big)(I) \\ &= \sum_{k,l} B_{kj} \delta_{il} A_{lk} - A_{kj} \delta_{il} B_{lk} \\ &= \sum_k A_{ik} B_{kj} - \sum_k B_{ik} A_{kj} \\ &= (AB - BA)_{ij} \end{aligned}$$

So, $[A, B] = AB - BA$.

3. Lecture 3

3.1. Lie Group Homomorphisms.

Definition 3.1.1. Let G and H be Lie groups. A map $\rho : G \to H$ is a *Lie group homomorphism* if

(1) ρ is a C^∞ map of manifolds and
(2) ρ is a group homomorphism.

Furthermore, we say ρ is a *Lie group isomorphism* if it is a group isomorphism and a diffeomorphism.

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, a *Lie algebra homomorphism* $\tau : \mathfrak{g} \to \mathfrak{h}$ is a map such that

(1) τ is linear and
(2) $\tau([X, Y]) = [\tau(X), \tau(Y)]$ for all $X, Y \in \mathfrak{g}$.

Now, suppose V is an n-dimensional vector space over $\mathbb{R}$. We define

$$GL(V) = \{A : V \to V \mid A \text{ is a linear isomorphism}\}$$

Since $V \cong \mathbb{R}^n$, $GL(V) \cong GL(n, \mathbb{R})$. Note that $GL(V) \subseteq \mathrm{Hom}(V, V)$ as an open subset.

Definition 3.1.2. A (real) representation of a Lie group G is a Lie group homomorphism $\rho : G \to GL(V)$.

We may similarly define $GL(W)$ for a complex vector space W and thus the notion of a complex representation.

There are two basic problems in Lie Theory:

(1) classify all Lie groups (and Lie algebras),
(2) classify all representations of Lie groups.

One step in this direction is the association between Lie group homomorphisms and homomorphisms of Lie algebras.

Theorem 3.1.3. *Suppose $\rho : G \to H$ is a Lie group homomorphism. Write $d\rho_1 = \delta\rho$. Then, $\delta\rho : T_1 G \to T_1 H$ is a Lie algebra homomorphism.*

Proof. It is enough to show that any two left invariant vector fields on G and H are ρ-related. So, let $X \in \Gamma(TG)^G$ and $\tilde{X} \in \Gamma(TH)^H$. Then, for each $a, g \in G$, we have

$$\begin{aligned} (\rho \circ L_a)(g) &= \rho(ag) \\ &= \rho(a)\rho(g) \\ &= (L_{\rho(a)} \circ \rho)(g) \end{aligned}$$

so that $\rho \circ L_a = L_{\rho(a)} \circ \rho$. Now,

$$\begin{aligned} d\rho_a(X(a)) &= d\rho_a(L_a(X(1))) \\ &= d(\rho \circ L_a)(X(1)) \\ &= d(L_{\rho(a)} \circ \rho)(X(1)) \\ &= dL_{\rho(a)}(d\rho(X(1))) \\ &= dL_{\rho(a)}(\delta\rho(X(1))) \\ &= dL_{\rho(a)}(\tilde{X}(1)) \\ &= \tilde{X}(\rho(a)) \quad \text{(since } \tilde{X} \text{ is left invariant)} \end{aligned}$$

and thus, X and $\tilde{X}$ are ρ-related. □

3.2. **Lie Subgroups and Lie Subalgebras.**

Definition 3.2.1. Let G be a Lie group. A subset H of G is a *Lie subgroup* if

(1) H is an abstract subgroup of G,
(2) H is a Lie group and
(3) the inclusion $\iota : H \hookrightarrow G$ is an immersion.

A linear subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a *Lie subalgebra* if $\mathfrak{h}$ is closed under the Lie bracket in $\mathfrak{g}$.

Example 3.2.2. Let $\mathfrak{g}$ be a Lie algebra, and $v \in \mathfrak{g}$ a nonzero vector. Then $\mathbb{R}v$, the span of v, is a Lie subalgebra of $\mathfrak{g}$.

Example 3.2.3. Let $H \hookrightarrow G$ be a Lie subgroup. By Theorem 3.2, $T_1H \hookrightarrow T_1G$ is a Lie subalgebra.

Example 3.2.4. Consider the Lie group $Gl(2, \mathbb{R})$. The subgroup

$$SO(2) = \left\{ \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \mid t \in \mathbb{R} \right\}$$

is a Lie subgroup. We will compute the Lie algebra of $SO(2)$. By Example 4.2, it will be sufficient to find a nonzero vector in $T_1SO(2)$ since this Lie algebra is of codimension 1 in $T_1GL(n, \mathbb{R})$. Now,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} &= \left.\begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix}\right|_{t=0} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

Thus, the Lie algebra $\mathfrak{so}(2)$ of $SO(2)$ is

$$\mathfrak{so}(2) = \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}$$

Example 3.2.5. Consider the Lie subgroup $O(n)$ of $GL(n,\mathbb{R})$. We compute $\mathfrak{o}(n)$, the Lie algebra of $O(n)$. Let $A(t)$ be a path in $O(n)$ with $A(0) = I$. Then, since $BB^T = I$ for all $B \in O(n)$, we have $A(t)A(t)^T = I$ for every t. Thus,

$$\begin{aligned} 0 = \frac{d}{dt}\bigg|_{t=0} I &= \frac{d}{dt}\bigg|_{t=0} A(t)A(t)^T \\ &= \Big(\frac{d}{dt}\bigg|_{t=0} A(t)\Big)A(0)^T + A(0)\Big(\frac{d}{dt}\bigg|_{t=0} A(t)^T\Big) \\ &= A'(0) + A'(0)^T \end{aligned}$$

Thus, we have $\mathfrak{o}(n) \subseteq S$ where

$$S = \left\{X \in M_n(\mathbb{R}) \mid X + X^T = 0\right\}$$

To prove equality, we proceed by dimension count. Now, $\dim S = \frac{n^2-n}{2}$ (the easiest way to see this is to write down the form of a general element of S and determine where to place 1's). View $I \in Sym^2(\mathbb{R}^n)$, the set of all symmetric $n \times n$ real matrices. Then, I is a regular value of the map

$$A \mapsto AA^T$$

where $A \in GL(n\mathbb{R})$. Thus,

$$\begin{aligned} \dim \mathfrak{o}(n) &= \dim GL(n,\mathbb{R}) - \dim Sym^2(\mathbb{R}^n) \\ &= n^2 - (\frac{n^2-n}{2} + n) \\ &= \frac{n^2-n}{2} \end{aligned}$$

We can conclude, therefore, that $\mathfrak{o}(n) = S$.

We conclude this section by discussing the induced maps δm and $\delta\mathrm{inv}$ on the Lie algebra of a Lie group G.

Proposition 3.2.6. *Let G be a Lie group. Then, for all $X, Y \in \mathfrak{g}$,*

(1) $\delta m(X,Y) = X + Y$.
(2) $\delta\mathrm{inv}(X) = -X$.

Proof. 1. First note that since $\delta m = (dm)_1$ is linear, it is enough to prove that $(dm)_1(X,0) = X$. So, let $\gamma : (a,b) \to G$ be a curve with $\gamma(0) = 1$ and $\frac{d}{dt}\gamma(t) = X$. Then,

$$\begin{aligned} \delta m(X,0) &= (dm)_1(X,0) \\ &= \frac{d}{dt}\bigg|_{t=0} m(\gamma(t),1) \\ &= \frac{d}{dt}\bigg|_{t=0} \gamma(t) \\ &= X \end{aligned}$$

2. Now, $m(\gamma(t), inv(\gamma(t))) = 1$ for each $t \in (a,b)$. Consider $F : G \to G$ defined by $F(g) = gg^{-1}$. Denote by Δ the diagonal map $\Delta(g) = (g,g)$ for each $G \in G$. Then, $F =$

$m \circ (\mathrm{id}_G \times \mathrm{inv}) \circ \Delta$. Thus,

$$\begin{aligned} 0 = (dF)_1(X) &= ((dm)_1 \circ (d\,\mathrm{id}_G)_1 \times (d\,\mathrm{inv})_1 \circ (d\Delta)_1)(X) \\ &= X + (d\,\mathrm{inv})_1(X) \end{aligned}$$

and so $\delta\mathrm{inv}(X) = -X$. $\square$

4. Lecture 4

4.1. Topological Groups.

Definition 4.1.1. A *topological group* G is a topological space which is a group and has the properties that the group operations are continuous.

Lemma 4.1.2. *Let G be a connected topological group. Suppose H is an abstract open subgroup of G. Then $H = G$.*

Proof. For any $a \in G$, $L_a : G \to G$ given by $g \mapsto ag$ is a homeomorphism. Thus, for each $a \in G$, $aH \subseteq G$ is open. Since the cosets partition G, and G is connected, we must have $|G/H| = 1$. $\square$

Lemma 4.1.3. *Let G be a connected topological group, $U \subseteq G$ a neigborhood of 1. Then U generates G.*

Proof. For a subset $W \subseteq G$, write $W^{-1} = \{g^{-1} \in G \mid g \in W\}$. Also, if k is a positive integer, we set $W^k = \{a_1 \dots a_k \mid a_i \in W\}$. Let U be as above, and $V = U \cap U^{-1}$. Then, V is open and $v \in V$ implies that $v^{-1} \in V$. Let $H = \cup_{n=1}^{\infty} V^n$. Then, H is a subgroup and we claim that H is open. Notice that H is precisely the subgroup generated by U, so if we prove that H is open, then $H = G$ and the Lemma is proved.

If V^k is open, then $V^{k+1} = \cup_{a \in V}(aV^k)$ is open since left multiplication is a homeomorhism. By induction, V^n is open for every n. Thus H is open. $\square$

We will use these results to prove that Lie subalgebras correspond to connected Lie subgroups. But, first, we'll need to develop some more terminology and recall some results from Differential Geometry.

Definition 4.1.4. A *d-dimensional distribution* $\mathcal{D}$ on a manifold M is a subbundle of TM of rank d.

Question: Given a distribution $\mathcal{D} \subseteq TM$, does there exist for each $x \in M$ an immersed submanifold $\mathcal{L}(x)$ of M such that $T_y\mathcal{L}(x) = \mathcal{D}_y$ for every $y \in \mathcal{L}(x)$? A necessary condition for this question to be answered in the affirmative is: If $X, Y \in \Gamma(\mathcal{D})$, then $[X, Y] \in \Gamma(\mathcal{D})$.

Definition 4.1.5. A distribution $\mathcal{D}$ on a manifold M is *integrable* (or *involutive*) if for every $X, Y \in \Gamma(\mathcal{D})$, $[X, Y] \in \Gamma(\mathcal{D})$. An immersed submanifold $\mathcal{L} \subseteq M$ is an *integral manifold of* $\mathcal{D}$ if $T_x\mathcal{L} = \mathcal{D}_x$ for every $x \in \mathcal{L}$.

We'll get some mileage out of the following theorem and proposition for which we omit the proofs.

Theorem 4.1.6. *(Frobenius) Let $\mathcal{D}$ be a d-dimensional integrable distribution on a manifold M. Then, for all $x \in M$, there exists a unique maximal, connected,immersed integral submanifold $\mathcal{L}(x)$ of $\mathcal{D}$ passing through x.*

Proposition 4.1.7. *Suppose $\mathcal{D} \subseteq TM$ is an integrable distribution and $L \subseteq M$ an immersed submanifold such that $T_yL = \mathcal{D}_y$ for every $y \in L$. Suppose $f : N \to M$ is a smooth map of manifolds and $F(N) \subseteq L$. Then, $f : N \to L$ is C^∞.*

Assuming both of these results, we'll prove

Theorem 4.1.8. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra. Then, there exists a unique connected Lie subgroup H of G with $T_1 H = \mathfrak{h}$.*

Proof. Consider $\mathcal{D} \subseteq TG$ given by $\mathcal{D}_a = dL_a(\mathfrak{h})$ for $a \in G$. Then, $\mathcal{D}$ is a distribution. We claim $\mathcal{D}$ is integrable. To prove this, let $v_1, \ldots, v_k$ be a basis of $\mathfrak{h}$. Let $V_1, \ldots, V_k$ be the corresponding left invariant vector fields on G. Then, $\{V_1(g), \ldots, V_k(g)\}$ is a basis of $\mathcal{D}_g$. Also, we have

$$[V_i(g), V_j(g)] = dL_g([V_i, V_j](g))$$

since the bracket of left invariant vector fields is left invariant.

Now, for arbitrary sections X, Y of $\mathcal{D}$, write them (locally) as

$$X = \sum_i x_i V_i$$
$$Y = \sum_j y_j V_j$$

where $x_i, y_j \in C^\infty(G)$ for each i, j. So,

$$[X, Y] \;=\; \sum_{i,j} x_i V_i(y_j) V_j + \sum i, j x_i y_j [V_i, V_j] - \sum_{i,j} V_j(x_i) y_j V_i$$

each term of which is in $\Gamma(\mathcal{D})$, and hence $[X, Y] \in \Gamma(\mathcal{D})$.

If we now apply the Frobenius Theorem, we get an immersed, connected, maximal submanifold H of G such that $1 \in H$ and $T_1 H = \mathfrak{h}$. The claim is that H is a Lie subgroup of G. To show H is a subgroup, fix some $x \in H$. Consider $x^{-1}H = L_{x^{-1}}(H)$. Then, $1 = xx^{-1} \in x^{-1}H$ and for all $a \in G$, we have

$$\begin{aligned} T_{x^{-1}a}(x^{-1}H) &= dL_{x^{-1}}(T_a H) \\ &= dL_{x^{-1}}(dL_a \mathfrak{h}) \\ &= dL_{x^{-1}a}\mathfrak{h} \\ &= \mathcal{D}_{x^{-1}a} \end{aligned}$$

So, $x^{-1}H$ is tangent to $\mathcal{D}$. Since H is connected, $x^{-1}H$ is connected, and by maximality and uniqueness of H, we have $x^{-1}H \subset H$. Therefore, H is a subgroup of G.

Finally, we need to show that $m|_{H \times H}$ and $\operatorname{inv}|_H$ are C^∞. But, $m : H \times H \to G$ is C^∞ and $m(H \times H) \subseteq H$. By Proposition 5.7, multiplication is a smooth binary operation on H. Similarly, inv is smooth on H and thus H is a Lie subgroup. $\square$

5. Lecture 5

5.1. **Simply Connected Lie Groups.** Recall now Theorem 3.2, which states that if $\rho : G \to H$ is a Lie group morphism, then $\delta\rho : \mathfrak{g} \to \mathfrak{h}$ is a map of Lie algebras. Is the conerse true? That is, if G, H are Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively and $\tau : \mathfrak{g} \to \mathfrak{h}$ is a map of Lie algebras, does there exist a Lie group morphism $\rho : G \to H$ with $\delta\rho = \tau$? Unfortunately, the answer is "not always". We can answer affirmitively when G is connected and simply connected however. Let's recall a couple of definitions from basic topology.

Definition 5.1.1. A connected topological space T is *simply connected* if T is arcwise connected and every pointed map $f : (S^1, 1) \to (T, *)$ is homotopically trivial.

Definition 5.1.2. A continuous map $p : X \to Y$ is a *covering map* if for each $y \in Y$, there exists a neighborhood U of y such that $p^{-1}U = \coprod U_\alpha$ where $U_\alpha \subset X$ is open for each α, and $p|_{U_\alpha}$ is a homeomorhism.

Lemma 5.1.3. *Let $\phi : A \to B$ be a map of Lie groups with $(d\phi)_1 : \mathfrak{a} \to \mathfrak{b}$ an isomorphism. Then,*

a) *ϕ is a local diffeomorphism and*
b) *If B is connected, ϕ is onto.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "L_a"'] & B \arrow[d, "L_{\phi(a)}"] \\ A \arrow[r, "\phi"'] & B \end{tikzcd}$$

which can also be viewed element-wise as

$$\begin{tikzcd} 1_A \arrow[r, mapsto, "\phi"] \arrow[d, mapsto, "L_a"'] & 1_B \arrow[d, mapsto, "L_{\phi(a)}"] \\ a \arrow[r, mapsto, "\phi"'] & \phi(a) \end{tikzcd}$$

From this we can conclude that

$$(d\phi)_1 = (dL_{\phi(a)})^{-1}_{\phi(a)} \circ (d\phi)_a \circ (dL_a)_1$$

Now, since $(d\phi)_1$ is an isomorphism, $(d\phi)_a$ is an isomorphism for every $a \in A$. Invoking the Inverse Function Theorem, we see then that ϕ is a local diffeomorphism. In particular, ϕ is an open map, so $\phi(A)$ is an open subgroup of B. Now if B is connected, Lemma 5.2 yields $\phi(A) = B$ and thus ϕ is onto. □

Lemma 5.1.4. *Let $\phi : A \to B$ be a surjective Lie group map that is a local diffeomorphism. Then, ϕ is a covering map.*

Proof. Let $\Lambda = \ker \phi$. Since ϕ is a local diffeomorphism, there exists an open neighborhood U of 1_A such that $\phi|_U$ is injective and so $U \cap \Lambda = 1_A$. Since A is a Lie group, the multiplication map $m : A \times A \to A$ is continuous and so there exists an open neighborhood V of 1_A such that

$m(V \times V) \subseteq U$. That is, $VV \subseteq U$. Let $W = V \cap V^{-1}$, then $WW^{-1} \subseteq U$. We claim that for every $\lambda, \lambda' \in \Lambda$, $\lambda W \cap \lambda' W = \emptyset$ if and only if $\lambda \neq \lambda'$.

To prove this claim, suppose $\lambda W \cap \lambda' W = \emptyset$ for some $\lambda, \lambda' \in \Lambda$. Then, there exists $w, w' \in W$ so that $\lambda w = \lambda' w'$. But then, $(\lambda')^{-1}\lambda = w'(w)^{-1}$ and so $(\lambda')^{-1}\lambda \in \Lambda \cap U$ and thus $(\lambda')^{-1}\lambda = 1$.

Now, what we've just proved is that $\ker \phi = \Lambda$ is discrete, so

$$\phi^{-1}(\phi(W)) = \Lambda W = \coprod_{\lambda \in \Lambda} \lambda W$$

and we have a homeomorphism $\phi|_{\lambda W} : \lambda w \mapsto \phi(w)$. Thus, for each $b \in B$ and $a \in \phi^{-1}(b)$, $\phi^{-1}(b\phi(W)) = \coprod_{\lambda \in \Lambda} a\lambda W$. Therefore, the fibers of ϕ are discrete and $\phi : A \to B$ is a covering map. $\square$

We also have the following fact from topology, stated without proof:

Lemma 5.1.5. *Let $\phi : A \to B$ be a covering map of topological spaces with B simply connected. Then ϕ is a homeomorphism.*

We are now in a position to answer the question posed in the beginning of this section.

Theorem 5.1.6. *Let G be a connected and simply connected Lie group with lie algebra $\mathfrak{g}$ and H a Lie group with Lie algebra $\mathfrak{h}$. Given a Lie algebra morphism $\tau : \mathfrak{g} \to \mathfrak{h}$, there exists a unique Lie group morphism $\rho : G \to H$ such that $\delta\rho = \tau$.*

Proof. First note that

$$\operatorname{graph}(\tau) = \{(X, \tau(X)) \in \mathfrak{g} \times \mathfrak{h} \mid X \in \mathfrak{g}\}$$

is a subalgebra of $\mathfrak{g} \times \mathfrak{h}$ since

$$\begin{aligned} [(X_1, \tau(X_1)), (X_2, \tau(X_2))] &= ([X_1, X_2], [\tau(X_1), \tau(X_2)]) \\ &= ([X_1, X_2], \tau[X_1, X_2]) \end{aligned}$$

So, by Theorem 5.8, there exists a connected Lie subgroup Γ of $G \times H$ so that $T_1\Gamma = \operatorname{graph}(\tau)$. The claim is that Γ is the graph of the Lie group morphism ρ we are trying to construct, and hence it is sufficient to show that Γ is in fact a graph. Formally, if Γ is a graph, then we have

$$\begin{tikzcd} & G \times H \arrow[ddl, "\pi_1"'] \arrow[ddr, "\pi_2"] & \\ & \Gamma \arrow[u, hook] \arrow[dl, "\pi_1|_\Gamma"] \arrow[dr, "\pi_2|_\Gamma"'] & \\ G & & H \end{tikzcd}$$

and can simply define $\rho = \pi_2 \circ (\pi_1|_\Gamma)^{-1}$.

Now, $(d\pi_1|_\Gamma)_{(1,1)} : \operatorname{graph}(\tau) \to \mathfrak{g}$ is an isomorphism, so $\pi_1|_\Gamma$ is a local diffeomorphism by Lemma 6.3, and evidently $\pi_1|_\Gamma : \Gamma \to G$ is a surjective group homomorphism. By Lemma 6.4, $\pi_1|_\Gamma$ is a covering map. Since G is simply connected, $\pi_1|_\Gamma$ is a homeomorphism.

Finally, define $\rho : G \to H$ by $\rho = \pi_2 \circ (\pi_1|_\Gamma)^{-1}$. Since Γ is a subgroup, ρ is a homomorphism and $\operatorname{graph}(\rho) = \Gamma$. This gives us the Lie group morphism we want.

We now have to establish the uniqueness of such a Lie group homomorphism. Suppose $\tilde{\rho} : G \to H$ is another such Lie group morphism, then

$$T_{(1,1)}(\mathrm{graph}(\tilde{\rho})) = \mathrm{graph}(\tau) = T_{(1,1)}(\mathrm{graph}(\rho))$$

Since $\mathrm{graph}(\tilde{\rho})$ and $\mathrm{graph}(\rho)$ are connected subgroups of $G \times H$ with the same Lie algebra, they must be identical. Therefore, $\tilde{\rho} = \rho$. $\square$

6. Lecture 6

6.1. The Exponential Map. Given a Lie group G and its Lie algebra $\mathfrak{g}$, we would like to construct an *exponential map* from $\mathfrak{g} \to G$, which will help to give some information about the structure of $\mathfrak{g}$.

Proposition 6.1.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then, for each $X \in \mathfrak{g}$, there exists a map $\gamma_X : \mathbb{R} \to G$ satisfying*

(1) $\gamma_X(0) = 1_G$,
(2) $\left.\frac{d}{dt}\right|_{t=0} \gamma_X(t) = X$ *and*
(3) $\gamma_X(s+t) = \gamma_X(s)\gamma_X(t)$.

Proof. Consider the Lie algeba map $\tau : \mathbb{R} \to \mathfrak{g}$ defined by $\tau : t \mapsto tX$ for all $X \in \mathfrak{g}$. Now, $\mathbb{R}$ is connected and simply connected, so by Theorem 6.6. there exists a unique Lie group map $\gamma_X : \mathbb{R} \to G$ such that $(d\gamma_X)_0 = \tau$; which is to say

$$\left.\frac{d}{dt}\right|_{t=0} \gamma_X(t) = X$$

□

This motivates the following definition:

Definition 6.1.2. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Define the *exponential map* $\exp : \mathfrak{g} \to G$ by $\exp(X) = \gamma_X(1)$.

Lemma 6.1.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $X \in \mathfrak{g}$. Write $\tilde{X}$ for the left invariant vector field on $\mathfrak{g}$ with $\tilde{X}(1) = X$. Then,*

$$\phi_t(a) = a\gamma_X(t)$$

is the flow of $\tilde{X}$. In particular, $\tilde{X}$ is complete; i.e. the flow exists for all $t \in \mathbb{R}$.

Proof. For $a \in G$, we have

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=s} a\gamma_X(t) &= (dL_a)_{\gamma_X(s)}\Big(\left.\frac{d}{dt}\right|_{t=s} \gamma_X(t)\Big) \\
&= (dL_a)_{\gamma_X(s)}\Big(\left.\frac{d}{dt}\right|_{t=0} \gamma_X(t+s)\Big) \\
&= (dL_a)_{\gamma_X(s)}\Big(\left.\frac{d}{dt}\right|_{t=0} \gamma_X(s)\gamma_X(t)\Big) \\
&= (dL_a)_{\gamma_X(s)}\Big(\left.\frac{d}{dt}\right|_{t=0} L_{\gamma_X(s)}(\gamma_X(t))\Big) \\
&= (dL_{a\gamma_X(s)})_1\Big(\left.\frac{d}{dt}\right|_{t=0} \gamma_X(t)\Big) \\
&= (dL_{a\gamma_X(s)})_1(X) \\
&= \tilde{X}(a\gamma_X(s)) \qquad \text{(since } \tilde{X} \text{ is left-invariant)}
\end{aligned}$$

So, $a\gamma_X(t)$ is the flow of $\tilde{X}$ and exists for all t. □

Lemma 6.1.4. *The exponential map is C^∞.*

Proof. Consider the vector field V on $G \times \mathfrak{g}$ given by

$$V(a, X) = (dL_a(X), 0)$$

Then, $V \in C^\infty(G, \mathfrak{g})$ and the claim is that the flow of V is given by $\psi_t(g, X) = (g\gamma_X(t), X)$. To prove this claim, consider the following:

$$\left.\frac{d}{dt}\right|_{t=s} (g\gamma_X(t), X) = (dL_{g\gamma_X(s)}(X), 0)$$
$$= V(g\gamma_X(s), X)$$

from which we can conclude that γ_X depends smoothly on X

Now, we note that the map $\phi : \mathbb{R} \times G \times \mathfrak{g}$ defined by $\phi(t, a, X) = (a\gamma_X(t), X)$ is smooth. Thus, if $\pi_1 : G \times \mathfrak{g} \to G$ is projection on the first factor, $(\pi_1 \circ)(1_G, X) = \gamma_X(1) = \exp(X)$ is C^∞. □

Lemma 6.1.5. *For all $X \in \mathfrak{g}$ and for all $t \in R$, $\gamma_{tX}(1) = \gamma_X(t)$.*

Proof. The intent is to prove that for all $s \in \mathbb{R}$, $\gamma tX(s) = \gamma_X(ts)$. Now, $s \mapsto \gamma_{tX}(s)$ is the integral curve of the left invariant vector field $\tilde{tX}$ through 1_G. But, $\tilde{tX} = t\tilde{X}$, so if we prove that $\gamma_X(ts)$ is an integral curve of $t\tilde{X}$ through 1_G, by uniqueness the Lemma will be established.

To prove this, first let $\sigma(s) = \gamma_X(ts)$. Then, $\sigma(0) = \gamma_X(0) = 1_G$. We also have

$$\begin{aligned}
\frac{d}{ds}\sigma(s) &= \frac{d}{ds}\gamma_X(ts) \\
&= d\left.\frac{d}{du}\right|_{u=ts} \gamma_X(u) \\
&= t\tilde{X}(\gamma_X(ts)) \\
&= t\tilde{X}(\sigma(s))
\end{aligned}$$

So, $\sigma(s)$ is also an integral curve of $t\tilde{X}$ through 1_G. Thus, $\gamma_{tX}(s) = \gamma_X(ts)$ and, in particular, when $s = 1$ we have $\gamma_{tX}(1) = \gamma_X(t)$. □

We'll now use this to prove a rather nice fact about the exponential map.

Proposition 6.1.6. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Identify both $T_0\mathfrak{g}$ and T_1G with $\mathfrak{g}$. Then, $(d\exp)_0 : T_0\mathfrak{g} \to T_1G$ is the identity map.*

Proof. Using the result established in Lemma 7.5, we have

$$\begin{aligned}
(d\exp)_0(X) &= \left.\frac{d}{dt}\right|_{t=0} \exp(0 + tX) \\
&= \left.\frac{d}{dt}\right|_{t=0} \gamma_{tX}(1) \\
&= \left.\frac{d}{dt}\right|_{t=0} \gamma_X(t) \\
&= X
\end{aligned}$$

□

Corollary 6.1.7. a) *For all* $t_1, t_2 \in \mathbb{R}$, $\exp((t_1 + t_2)X) = \exp t_1 X + \exp t_2 X$.
b) $\exp(-tX) = (\exp(tX))^{-1}$.

7. Lecture 7

7.1. **Naturality of** exp**.** In this section, we reveal a property that will be used liberally in discussions to come and provides an important relationship between morphisms of Lie groups and morphisms of Lie algebras.

Theorem 7.1.1. *Let $\phi : H \to G$ be a morphism of Lie groups. Then, the following diagram commutes:*

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "\delta\phi"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ H \arrow[r, "\phi"'] & G \end{tikzcd}$$

That is to say, exp *is natural.*

Proof. Fix $X \in \mathfrak{g}$. Consider the curves

$$\begin{aligned} \sigma(t) &= \phi(\exp(tX)) \\ \tau(t) &= \exp(\delta\phi(tX)) \end{aligned}$$

Now, $\sigma, \tau : \mathbb{R} \to G$ are Lie group homomorphisms with $\sigma(0) = \tau(0) = 1$. So,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \sigma(t) &= (d\phi)_1 \left(\left.\frac{d}{dt}\right|_{t=0} \exp(tX) \right) \\ &= (\delta\phi)(X) \\ &= \left.\frac{d}{dt}\right|_{t=0} \tau(t). \end{aligned}$$

So, $\sigma(t) = \tau(t)$ for all t. □

Corollary 7.1.2. *Let $H \subseteq G$ be a Lie subgroup of a Lie group G. Then, for all $X \in \mathfrak{h}$,*

$$\exp_G(X) = \exp_H(X).$$

In particular, $X \in \mathfrak{h}$ if and only if $\exp(tX) \in \mathfrak{h}$ *for all t.*

Theorem 7.1.3. *Every connected Lie group G is a quotient $\tilde{G}/N$ where $\tilde{G}$ is a simply connected Lie group of the same dimension as G and N is a central discrete normal subgroup of $\tilde{G}$. Both $\tilde{G}$ and N are unique up to isomorphism.*

Proof. Recall that the *universal covering space* of a topological space is the unique (up to deck isomorphism) simply connected covering space. We will use, but not prove, the fact that every connected Lie group has a universal covering space.

Let $\tilde{G}$ be the universal covering space of G, and denote by p the covering map. Let $\tilde{1} = p^{-1}(1)$. Denote by $\tilde{m}$ the lift of the multiplication map $m : G \times G \to G$ to $\tilde{G}$ uniquely determined by $\tilde{m}(\tilde{1}, \tilde{1}) = \tilde{1}$. Similarly, inv : $G \to G$ lifts to $\tilde{G}$ as well. Thus, $\tilde{G}$ is a group. Also, p is a Lie group homomorphism by definition of $\tilde{m}$:

$$p(\tilde{m}(a, b)) = m(p(a), p(b)).$$

Now, kernels of covering maps are discrete, and evidently, $G \cong \tilde{G}/\ker p$.

It remains to prove that $N = \ker p$ is central, that is for all $g \in \tilde{G}$ and $n \in N$, $gng^{-1} = n$. Fix $n \in N$. Define $\phi : \tilde{G} \to \tilde{G}$ by $\phi(g) = gng^{-1}$. Since N is normal $\phi(G) \subseteq N$. Now, $\tilde{G}$ is connected, so $\phi(G)$ is connected since ϕ is continuous. But N is discrete, so $\phi(G)$ is a single point. We have $\phi(1) = n$ and hence $\phi(G) = n$. Therefore, N is central. $\square$

Corollary 7.1.4. *If G is a connected topological group, then the fundamental group $\pi_1(G)$ is abelian.*

Our last result for this section deals with subgroups of Lie groups.

Proposition 7.1.5. *Lie groups have no small subgroups, i.e. if G is a Lie group, then there exists a neighborhood U of the identity so that for all $g \in U$ there exists a positive integer N (depending on g) having the property that $g^N \notin U$.*

Proof. recall first that $(d\exp)_0 : \mathfrak{g} \to \mathfrak{g}$ is the identity. By the Inverse Function Theorem, there exists neighborhoods V' of 0 in $\mathfrak{g}$ and U' of $1 \in G$ so that $\exp : V' \to U'$ is a diffeomorphism. Let $U = \exp(\frac{1}{2}V')$. We claim that U is the desired neighborhood.

If $g \in U$, then $g = \exp(\frac{1}{2}v)$ for some $v \in V'$. Then,

$$g^n = \exp\left(\frac{1}{2}v\right)\dots\exp\left(\frac{1}{2}v\right) \ \ (n \text{ times})$$

for any positive integer n. Now, given v, pick N so that $\frac{N}{2}v \in V' \setminus \frac{1}{2}V'$. Then, $g^N \in \exp(V') \setminus \exp\left(\frac{1}{2}V'\right) = U' \setminus U$. $\square$

8. Lecture 8

8.1. Ad, ad and exp.

Definition 8.1.1. Let G be a Lie group and V a vector space. A *representation* of a Lie group is a map $\rho : G \to GL(V)$ of Lie groups.

For a Lie group G, consider the action of G on itself by conjugation: for each $g \in G$ we have a diffeomorphism $c_g : G \to G$ given by $c_g(a) = gag^{-1}$. Notice that $c_g(1) = 1$, and we have an invertible linear map $(dc_g)_1 : \mathfrak{g} \to \mathfrak{g}$. Now, $c_{g_1 g_2} = c_{g_1} \circ c_{g2}$ for all $g_1, g_2 \in G$, and hence $(dc_{g_1})_1 \circ (dc_{g_2})_1 = (dc_{g_1 g_2})_1$.

Definition 8.1.2. The *Adjoint Representation* of a Lie group G is the representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ defined by

$$\mathrm{Ad}(g) = (dc_g)_1$$

The *adjoint representation* of a Lie algebra $\mathfrak{g}$ is the representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) = \mathrm{Hom}(\mathfrak{g}, \mathfrak{g})$ defined by

$$\mathrm{ad}(X) = (d\,\mathrm{Ad})_1(X)$$

Proposition 8.1.3. *Suppose G is a Lie group. Then, for all $t \in \mathbb{R}, g \in G$ and $X \in \mathfrak{g}$ we have*

(1) $g \exp(tX)g^{-1} = \exp(t\,\mathrm{Ad}(g)(X))$ *and*
(2) $\mathrm{Ad}(\exp(tX)) = \exp(t\,\mathrm{ad}(X))$.

Proof. For the first statement, apply naturality of exp to the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\mathrm{Ad}(g)"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\
G \arrow[r, "c_g"] & G
\end{tikzcd}$$

Similarly, to prove 2, apply the naturality of exp to the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] & \mathfrak{gl}(\mathfrak{g}) \arrow[d, "\exp"] \\
G \arrow[r, "\mathrm{Ad}"] & GL(\mathfrak{g})
\end{tikzcd}$$

□

Example 8.1.4. We compute what Ad and ad are as maps when $G = GL(n, \mathbb{R})$. Recall that for any $A, g \in G$ we have the conjugation map $c_g(A) = gAg^{-1}$. Note that congugation is linear.

Thus, for $X \in \mathfrak{g}$ we have

$$\begin{aligned}
\mathrm{Ad}(g)(X) &= (dc_g)_I(X) \\
&= \left.\frac{d}{dt}\right|_{t=0} c_g(\exp(tX)) \\
&= \left.\frac{d}{dt}\right|_{t=0} g\exp(tX)g^{-1} \\
&= g\left(\left.\frac{d}{dt}\right|_{t=0} \exp(tX)\right)g^{-1} \\
&= gXg^{-1}.
\end{aligned}$$

Also, for $X, Y \in \mathfrak{g}$

$$\begin{aligned}
\mathrm{ad}(X)Y &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp(tX))Y \\
&= \left.\frac{d}{dt}\right|_{t=0} \exp(tX)Y\exp(-tX) \\
&= \left(\left.\frac{d}{dt}\right|_{t=0} \exp(tX)\right)Y\exp(-0X) + \exp(0X)Y\left(\left.\frac{d}{dt}\right|_{t=0} \exp(-tX)\right) \\
&= XY + Y(-X) \\
&= [X, Y]
\end{aligned}$$

the commutator of the matrices X, Y.

The second result of the previous example generalizes nicely to any Lie group.

Theorem 8.1.5. *Let G be a Lie group. Then, for any $X, Y \in \mathfrak{g}$,*

$$\mathrm{ad}(X)Y = [X, Y]$$

Proof. First note that

$$\begin{aligned}
\mathrm{ad}(X)Y &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp tx)Y \\
&= \left.\frac{d}{dt}\right|_{t=0} d(C_{\exp tx})_1(Y).
\end{aligned}$$

Also, recall that we have shown:

(1) $c_g(a) = gag^{-1} = (R_{g^{-1}} \circ L_g)(a)$,
(2) $(dL_g)_1(Y) = \tilde{Y}(g)$, where $\tilde{Y}$ is the left invariant vector field with $\tilde{Y}(1) = Y$,
(3) the flow $\phi_t^{\tilde{Y}}$ of $\tilde{Y}$ is given by

$$\phi_t^{\tilde{Y}}(a) = a(\exp tX) = R_{\exp tX}(a),$$

(4) $[\tilde{X}, \tilde{Y}](a) = \left.\frac{d}{dt}\right|_{t=0} d(\phi_{-t}^{\tilde{X}})(\tilde{Y}(\phi_t^{\tilde{X}}(a)))$,
(5) $(\exp tX)^{-1} = \exp(-tX)$.

We now put 1-5 together:

$$\begin{aligned}
\mathrm{ad}(X)Y &= \left.\frac{d}{dt}\right|_{t=0} dR_{\exp(-tX)}(dL_{\exp tX}Y) \quad \text{(by 1,5)} \\
&= \left.\frac{d}{dt}\right|_{t=0} dR_{\exp(-tX)}(\tilde{Y}(\exp tX) \quad \text{(by 2)} \\
&= \left.\frac{d}{dt}\right|_{t=0} d(\phi_t^{\tilde{X}})(\tilde{Y}(\phi_t^{\tilde{X}}(1)) \quad \text{(by 3)} \\
&= [\tilde{X}, \tilde{Y}](1) \quad \text{(by 4)}
\end{aligned}$$

□

9. Lecture 9

9.1. **Normal Lie Subgroups.** Problem: Given a Lie group G and $N \subset G$ a normal Lie subgroup, what properties does the Lie algebra $\mathfrak{n}$ of N have?

Since N is normal in G, for all $g \in G$ and for all $n \in N$, we have

$$\begin{aligned} gng^{-1} \in N &\implies c_g(N) \subseteq N \\ &\implies d(c_g)_1(\mathfrak{n}) \subseteq \mathfrak{n} \\ &\implies \mathrm{Ad}(g)\mathfrak{n} \subseteq \mathfrak{n} \end{aligned}$$

Thus, for any $X \in \mathfrak{n}$, any $Y \in \mathfrak{g}$ and any $t \in \mathbb{R}$, we have

$$\mathrm{Ad}(\exp tY)X \in \mathfrak{n} \implies \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp tY)X \in \mathfrak{n}$$

But, $\left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp tY)X = \mathrm{ad}(Y)X = [Y,X]$ and hence $[Y,X] \in \mathfrak{n}$.

Definition 9.1.1. A Lie subalgebra $\mathfrak{n}$ of a Lie algebra $\mathfrak{g}$ is an *ideal* if $[\mathfrak{g},\mathfrak{n}] \subseteq \mathfrak{n}$.

So, the above discussion proves

Proposition 9.1.2. *If N is a normal Lie subgroup of a Lie group G, then its Lie algebra $\mathfrak{n}$ is an ideal in $\mathfrak{g}$.*

It's natural to wonder whether the converse of this proposition is true. That is, if G is a Lie group and $\mathfrak{n}$ is an ideal in $\mathfrak{g}$, is it necessarily true that the unique subgroup N of G corresponding to $\mathfrak{n}$ is normal? We need an extra hypothesis on G:

Proposition 9.1.3. *Suppose G is a connected Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{n} \subseteq \mathfrak{g}$ an ideal and N the unique connected Lie subgroup with $\mathfrak{n} \subseteq T_1N$. Then N is a normal subgroup of G.*

Proof. For all $X \in \mathfrak{n}, Y \in \mathfrak{g}$ we have $[Y,X] \in \mathfrak{n}$. So, $[Y,[Y,X]] = (\mathrm{ad}(Y) \circ \mathrm{ad}(Y))(X) \in \mathfrak{n}$. It follows that $\mathrm{ad}(Y)^n(X) \in \mathfrak{n}$. But then,

$$\begin{aligned} \sum \frac{1}{n!} \mathrm{Ad}\, y^n(X) &= \exp(t\,\mathrm{ad}(Y))(X) \\ &= \mathrm{Ad}(\exp tY)X \in \mathfrak{n} \end{aligned}$$

for all $t \in \mathbb{R}$. Since G is connected, $\exp \mathfrak{g}$ generates G and so $\mathrm{Ad}(g)X \in \mathfrak{n}$ for all $g \in G, X \in \mathfrak{n}$. This implies that $c_g(\exp X) = \exp(\mathrm{Ad}(g)X) \in N$. But N is connected, and so $\exp \mathfrak{n}$ generates N. Hence, $gag^{-1} \in N$ for any $g \in G, a \in N$. □

9.2. **The Closed Subgroup Theorem I.** We now start a proof of The Closed Subgroup Theorem:

Theorem 9.2.1. *If G is a Lie group, $H \subseteq G$ a closed subgroup, then H is a Lie subgroup of G.*

What we will actually prove is that H (as above) is an embedded submanifold of G and the theorem will follow from a couple of results we will now establish.

Lemma 9.2.2. *Let G be a Lie group. Suppose $H \subseteq G$ is an abstract subgroup and an embedded submanifold. Then, H is a Lie subgroup.*

Proof. It's enough to show that the multiplication map $m : H \times H \to H$ is smooth. Fix $(x, y) \in H$. Since $m : H \times H \to G$ is continuous, there exists a neighborhood U of $xy = m(x, y)$ in G and a neighborhood V of $(x, y) \in H$ so that $m(V) \subseteq U$. Since H is embedded in G, there exists a neighborhood $U' \subseteq U$ of xy in G and a coordinate map $\phi : U' \to \mathbb{R}^n$ ($n = \dim G$) such that

$$\phi(U' \cap H) = \phi(U') \cap (\mathbb{R}^k \times \{0\})$$

where $k = \dim H$. Now, $\phi \circ m : m^{-1}(U') \cap V \to \mathbb{R}^n$ is C^∞ and the image of this composition lies in $\mathbb{R}^k \times \{0\}$. Letting $p : \mathbb{R}^n \to \mathbb{R}^k$ be the standard projection, we see that $p \circ \phi \circ m : m^{-1}(U') \cap V \to \mathbb{R}^k$ is C^∞. Hence, m is C^∞. □

Theorem 9.2.3. *An abstract subgroup H of a Lie group G is an embedded submanifold if and only if H is closed in G.*

Proof. We first prove that if H is an embedded submanifold, it is closed in G. Since H is embeddced, there exists a neighborhood U of $1 \in G$ such that $U \cap H$ is closed in U. Let $\overline{H}$ be the closure of H and suppose $y \in \overline{H}$. Then, $yU^{-1} \subseteq G$ is a neighborhood of y in G. We conclude that $yU^{-1} \cap H \neq \emptyset$. Pick a point $x \in yU^{-1} \cap H$. Then, $x = yu^{-1}$ for some $u \in U$. Thus, $x^{-1}y = u \in U$.

Now, $L_{x^{-1}} : G \to G$ is a homemorphism so that $L_{x^{-1}}(H) = H$. By continuity, $L_{x^{-1}}(\overline{H}) = \overline{H}$. Thus, $x^{-1}y \in \overline{H}$. We now argue that $x^{-1}y \in H \cap U$. Since $x^{-1}y \in \overline{H} \cap U$, there exists a sequence $\{h_n\} \subset H \cap U$ so that $h_n \to x^{-1}y$. But recall that $H \cap U$ is closed in U and so $x^{-1}y \in H \cap U$. Since this implies $y \in xH = H$, we must have $\overline{H} \subseteq H$. Thus, H is closed in G.

□

10. Lecture 10

10.1. **The Closed Subgroup Theorem II.** We have yet to complete the proof of the Closed Subgroup Theorem. What remains is to show that an abstract closed subgroup H of a Lie group G is an embedded submanifold. By Lemma 9.5, H is then a Lie subgroup of G.

Proof. Fix a norm $\|\cdot\|$ on $\mathfrak{g}$. Choose neighborhoods W' of $0 \in \mathfrak{g}$ and W of $1 \in G$ so that $\exp : W' \to W$ is a diffeomorphism. Let $V' = W' \cap (-W')$. Take $V = \exp(V')$ and note that $a \in V \implies a^{-1} \in V$. Define

$$\log := (\exp|_V)^{-1} : V \to V'$$

and let

$$\mathfrak{h} = \left\{ X \in \mathfrak{g} \mid \begin{array}{c} \exists \text{ sequences } \{h_n\} \subset H \cap V, \{t_n\} \subset \mathbb{R}^{\geq 0} \text{ with} \\ (1) \lim_{n\to\infty} h_n = 1 \text{ and } (2) \lim_{n\to\infty} t_n \log h_n = X \end{array} \right\}$$

Claim 1: There exists a neighborhood U' of $0 \in \mathfrak{h}$ such that $\exp(U') \subseteq H$.

To prove this claim, first take $U' = V' \cap \mathfrak{h}$. Then, for any $X \in U'$ there are sequences $\{h_n\}$ and $\{t_n\}$ so that (1) and (2) hold. Since $\lim_{n\to\infty} h_n = 1$, $\lim_{n\to\infty} \log h_n = 0$. Denote by $\lfloor t_n \rfloor$ the largest integer less than or equal to t. We then have

$$\lim_{n\to\infty} (t_n - \lfloor t_n \rfloor) \log h_n = 0$$

so that $X = \lim_{n\to\infty} t_n \log h_n = \lim_{n\to\infty} \lfloor t_n \rfloor \log h_n$. In light of this, we see that

$$\begin{aligned} \exp(X) &= \lim_{n\to\infty} \exp(\lfloor t_n \rfloor \log h_n) \\ &= \lim_{n\to\infty} (\exp(\log h_n))^{\lfloor t_n \rfloor} \\ &= \lim_{n\to\infty} h_n^{\lfloor t_n \rfloor} \in H \end{aligned}$$

since $h_n^{\lfloor t_n \rfloor} \in H$ for all n and H is closed. But this proves the claim that $\exp(U') \subseteq H$.

Claim 2: $\mathfrak{h}$ is a linear subspace of $\mathfrak{g}$.

Piick $X \in \mathfrak{h}$. As before, we then have sequences $\{h_n\}$ and $\{t_n\}$ satisfying the conditions (1) and (2) above. Now, $\{h_n^{-1}\} \subset V \cap H$ and

$$\begin{aligned} \lim_{n\to\infty} (h_n^{-1}) &= (\lim_{n\to\infty} h_n)^{-1} \\ &= 1^{-1} = 1 \end{aligned}$$

while

$$\begin{aligned} \lim_{n\to\infty} t_n \log h_n^{-1} &= -\lim_{n\to\inf} t_n \log h_n \\ &= -X \in \mathfrak{h}. \end{aligned}$$

Also, for all $t \geq 0$, we have $\lim_{n\to\infty}(t(t_n) \log h_n) \to tX$. Hence, $tX \in \mathfrak{h}$ for all $t \in \mathbb{R}$.

Now, if $X, Y \in \mathfrak{h}$, then for t sufficiently small, $tX, tY \in U'$ and so $\exp tX \exp tY \in H$. In addition, since $\lim_{t\to 0} \exp tX \exp tY = 1$, $\exp tX \exp tY \in V$. Hence for t sufficiently small, $\exp tX \exp tY \in V \cap H$. But,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \log(\exp tX \exp tY) &= \lim_{t\to 0} \frac{1}{t} \log(\exp tX \exp tY) \\ &= X + Y \end{aligned}$$

Let $t_n = 1/n$ and $h_n = \exp t_n X \exp t_n Y$. Then $h_n \in H$ and $\lim_{n\to\infty} h_n = 1$. It follows that $\lim_{n\to\infty} t_n \log h_n = X + Y \in \mathfrak{h}$.

Claim 3: For any neighborhoos U' of $0 \in \mathfrak{h}$, $\exp(U')$ is a neighborhood of $1 \in H$.

Suppose the claim doesn't hold. Then, there exists a neighborhood U' of $0 \in \mathfrak{h}$ and a sequence $\{h_n\} \subset H \setminus \exp(U')$ such that $h_n \to 1$. Choose a linear subspace $\mathfrak{k}$ in $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$. Consider $\phi : \mathfrak{h} \oplus \mathfrak{k} \to G$ defined by $\phi(X \oplus Y) = \exp X \exp Y$. Then $(d\phi)_0 = X + Y$ and so ϕ is a diffeomorphism in a neighborhood of $0 \in \mathfrak{h} \oplus \mathfrak{k}$. Thus, there are sexuences $\{X_n\} \subset \mathfrak{h}$, $\{Y_n\} \subset \mathfrak{k}$ so that $h_n = \exp X_n \exp Y_n$ for n sufficiently large. Note that $Y_n \neq 0$ since $h_n \notin \exp(U')$.

Now, $h_n \in H$ implies that $\exp(X_n) \in H$ and so $\exp(X_n)^{-1} h_n \in H$. On the other hand, $1/\|Y_n\| Y_n$ is bounded. By passing to a subsequence, we may assume that $1/\|Y_n\| Y_n \to Y \in \mathfrak{k}$ with $\|Y\| = 1$. Let $h_n = \exp(Y_n)$ so that for n large, $h_n \in V \cap U$ and $Y_n = \log(h_n)$. Since $Y_n \to 0$, $h_n \to 1$ and $1/\|Y_n\| \log h_n \to Y \in \mathfrak{h}$, a contradiction.

From the three claims above, we can conclude that there exists a linear subspace $\mathfrak{h}$ of $\mathfrak{g}$, a neighborhood V' of $0 \in \mathfrak{g}$ and a neighborhood V of $1 \in G$ such that

(1) $\exp : V' \to V$ is a diffeomorphism,
(2) $\exp(V' \cap \mathfrak{h})$ is a neighborhood of 1 in H.

So, $log : V \to V'$ are coordinates on G adapted to H. This in turn implies that for all $x \in U$, $\log \circ L_{x^{-1}} : L_x(V) \to \mathfrak{g}$ are coordinates adapted to H. Hence H is an embedded submanifold of G. $\square$

11. Lecture 11

11.1. **Applications of the Closed Subgroup Theorem.** We begin with a remark about the Lie algebra of a closed subgroup of a Lie group. If H is a closed subgroup of a Lie group G, then the Lie algebra $\mathfrak{h}$ of H is

$$\{X \in \mathfrak{g} \mid \exp tX \in H, \forall t \in \mathbb{R}\}$$

Since exp is natural, $\exp tX \in H$ for all t. Conversely, if $\exp tX \in H$ for all t,

$$X = \left.\frac{d}{dt}\right|_{t=0} \exp tX \in T_1 H = \mathfrak{h}$$

Our first application of the Closed Subgroup Theorem is a rather surprising result about continuous group homomorphisms.

Theorem 11.1.1. *Suppose H and G are Lie groups and $f : H \to G$ is a continuous group homomorphism. Then, f is smooth.*

For the proof of Theorem 11.1, we'll need Sard's Theorem.

Definition 11.1.2. Let $f : M \to N$ be a C^∞ map of manifolds. A point $y \in N$ is a *regular value* of f if for any $x \in f^{-1}(y)$, $(df)_x : T_x M \to T_y N$ is onto.

Theorem 11.1.3 (Sard's Theorem)**.** *Let $f : M \to N$ be a smooth map of manifolds. Then, the set of regular values of f is dense in N.*

Remark 11.1.4. If $f^{-1}(y) = \emptyset$, y is still a regular value. Hence, if $f(M)$ is a single point in N, the complement $N \setminus f(M)$ is still dense and consists of regular values.

Proposition 11.1.5. *Suppose $f : A \to B$ is a Lie group homomorphism. Then,*

(i) *if f is onto, $(df)_a : T_a A \to T_{f(a)} B$ is onto for all $a \in A$ and*
(ii) *if f is 1-1, $(df)_a : T_a A \to T_{f(a)} B$ is 1-1 for all $a \in A$.*

Proof. Recall first for any $a \in A$, $L_{f(a)} \circ f = f \circ L_a$. So, $(dL_{f(a)})_1 \circ (df)_1 = (df)_a \circ (dL_a)_1$. Consequently, using the fact that L_g is always a diffeomorphism,

$$\begin{aligned} \dim\ker(df)_a &= \dim\ker(df)_1 \quad \text{and} \\ \dim\operatorname{im}(df)_a &= \dim\operatorname{im}(df)_1 \end{aligned}$$

for all $a \in A$.

We are now in a position to prove (i). By Sard's Theorem, the set of regular values of f is dense in B. By assumption, $B = f(A)$ and so there is $b \in f(A)$ which is a regular value. Hence, there is $a_0 \in A$ so that $(df)_{a_0} : T_a A \to T_b B$ is onto. Thus, $(df)_a$ is onto for all $a \in A$.

To prove (ii), first suppose that $(df)_1(X) = 0$ for some $X \in T_1 A$. Then, $f(\exp tX) = \exp(t(df)_1(X)) = 1$ for all $t \in \mathbb{R}$. Thus, $\{\exp tX\} \subset \ker f = \{1\}$. So, $X = 0$ and $\ker(df)_a = 0$ for all $a \in A$. $\square$

We now prove Theorem 11.1:

Proof. Since f is continuous, its graph

$$\Gamma_f = \{(a, f(a)) \in H \times G \mid a \in H\}$$

is a closed subgroup of $H \times G$. By the Closed Subgroup Theorem, Γ_f is a Lie subgroup. Consider the projections

$$\begin{tikzcd} & \Gamma_f \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ H & & G \end{tikzcd}$$

Now, p_1 is a group isomorphism and we can thus write $f = p_2 \circ p_1^{-1}$ So, it's enough to prove p_1^{-1} is smooth. But, by Propostition 11.5, dp_1 is everywhere onto and injective. By the Inverse Function Theorem, p_1^{-1} is smooth. $\square$

For another application of the Closed Subgroup Theorem, we look towards group actions.

Definition 11.1.6. Suppose a Lie group G acts on a manifold M. The *stabilizer* (or *isotropy*) group of $x \in M$ is

$$G_x = \{a \in G \mid a \cdot x = x\}$$

The *orbit* of $x \in M$ is

$$G \cdot x = \{a \cdot x \mid a \in G\}.$$

It is left as an exercise to prove that G_x is a subgroup of G for each $x \in M$.

Proposition 11.1.7. *Suppose a Lie group G acts on a manifold M. For each $x \in M$, the stabilizer group G_x is a Lie subgroup of G.*

Proof. Choose $x \in M$. We will show that G_x is closed in G. Let $A : G \times M \to M$ denote the action of G on M. Define $\iota_x : G \to G \times M$ by $\iota_x(a) = (a, x)$. Then,

$$G_x = \{a \in G \mid A(\iota_x(a)) = x\}$$

That is, $G_x = \iota_x^{-1}(A^{-1}(x))$. Noting that both A and ι_x are continuous, we're done. $\square$

We will denote the Lie algebra of G_x by $\mathfrak{g}_x$. Note that

$$\mathfrak{g}_x = \{X \in \mathfrak{g} \mid (\exp tX) \cdot x = x, \forall t \in \mathbb{R}\}$$

Example 11.1.8. $G = GL(n, \mathbb{R})$ acts on symmetric bilinear forms $Sym^2((\mathbb{R}^n)^*)$ by $(A \cdot b)(v, w) = b(A^{-1}v, A^{-1}w)$. Let $b_0 = \langle , \rangle$ be the standard inner product on $\mathbb{R}^n$. Then,

$$G_{b_0} = \{A \in GL(n, \mathbb{R}) \mid \langle Av, Aw\rangle = \langle v, w, \rangle\} = O(n)$$

Example 11.1.9. $G = GL(n, \mathbb{C})$ acts on $\mathcal{H}$ = hermitian forms on $\mathbb{C}^n$. Let $h_0(z, w) = \sum \overline{z}_j w_j$ be the standard Hermitian form. By a computation similar to the previous example, $G_{h_0} = U(n)$.

Example 11.1.10. Let $B =$ all bilinear forms on $\mathbb{R}^n$. Suppose $n = 2k$ and examine

$$\omega(v, w) = \left\langle v, \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} x \right\rangle = \sum_{i=1}^{k} v_i w_{i+k} - \sum_{i=k+1}^{2k} v_i w_{i-k}$$

The stabilizer group of ω is denoted $Sp(\mathbb{R}^{2k})$ (or $Sp(k, \mathbb{R})$) and is called the *symplectic group*.

12. Lecture 12

12.1. **Group Actions and Induced Vector Fields.** In this lecture we will see how group actions are used to find the Lie algebra of Lie groups given as the stabilizer of some group action.

Suppose a Lie group G acts on a manifold M. Fix $X \in \mathfrak{g}$. Then, $\Phi_t(x) = (\exp tX) \cdot x$ is a 1-parameter group of diffeomorphisms of M:

$$\begin{aligned}
\Phi_{t+s}(x) &= (\exp(t+s)X) \cdot x \\
&= ((\exp tX)(\exp sX)) \cdot x \\
&= (\exp tX) \cdot (\exp sX) \cdot x \\
&= \Phi_t(\Phi_s(x))
\end{aligned}$$

Given this 1-parameter group, the corresponding vector field (the *induced* vector field) is given by

$$X_m(x) = \left.\frac{d}{dt}\right|_{t=0} (\exp tX) \cdot x$$

Remark 12.1.1. This gives us a map $\mathfrak{g} \to \Gamma(TM)$, $X \mapsto X_M$. Is this a map of Lie algebras?

Lemma 12.1.2. *Suppose a Lie group G acts on a manifold M and $x \in M$. Then,*

$$\mathfrak{g}_x = \{X \in \mathfrak{g} \mid X_M(x) = 0\} = \left\{X \in \mathfrak{g} \mid \left.\frac{d}{dt}\right|_{t=0} (\exp tX) \cdot x = 0\right\}$$

Proof. If $X \in \mathfrak{g}$, then $\exp tX \in G_x$ for all t and hence $x = (\exp tX) \cdot x$ for all t. So, differentiating gives $0 = \left.\frac{d}{dt}\right|_{t=0} (\exp tX) \cdot x = X_M(x)$.

Conversely, suppose $Y \in \mathfrak{g}$ and $Y_M(x) = 0$. Then, the curve $\gamma(t) = x$ is an integral curve of Y_M through x. By definition, the map $t \mapsto (\exp tY \cdot x$ is also an integral curve of Y_M through x. But then, for all $t \in \mathbb{R}$, $(\exp tY) \cdot x = x$ implies $\exp tY \in G_x$ and so $Y \in \mathfrak{g}_x$. □

Example 12.1.3. Recall that $GL(n,\mathbb{R})$ acts on the vector space B of all bilinear forms on $\mathbb{R}^n$ by

$$(A \cdot b)(v,w) = b(A^{-1}v, A^{-1}w)$$

Now, if $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a bilinear form, then

$$\begin{aligned}
(db)_{(x,y)}(v,w) &= \left.\frac{d}{dt}\right|_{t=0} b(x+tv, y+tw) \\
&= \left.\frac{d}{dt}\right|_{t=0} (b(x,y) + tb(v,y) + tb(x,w) + t^2 b(v,w)) \\
&= b(x,w) + b(v,y).
\end{aligned}$$

What happens when we take the bilinear form to be $b_0 = \langle , \rangle$; the standard inner product on $\mathbb{R}^n$? Recall that $GL(n,\mathbb{R})_{b_0} = O(n)$ and let's compute the Lie Algebra $\mathfrak{o}(n)$ of $O(n)$. We have

$$\begin{aligned} 0 &= \left(\frac{d}{dt}\bigg|_{t=0} (\exp tX)\cdot b \right)(v,w) \\ &= \frac{d}{dt}\bigg|_{t=0} b(\exp(-tX)v, \exp(-tX)w) \\ &= (db)_{v,w}(-Xv,-Xw) \\ &= b(v,-Xw) + b(-Xv,w) \\ &= -b(v,Xw) - b(v,X^T w) \\ &= -b(v,(X+X^T)w). \end{aligned}$$

Since v, w were arbitrary, we have $\mathfrak{o}(n) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid X + X^T = 0\}$.

Now we turn to examine the orbits of a Lie group action on a manifold. Recall that the orbit of $x \in M$ is $G \cdot x = \{a \cdot x \mid a \in G\}$.

Example 12.1.4. $G = \mathbb{R}^+$ acts on $\mathbb{R}$ by left multiplication. The orbit of any real number is either $(-\infty, 0)$, $\{0\}$ or $(0,\infty)$.

Example 12.1.5. $SO(n)$ act on $\mathbb{R}^n$ in the following fashion: if $v \in \mathbb{R}^n$ and $A \in SO(n)$, then $A \cdot v = Av$ (matrix-vector multiplication). The orbit of v under this action is $SO(n)\cdot v = \{w \in \mathbb{R}^n \mid \|w\| = \|v\|\}$, i.e. the sphere of radius $\|v\|$.

An action of G on M defines an equivalence relation whose equivalence classes are orbits. The relation is $m \sim m'$ if and only if there exists $g \in G$ so that $m' = g \cdot m$. Since this is an equivalence relation (prove this!), if $x, x' \in M$ we have $G\cdot x \cap G \cdot x' \neq \emptyset \implies G\cdot x = G \cdot x'$. In particular, we may form the quotient space $M/G := M/\sim$ from this relation. The elements of this topological space are the orbits, but what is the topology?

Let $\pi : M \to M/G$ be defined by $\pi(x) = G\cdot x$. This map is, of course, onto. Call a set $U \subset M/G$ open if and only if $\pi^{-1}(U) \subset M$ is open. Check that this does indeed define a topology on M/G. We may now ask basic questions about the general topology of this quotient space. For example, when is M/G Hausdorff?

Proposition 12.1.6. *Let a Lie group G act on a space M. If the set*

$$R = \{(m,m') \in M \times M \mid m' = g\cdot m \text{ for some } g \in G\}$$

is closed in $M \times M$, then M/G is Hausdorff.

Proof. Choose points $\bar{x} \neq \bar{y} \in M/G$. Since the canonical projection is onto, there exists $x, y \in M$ so that $\pi(x) = \bar{x}$ and $\pi(y) = \bar{y}$. Since $\bar{x} \neq \bar{y}$, $(x,y) \notin R$. Since R is closed, there exists (open) neighborhoods U of $x \in M$ and V of $y \in M$ so that $(U \times V) \cap R = \emptyset$. Now, if $\pi(z) \in \pi(U) \cap \pi(V)$, then there exists $g_1, g_2 \in G$ so that $g_1 z \in U$ and $g_2 z \in V$. But then, we must have $(g_1 z, g_2 z) = (g_1 z, (g_2 g_2^{-1}) g_2 z) \in R$. This is a contradiction since then

$$\emptyset = \pi^{-1}(\pi(U) \cap \pi(V)) = \pi^{-1}(\pi(U)) \cap \pi^{-1}(\pi(V)).$$

$\square$

Corollary 12.1.7. *Let G be a Lie group, $H \subseteq G$ a closed subgroup. Then G/H is Hausdorff.*

Proof. In this case, $R = \{(g_1, g_2) \mid g_2 = g_1 h, h \in H\}$. The map $\psi : G \times G \to G \times G$ given by $\psi(a, b) = (a, ab)$ is a diffeomorphism and $R = \psi(G \times H)$. Since $G \times H$ is closed, R is as well. $\square$

13. Lecture 13

13.1. **More on Group Actions and Principal Bundles I.**

Definition 13.1.1. Let G be a Lie group and X a set. A *right action* of G on X is a map $X \times G \to X$ satisfying

(1) $x \cdot 1 = x$,
(2) $(x \cdot g) \cdot g' = x \cdot gg'$.

Example 13.1.2. Let $X = \mathrm{Hom}(V, W)$. Then, $GL(V)$ acts on the left and $GL(W)$ acts on the right.

Definition 13.1.3. Let H be a Lie group. A *principal H-bundle* is a manifold P equipped with a right action of H such that

(1) $B = P/H$ is a smooth manifold and the orbit map $\pi : P \to B$ is a submersion,
(2) For any $b \in B$, there is an open set $U \subset B$ with $b \in U$ and a smooth map

$$\psi : \pi^{-1}(U) \to U \times H$$

so that
 (i) the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times H \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

 (ii) $\psi(p \cdot a) = \psi(p) \cdot a$ for all $p \in P$, $a \in H$ where H acts on $U \times H$ by $(u, h) \cdot a = (u, ha)$.

Example 13.1.4. 0) Let $P = B \times H$, $\pi =$ projection on the first factor and $(b, h) \cdot a = (b, ha)$ for all $(h, a) \in P$, $a \in H$. This is called the *trivial* principal bundle.

1) $S^{2n-1} \to \mathbb{C}P^{n-1}$ is a principal S^1 bundle. S^1 acts on $S^{2n-1} \subset \mathbb{C}^n$ by $(z_1, \dots, z_n) \cdot \lambda = (\lambda z_1, \dots, \lambda z_n)$. The bundle projection $\pi : S^{2n-1} \to \mathbb{C}P^{n-1}$ maps $(z_1, \dots, z_n)$ to the equivalence class $[z_1, \dots, z_n]$.
2) Let M be an n-dimensional manifold. The *frame bundle* of M is $Fr(M) = \coprod_{x \in M} Iso(\mathbb{R}^n, T_x M) \subset \mathrm{Hom}(M \times \mathbb{R}^n, TM)$. This is a principal $GL(n, \mathbb{R})$ bundle: $f \cdot A = f \circ A$.
3) Let G be a Lie group, $H \subseteq G$ a closed subgroup. Then, $\pi : G \to G/H$ is a principal H-bundle.

Remark 13.1.5. If P is a principal H-bundle, P is called the *total space* of the bundle, $B = P/H$ is the *base space*, H is the *structure group*; for $b \in B$, $\pi^{-1}(b) = P_b$ is the *fiber* above b. The bundle is often written $H \to P \xrightarrow{\pi} B$.

Remark 13.1.6. A right action of a Lie group G on a manifold M is *free* if for all $x \in M$, $x \cdot g = x \implies g = 1$. A right action is *proper* if the map $f : M \times G \to M \times M$, $(m, g) \mapsto (m, m \cdot g)$ is proper; i.e. preimages of compact sets are compact. We have the following result:

Theorem 13.1.7. *If a Lie group G acts freely and properly on a manifold P, then P/G is a manifold and $G \to P \to P/G$ is a principal G-bundle.*

The last example above of a principal bundle was slipped in without proof. Let's take care of that now.

Theorem 13.1.8. *let G be a Lie group, $H \subseteq G$ a closed subgroup. Then, G/H is a smooth manifold and $H \to G \xrightarrow{\pi} G/H$ is a principal H-bundle. Moreover, $T_{1H}(G/H) \cong \mathfrak{g}/\mathfrak{h}$.*

Proof. We will show only that G/H is a smooth manifold and that $T_{1H}(G/H) \cong \mathfrak{g}/\mathfrak{h}$. The proof that $G \to G/H$ is a principal $H-$bundle is left to the reader. Recall from the proof of the Closed Subgroup Theorem that if we choose a linear subspace $\mathfrak{m} \subset \mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{h} \times \mathfrak{m}$ then the map $\phi : \mathfrak{m} \times \mathfrak{h} \to G$, $\phi(Y, X) = \exp Y \exp X$ has the properties that

(1) ϕ is a diffeomorphism on a neighborhood $V \times U$ of $(0, 0)$ in $\mathfrak{m} \times \mathfrak{h}$,
(2) If U and V are sufficiently small, then

$$\phi(V \times U) \cap H = \phi(\{0\} \times U) = \exp(U)$$

Choose $V' \subset V$ open so that $Y_1, Y_2 \in V'$ implies $Y_1 - Y_2 \in V$. Let $\Sigma = \exp(V')$. Then, $s_1, s_2 \in \Sigma \implies s_2^{-1}s_1 \in \exp(V)$.

Claim 1: The map $\psi : \Sigma \times H \to G$, $\psi(s, h) = sh$ is injective.

To prove this claim, first suppose that $s_1, s_2 \in \Sigma$, $h_1, h_2 \in H$ and $s_1h_1 = s_2h_2$. Then

$$\phi(V \times U) \supset \phi(V \times \{0\}) = \exp(V) \ni s_2^{-1}s_1 = h_2h_1^{-1} \in H$$

But $H \cap \phi(V \times U = \phi(\{0\} \times U)$ and $\phi(\{0\} \times U) \cap \phi(V \times \{0\}) = \phi(0, 0) = 1$. Hence, $s_1 = s_2$ and $h_1 = h_2$.

Claim 2: If V' is sufficiently small, then

$$\psi : \exp(V') \times H = \Sigma \times H \to G$$

is an open embedding.

Let's compute $(d\psi)_{(1,1)} : T_1\Sigma \times T_1H \to T_1G = \mathfrak{g}$. Write $T_1\Sigma = \mathfrak{m}$ and $T_1H = \mathfrak{h}$. For all $Y \in \mathfrak{m}$ and $X \in \mathfrak{h}$ we have

$$\begin{aligned} (d\psi)_{(1,1)}(Y, 0) &= \left.\frac{d}{dt}\right|_{t=0} \psi(\exp Y \cdot 1) = Y \\ (d\psi)_{(1,1)}(0, X) &= \left.\frac{d}{dt}\right|_{t=0} \psi(1 \cdot \exp X) = X \end{aligned}$$

and hence $(d\psi)_{(1,1)}(Y, X) = Y + X$. So, if V' is sufficiently small, $(d\psi)_{(s,1)} : T_s\Sigma \times T_1H \to T_sG$ is an isomorhism for all $s \in \Sigma$.

Now, the diagram

$$\begin{tikzcd} \Sigma \times H \arrow[r, "\psi"] & G \\ \Sigma \times H \arrow[r, "\psi"] \arrow[u, "\mathrm{id} \times R_h"] & G \arrow[u, "R_n"'] \end{tikzcd}$$

commutes: i.e. $\psi(s, ah) = sah = R_hsa = (R_h \circ \psi)(s, a)$ and it follows that, for all $(s, h) \in \Sigma \times H$, that

$$(d\psi)_{(s,h)} : T_s\Sigma \times T_hH \to T_{sh}G$$

is an isomorphism. Hence, ψ is an open embedding. $\square$

14. Lecture 14

14.1. **Principal Bundles II.** In this lecture we continue the proof of Theorem 13.8. Recall we have the embedded submanifold $\Sigma = \exp(V')$ of G and an open embedding $\psi : \Sigma \times H \to G$, $\psi(s,h) = sh$. Note that $\psi(s,ha) = sha = \psi(s,h)a$. That is, ψ is equivariant.

Since H is a closed subgroup of G, we know that G/H is a Hausdorff topological space. What remains is to produce charts on G/H and prove that it is a principal H-bundle (exercise).

Proof. Since ψ is an open embedding, for all $a \in G$ $\psi_a = L_a \circ \psi : \Sigma \times H \to G$, $\psi_a(s,h) = ash$ is also an open embedding which is H-equivariant. So, the following hold:

1. For any orbit gH of H, $gH \cap a\Sigma$ $(a \in G)$ is a single point.
2. For any $a \in G$, $\pi \circ L_a : \Sigma \to G/H$ is 1-1. Note that
$$\begin{tikzcd} \Sigma \times H \arrow[r, "\psi_a"] \arrow[d, "p_1"'] & G \arrow[d, "\pi"] \\ \Sigma \arrow[r, "\pi \circ L_a"] & G/H \end{tikzcd}$$
commutes.
3. The map $\pi \circ L_a$ is <u>open</u>:
   Suppose $U \subseteq \Sigma$ is open. Then $U \times H \subseteq \Sigma \times H$ is open. So, $\psi_a(U \times H) = aUH \subseteq G$ is open and we can conclude that $\pi(aUH) \subset G/H$ is open as well. But, $\pi(aUH) = \pi(aU) = (\pi \circ L_a)(U)$.
   We conclude from this that $\{\pi(a\Sigma) \mid a \in G\}$ is an open cover of G/H and $\pi \circ L_a : \Sigma \to \pi(a\Sigma) \subset G/H$ are homeomorphisms.
4. We define an atlas on G/H to be
$$\{\pi(a\Sigma), \tau_a = (\pi \circ L_a)^{-1} : \pi(a\Sigma) \to \Sigma\}$$
We need to check that for any $a, b \in G$, if $W := \pi(a\Sigma) \cap \pi(b\Sigma) \neq \emptyset$, then $\tau_b \circ \tau_a^{-1} : \tau_a(W) \to \Sigma$ is C^∞. Consider the following commutative diagram:
$$\begin{tikzcd} \Sigma \times H \arrow[r, shift left, "\psi_b"] \arrow[dd, "p_1"'] & b\Sigma H \subseteq G \arrow[l, shift left, "\psi_b^{-1}"] \arrow[dd, "\pi"] & \\ & & \tau_a(W) \subseteq \Sigma \arrow[ul, "L_a"'] \arrow[dl, "\pi \circ L_a"] \\ \Sigma \arrow[r, shift left, "\pi \circ L_b"] & \pi_{b\Sigma} \arrow[l, shift left, "\tau_b"] & \end{tikzcd}$$
Now, by definition, $\tau_a^{-1} = \pi \circ L_a$. Since $\pi \circ \psi_b = p_1 \circ (\pi \circ L_b)$, we must have $\tau_b \circ \pi = (\pi \circ L_b)^{-1} \circ \pi = p_1 \circ \psi_b^{-1}$. Hence,
$$\tau_b \circ \tau_a^{-1} = \tau_b \circ (\pi \circ L_a) = p_1 \circ \psi_b^{-1} \circ L_a$$
which is a smooth map. Thus, G/H is a smooth manifold and $\pi : G \to G/H$ is a C^∞ submersion.

Note finally that

$$\ker\{(d\pi)_1 : T_1G \to T_{\pi(1)}G/H\} = T_1H$$

and so $T_{\pi(1)}G/H \cong T_1G/T_1H = \mathfrak{g}/\mathfrak{h}$.

$\square$

Remark 14.1.1. If $f : G/H \to M$ is a C^∞ map of manifolds, then $\pi \circ f : G \to M$ is also C^∞. The converse is true as well: if $\pi \circ f : G \to M$ is C^∞, then so is $f : G/H \to M$. Why? Since f is C^∞, for all $a \in G$, $f \circ \tau_a^{-1} : \Sigma \to M$ is C^∞. But, $f \circ \tau_a^{-1} = (f \circ \pi) \circ L_a$.

Similarly, $f : N \times G/H \to M$ is smooth if and only if $f \circ (\mathrm{id} \times \pi) : N \times G \to M$ is.

15. Lecture 15

15.1. **Transitive Actions.** In this section, we will use the notion of a transitive group action to determine the uniqueness of the smooth structure on G/H.

Definition 15.1.1. An action of a Lie group G on a manifold M is *transitive* if for every $m, m' \in M$, there exists $g \in G$ so that $g \cdot x = x'$. If the action of G on M is transitive, we say M is a *homogeneous G-space.*

Example 15.1.2. The standard action of $SO(n)$ on $S^{n-1} \subset R^n$ is transitive.

Example 15.1.3. For a closed subgroup H of a Lie group G, the action of G on G/H given by $g \cdot aH = gaH$ is transitive.

Example 15.1.4. The standard action of $SU(n)$ on $S^{2n-1} \subset \mathbb{C}^n$ is transitive:
given $v \in S^{2n-1}$, extend to an orthonormal basis $(v = v_1, v_2, ldots, v_n)$ of $\mathbb{C}^n$. Let $\lambda = \det[v_1|\dots|v_n]$. Define $A = [v_1|\dots|\lambda^{-1}v_n] \in SU(n)$ and note that $Ae_1 = v$ where $e_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$.

Theorem 15.1.5. *Suppose a Lie group G acts transitively on a manifold M. Fix $x \in M$. Then, the evaluation map*

$$\mathrm{ev}_x : G \to M, \mathrm{ev}_x(g) = g \cdot x$$

induces a diffeomorphism $\beta : G/H \to M$, $\beta(aH) = a \cdot x$, where $H = G_x$ (the stabilizer of x).

Proof. We first check that β is well-defined: suppose $aH = a'H$ for some $a, a' \in G$. But, this is if and only if $a = a'h$ for some $h \in H$. Now, $\beta(aH) = a \cdot x = (a'h) \cdot x = a' \cdot (hx) = \beta(a'H)$.

Since the action of G on M is transitive, β is onto. Also, β is 1-1:

$$\begin{aligned}
\beta(aH) = \beta(a'H) &\iff a \cdot a = a' \cdot x \\
&\iff (a^{-1}a') \cdot x = x \\
&\iff a^{-1}a' \in H \\
&\iff aH = a'H
\end{aligned}$$

Next, observe that

$$\begin{tikzcd}
G \arrow[d, "\pi"'] \arrow[dr, "\mathrm{ev}_x"] & \\
G/H \arrow[r, "\beta"] & M
\end{tikzcd}$$

commutes. Hence, for any open set $U \subset M$, $\pi^{-1}(\beta^{-1}(U)) = \mathrm{ev}_x^{-1}(U)$ is open. Since ev_x is continuous, so it β. Finally, since $\mathrm{ev}_x = \beta \circ \pi$ is C^∞, β is a smooth map.

Since β is onto, Sard's Theorem implies that

$$(d\beta)_{aH} : T_{aH}(G/H) \to T_{a \cdot x}M$$

is onto for *some* aH. Now,

$$\begin{tikzcd} T_1G \arrow[d, "(d\pi)_1"'] \arrow[dr, "(d\,\mathrm{ev}_x)_1"] & \\ T_{1H}G/H \arrow[r, "(d\beta)_{1H}"'] & T_xM \end{tikzcd}$$

commutes and $(d\pi)_1$ is onto. Thus,

$$\begin{aligned} \ker\,(d\beta)_{1H} &\cong \ker d(\beta\circ\pi)_1/\ker\,(d\pi)_1 \\ &= \ker\,(d\,\mathrm{ev}_x)_1/\mathfrak{h} \end{aligned}$$

Now,

$$\begin{aligned} \ker\,(d\,\mathrm{ev}_x)_1 &= \left\{X\in\mathfrak{g}\,|\,0=\left.\frac{d}{dt}\right|_{t=0}\mathrm{ev}_x(\exp tX)\right\} \\ &= \left\{X\in\mathfrak{g}\,|\,0=\left.\frac{d}{dt}\right|_{t=0}(\exp tX)\cdot x\right\} \\ &= \mathfrak{h} \end{aligned}$$

Thus, $(d\beta)_1$ is 1-1. We conclude that β is a diffeomorphism. □

Remark 15.1.6. Whenever a Lie group acts on a manifold M, we get $G/G_x \to G\cdot x \subseteq M$ for any $x\in M$, 1-1 immersions. In general, these maps are not embeddings, but if G is compact they are.

We now return to some examples.

Example 15.1.7. $SO(n)$ acts transitively on S^{n-1} and by the theorem, $S^{n-1}\cong SO(n)/H$. But what is H?

$$\begin{aligned} H &= \{A\in SO(n)\,|\,Ae_1=e_1\} \\ &= \left\{A\in SO(n)\,|\,A=\begin{bmatrix}1 & 0 & \dots & 0\\ 0 & * & * & *\\ \vdots & * & * & *\\ 0 & * & * & *\end{bmatrix},\ \begin{bmatrix}* & * & *\\ * & * & *\\ * & * & *\end{bmatrix}\in SO(n-1)\right\} \\ &\cong SO(n-1) \end{aligned}$$

Thus, $S^{n-1}\cong SO(n)/SO(n-1)$. In particular, $S^1\cong SO(2)/SO(1)=SO(2)$.

Example 15.1.8. In a similar fashion, we see that from the transitive action of $SU(n)$ on S^{2n-1}, $S^{2n-1}\cong SU(n)/SU(n-1)$. In particular, $S^3\cong SU(2)$.

What do these examples tell us about the topological properties of $SO(n)$ and $SU(n)$? It would seem that we can at least assert each is connected. First, a result from general topology.

Lemma 15.1.9. *Suppose that $f: X\to Y$ is a continuous map of topological spaces which is surjective and open. If the fiber $f^{-1}(y)$ is connected for any $y\in Y$, and Y itself is connected, then X is connected.*

Proof. Suppose U, V are two open sets in X such that $U \cup V = X$. We wish to show that $U \cap V \neq \emptyset$. Since $U \cup V = X$ and f is surjective, $f(U) \cup f(V) = Y$. Also, both $f(U)$ and $f(V)$ are open, since f is an open mapping. Since Y is connected, $f(U) \cap f(V) \neq \emptyset$. Let y be a point in this intersection. Then, $f^{-1}(y) \cap U \neq \emptyset$ and $f^{-1}(y) \cap V \neq \emptyset$. Since $f^{-1}(y)$ is connected,

$$(f^{-1}(y) \cap U) \cap (f^{-1}(y) \cap V) \subseteq U \cap V \neq \emptyset.$$

□

Note now that is $F \to P \xrightarrow{\pi} B$ is a principal bundle, π is a surjective, open map.

Proposition 15.1.10. *$SO(n)$ and $SU(n)$ are connected.*

Proof. We proceed by induction on n. The base cases are handles in the above examples, as both S^1 and S^3 are connected. Now, suppose $n - 1 \geq 2$ and consider the principal $SO(n-1)$ bundle $SO(n-1) \to SO(n) \xrightarrow{\pi} SO(n)/SO(n-1) \cong S^{n-1}$. Assuming $SO(n-1)$ is connected, using the lemma and the fact that S^1 is connected, $SO(n)$ is connected as well.

A similar argument works for $SU(n)$. □

16. Lecture 16

16.1. **Fiber Bundles.**

Definition 16.1.1. A (C^∞) *fiber bundle* $\pi : E \to B$ with *total space* E, *base space* B and *typical fiber* F is a submersion such that for any $b \in B$, there exists an open neighborhood U of b and a diffeomorphism $\phi : \pi^{-1}U \to U \times F$ so that

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "p_1"] \\ & U & \end{tikzcd}$$

commutes.

We already know some examples of fiber bundles:

(1) vector bundles are fiber bundles having finite dimensional vector spaces as typical fibers,
(2) principal bundles are fiber bundles having Lie groups as typical fibers,
(3) covering spaces of manifolds are fiber bundles having discrete sets of points as typical fibers.

Our motivation for introducing the concept of a fiber bundle is to compute some topological invariants of Lie groups. In fact, we have already done so when it was proved that both $SO(n)$ and $SU(n)$ are connected via the fiber bundles $SO(n-1) \to SO(n) \to S^{n-1}$ and $SU(n-1) \to SU(n) \to S^{2n-1}$.

Given a topological space X, we may (try) to compute it's fundamental group $\pi_1 X$. We will show:

Proposition 16.1.2. *$\pi_1 SU(n)$ is trivial for all n.*

Proposition 16.1.3. *$\pi_1 SO(n) \cong \pi_1 SO(3)$ for all $n \geq 3$.*

Remark 16.1.4. Later, we will use the Adjoint representation to show that $\pi_1 SO(3) \cong \mathbb{Z}/2\mathbb{Z}$.

16.2. **Prerequisites from Homotopy Theory.**

Definition 16.2.1. Let $f_0, f_1 : X \to Y$ be continuous maps of topological spaces. We say f_0 and f_1 are *homotopic* if there exists a continuous map $F : X \times [0,1] \to Y$ satisfying

$$\begin{aligned} F(x,0) &= f_0(x) \\ F(x,1) &= f_1(x) \end{aligned}$$

We may also fix a subset $A \subset X$ and require $F(x,t) = y_0$ for some $y_0 \in Y$ and *all* $t \in [0,1]$ *and all* $x \in A$ IN this case we say f_0 and f_1 are *homotopic relative to* A.

One can prove that continuous maps being homotopic is an equivalence relation. The equivalence classes are written as $[(X,A),(Y,y_0)]$.

Example 16.2.2. Let $X = S^1$ and $A = \{x_0\} \in S^1$. Then, the equivalence classes are $[(S^1,x_0),(Y,y_0)] = \pi_1 Y$, the fundamental group of Y. The group operation is "concatenation" of loops.

Letting I denote the unit interval and ∂I its boundary, we can also think of the fundamental group as $[(I, \partial I), (Y, y_0)]$. At first, this does not seem too advantageous, but it allows us a way to think of the *higher homotopy groups*: $\pi_k Y := [(I^k, \partial I^k), (Y, y_0)]$.

Fact: $\pi_k S^n$ is trivial for $k < n$.

The homotopy groups are functors on the category of pointed spaces and continuous maps; in particular if $f : (X, x_0) \to (Y, y_0)$ is a continuous map, it induces a group homomorphism $f_* : \pi_k(X, x_0) \to \pi_k(Y, y_0)$. This comes into play in the following theorem.

Theorem 16.2.3. *Let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fiber bundle. Then, there is an exact sequence*

$$\cdots \longrightarrow \pi_2(B, b_0) \xrightarrow{\partial_2} \pi_1(F, f_0) \xrightarrow{i_*} \pi_1(E, e_0) \xrightarrow{\pi_*} \pi_1(B, b_0) \xrightarrow{\partial_1} \pi_0(F)$$

where $i_*, \pi_*, \partial_2 \ldots$ are group homomorphisms and ∂_1 is a map of sets. Exactness at $\pi_1(B, b_0)$ means the fibers of ∂_1 are cosets of $\pi_*(\pi_1(E, e_0))$ in $\pi_1(B, b_0)$.

Idea of where ∂_2 comes from in the exact sequence:

Suppose $\sigma : (I^2, \partial I^2) \to (B, b_0)$ represents a class in $\pi_2(B, b_0)$ Lift σ to a map $\tilde{\sigma} : I^2 \to E$ (we are using the fact that $E \to B$ is a fiber bundle here). Then, $\tilde{\sigma}(\partial I^2) \subset \pi^{-1}(b_0) \subset F$. This gives a map $\alpha : \partial I^2 = S^1 \to F$, and hence an element of $\pi_1 F$. Define $\partial_2[\sigma] = [\alpha]$.

Let's apply this theorem and prove Propositions 16.2 and 16.3. Consider first the fiber bundle $SU(n-1) \to SU(n) \to S^{2n-1}$. We proceed by induction on n. For $n \geq 3$, the portion of the homotopy long exact sequence we are interested in looks like

$$\pi_1 SU(n-1) \to \pi_1 SU(n) \to \pi_1 S^{2n-1}$$

By induction, and the fact listed above, we may rewrite this sequence as

$$\{1\} \to \pi_1 SU(n) \to \{1\}$$

and hence $\pi_1 SU(n) \cong \{1\}$

In a similar manner, considering the fiber bundle $SO(n-1) \to SO(n) \to S^{n-1}$ we look at

$$\pi_2 S^{n-1} \to \pi_1 SO(n-1) \to \pi_1 SO(n) \to \pi_1 S^{n-1}$$

Since $\pi_1 S^{n-1} = \{1\}$ for $n > 2$ and $\pi_2 S^{n-1} = \{1\}$ for $n > 3$, we see that

$$\pi_1 SO(n) \cong \pi_1 SO(n-1), \;\; n-1 \geq 3$$

17. Lecture 17

17.1. **Deformation Retracts of Classical Lie Groups.**

Definition 17.1.1. A space *deformation retracts* onto a subspace $A \hookrightarrow X$ if the identity map on X is homotopic to a map $r : X \to X$ such that $r(X) \subseteq A$ and $r|_A = \mathrm{id}_A$.

Theorem 17.1.2. *Each of the following Lie groups admits a deformation retraction onto the indicated subspace:*

$$\begin{array}{rcl} GL(n,\mathbb{R}) & \to & O(n) \\ SL(n,\mathbb{R}) & \to & SO(n) \\ GL(n,\mathbb{C}) & \to & U(n) \\ SL(n,\mathbb{C}) & \to & SU(n) \end{array}$$

Proof. (1) The main idea behind each of these proofs is the Gram-Schmidt method from linear algebra. Let

$$T^+(n,\mathbb{R}) = \{(a_{ij}) \in GL(n\mathbb{R}) \mid a_{ii} > 0, a_{ij} = 0 \text{ for } i < j\}$$

Then $T^+(n,\mathbb{R})$ is a Lie group and is diffeomorphic to $(\mathbb{R}^{>0})^n \times \mathbb{R}^{(n^2-n)/2}$ and is hence contractible.

Claim 1: Any $B \in GL(n,\mathbb{R})$ can be written uniquely as $B = AT$ where $A \in O(n)$ and $T \in T^+(n,\mathbb{R})$.

To prove this, suppose that $B = [v_1|\ldots|v_n]$ and apply Gram-Schmidt to $\{v_1,\ldots,v_n\}$ to obtain an orthonormal set of vectors $\{w_1,\ldots,w_n\}$. Let $W = [w_1|\ldots|w_n] \in O(n)$. Note that, by construction, we have

$$\begin{array}{rcl} v_1 & = & \|v_1\|w_1 \\ v_2 & = & \|v_2 - \langle v_2, w_1\rangle\|w_1\|w_2 + \langle v_2, v_1\rangle w_1 \\ & \vdots & \end{array}$$

That is,

$$\begin{array}{rcl} v_1 & = & a_{11}w_1 \\ v_2 & = & a_{12}w_1 + a_{22}w_2 \\ & \vdots & \end{array}$$

where $a_{ij} > 0$. But this says exactly that we have obtained the factorization $B = WA$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ 0 & a_{22} & \ldots & a_{2n} \\ 0 & 0 & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

is in $T^+(n,\mathbb{R})$.

What about uniqueness? Suppose $W_1A_1 = W_2A_2$ where $W_i \in O(n)$ and $A_i \in T^+(n,\mathbb{R})$. Then, $X := W_2^{-1}W_1 = A_2A_1^{-1} \in O(n) \cap T^+(n,\mathbb{R})$. Since X is orthogonal, $X^{-1} = X^T$. Since $X \in T^+(n,\mathbb{R})$, so is X^{-1} and hence X^T. But then X is a diagonal matrix with positive entries so that $X^T = X^{-1} = X$. Hence, X is the identity.

Now, existence and uniqueness of the factorization $B = WA$ is equivalent to the map

$$\begin{aligned} \phi : O(n) \times T^+(n,\mathbb{R}) &\to GL(n,\mathbb{R}) \\ (W,A) &\mapsto WA \end{aligned}$$

Now, ϕ is a smooth map since it's a polynomial. Is it a diffeomorphism? Note that

$$\begin{aligned} \mathfrak{o}(n) &= \left\{X \in \mathfrak{gl}(n,\mathbb{R}) \,|\, X + X^T = 0\right\} \\ \mathfrak{t}^+ &= \{Y \in \mathfrak{gl}(n,\mathbb{R}) \,|\, Y \text{ is upper triangular}\} \end{aligned}$$

Let E_{ij} be the matrix with a 1 in the ijth entry and zeros elsewhere. Then,

$$E_{ji} = (E_{ji} - E_{ij}) + E_{ij} \implies \mathfrak{gl}(n,\mathbb{R}) = \mathfrak{o}(n) \oplus \mathfrak{t}^+$$

and the map $\delta\phi : T_IO(n) \times T_IT^+(n,\mathbb{R}) \to T_1GL(n,\mathbb{R})$ is an isomorphism. But ϕ is also equivariant: for all $W_1, W_2 \in O(n)$ and $A_1, A_2 \in T^+(n,\mathbb{R})$, we have

$$\phi(W_1W_2, A_1A_2^{-1}) = W_1W_2A_1A_2^{-1} = W_1\phi(W_2,A_1)A_2^{-1}$$

Hence, ϕ is a diffeomorphism and $GL(n,\mathbb{R})$ deformation retracts onto $O(n)$. A similar proof works to show that $GL(n,\mathbb{C})$ retracts onto $U(n)$.

(2) Now, suppose that $B \in SL(n,\mathbb{R})$. Then, as above there is a unique factorization $B = WA$. Note that $1 = \det B = \det W \det A = \pm 1 \det A$. and $\det A$ is positive. Hence, $A \in SL(n,\mathbb{R}) \cap T^+(n,\mathbb{R})$. Now,

$$\begin{aligned} T^+(n,\mathbb{R}) \cap SL(n,\mathbb{R}) &= \left\{(a_{ij}) \,|\, a_{ii} > 0, \prod a_{ii} = 1\right\} \\ &\approx \left\{(e^{x_1}, \ldots, e^{x_n}) \,|\, x_i \in \mathbb{R}, \sum x_i = 0\right\} \times \mathbb{R}^{(n^2-n)/2} \\ &\approx \mathbb{R}^{n+(n-2)/2} \end{aligned}$$

Therefore, $SL(n,\mathbb{R}) \approx SO(n) \times R^{n+(n-2)/2}$ and from this we see $SO(n)$ is a deformation retract of $SL(n,\mathbb{R})$. A similar proof works in showing that $SL(n,\mathbb{C})$ deformation retracts onto $SU(n)$.

□

Remark 17.1.3. From the above computations we can say a few more things about the classical Lie groups:

(1) $O(n) = \begin{bmatrix} -1 & 0 & \ldots & 0 \\ 0 & 1 & \vdots & 0 \\ 0 & \ldots & \ldots & 1 \end{bmatrix} SO(n) \coprod SO(n)$ and hence $O(n)$ has exactly two connected components.

(2) $SL(n,\mathbb{R})$ is connected and $\pi_1 SL(n,\mathbb{R}) = \pi_1 SO(n)$.

(3) The map

$$\begin{array}{rcl} \phi : U(1) \times SU(n) & \to & U(n) \\ (\lambda, A) & \mapsto & \begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix} A \end{array}$$

is onto and $\ker \phi \cong \{\lambda \mid \lambda^n = 1\}$. Hence, $U(n)$ is connected and $\pi_1 U(n) = \pi_1 U(1) \cong \mathbb{Z}$. Furthermore, $GL(n, \mathbb{C})$ is connected and $\pi_1 GL(n, \mathbb{C}) \cong \mathbb{Z}$ as well.

(4) $SL(n, \mathbb{C})$ is connected and $\pi_1 SL(n, \mathbb{C}) = \pi_1 SU(n) = \{1\}$.

Theorem 17.1.4. *Let $\widetilde{SL}(2, \mathbb{R}) \xrightarrow{\pi} SL(2, \mathbb{R})$ denote the double cover of $SL(2, \mathbb{R})$. Then, any Lie group homomorphism $\rho : \widetilde{SL}(2, \mathbb{R}) \to GL(n, \mathbb{R})$ has a nontrivial kernel.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathfrak{sl}(2,\mathbb{C}) \arrow[r, "(\delta\rho)_{\mathbb{C}}"] \arrow[dddd, bend right=60, "\exp"'] & \mathfrak{gl}(n,\mathbb{C}) \arrow[dddd, bend left=60, "\exp"] \\
\mathfrak{sl}(2,\mathbb{R}) \arrow[u, hook] \arrow[r, "\delta\rho"] \arrow[d, "\exp"'] & \mathfrak{gl}(n,\mathbb{R}) \arrow[u, hook] \arrow[d, "\exp"] \\
\widetilde{SL}(2,\mathbb{R}) \arrow[r, "\rho"] \arrow[d, "\pi"'] \arrow[ddr, "\tau\circ\pi"] & GL(n,\mathbb{R}) \arrow[dd, hook] \\
SL(2,\mathbb{R}) \arrow[d, hook] & \\
SL(2,\mathbb{C}) \arrow[r, "\tau"] & GL(n,\mathbb{C})
\end{tikzcd}$$

where

(i) $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{R}) \oplus i\mathfrak{sl}(2, \mathbb{R})$ (similarly for $\mathfrak{gl}(2, \mathbb{C})$),

(ii) $(\delta\rho)_{\mathbb{C}}$ is the extension of $\delta\rho$ with respect to the decomposition in (i), and

(iii) since $\pi_1 \mathfrak{sl}(2, \mathbb{C}) = \{1\}$, there is a unique $\tau : SL(2, \mathbb{C}) \to GL(n, \mathbb{C})$ such that $(d\tau)_I = \delta\rho$.

Now, since $\widetilde{SL}(2, \mathbb{R})$ is connected, $\tau \circ \pi = \rho$. But π is a covering map with two sheets and $\ker \rho \supseteq \ker \pi \cong \mathbb{Z}/2\mathbb{Z}$. $\square$

18. Lecture 18

18.1. **Compact Connected Abelian Groups.** In this lecture, we prove that all compact, connected, abelian Lie groups are isomorphic to tori. This will aid us greatly when we examine representations of Lie groups.

Theorem 18.1.1. *Any compact, connected, abelian Lie group is isomorphic to $\mathbb{T}^n := (S^1)^n \cong \mathbb{R}^n/\mathbb{Z}^n$ for some n.*

Before we prove this theorem, we'll need a few results about discrete subgroups of finite dimensional real vector spaces.

Lemma 18.1.2. *Let V be a finite dimensional real vector space and $\Lambda \subset V$ a discrete subgroup. Then, there is a set $\{v_1, \ldots, v_n\}$ which is linearly independent such that for all $\lambda \in \Lambda$, there exists $n_1, \ldots, n_k \in \mathbb{Z}$ so that $\lambda = \sum n_i v_i$. In other words, $\Lambda = span_{\mathbb{Z}}\{v_1, \ldots, v_n\}$.*

Proof. We proceed by induction on $\dim V$. Suppose that $\dim V = 1$. We may assume $V = \mathbb{R}$. Let $v_1 = \inf\{\lambda \in \Lambda \mid \lambda > 0\}$. Since Λ is discrete, $v_1 \neq 0$ and $\Lambda = \mathbb{Z}v_1$.

Now, fix an inner product on V. Since Λ is discrete, Λ has a shortest nonzero vector, call it v_1. Then,

$$0 < \|v_1\| \leq \|\lambda\| \text{ for all } \lambda \in \Lambda, \lambda \neq 0.$$

Let $W = (\mathbb{R}v_1)^\perp$, the orthogonal complement so that $V = \mathbb{R}v_1 \oplus W$. Let $p_1 : V \to \mathbb{R}v_1$ and $p_2 : V \to W$ be the corresponding orthogonal projections. If we can show that $p_2(\Lambda)$ is discrete, we are done by induction.

Since $p_2(\Lambda)$ is a subgroup of W, it is enough to show: for any $\lambda \in \Lambda$ with $p_2(\lambda) \neq 0$, $\|p_2(\lambda)\| \geq 1/2\|v_1\|$. Suppose not, that is there exists $\lambda \in Lambda$ so that $0 < \|p_2(\lambda)\| < 1/2\|v_1\|$. Now, for some $N \in \mathbb{Z}$,

$$Nv_1 \leq p_1(\lambda) \leq (N+1)v_1$$

and thus there exists an integer m such that

$$\|p_1(\lambda - mv_1)\| = \|p_1(\lambda) - mv_1\| \leq 1/2\|v_1\|$$

This implies

$$\begin{aligned} \|\lambda - mv_1\| &= (\|p_1(\lambda - mv_1\|)^2 + \|p_2(\lambda - mv_1)\|^2)^{1/2} \\ &\leq (\|p_2(\lambda)\|^2 + 1/4\|v_1\|^2)^{1/2} \\ &< ((1/2\|v_1\|)^2 + (1/2\|v_1\|)^2)^{1/2} \\ &= 1/\sqrt{2}\|v_1\| < \|v_1\| \end{aligned}$$

But this contradicts the choice of v_1. ∎

Corollary 18.1.3. *Let V be an n-dimensional real vector space and $\lambda \subset V$ a discrete subgroup. Then, $V/\Lambda \cong (\mathbb{R}^k/\mathbb{Z}^k) \times \mathbb{R}^{n-k}$.*

Proof. Choose $\{v_1, \ldots, v_k\} \subset V$ such that $\operatorname{span}_{\mathbb{Z}}\{v_1, \ldots, v_k\} = \Lambda$. Complete this to a basis $\{v_1, \ldots, v_n\}$ of V. The isomorpism $R^k \times \mathbb{R}^{n-k} \to V$, $(a_1, \ldots, a_n) \mapsto \sum a_i v_i$ identifies $\mathbb{Z}^n$ with Λ. Hence, $V/\Lambda \cong \mathbb{R}^k/\mathbb{Z}^k \times \mathbb{R}^{n-k}$. ∎

We are now in a position to prove Theorem 18.1:

of Theorem 18.1. We first argue that $\exp : \mathfrak{g} \to G$ is a surjective Lie group homomorphism. Since G is abelian, the multiplication map $m : G \times G \to G$ is a homomorphism. Since exp is natural, the diagram

$$\begin{tikzcd} \mathfrak{g} \times \mathfrak{g} \arrow[r, "\delta m"] \arrow[d, "\exp \times \exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

commutes. Since $\delta m(X,Y) = (dm)_{(1,1)}(X,Y) = X + Y$ we get

$$\exp(X+Y) = \exp X \exp Y$$

and so $\exp : \mathfrak{g} \to G$ is a homomorphism. Now, $\exp(\mathfrak{g})$ is a connected subgroup containing an open neighborhood of $1 \in G$. Since G is connected, it follows that exp is onto.

Since $(d\exp)_0 = \mathrm{id}$, we see from the above that exp is a covering map. Hence, $\Lambda = \ker\{\exp\} \subset \mathfrak{g}$ is a discrete subgroup.

Since $\mathfrak{g}/\Lambda = G$ is compact, there is a basis $\{v_1, \dots, v_n\}$ of $\mathfrak{g}$ so that $\Lambda = \operatorname{span}_{\mathbb{Z}}\{v_1, \dots, v_n\}$. Hence, $G \cong \mathfrak{g}/\Lambda \cong \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$. $\square$

19. Lecture 19

19.1. **Representations of Lie Groups.** Recall that a representation of a Lie group G is a homomorphism from G to the general linear group of some finite dimensional vector space. Let $\rho : G \to GL(V)$ be such a representation.

Definition 19.1.1. A *subrepresentation* is a subspace $W \subset V$ such that for any $g \in G$, $w, \in W$ we have $\rho(g)w \in W$. In other words, W is a G-invariant subspace.

if $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ are two representations of G, then we have

$$\begin{aligned} \rho_1 \oplus \rho_2 : G &\to GL(V_1 \oplus V_2) \\ (\rho_1 \oplus \rho_2)(g(v_1 + v_2) &= \rho_1(g)v_1 + \rho_2(g)v_2 \end{aligned}$$

A representation (real or complex) $\rho : G \to GL(V)$ is *irreducible* if it has no nontrivial invariant subspaces: if $W \subset V$ is such that $\rho(g)W \subset W$ for all $g \in G$, then either $W = \{0\}$ or $W = V$.

Example 19.1.2. $\rho : S^1 \to GL(1, \mathbb{C})$ given by $\rho(\lambda) = lambda$ is irreducible since $\dim \mathbb{C} = 1$.

Example 19.1.3. $\rho : SU(2) \to GL(2, \mathbb{C})$ is irredducible: $SU(2)$ acts transitively on S^3 and $\operatorname{span}_{\mathbb{C}}(S^3) = \mathbb{C}^2$.

Example 19.1.4. $\rho : \mathbb{R} \to G : (2, \mathbb{R})$ given by $\rho(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ is not irreducible: $\mathbb{R} \times \{0\}$ is invariant.

Definition 19.1.5. A complex representation $\rho : G \to GL(V)$ is *unitary* if there is a Hermitian inner product $\langle , \rangle$ on V such that

$$\langle \rho(g)v, \rho(g)w \rangle = \langle v \rangle w, \quad \forall g \in G, \forall v, w, \in V$$

That is, the representation of G on V preserves $\langle , \rangle$.

Example 19.1.6. $\rho : S^1 \to GL(2, \mathbb{C})$ given by $\rho(e^{i\theta})z = e^{i\theta}z$ is unitary but not irreducible ($\mathbb{C}z$ is invariant for any z).

Lemma 19.1.7. *Let $\rho : G \to GL(V)$ be a unitary represenatation. Then, ρ is a direct sum of irreducible representations.*

Proof. We proceed by induction on $\dim_{\mathbb{C}} V$. If $\dim_C V = 1$, then ρ is irreducible. Suppose $\dim_{\mathbb{C}} V > 1$ and V is not irreducible. Then, there exists an G-invariant subspace W, $\dim_{\mathbb{C}} W \neq \dim_{\mathbb{C}} V$. Let $W^\perp$ denote the orthogonal complement of W with respect to the Hermitian inner product on V.

<u>claim</u>: $W^\perp$ is an invariant subspace.

To prove this, take $v \in W^\perp$. Then, for any $w \in W$ and any $g \in G$

$$\begin{aligned} \langle \rho^{-1}(G)w, v \rangle &= \langle \rho(g)\rho(g^{-1})w, \rho(g)v \rangle \\ &= \langle w, \rho(g)v \rangle \end{aligned}$$

Since W is invariant, $\rho(g^{-1})w \in W$ and so $0 = \langle \rho(g^{-1})w, v\rangle = \langle w, \rho(g)v\rangle$. Hence, $\rho(g)v \in W^\perp$ for all $g \in G$. Thus, $V = W \oplus W^\perp$ where (by induction and assumption) both W and $W^\perp$ are invariant. $\square$

Definition 19.1.8. A representation $\rho : G \to GL(V)$ is *completely reducible* if it is a direct sum of irreducible representations.

Remark 19.1.9. We have thus proved that any unitary representation is completely reducible.

Example 19.1.10. The representation $rho : \mathbb{R} \to GL(2, \mathbb{C})$ from Example 19.4 is neither irreducible nor completely reducible. Note that $\mathbb{C} \times \{0\}$ is invariant. If $w \notin \mathbb{C} \times \{0\}$ say $w = (w_1, w_2)$ then $w_2 \neq 0$. So

$$\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1 + tw_2 \\ w_2 \end{bmatrix}$$

Set $t = -w_1/w_2$ so that $\rho(t)^2 w = w_2 \begin{bmatrix} t \\ 1 \end{bmatrix}$. Hence

$$\operatorname{span}_{\mathbb{C}} \left\{ \rho(t) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \mid t \in \mathbb{C} \right\} = \mathbb{C}^2$$

The following proposition is a glance at results to come:

Proposition 19.1.11. *Any complex representation of a finite group G is unitary.*

Proof. let $\rho : G \to GL(V)$ be a representation. Pick a hermitian inner product $\langle , \rangle$ on V. It need not be invariant. Now, define

$$\ll v, w \gg = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle$$

where $|G|$ is the number of elements in G.

Then, for any $a \in G$, $v, w \in V$

$$\begin{aligned} \ll \rho(a)v, \rho(a)w \gg &= \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)\rho(a)v, \rho(g)\rho(a)w \rangle \\ &= \frac{1}{|G|} \sum_{g \in G} \langle \rho(ga)v, \rho(ga)w \rangle \end{aligned}$$

But, $R_a : G \to G$ is a bijection. So, let $g' = ga$. Then, the last equality becomes

$$\frac{1}{|G|} \sum_{g' \in G} \langle \rho(g')v, \rho(g')w \rangle = \ll v, w \gg$$

It follows that $\ll , \gg$ is invariant. It is clear that $\ll , \gg$ is sesquilinear and moreover, for $v \neq 0$, $\ll v, v \gg > 0$. Hence, $\ll , \gg$ is an invariant Hermitian inner product. $\square$

We'd like to do the same thing for Lie groups: if $\langle , \rangle$ is a Hermitian inner product on a representation $\rho : G \to GL(V)$ of G, then for fixed $v, w \in V$ $g \mapsto \langle \rho(g)v, \rho(g)w \rangle$ is a function on G.

Also, for $v \neq 0$, $f_v(g) = \langle \rho(g)v, \rho(g)v \rangle > 0$ for all g. Thus, for an appropriate measure $d\mu_g$, we have

$$\int_G f_v(g) d\mu_g > 0$$

If $|G| := \displaystyle\int_G \mu_g < \infty$, then

$$\ll v, w \gg = \frac{1}{|G|} \int_G \langle \rho(g)v, \rho(g)w \rangle d\mu_g$$

makes sense and is a Hermitian inner product. Hence,

Theorem 19.1.12. *Any representation of a compact Lie group is completely reducible.*

20. Lecture 20

20.1. Schur's Lemma.

Definition 20.1.1. let V_1, V_2 be two representations of Lie group G. A linear map

$$T : V_1 \to V_2$$

intertwines the two representations if $T(g \cdot v) = g \cdot T(v)$ for all $g \in G$, $v \in V_1$. We write

$$\mathrm{Hom}_G(V_1, V_2) = \{T : V_1 \to V_2 \,|\, T \text{ intertwines } V_1, V_2\}$$

Proposition 20.1.2 (Schur's Lemma, Version 1)**.** *Suppose T_1, T_2 are two irreducible representations and $T \in Hom_G(V_1, V_2)$. Then, either $T = 0$ or T is an isomorphism.*

Proof. Note that since T is intertwining, $\ker T$ and $\operatorname{im} T$ are invariant subspaces of V_1 and V_2 respectively.

(i) $Tv = 0 \implies T(g \cdot 0) = g \cdot (TV) = 0$
(ii) $g \cdot (Tv) = T(g \cdot v) \in \operatorname{im} T$.

Suppose $T \neq 0$. Them $\operatorname{im} T \neq 0$ and hence $\operatorname{im} T = V_2$. Since $\operatorname{im} T \neq 0$, $\ker T \neq V_1$ and T is an isomorphism. □

Proposition 20.1.3 (Schur's Lemma, Version 2)**.** *Suppose V_1 and V_2 are two irreducible representations. Suppose further V is complex. Then,*

$$Hom_G(V, V) \cong \mathbb{C}$$

That is to say, if $T : V \to V$ intertwines then there is $\lambda \in \mathbb{C}$, depending on T, such that $Tv = \lambda v$ for all $v \in V$.

Proof. Let λ be an eigenvalue of T. Then, $T - \lambda \,\mathrm{id} : V \to V$ is an intertwining map and $\ker(T - \lambda \,\mathrm{id}) \neq 0$. Thus, $T - \lambda \,\mathrm{id} = 0$. □

Corollary 20.1.4. *Any complex irreducible representation of a compact abelian group is* 1*-dimensional.*

Proof. Let $\rho : G \to GL(V)$ be a complex irreducible representation. For any $a \in G$, $\rho(a) : V \to V$ is an intertwining map. Thus, $\rho(a) = \lambda_a$ for some $\lambda_a \in \mathbb{C}$. Since a is arbitrary, $\rho(a) = \lambda_a$ for all $a \in G$. Hence, any 1-dimensional subspace is ireeducible and $\dim V = 1$. □

20.2. Irreducible Representations of $\mathbb{T}^n$.

Lemma 20.2.1. *Any complex irreducible representation $\rho : \mathbb{T}^n \to GL(V)$ is of the form*

$$\rho(\exp v) = \rho(v \mod \mathbb{Z}^n) = e^{2\pi i \delta\rho(v)}$$

where $\delta\rho : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\delta\rho(\mathbb{Z}^n) \subset \mathbb{Z}^n$ (i.e. $\delta\rho \in Hom_{\mathbb{Z}}(\mathbb{Z}^n, Z)$).

Proof. If $\mu \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}) \subset \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n, \mathbb{R})$, then

$$\begin{aligned} \xi_\mu : \mathbb{R}^n/\mathbb{Z}^n &\to \mathbb{R}/\mathbb{Z} \\ \xi_\mu(v \mod \mathbb{Z}^n) &= \mu(v) \mod \mathbb{Z} \end{aligned}$$

is well-defined. We identify $\mathbb{R}/\mathbb{Z}$ with S^1: $a \mod Z \mapsto e^{2\pi i a}$. So, ξ_μ is a representation

$$\begin{aligned} \xi_\mu : \mathbb{T}^n &\to Gl(\mathbb{C}) \\ \xi_\mu(v \mod \mathbb{Z}^n) &= e^{2\pi i \mu(v)} \end{aligned}$$

Conversely, suppose that $\rho : \mathbb{T}^n \to GL(\mathbb{C}) \cong \mathbb{C} \setminus \{0\}$ is a representation. Since $\rho(\mathbb{T}^m) \subseteq \mathbb{C}^\times$ is compact, $\rho(\mathbb{T}^n) \subseteq S^1$. We have a commuting diagram

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\delta\rho"] \arrow[d, "\exp"'] & \mathbb{R} \arrow[d, "\exp"] \\ \mathbb{T}^n \arrow[r, "\rho"] & S^1 \end{tikzcd}$$

so $\delta\rho(\ker \exp) \subseteq \ker \exp = \mathbb{Z}$. Therefore, $\delta\rho \in \operatorname{Hom}_Z(\mathbb{Z}^n, \mathbb{Z})$ □

Remark 20.2.2. If G is any compact, connected abelian Lie group (i.e. a torus),

$$\mathbb{Z}_G := \ker\{\exp : \mathfrak{g} \to G\}$$

is called the *integral lattice*. The set

$$\mathbb{Z}_G^* := \operatorname{Hom}_Z(\mathbb{Z}_G, \mathbb{Z})$$

is called the *weight lattice*

The above lemma proves: irreducible, unitary represenations of G are in one-to-one correspondence with elements of the weight lattice.

20.3. **Representations of $SU(2)$.** We start by constructing complex irreducible representations of $SU(n)$. Let V_n be the set of all complex homogeneous polynomials of degree n in two variables. That is

$$v_n = \operatorname{span}_{\mathbb{C}}\{z_1^n, z_1^{n-1}z_2, \dots, z_2^n\}$$

Note that $V_0 = \mathbb{C}$ and $V_1 \cong \mathbb{C}^2$.

We have an action of $GL(2, \mathbb{C})$ on V_n.

$$(A \cdot f)(z_1, z_2) = f((z_1, z_2)A)$$

where $(z_1, z_2)A$ is regarded as matrix multiplication.

It is left as an exercise to the reader to prove this indeed defines an action. This also gives us a representation:

$$A \cdot (\lambda f + \mu g) = \lambda(A \cdot f) + \mu(A \cdot g)$$

for all $\lambda, \mu \in \mathbb{C}$, $f, g \in V_n$ and $A \in GL(2, \mathbb{C})$.

Theorem 20.3.1. *Let V_n be as above. Then,*

(i) *V_n is an irreducible representation of $SU(n)$ for all $n \geq 0$.*
(ii) *If V is an irreducible representation of $SU(n)$ of dimension $n+1$ then $V \cong V_n$ as representations.*

We defer the proof of this theorem for now.

21. Lecture 21

21.1. **Complexification.**

Definition 21.1.1. let V be a finite dimensional vector space over $\mathbb{R}$. The *complexification* $V_{\mathbb{C}}$ of V is $V \otimes \mathbb{C}$.

Note that $V_{\mathbb{C}}$ is a complex vector space: for any $a, b \in \mathbb{C}$, $v \in V$ we have $a(v \otimes b) = v \otimes ab$. Also, V is embedded in $V_{\mathbb{C}}$ as a real subspace

$$V \hookrightarrow V_{\mathbb{C}},\ v \mapsto v \otimes 1$$

We now identify V with $V \otimes 1 \subset V_{\mathbb{C}}$ and we write av for $v \otimes a$, $v \in V$, $a \in \mathbb{C}$.

As a real vector space, $V_{\mathbb{C}} = V \oplus iV$, where $iV = \{v \otimes i \,|\, v \in V\}$.

Remark 21.1.2. If $\{v_1, \ldots, v_n\}$ is a basis of V, then it is also a complex basis of $V_{\mathbb{C}}$. Considered as a real vector space, $V_{\mathbb{C}}$ has as a basis the set $\{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$.

Lemma 21.1.3. *let V be a real vector space, W a complex vector space and $T : V \to W$ an $\mathbb{R}$-linear map. Then, there exists a unique $\mathbb{C}$-linear map $T_{\mathbb{C}} : V_{\mathbb{C}} \to W$ extending T.*

Proof. (Uniqueness) For any $v, w \in V$, we have $T_{\mathbb{C}}(v + iw) = T_{\mathbb{C}}(v) + iT_{\mathbb{C}}(w) = T(v) + iT(w)$.

(Existence) Let $\{v_1, \ldots, v_n\}$ be a basis for V. Define

$$T_{\mathbb{C}}\left(\sum a_i v_i\right) = \sum a_i T(v_i)$$

for $a_i \in \mathbb{C}$. Then, $T_{\mathbb{C}}$ is complex linear and extends T. By uniqueness, $T_{\mathbb{C}}$ doesn't depend on the choice of basis. □

Lemma 21.1.4. *Let V be a real vector space, W a complex vector space and $b : V \times V \to W$ and $\mathbb{R}$-bilinear map. Then, there exists a unique $\mathbb{C}$-bilinear map $b_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to W$ extending b.*

The proof of this lemma is left as an exercise. As a corollary, we get

Corollary 21.1.5. *If $\mathfrak{g}$ is a real Lie algebra, then the Lie bracket on $\mathfrak{g}$ extends to a unique $\mathbb{C}$-bilinear map*

$$[,]_{\mathbb{C}} : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$$

such that $(\mathfrak{g}_{\mathbb{C}}, [,]_{\mathbb{C}})$ is a Lie algebra.

Example 21.1.6. $\mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2, \mathbb{C})$. To see this, first let $T : \mathfrak{su}(2) \to \mathfrak{sl}(2, \mathbb{C})$ denote the inclusion. Then, there exists a unique $T_{\mathbb{C}} : \mathfrak{su}(2)_{\mathbb{C}} \to \mathfrak{sl}(2, \mathbb{C})$ extending T Now

$$\mathfrak{su}(2) = \operatorname{span}_{\mathbb{R}} \left\{ \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\}$$

so that $\mathfrak{su}(2)_{\mathbb{C}}$ is the complex span of the same matrices.

Let $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then, $\mathfrak{sl}(2, \mathbb{C}) = \operatorname{span}_{\mathbb{C}}\{H, E, F\}$. But,

$$\begin{aligned} H &= (-i)\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \in T_{\mathbb{C}}(\mathfrak{su}(2)_{\mathbb{C}}) \\ E &= \frac{1}{2}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - i\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\right) \in T_{\mathbb{C}}(\mathfrak{su}(2)_{\mathbb{C}}) \\ F &= \frac{1}{2}\left((-i)\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \in T_{\mathbb{C}}(\mathfrak{su}(2)_{\mathbb{C}}) \end{aligned}$$

Thus, $T_{\mathbb{C}}$ is onto and hence an isomorphism (by dimension count).

21.2. Representations of $\mathfrak{sl}(2,\mathbb{C})$ I.

Lemma 21.2.1. *There is a bijection between complex representations of $SU(2)$ and of $\mathfrak{sl}(2,\mathbb{C})$.*

Proof. Since $\pi_1 SU(2)$ is trivial, there is a bijection between representations of $SU(2)$ and $\mathfrak{su}(2)$. Any complex representation of $\mathfrak{su}(2)$ extends to a unique complex representation of $\mathfrak{sl}(2)_{\mathbb{C}} = \mathfrak{sl}(s,\mathbb{C})$.

Conversely, a representation of $\mathfrak{sl}(2,\mathbb{C})$ restricts to a representation of $\mathfrak{su}(2) \subset \mathfrak{sl}(2,\mathbb{C})$. □

Corollary 21.2.2. *Any finite dimensional complex representation of $\mathfrak{sl}(2,\mathbb{C})$ is completely reducible.*

Proof. representations of $SU(2)$ are completely reducible. □

We now start proving:

Theorem 21.2.3. *Irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ are classified by non-negative integers: for any $n = 0, 1, 2, \ldots$ there exists a unique representation of $\mathfrak{sl}(2,\mathbb{C})$ of dimension $n+1$*

We first observe

$$\begin{aligned} [H,E] &= 2E \\ [H,F] &= -2F \\ [E,F] &= H \end{aligned}$$

Lemma 21.2.4. *Let $\tau : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V)$ be a representation. Suppose $\tau(H)v = cv$ for some $c \in \mathbb{C}$. Then,*

$$\begin{aligned} \tau(H)(\tau(E)v) &= (c+2)(\tau(E)v) \\ \tau(H)(\tau(F)v) &= (c-2)(\tau(F)v) \end{aligned}$$

Proof.

$$\begin{aligned} 2\tau(E)v &= \tau([H,E])v \\ &= \tau(H)\tau(H)\tau(E)v - \tau(E)\tau(H)v \\ &= \tau(H)(\tau(E)v) - c\tau(E)v \end{aligned}$$

and so, $\tau(H)(\tau(E)v) = (c+2)\tau(E)v$

Similarly,

$$\begin{aligned} -2\tau(F)v &= \tau([H,F]v) \\ &= \tau(H)(\tau(F)v) - c\tau(F)v \end{aligned}$$

and thus $\tau(H)(\tau(F)v) - (c-2)\tau(F)v$ □

Remark 21.2.5. By induction, for $k = 1, 2, \ldots$ we see that

$$\tau(H)(\tau(E)^k v) = (c + 2k)(\tau(E)^k v).$$

But, $\tau(H)$ has only finitely many eigenvalues and so there exists $k \geq 1$ such that $\tau(E)^k v = 0$, $\tau(E)^{k-1} v \neq 0$. We conclude that there exists a $v_0 \in V$ such that $v_0 \neq 0$, $\tau(E)v_0 = 0$ and $\tau(H)v_0 = \lambda v_0$ for some $\lambda \in \mathbb{C}$.

22. Lecture 22

22.1. Representations of $\mathfrak{sl}(2,\mathbb{C})$ II.

Lemma 22.1.1. *Let $\tau : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V)$ be a representation, $v_0 \in V$ be as above. Let*

$$\begin{aligned} v_k &= \frac{1}{k!}\tau(F)^k v_0 \\ v_{-1} &\equiv 0 \end{aligned}$$

Then,

(i) $\tau(H)v_k = (\lambda - 2k)v_k$,
(ii) $\tau(F)v_k = (k+1)v_{k+1}$,
(iii) $\tau(E)v_k = (\lambda - k + 1)v_{k-1}$.

Proof. (i) This is a restatement of Lemma 21.10.
(ii) $\tau(F)v_k = \tau(F)(1/k!\tau(F)^k v_0) = (k+1)(1/(k+1)!\tau(F)^{k+1}v_0)$.
(iii) Proceed by induction on k: $\tau(E)v_0 = (\lambda+1)v_{-1} = 0$. Now,

$$\begin{aligned} k\tau(E)v_k &= \tau(E)(\tau(F)v_{k-1}) \\ &= ((\tau(E)\tau(F) - \tau(F)\tau(E)) + \tau(F)\tau(E))v_{k-1} \\ &= \tau(H)v_{k-1} + \tau(F)(\tau(E)v_{k-1}) \\ &= (\lambda - 2(k-1))v_{k-1} + \tau(F)((\lambda - (k-1) + 1)v_{k-2}) \\ &= ((\lambda - 2k + 2) + (\lambda - k + 2)(k-1))v_{k-1} \\ &= k(\lambda - k + 1)v_{k-1} \end{aligned}$$

So, $\tau(E)v_k = (\lambda - k + 1)v_{k-1}$.

□

Lemma 22.1.2. *Let $\tau : \mathfrak{al}(2,\mathbb{C}) \to \mathfrak{gl}(V)$ be a representation and $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then, $\tau(H)$ is diagonalizable, i.e. V has a basis of eigenvectors.*

Proof. The vector v is an eigenvector of $\tau(H)$ if and only if v is an eigenvector of $\tau(iH)$ which happens if and only if $e^{t\tau(tiH))=\rho(e^{tiHt}}$ where $\rho : SL(2,\mathbb{C}) \to GL(V)$ is a representation with $(d\rho)_1 = \tau$.

But,

$$\{e^{iHt}\} = \left\{ \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \right\} \cong S^1$$

and any representation of S^1 is completely reducible. Moreover, all irreducible representations of S^1 are 1-dimensional. Hence, $\rho(e^{itH})$ has a basis of eigenvalues. □

Since the list of eigenvalues of $\tau(H)$ is finite, there exists an $n \geq 0$ such that $v_{n+1} = 0$ and $v_v \neq 0$. Then, $0 = \tau(E)v_{n+1} = (\lambda - n)v_n$. and so $\lambda = n$. Also, if we let

$$W = \operatorname{span}_C \{v_i \,|\, 1 \leq i \leq n\}$$

we know that W is a representation of $\mathfrak{sl}(2,\mathbb{C})$.

Comclusion: Suppose $\tau : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V)$ is a complex irreducible representation, $n = \dim_{\mathbb{C}} V - 1$. Then, there is a basis $\{v_0, \ldots, v_n\}$ such that the action of $\tau(H), \tau(E)$ and $\tau(F)$ is given by Lemma 22.1 (i)-(iii) with $\lambda = n$. Hence, if V_1, V_2 are two representations of $\mathfrak{sl}(2,\mathbb{C})$, and $\dim V_1 = \dim V_2$, then $V_1 \cong V_2$ as represenations. That is, there exists an isomorphism $T : v_1 \to V_2$ such that for any $X \in \mathfrak{sl}(2,\mathbb{C})$ we have

$$T(X \cdot V) = X \cdot T(v)$$

To complete the proof that irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ are classified by non-negative integers, we check that

$$V_n = \operatorname{span}_{\mathbb{C}} \left\{ z_1^{n-k} z_2^k \,|\, 0 \le k \le n \right\}$$

is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$.

We note that

$$\begin{aligned}
e^{tH} &= \begin{bmatrix} e^t & 0 \\ 0 & e^{-1} \end{bmatrix} \\
e^{tE} &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \\
e^{tF} &= \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}
\end{aligned}$$

So, $(z_1 z_2)e^{tH} = (e^t z_1, e^{-t} z_2)$. Thus, we get

$$\begin{aligned}
e^{tH} \cdot (z_1^{n-k} z_2^{n-k} &= (e^t)^{n-k}(e^{-t})^k z_1^{n-k} z_2^k \\
&= e^{(n-2k)t} z - 1^{n-k} z_2^k
\end{aligned}$$

which implies

$$\begin{aligned}
H \cdot (z_1^{n-k} z_2^k) &= \left.\frac{d}{dt}\right|_{t=0} e^{tz} \cdot (z_1^{n-k} z_2^k) \\
&= (n-2k) z_1^{n-k} z_2^k.
\end{aligned}$$

Also, the fact that $(z_1, z_2) \cdot e^{tE} = (z_1, tz_1 + z_2)$ gives

$$\begin{aligned}
E \cdot (z_1^{n-k} z_2^k) &= \left.\frac{d}{dt}\right|_{t=0} z_1^{n-k}(tz_1 + z_2)^k \\
&= z_1^{n-k} k (tz_1 + z_2)^{k-1} z_1 \\
&= k z_1^{n-k+1} z_2^{k-1}.
\end{aligned}$$

Finally, since $(z_1, z_2)(e^{Tf}) = (z_1 + tz_2, z_2)$, we see that

$$\begin{aligned} F \cdot (z_1^{n-k} z_2^k) &= \left.\frac{d}{dt}\right|_{t=0} (z_1 + t z_2)^{n-k} z_2^k \\ &= (n-k) z_1^{n-(k+1)} z_2^{k+1} \end{aligned}$$

Letting

$$\begin{aligned} v_0 &= z_1^n \\ v_1 &= F \cdot z_1^n = n z_1^{n-1} z_2 \\ v_2 &= \frac{1}{2!} n(n-1) z_1^{n-2} z_2^2 = \binom{n}{2} z_1^{n-2} z_2^2 \\ \vdots &\;\;\vdots\;\;\vdots \\ v_k &= \binom{n}{k} z_1^{n-k} z_2^k \\ \vdots &\;\;\vdots\;\;\vdots \end{aligned}$$

thus completes the classification.

23. Lecture 23

23.1. **Representation Theory of Compact Lie Groups.** Let's list some generalities and give a quick review:

(1) If V_1, V_2 are two complex representations of a Lie group G, then so are
  (i) $V_1 \oplus V_2$; $g \cdot (v_1 \oplus v_2) = (g \cdot v_1) \oplus (g \cdot v_2)$,
  (ii) $V_1 \otimes_{\mathbb{C}} V_2$; $g \cdot (v_1 \otimes v_2) = (g \cdot v_1) \otimes (g \cdot v_2)$,
  (iii) $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$; $(g \cdot T)v = g \cdot (T(g^{-1} \cdot v))$.
  In particular,
  (iv) $V^* = \mathrm{Hom}(V, \mathbb{C})$ is the *dual* representation $(g \cdot l)v = l(g^{-1} \cdot v)$.

(2) If V is a representation of G, we define

$$V^G = \{v \in V \mid g \cdot v = v, \forall g \in G\}$$

the subspace of G-fixed vectors. It's a subrepresentation.
In particular, if V_1, V_2 are two representations of G, then

$$\begin{aligned} \mathrm{Hom}(V_1, V_2)^G &= \left\{T : V_1 \to V_2 \mid g \cdot T(g^{-1} \cdot v) = T(v)\right\} \\ &= \text{intertwining maps} \\ &= \mathrm{Hom}_G(V_1, V_2) \\ &= \text{morphisms of representations from } V_1 \text{ to } V_2 \end{aligned}$$

(3) Schur's Lemma: Let V, W be two complex irreducible representations of G. Then,
  (i) A morphism $T : V \to W$ is either indentically the zero map or an isomorphism,
  (ii) every morphism $T : V \to V$ has the form

$$T(v) = \lambda_T v$$

  for some $\lambda_T \in \mathbb{C}$.
  (iii) we have

$$\mathrm{Hom}_G(V, W) \cong \begin{cases} \mathbb{C} & V \cong W \\ 0 & V \ncong W \end{cases}$$

(4) Let U, V be two representations of$G$. If U is isomorphic to a G-invariant subspace of W, we say that U is *contained* in W.

Example 23.1.1. Let $G = S^1$. Then $\rho_1 : S^1 \to GL(1, \mathbb{C})$, $\rho_1(\lambda) = \lambda^2$ is contained in $\rho_2 : S^1 \to GL(2, \mathbb{C})$, $\rho_2(\lambda) = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$.

Definition 23.1.2. let W be a representation of G, U an irreducible representation of G. We call

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(U, W)$$

the *multiplicity* of U in W.

The reason for this definition: write $W = \oplus_{j=1}^{d} W_j$ where W_j is irreducible. Then,

$$\mathrm{Hom}_G(U, W) = \mathrm{Hom}_G(U \oplus W_j) = \bigoplus \mathrm{Hom}_G(U, W_j)$$

By Schur's Lemma,

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(U, W_j) = \begin{cases} 1 & U \cong W_j \\ 0 & U \not\cong W_j \end{cases}.$$

So,

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(U, W) = |\{W_j \mid W_j \cong U\}|$$

We will write $\mathrm{Irr}_{\mathbb{C}}(G)$ for the irreducible representations of G.

Example 23.1.3. $\mathrm{Irr}(\mathbb{T}^n) \cong \mathbb{Z}^n$.

Suppose that W is a complex representation of a compact Lie group G. Then, since W is completely reducible, we have a decomposition $W = \sum_j W_j$ where the summands are irreducible representations.

We can rewrite this as

$$W = \bigoplus_{U \in \mathrm{Irr}(G)} \left(\bigoplus_{W_j \cong U} W_j \right) = \bigoplus_{U \in \mathrm{Irr}(G)} W_U \tag{1}$$

where we call W_U a *U-isotypical* summand of W.

Example 23.1.4. Let $W = \mathbb{C}^3$, $G = S^1$ and let G act on W by $\lambda \cdot (z_1, z_2, z_3) = (\lambda z_1, \lambda z_2, \lambda^{-1} z_3)$. Then,

$$\mathbb{C}^3 \cong \{\lambda \cdot z = \lambda z\} \oplus \{\lambda \cdot z = \lambda^{-1} z\} = \mathbb{C}^2 \oplus \mathbb{C}$$

Lemma 23.1.5. *The outer direct sum in* (1) *is a canonical decomposition.*

Proof. Given $U \in \mathrm{Irr}(G)$, consider

$$\begin{aligned} d_U : (\mathrm{Hom}_G(U, W)) \otimes_{\mathbb{C}} U &\to W \\ T \otimes u &\mapsto T(u). \end{aligned}$$

Then, d_U is an intertwining map:

$$\begin{aligned} d_U(g \cdot (T \otimes u)) &= d_U((g \cdot T g^{-1})(\cdot) \otimes g \cdot u) \\ &= g \cdot T(g^{-1} \cdot g \cdot u) \\ &= g \cdot T(u) \end{aligned}$$

if $V \subseteq W$ is an irreducible representation, then $\mathrm{Hom}_G(U, V) \hookrightarrow \mathrm{Hom}_G(U, W)$. By Schur's Lemma, $\mathrm{Hom}_G(U, V) \cong \mathbb{C}$ if and only if $U \cong V$. So, $V = d_U(\mathrm{Hom}_G(U, V) \otimes U) \subseteq \mathrm{im}\, d_U$. In other words, $\mathrm{im}\, d_U$ contains all irreducible represenataions isomorphis to U.

Since $W_U = \oplus_{W_j \cong U}$, $\mathrm{im}\, d_U \supseteq W_U$.

Coinversely, if $0 \neq v \in \operatorname{im} d_U$, then $v = T(u)$ for some $u \in U$ and $T \in \operatorname{Hom}_G(U, W)$. Since u is nonzero, $T \neq 0$ and so $T : U \to W$ is injective and $T(U)$ is a subrepresentation isomorphic to U. Thus, $\operatorname{im} d_U \subseteq W_U$.

Since we have shown $\operatorname{im} d_U = W_U$, W_U is canonically defined. □

Remark 23.1.6. The map

$$d := \oplus d_U : \bigoplus_{U \in \operatorname{Irr}(G)} \operatorname{Hom}_G(U, W) \otimes U \to W$$

is an isomorphism.

Proof. By the previous lemma, d is onto. Also, if $W = W_1 \oplus W_2$, then $\operatorname{Hom}_G(U, W_1 \oplus W_2) = \operatorname{Hom}_G(U, W_1) \oplus \operatorname{Hom}_G(U, W_2)$. So, $d = d^{(1)} + d^{(2)}$. By induction, both of these maps are isomorphisms, and it follows that d is as well. □

24. Lecture 24

24.1. **Invariant Integration.** Recall that if G is a Lie group of dimension n, $0 \neq \mu_1 \in \bigwedge^n(T_1^*G) = \bigwedge^n(\mathfrak{g}^*)$, then we may define a left invariant volume form $\mu \in \Omega^n(G)$ by

$$\mu_g = (L_{g^{-1}})^*(\mu_1).$$

That is,

$$\mu_g(v_1, \ldots, v_n) = \mu_1(dL_{g^{-1}}v_1, \ldots, dL_{g^{-1}}v_n)$$

for all $v_1, \ldots, v_n \in T_gG$. Then, it's clear that μ is left G-invariant: for any $h \in G$ $(L_h^*\mu)_g = \mu_g$ for all $g \in G$.

Question: Are there any bi-invariant (i.e., left and right invariant) volume forms?

We want: $L_h^*\mu = \mu$ and $R_h^*\mu = \mu$ for all $h \in G$. This would imply that

$$(L_h \circ R_{h^{-1}})^*\mu = \mu$$

and hence for all $v_1, \ldots, v_n \in T_1G$ we would have

$$\begin{aligned} \mu_1(v_1, \ldots, v_n) &= \mu_1(d(L_h \circ R_{h^{-1}})v_1, \ldots, d(L_h \circ R_{h^{-1}})v_n) \\ &= \mu_1(\mathrm{Ad}(h)v_1, \ldots, \mathrm{Ad}(h)v_n) \\ &= \det(\mathrm{Ad}(h))\mu_1(v_1, \ldots, v_n) \end{aligned}$$

We conclude that a necessary condition for bi-invariant forms to exist is if $\det(\mathrm{Ad}(h)) = 1$ for all $h \in G$. It can be shown that this is also a sufficient condition; the proof is left to the reader.

Remark 24.1.1. Note that $\det \circ \mathrm{Ad} : G \to GL(\mathfrak{g}) \to \mathbb{R}^\times$. If G is compact and connected, then $\det(\mathrm{Ad}(G)) \subset \mathbb{R}^\times$ is a compact, connected subgroup. Hence, $\det \circ \mathrm{Ad} \equiv 1$.

What if G is not connected?

Example 24.1.2. Let

$$G = O(2) = \left\{ \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \mid a^2 + b^2 = 1 \right\} \cup \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a^2 + b^2 = 1 \right\}$$

(note that the second set in the above union is $SO(2)$). We have

$$\mathfrak{g} = \mathfrak{o}(2) = \left\{ \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \mid x \in \mathbb{R} \right\} = \mathbb{R} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Since $SO(2)$ is abelian, $\mathrm{Ad}(g) = \mathrm{id}$ for all $g \in SO(2)$. Thus, $SO(2) \subseteq \ker\{\mathrm{Ad} : O(2) \to GL(\mathfrak{g}) = GL(1, \mathbb{R})\}$. passing to the quotient we get a map $\overline{\mathrm{Ad}} : O(s)/SO(2) \to GL(1, \mathbb{R})$. What is this map? Well,

$$\begin{aligned}
\mathrm{Ad}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} &= \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\\
&= \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\\
&= \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\\
&= -\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}
\end{aligned}$$

from which we conclude that $\mathrm{Ad}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = -1$. Thus, $O(2)$ has no bi-invariant 1-forms. It does, however have a bi-invariant measure: "$d\theta''$.

24.2. **Densities - a Crash Course.** Recall that volume forms on a manifold M of dimension d are nowhere zero sections of the line bundle $\bigwedge^d(T^*M)$. If $\{(U_\alpha, \phi_\alpha)\}$ are coordinate charts on M, then the transition maps for $\bigwedge^d(T^*M)$ are $\det(d(\phi_\beta \circ \phi_\alpha^{-1}))$.

We want objects that transform by

$$|\det(d(\phi_\beta \circ \phi_\alpha^{-1}))|. \tag{2}$$

These objects are called "densities". they are sections of a line bundle over M with transition maps given by (2). More concretely, The fibers of $\bigwedge^d(T^*M)$ are linear functionals $\mu_x : \bigwedge^d(T_xM) \to \mathbb{R}$. The fibers of the density bundle $\mathcal{D}(M) \to M$ are functions $\theta_x : \bigwedge^d(T_xM) \to \mathbb{R}$ satisfying

$$\theta_x(cv_1 \wedge \ldots \wedge v_n) = |c|\theta_x(v_1 \wedge \ldots \wedge v_n).$$

Densities can be integrated in the same way as top dimensional forms. We do not need, however, to worry about orientations. Moreover, if dg is a bi-invariant density on a Lie group G, then $|\det \mathrm{Ad}(g)| = 1$ for all $g \in G$. This is also sufficient.

Bi-invariant densities are unique up to scaling. We normalize a density dg by requiring

$$\int_G 1\, dg = 1$$

The upshot of all this is: given a compact Lie group G, we have a linear map

$$\begin{aligned}
\int_G \cdot\, dg : C^0(G, V) &\to \mathbb{R}\\
f &\mapsto \int_G f(g)\, dg
\end{aligned}$$

satisfying

(i) $\displaystyle\int_G f(ag)\, dg = \int_G f(g)\, dg$ and

(ii) $\displaystyle\int_G f(ga)\,dg = \int_G f(g)\,dg.$

for all $a \in G$. The measure dg is also known as the Haar measure.

Remark 24.2.1. If V is a finite dimensional vector space and $f : G \to V$ is continuous, we can define

$$\int_G f(g)\,dg \in V$$

So, if $T : V_1 \to V_2$ is a linear map, then

$$T\left(\int_G f(g)\,dg\right) = \int_G T(f(g))\,dg.$$

25. Lecture 25

25.1. **Group Characters.**

Definition 25.1.1. let G be a Lie group and $\rho : G \to GL(V)$ a complex representation. The *character* of the representation is the function

$$\begin{aligned} \chi_\rho = \chi_V : G &\to \mathbb{C} \\ \chi_V(g) = \operatorname{tr}(\rho(g)) \end{aligned}$$

Remark 25.1.2. If A, B are complex matrices such that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, then $\operatorname{tr}(ABA^{-1}) = \operatorname{tr}(A)$. So tr is independent of the chosen basis.

Also, if $T : V \to V$ is linear,$\{v_1, \dots, v_n\}$ is a basis of V, $v_1^*, \dots, v_n^*$ the corresponding dual basis of V^*, then $\operatorname{tr}(T) = \sum_i v_i^*(T(v_i))$.

If V is a representation of G, then $V^* = \operatorname{Hom}(V, \mathbb{C})$ is the dual representation of G. If G is compact, we may choose a G-invariant Hermitian inner product $\langle , \rangle$ on V. This gives a G-equivariant complex antilinear map

$$\begin{aligned} V &\to V^* \\ v &\mapsto \langle , v, \cdot \rangle \end{aligned}$$

This gives an isomorphism $V^* \cong \overline{V}$ where $\overline{V}$ is the complex vector space with the same addition as V and scalar multiplication is given by $\lambda \cdot V = \overline{\lambda} v$ for $\lambda \in \mathbb{C}$, $v \in V$.

Proposition 25.1.3. *Let G be a Lie group. Then,*

a) *a character of a representation of G is a C^∞ function on G,*
b) *if V and W are isomorphic representations of G, then $\chi_V = \chi_W$,*
c) $\chi_V(ghg^{-1}) = \chi_V(h)$, *for all $g, h \in G$,*
d) $\chi_{V \oplus W} = \chi_V + \chi_W$,
e) $\chi_{V \otimes W} = \chi_V \chi_W$,
f) $\chi_{V^*}(g) = \chi_{\overline{V}}(g) = \overline{\chi_V(g)} = \chi_V(g^{-1})$,
g) $\chi_V(1) = \dim_{\mathbb{C}} V$.

Proof. a) This is left as an exercise for the reader.

b) If $\rho_1, \rho_2 : G \to GL(n, \mathbb{C})$ are two representations and

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "T"] \arrow[d, "\rho_1(g)"'] & \mathbb{C}^n \arrow[d, "\rho_2(g)"] \\ \mathbb{C}^n \arrow[r, "T"] & \mathbb{C}^n \end{tikzcd}$$

commutes, then $\operatorname{tr}(\rho_2(g)) = \operatorname{tr}(T\rho_2(g)T^{-1}) = \operatorname{tr}(\rho_1(g))$.

c) $\operatorname{tr}(\rho(ghg^{-1})) = \operatorname{tr}(\rho(g)\rho(h)\rho(g^{-1})) = \operatorname{tr}(\rho(h))$.

d),e) recall from linear algebra that if $A : V \to W$ and $B : V \to V$ are linear, then

$$\begin{aligned} \operatorname{tr}(A \oplus B) &= \operatorname{tr}(A) + \operatorname{tr}(B) \\ \operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \operatorname{tr}(B). \end{aligned}$$

f) If $\rho : G \to GL(V)$ is a representation, $\{v_1, \ldots, v_n\}$ is a basis for V and $v_1^*, \ldots, v_n^*$ is the associated dual basis, then

$$\begin{aligned}
\chi_{\rho^*}(g) &= \operatorname{tr}(\rho^*(g)) \\
&= \sum_i v_i(\rho^*(g)v_i^*) \\
&= \sum_i v_i^*(\rho(g^{-1})v_i) \\
&= \chi_{\rho(g^{-1})}
\end{aligned}$$

If $\langle , \rangle$ is an invariant Hermitian inner product, and $\{v_i\}$ is an orthonormal basis, then

$$\begin{aligned}
\operatorname{tr} \rho^*(g) &= \sum_i \langle v_i, \cdot \rangle \circ \rho(g^{-1})(v_i) \\
&= \sum_i \langle v_i, \rho(g)^{-1} v_i \rangle \\
&= \sum_i \langle \rho(g) v_i, v_i \rangle \\
&= \sum \langle \rho(g)_{ji} v_j, v_i \rangle \\
&= \sum \overline{\rho(g)_{ji}} \langle v_j, v_i \rangle \\
&= \sum \overline{\rho(g)_{ii}} \\
&= \overline{\chi_\rho(g)}
\end{aligned}$$

g) $\chi_V(1) = \operatorname{tr}(\mathrm{id}) = \dim_{\mathbb{C}} V$. □

Proposition 25.1.4. *Let $\rho : G \to GL(V)$ be a representation of G and*

$$V^G = \{g \in V \mid g \cdot v = v\}.$$

Then,

$$\int_G \chi_V(g)\, dg = \dim_{\mathbb{C}} V$$

Proof. Consider $P : V \to V$ given by $P(v) = \displaystyle\int_G \rho(g) v\, dg$. We claim that P is a linear G-equivariant map such that $P(V) \subseteq V^G$ and $P|_{V^G} = \mathrm{id}_{V^G}$.

It's clear that P is linear. Now,

$$\begin{aligned}
P(\rho(a)v) &= \int_G \rho(g)\rho(a)v\,dg \\
&= \int_G \rho(ag)v\,dg \\
&= \int_G \rho(g)v\,dg \\
&= P(v) \\
&= \int_G \rho(ag)v\,dg \\
&= \rho(a)\int_G \rho(g)\,dg \\
&= \rho(a)P(v)
\end{aligned}$$

and so $P(V) \subseteq V^G$ and $P(\rho(a)\cdot v) = P(v)$ for all $g \in G$ and $v \in V$. Also, if $v \in V^G$, we have

$$P(v) = \int_G \rho(g)c\,dg = \int_G v\,dg = v$$

(since $\int_G 1\,dg = 1$).

This claim implies that $\mathrm{tr}(P) = \dim V^G$. On the other hand,

$$\begin{aligned}
\mathrm{tr}(P) &= \sum v_i^*(P(v_i)) \\
&= \sum v_i^*\left(\int_G \rho(g)v_i\,dg\right) \\
&= \int_G \left(\sum v_i^*(\rho(g)v_i)\right)\,dg \\
&= \int_G \chi_V(g)\,dg
\end{aligned}$$

This completes the proof. □

25.2. **Orthogonality of Characters.**

Theorem 25.2.1. *Let V, W be two compact representations of a compact Lie group G. Then,*

$$\langle \chi_V, \chi_W\rangle := \int_G \chi_V(g)\chi_W(g)\,dg = \dim_{\mathbb{C}} Hom_G(V, W)$$

In particular, if V, W are irreducible, then

$$\langle \chi_V, \chi_W\rangle = \begin{cases} 1 & V \cong W \\ 0 & V \ncong W \end{cases}$$

Proof. $\overline{\chi_V}\chi_W = \chi_{V^*}\chi_W = \chi_{W\otimes V^*} = \chi_{\mathrm{Hom}(V,W)}$. Since $W \otimes V^* \cong \mathrm{Hom}(V, W)$, $\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$. By Proposition 25.4,

$$\begin{aligned}\int_G (\overline{\chi_V}\chi_W)(g)\, dg &= \int_G \chi_{\mathrm{Hom}(V,W)}(g)\, dg \\ &= \mathrm{Hom}(V, W)^G \\ &= \mathrm{Hom}_G(V, W).\end{aligned}$$

The result now follows from Shur's Lemma. □

Example 25.2.2. Let $G = S^1$, $\rho_n : S^1 \to GL(1, \mathbb{C})$ be given by $\rho_n(e^{i\theta}) = e^{in\theta}$. Then, $dg = 1/(2\pi)d\theta$ and

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta}e^{in\theta}\, d\theta = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

26. Lecture 26

26.1. **Maximal Tori I.** Recall: We have proved that the characters of complex irreducible representations of compact Lie groups are lineraly independent as elements of $C^\infty(G)$.

We also have the notion of the *multiplicity* $m_{U,W}$ of an irreducible representation U in a representation W:

$$W = \bigoplus_{U \in \mathrm{Irr}(G)} m_{U,W} U$$

where nU is the direct sum of n copies of U.

Corollary 26.1.1. *Let G be a compact Lie group. Then, the map from representations of G to characters of G, $W \mapsto \chi_W$.*

Proof. Suppose W, W' are two representations of G such that $\chi_W = \chi_{W'}$. Write

$$W = \bigoplus_{U \in \mathrm{Irr}(G)} m_{U,W} U, \qquad W' = \bigoplus_{U \in \mathrm{Irr}(G)} m_{U,W'} U$$

Then,

$$\chi_W = \sum_{U \in \mathrm{Irr}(G)} m_{U,W} \chi_U = \sum_{U \in \mathrm{Irr}(G)} m_{U,W'} \chi_U$$

Since $\{\chi_U \,|\, U \in \mathrm{Irr}(G)\}$ is linearly dependent, $m_{U,W} = m_{U,W'}$ and hence $W = W'$. □

Example 26.1.2. Take $G = SU(2)$. Let

$$\mathbb{T} = \left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \,|\, |\lambda| = 1 \right\} \subset SU(2)$$

Recall that

$$V_n = \left\{ \sum_{k=0}^{n} a_k z_1^{n-k} z_2^k \,|\, a_n \in \mathbb{C} \right\}$$

is an irreducible representation of $SU(2)$ and $A \in SU(2)$ acts on $f(z_1, z_2) \in V_n$ by $(A \cdot f)(z_1, z_2) = f((z_1, z_2)A)$. So

$$\begin{aligned} \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \cdot z_1^{n-k} z_2^k &= (\lambda z_1)^{n-k} (\lambda^{-1} z_2)^k \\ &= \lambda^{n-2k} z_1^{n-k} z_2^k \end{aligned}$$

Hence, $\chi_n \left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \right) = \sum_{k=0}^{n} \lambda^{n-2k}$.

What about $\chi_n(A)$ where A is an arbitrary element of $SU(2)$? Well, any unitary matrix is diagonalizable, $\chi(BAB^{-1}) = \chi(A)$ and the eigenvalues of any unitary matrix lie on S^1. Hence, χ_n is completely determined by its values on $\mathbb{T}$.

Since for any character $\chi : G \to \mathbb{C}$ of a Lie group G, $\chi(aga^{-1}) = \chi(g)$ for all $a, g \in G$, in order to understand characters of G it's important to understand the congugacy classes of G, i.e. the orbit space G/G where G acts on itself by congugation. Let's consider an example to see what we should expect.

Example 26.1.3. Consider the Lie group $U(n)$. As mentioned above, every element of $U(n)$ is congugate to a diagonal matrix whose nonzero entries have norm 1. Note, however, that the set of eigenvalues for a particular element of $U(n)$ is an *unordered* set! So

$$U(n)/U(n) \cong \{D \in U(n) \mid D \text{ is diagonal}\}/\Sigma_n = \mathbb{T}^n/\Sigma_n$$

where Σ_n is the group of permutations of n letters.

Definition 26.1.4. A *torus* is a compact, connected abelian Lie group. A *maximal torus* T of a Lie group G is a torus subgroup of G such that if T' is any other torus subgroup of G and $T \subseteq T'$ then $T = T'$.

Example 26.1.5. Consider again $G = U(n)$. The torus

$$T = \left\{ \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \,\middle|\, |\lambda_j| = 1 \right\} \cong \mathbb{T}^n$$

is a maximal torus.

Reason: Suppose A is a matrix in $U(n)$ that commutes with all elements of T. But every element of T is diagonal, and matrix theory tells us that a matrix that commutes with an arbitrary diagonal matrix is itself diagonal. Hence, $A \in T$.

We will prove over the next two lectures:

(1) maximal tori exist,
(2) if G is compact and T_1, T_2 are two maximal tori, then T_1 and T_2 are conjugate, i.e. there exists $a \in G$ so that $aT_1a^{-1} = T_2$,
(3) if G is compact and connected, then for any $g \in G$ there is a maximal torus T such that $g \in T$,
(4) if $T \leq G$ is a maximal torus, G compact and connected, then

$$N(T) = \left\{ g \in G \mid gTg^{-1} \subseteq T \right\},$$

the *normalizer* of T in G satisfies:
   a) $W := N(T)/T$ is finite ($N(T)/T$ is called the *Weyl group* of T),
   b) $T/W \cong G/G$.

Proposition 26.1.6. *Let G be a Lie group, $K \subseteq G$ a connected abelian Lie subgroup. Then, the closure $\overline{K}$ of K in G is also a connected abelian subgroup.*

Proof. Since the closure of a connected set is connected, $\overline{K}$ is connected. We have only to argue that $\overline{K}$ is an abelian subgroup. Since $f : G \times G \to G$, $f(a, b) = ab^{-1}$ is continuous and since $f(K \times K) \subset K$, $f(\overline{K} \times \overline{K}) = f(\overline{K \times K}) \subset \overline{f(K)} \subset \overline{K}$, we see that $\overline{K}$ is a closed subgroup of G.

It remains to show that $\overline{K}$ is abelian. For any $x, y \in K$, $xyx^{-1} = y$ which also certainly holds if $y \in \overline{K}$. But then, yxy^{-1} holds as well. By the same argument, this relation holds for all $x, y \in \overline{K}$. □

Corollary 26.1.7. *Any compact Lie group G with $\dim G > 0$ has a torus T with $\dim T > 0$.*

Proof. Pick any $X \in \mathfrak{g}$, then $\{\exp tX \mid t \in \mathbb{R}\} \subseteq G$ is a connected abelian group. Hence, its closure is a closed connected abelian subgroup of G. Since G is compact, $\overline{\{\exp tX \mid t \in \mathbb{R}\}}$ is a torus. □

So, at least tori exist as subgroups of compact Lie groups.

27. Lecture 27

27.1. **Maximal Tori II.** In this lecture we begin by proving that maximal tori exist in a compact, connected Lie group. We then examine the geometry of the adjoint representation to begin our proof that maximal tori are conjugate.

Lemma 27.1.1. *let G be a compact Lie group and $\mathfrak{g}$ its Lie algebra. The maximal tori of G are in one to one correspondence with maximal abelian Lie subalgebras of $\mathfrak{g}$ (such subalgebras are called* Cartan *subalgebras).*

Proof. Suppose $\mathfrak{a} \subset \mathfrak{g}$ is a maximal abelian subalgebra. Then, $A := \exp \mathfrak{a} \subseteq G$ is a connected, abelian Lie subgroup. Hence, the closure $\overline{A}$ of A is a torus and $\overline{\mathfrak{a}} \supset \mathfrak{a}$. Since $\mathfrak{a}$ is maximal, $\overline{\mathfrak{a}} = \mathfrak{a}$. Since $\overline{A}$ is a torus, $\overline{A} = \exp \overline{\mathfrak{a}} = \exp \mathfrak{a} = A$. This proves that if $\mathfrak{a}$ is a maximal abelian subalgebra, then A is a torus. We now argue that A is maximal. Let $T \supset A$ be another torus. Then, $\mathfrak{a} \subseteq Lie(T)$ and since $\mathfrak{a}$ is maximal, $\mathfrak{a} = Lie(T)$. Hence, $A = T$.

Conversely, suppose $T \subseteq G$ is a maximal torus. We argue that the Lie algebra $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$. If $\mathfrak{a} \supset \mathfrak{t}$ is an abelian subalgebra of $\mathfrak{g}$, then by the above $\overline{\exp \mathfrak{a}} \supset \exp \mathfrak{a} \supset T$. Since $\overline{\exp \mathfrak{a}}$ is a torus, and T is maximal, $T = \overline{\exp \mathfrak{a}}$ and hence $\mathfrak{t} = \mathfrak{a}$. □

As an exercise, prove that Cartan subalgebras, and hence maximal tori, exist.

Lemma 27.1.2. *Let G be a torus, $G = \mathfrak{g}/\mathbb{Z}_G$, where $\mathbb{Z}_G$ is the integral lattice. Let $\mathbb{Z}_G^*$ denote the weight lattice. For $X \in \mathfrak{g}$, $\overline{\{\exp tX \mid t \in \mathbb{R}\}} = G$ if and only if $\eta(X) \neq 0$ for any $0 \neq \eta \in \mathbb{Z}_G$.*

Proof. Suppose first that $\eta(X) = 0$ for some $0 \neq \eta \in \mathbb{Z}_G$. Then $\xi_\eta : G \to S^1$ given by $\xi_\eta(\exp Y) = e^{2\pi i \eta(Y)}$ is a well-defined Lie group map. Since η is nonzero, $\xi_\eta \neq 1$, i.e. $\ker \xi_\eta \neq G$. If $X = \ker \eta$, then $\xi_\eta(\exp tX) = 1$ for all t. But then, $\{\exp tX \mid t \in \mathbb{R}\} \subseteq \ker \xi_\eta$ implies that $\overline{\{\exp tX\}} \leq \ker \xi_\eta \neq G$.

Conversely, suppose $H := \overline{\{\exp tX \mid t \in \mathbb{R}\}} \neq G$. Note that H, being a compact, connected abelian subgroup of G is a torus. Since $H \neq G$, G/H is a nontrivial compact connected abelian Lie group, i.e. another torus. Since $G/H \neq \{1\}$, there exists $\rho : G/H \to S^1$, $\rho \not\equiv 1$ (why?). Therefore, if $\pi : G \to G/H$ denotes the quotient map, $\rho \circ \pi : G \to S^1$ is a nontrivial Lie group map whose kernel contains H.

Set $\eta = d(\rho \circ \pi)_1 : \mathfrak{g} \to \mathbb{R}$. Then $\eta \in \mathbb{Z}_G^*$ and is nonzero. Since $H \in \ker(\rho \circ \pi)$, $X \in \ker \eta$. □

Corollary 27.1.3. *Let G be a torus. Then,*

$$X \in \mathfrak{g} \setminus \left(\bigcup_{0 \neq \eta \in \mathbb{Z}_G^*} \ker \eta \right) \iff \overline{\{\exp tX \mid t \in \mathbb{R}\}} = G$$

Hence, "for almost all" $X \in \mathfrak{g}$, $\overline{\{\exp tX \mid t \in \mathbb{R}\}} = G$.

Proof. The set $\mathbb{Z}_G^*$ is countable and for $0 \neq \eta \in \mathbb{Z}_G^*$, $\ker \eta \subseteq \mathfrak{g}$ is nowhere dense. Hence, $\bigcup_{0 \neq \eta \in \mathbb{Z}_G^*} \ker \eta$ is nowhere dense. □

27.2. **Geometry of the Adjoint Representation.**

Example 27.2.1. Consider $G = SO(3)$. We have seen that $\mathfrak{so}(3)$ is the set of 3×3 skew-symmetric real matrices. For $w = (w_1, w_2, w_3) \in \mathbb{R}^3$ and $x \in \mathbb{R}^3$, define

$$w \times x = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}.$$

For $A \in SO(3)$, we have $A(w \times x) = Aw \times Ax$. So,

$$(Aw) \times x = A \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} A^{-1}x.$$

Thus, the map $\phi : \mathbb{R}^3 \to \mathfrak{so}(3)$, $\phi(w) = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$ satisfies: $\phi(AW) = A\phi(w)A^{-1} = \mathrm{Ad}(A)\phi(w)$. Therefore the orbirts of the action of $SO(3)$ are 2-spheres (and 0).

Now, consider the Adjoint action of G on $\mathfrak{g}$ given by $g \cdot X = \mathrm{Ad}(g)X$. If $xi \in \mathfrak{g}$, we have $\mathrm{Ad}(\exp t\xi)X = e^{t\,\mathrm{ad}(\xi)}X$ and so, denoting the induced vector field of this action by $\xi_{\mathfrak{g}}$, we have

$$\xi_{\mathfrak{g}}(X) = \frac{d}{dt}\bigg|_{t=0} e^{t\,\mathrm{ad}(\xi)}X = \mathrm{ad}(\xi)X = [\xi, X]$$

We then calculate the following:

$$\begin{aligned} G_X &= \{g \in G \mid \mathrm{Ad}(g)X = X\} \\ \mathfrak{g}_X &= \{\xi \in \mathfrak{g} \mid [\xi, X] = 0\} \\ &= \ker\{\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}\} \\ T_x(G \cdot X) &= \{\xi_M(X) \mid \xi \in \mathfrak{g}\} \\ &= \{[\xi, X] \mid \xi \in \mathfrak{g}\} \\ &= \mathrm{im}\{\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}\} \end{aligned}$$

Lemma 27.2.2. *Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. Suppose there is an $\mathrm{Ad}(G)$-invariant inner product $(\cdot, cdot)$ on $\mathfrak{g}$. Then*

$$(\ker \mathrm{ad}(X))^{\perp} = \mathrm{im}\,\mathrm{ad}(X)$$

for any $X \in \mathfrak{g}$.

Proof. Since the inner product is $\mathrm{Ad}(G)$-invariant, for any $X, Y, Z \in \mathfrak{g}$ we have

$$(\mathrm{Ad}(\exp tX)Y, \mathrm{Ad}(\exp tX)Z) = (Y, Z)$$

and so

$$0 = \frac{d}{dt}\bigg|_{t=0} (\mathrm{Ad}(\exp tX)Y, \mathrm{Ad}(\exp tX)Z) = ([X, Y], Z) + (Y, [X, Z])$$

(the converse is also true if G is connected, i.e. $([X, Y], Z) + (Y, [X, Z]) = 0$ implies $(,)$ is $\mathrm{Ad}(G)$-invariant).

Now, $Y \in \ker \mathrm{ad}(X)$ if and only if for all $Z \in \mathfrak{g}$, $0 = (\mathrm{ad}(X)Y, Z) = -(Y, \mathrm{ad}(X)Z)$. But, this happens if and only if $Y \in (\mathrm{im}\,\mathrm{ad}(X))^{\perp}$. ∎

28. Lecture 28

28.1. **Maximal Tori III.** Our current goal is to prove the following:

Theorem 28.1.1. *Let G be a compact Lie group, T, T' two maximal tori in G. Then, there exists $g \in G$ such that $gTg^{-1} = T$.*

Recall from the last lecture we proved that if we have an $\mathrm{Ad}(G)$-invariant inner product on the Lie algebra $\mathfrak{g}$ of a Lie group G (such an inner product always exists if G is compact), then we get a splitting

$$\mathfrak{g} = \ker \operatorname{ad} X \oplus \operatorname{im} \operatorname{ad} X.$$

Notation: For $X \in \mathfrak{g}$, we write $\mathfrak{g}_X$ for the Lie group of $G_X = \{g \in G \mid \mathrm{Ad}(g)X = X\}$. We've seen that $\mathfrak{g}_X = \ker \operatorname{ad} X$.

Lemma 28.1.2. *Suppose G is a compact Lie group, $X \in \mathfrak{g}$ is such that $T := \overline{\{\exp tX\}}$ is a maximal torus. Denote by $\mathfrak{t}$ the Lie algebra of T. Then, $\mathfrak{g}_X = \mathfrak{t}$. Hence, if $\mathfrak{h} \subseteq \mathfrak{g}$ is a maximal abelian subalgebra, then for a "generic" $X \in \mathfrak{h}$, $\ker \operatorname{ad} X = \mathfrak{h}$.*

Proof. Since T is a torus, $\mathfrak{t}$ is abelian. So, $X \in \mathfrak{t}$ implies that $[Y, X] = 0$ for all $Y \in \mathfrak{t}$. Thus, $\mathfrak{t} \subseteq \mathfrak{g}_X$.

Conversely, suppose that $Y \in \mathfrak{g}_X$. We want to show that $Y \in \mathfrak{t}$. Since $Y \in \mathfrak{g}_X$, $[Y, X] = 0$ and the flows ϕ_t^X and ϕ_t^Y of the corresponding left invariant vector fields commute. But, $\phi_1^Y(a) = a \exp tY$, $\phi_1^Y(b) = b \exp tY$ and so

$$\phi_t^Y(\phi_s^X(1)) = \phi_s^X(\phi_t^Y(1))$$

and hence $\exp sX \exp tY = \exp tY \exp sX$.

But, by definition of T, for all $a \in T$,

$$\begin{aligned} a \exp tY = (\exp tY)a \quad &\Longrightarrow \quad a(\exp tY)a^{-1} = \exp tY \\ &\Longrightarrow \quad \mathrm{Ad}(a)Y = Y \\ &\Longrightarrow \quad [Z, Y] = 0 \qquad \forall Z \in \mathfrak{t} \end{aligned}$$

which implies $\mathfrak{t} + \mathbb{R}Y$ is an abelian subalgebra of $\mathfrak{g}$. But T is a maximal torus, and $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$. Therefore, $\mathfrak{t} + \mathbb{R}Y = \mathfrak{t}$, i.e. $Y \in \mathfrak{t}$. So we have proved that $\mathfrak{g}_X \subseteq \mathfrak{t}$. ∎

We are now in a position to prove Theorem 28.1:

Proof. Pick $X, X' \in \mathfrak{g}$ so that $\overline{\{\exp tX\}} = T$ and $\overline{\{\exp tX'\}} = T'$ Then, $\ker \operatorname{ad} X = \mathfrak{t}$ and $\ker \operatorname{ad} X' = \mathfrak{t}'$. Suppose we can find $g \in G$ such that $\mathrm{Ad}(g)X \in \ker \operatorname{ad} X'$. Then, $\exp(t \,\mathrm{Ad}(g)X) = g(\exp tX)g^{-1} \in T'$ for all t.

So, $gTg^{-1} = g\overline{\{\exp tX\}}g^{-1} \subseteq \overline{T'} = T'$. Since both T and T' are maximal, we actually have equality: $gTg^{-1} = T'$.

Now we must find such a $g \in G$. Fix an $\mathrm{Ad}(G)$-invariant inner product $(,)$ on $\mathfrak{g}$. Consider

$$\begin{array}{rcl} f : \mathrm{Ad}(G)X = G \cdot X & \to & \mathbb{R} \\ f(Y) & = & \|Y - X'\|^2 = \|Y\|^2 - 2(Y, X') + \|X'\|^2 \end{array}$$

We claim that $Y \in \mathrm{Ad}(G)X$ is a critical point of f if and only if $Y \in \mathrm{Ad}(G)X \cap \ker \mathrm{ad}\, X'$. To show this, note that Y is critical if and only if for all $Z \in \mathfrak{g}$

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} f(\mathrm{Ad}(\exp Z)Y) \\ &= \left.\frac{d}{dt}\right|_{t=0} (\|\mathrm{Ad}(\exp tZ)Y\|^2 - 2(\mathrm{Ad}(\exp tZ)Y, X') + \|X\|^2) \\ &= -2 \left.\frac{d}{dt}\right|_{t=0} (\mathrm{Ad}(\exp tZT, X') \\ &= (-2)([Z, Y], X') \\ &= 2(Y, [Z, X']). \end{aligned}$$

We conclude that Y is critical for f if and only if $Y \in (\mathrm{im}\, \mathrm{ad}\, X')^\perp = \ker \mathrm{ad}\, X'$. Since G is compact, $\mathrm{Ad}(G)X$ is compact and so f does indeed have critical points. This establishes the theorem. $\square$

29. Lecture 29

29.1. **Maximal Tori IV.** Our next goal is to prove that every element of a compact, connected Lie group lies in some maximal torus. Suppose we know that $\exp : \mathfrak{g} \to G$ is onto. Then, if $g \in G$, we see that $g = \exp X$ for some $X \in \mathfrak{g}$. Now, $\mathbb{R}X$ is an abelian subalgebra of $\mathfrak{g}$ and therefore lies in a maximal abelian subalgebra $\mathfrak{h}$. Then, $\exp \mathfrak{h}$ is a maximal torus in G containing g. To prove that exp is onto, we will appeal to familiar tools from Riemannian geometry.

Lemma 29.1.1. *let G be a compact Lie group. Then G has a bi-invariant Riemannian metric.*

Proof. On a Lie group G, bi-invariant metrics correspond to Ad-invariant inner products on $\mathfrak{g}$: if g is a bi-invariant metric, g_1 on T_1G is Ad-invariant. If g_1 is an Ad-invariant inner product on T_1G, then its left translation is a bi-invariant metric. If G is compact, then T_1G has an Ad-invariant inner product: take an arbitrary positive definite inner product and average it over G. □

Before we proceed, we shall review some facts about Riemmanian manifolds. In particular, we need to use the notions of connections and geodesics on a manifold.

Definition 29.1.2. A *connection* ∇ on a manifold M is an $\mathbb{R}$-bilinear map

$$\begin{aligned} \nabla : \Gamma(TM) \times \Gamma(TM) &\to \Gamma(TM) \\ (X, Y) &\mapsto \nabla_X Y \end{aligned}$$

such that

(1) $\nabla_{fX} Y = f \nabla_X Y$ and
(2) $\nabla_X (fY) = (Xf)Y + f\nabla_X Y$ for any $f \in C^\infty(M)$ and $X, Y \in \Gamma(TM)$.

Theorem 29.1.3 (Levi-Civita)**.** *Let (M, g) be a Riemannian manifold. Then, there is a unique connection $\nabla = \nabla^g$ on M such that*

(1) *$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ and*
(2) *$\nabla_X Y - \nabla_Y X = [X, Y]$.*

Moreover,

$$\begin{aligned} 2g(X, \nabla_Z Y) &= Z(g(X,Y)) + Y(g(X,Z)) - X(g(Y,Z)) \\ &\quad + g(Z, [X,Y]) + g(Y, [X,Z]) - g(X, [Y,Z]) \end{aligned}$$

Theorem 29.1.4. *Let M be a manifold with a connection ∇ and $\gamma : (a, b) \to M$ a curve. Then there exists a unique $\mathbb{R}$-linear map*

$$\frac{\nabla}{dt} : \Gamma(\gamma^* TM) \to \Gamma(\gamma^* TM)$$

such that

(1) *$\frac{\nabla}{dt}(fV) = \frac{df}{dt} V + f \frac{\nabla}{dt} V$ for all $f \in C^\infty(a, b)$ and $V \in \Gamma(TM)$.*
(2) *if $X \in \Gamma(TM)$, then*

$$\frac{\nabla}{dt}(X \circ Y) = \nabla_{\dot{Y}} X$$

Definition 29.1.5. A curve $\gamma : (a, b) \to M$ is a *geodesic* for a connection ∇ if

$$\frac{\nabla}{dt}\dot{\gamma} = 0$$

Recall that if $x \in M$, $v \in T_xM$, then there is a unique geodesic γ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = v$.

Theorem 29.1.6. *Let G be a Lie group, g a bi-invariant metric on G and ∇ the corresponding Levi-Civita connection. Then, for any left invariant vector fields Z and Y*

$$\nabla_Z Y = \frac{1}{2}[Y, Y]$$

Proof. Let X, Y, Z be left invariant vector fields. Then, $(g(X, Y))(a) = (g(X, Y))(1)$ for any $a \in G$. Consequently, the map $a \mapsto (g(X, Y))(a)$ is a constant function. Also, since g is bi-invariant, we see that $g([X, Y], Z) + g(X, [Y, Z]) = 0$. These two facts together, along with the formula for the Levi-Civita connection in the above theorem show that $2g(X, \nabla_Z Y) = g(X, [Z, Y])$. Since X is arbitrary and the metric is nondegenerate, $2\nabla_Z Y = [Z, Y]$ ☐

Corollary 29.1.7. *For any $X \in \mathfrak{g}$, $a \in G$, $\gamma(t) = a \exp tX$ is a geodesic. Moreover, all the geodesics are of this form.*

Proof. If $\gamma(t) = a \exp tX$, then

$$\begin{aligned} \dot{\gamma}(t) &= (dL_{a \exp tX})X(1) \\ &= X(\gamma(t)) \end{aligned}$$

and so

$$\begin{aligned} \frac{\nabla}{dt}\dot{\gamma} &= \nabla_X X \\ &= \frac{1}{2}[X, X] = 0. \end{aligned}$$

Thus, $\gamma(t)$ is a geodesic. Moreover, for all $a \in G$ and for all $v \in T_aG$, there is $X \in \mathfrak{g}$ such that $X(a) = v$. Therefore, $\gamma(t) = a \exp tX$ is a geodesic with $\gamma(0) = a$, $\dot{\gamma}(0) = X(a) = v$. ☐

The following theorem from Riemannian geometry is integral to our goal:

Theorem 29.1.8 (Hopf-Rinow)**.** *If (M, g) is a complete, connected Riemannian manifold, then any two points can be joined by a geodesic.*

As a consequence, we recover

Theorem 29.1.9. *Let G be a compact, connected Lie group. Then, exp : $\mathfrak{g} \to G$ is onto.*

Proof. Any point $g \in G$ can be connected to $1 \in G$ by a geodesic which is of the form $t \mapsto \exp tX$ for some $X \in \mathfrak{g}$. ☐

By the remark at the beginning of this lecture, we see that we have proved any element of a compact, connected Lie group lies in a maximal torus.

30. Lecture 30

30.1. The Weyl Group.

Definition 30.1.1. Let $H \leq G$ be a subgroup of a group G. The *normalizer* $N_G(H) = N(H)$ of H in G is

$$N(H) = \{g \in G \,|\, gHg^{-1} = H\}$$

Note that $H \subseteq N(H)$ and $N(H)$ is a subgroup of G.

Proposition 30.1.2. *Let G be a Lie group and $H \leq G$ a closed subgroup. Then, $N(H)$ is closed in G and is hence a Lie subgroup of G.*

Proof. For $b \in H$ consider the map

$$\begin{array}{rcl} \psi_b : G & \to & G \\ \psi_b(g) & = & gbg^{-1} \end{array}$$

Since H is closed and ψ_b is smooth, $\psi_b^{-1}(H)$ is closed for all $b \in H$. Now,

$$\begin{array}{rcl} N(H) & = & \displaystyle\bigcap_{b \in H} \{g \in G \,|\, gbg^{-1} \in H\} \\ & = & \displaystyle\bigcap_{b \in H} \psi_b^{-1}(H) \end{array}$$

and so $N(H)$ is closed. □

Definition 30.1.3. Let G be a compact Lie group, $T \subseteq G$ a maximal torus. The *Weyl group* $W = W(T, G)$ is $W = N(T)/T$.

Note that W acts on T: $(nT) \cdot a = nan^{-1}$ for all $a \in T$, $nT \in W$. We will see that

$$T/W = G/\sim$$

where $G/tilde$ is the quotient of G by the conjugation action.

Theorem 30.1.4. *let G be a compact Lie group, $T \subseteq G$ a maximal torus. Then, the Weyl group W is finite.*

Proof. We will argue that the connected component $N(T)_0$ of 1 in $N(T)$ is T. This will be enough since $|N(T)/N(T)_0|$ is the number of connected components of $N(T)$.

Now, $N(T)_0$ acts on T by conjugation" for all $g \in N(T)_0$ and $a \in T$, we have $gag^{-1} = c_g(a) \in T$. Hence,

$$dg_{g1}(\mathfrak{t}) \subseteq \mathfrak{t}.$$

In other words, $\mathrm{Ad}(g)(\mathfrak{t}) \subseteq \mathfrak{t}$. Thus, we get a Lie group map

$$\begin{array}{rcl} \mathrm{Ad}(\cdot)|_{\mathfrak{t}} : N(T_0) & \to & GL(\mathfrak{t}) \\ g & \mapsto & \mathrm{Ad}(g)|_{\mathfrak{t}} \end{array}$$

Also, for any $g \in N(T)_0$,

$$\begin{tikzcd} \mathfrak{t} \arrow[r, "\mathrm{Ad}(g)"] \arrow[d, "\exp"'] & \mathfrak{t} \arrow[d, "\exp"] \\ T \arrow[r, "c_g"] & T \end{tikzcd}$$

commutes since c_g is a Lie group map. Therefore, $\mathrm{Ad}(g)(\ker \exp) \subset \ker \exp$ for all $g \in N(T)_0$

Recall that

$$\mathbb{Z}_T := \ker\{\exp : \mathfrak{t} \to T\} \cong \mathbb{Z}^n$$

where $n = \dim T$. So, the image of $\mathrm{Ad}|_{\mathfrak{t}}$ in $GL(\mathbb{Z}_T) \cong GL(n, Z)$ is discrete. But, $N(T)_0$ is connected and so for all $g \in N(T)_0$, $\mathrm{Ad}(g)|_{\mathfrak{t}} = \mathrm{id}$. Thus, for all $X \in Lie(N(T)_0)$ and $Y \in \mathfrak{t}$ we have $\mathrm{Ad}(\exp X)Y = Y$ and so $[X, Y] = 0$. Since $\mathfrak{t}$ is maximal abelian, we must have $Lie(N(T)_0) \subseteq \mathfrak{t}$.

On the other hand, $T \subseteq N(T)_0$ and so $Lie(N(T)_0) = \mathfrak{t}$. Since both $N(T)_0$ and T are connected, they must therefore be equal. □

Remark 30.1.5. In fact, $Aut(T) = \{\phi : T \to T \mid \phi \text{ is a Lie group map}\} = GL(\mathbb{Z}_T)$.

Definition 30.1.6. let G be a Lie group. A function $f \in C^\infty(G)$ is a *class function* if

$$f(x) = f(gxg^{-1})$$

for all $x, g \in G$. We denote the space of all class functions by $C^0(G)^G$.

Lemma 30.1.7. *Let G be a compact Lie group. Two elements x_1, x_2 of a maximal torus T are conjugate in G if and only if there is $w \in W = N(T)/T$ so that $w \cdot x_1 = x_2$.*

Before we prove this lemma, let's make a couple observations.

Remark 30.1.8. Recall that W acts on T "by conjugation": for all $gT \in W$ and $a \in T$ we have $(gT) \cdot a = gag^{-1}$.

Definition 30.1.9. let G be a group and $H \subseteq G$ a subgroup. The *centralizer* $Z(G)$ of H in G is

$$Z(H) = \left\{g \in G \mid ghg^{-1} = H, \forall h \in H\right\}$$

Remark 30.1.10. If G is a Lie group and $H \subseteq G$ is closed, then

$$Z(H) = \bigcap_{h \in H} \psi_h^{-1}(h)$$

where $\psi_h : G \to G$ is given by $\psi_h(g) = ghg^{-1}$.

proof of Lemma 30.7. Suppose $x, y \in T$ and $y = gxg^{-1}$ for some $g \in G$. Then, $Z(y) = gZ(y)g^{-1} = c_g(Z(x))$. Since $x \in T$ and $T \subseteq Z(x)$ we have $c_g(T) \subseteq Z(y)$.

Now, $Z(y)_0$ is compact and connected and $T, c_g(T) \subseteq Z(y)_0$ are tori. Since both tori are maximal in G, they are maximal in $Z(y)_0$. Thus, there exists $h \in Z(y)_0$ such that $c_h(g_g(T)) = T$ and so $hg \in N(T)$.

Also, $c_{hg}(x) = hgxg^{-1}h^{-1} = hyh^{-1} = y$ since $h \in Z(y)$. We conclude that $hgT \in N(T)/T$ and $(hgT) \cdot x = y$. If $x_1, x-2 \in T$ and $w = gT \in W$ with $w \cdot x_1 = x_2$, then $gx_1g^{-1} = x_2$. So x_1 and x_2 are conjugate in G. $\square$

The previous lemma implies that the induced map $T/W \to G/\sim$ is a continuous bijection. Since T/W is compact and $G/\sim$ is Hausdorff, this map is actually a homeomorphism. Hence,

Lemma 30.1.11. *Let G be a compact Lie group, $T \subseteq G$ a maximal torus and W the Weyl group. Then,*

(i) *$G/\sim \approx T/W$ and*
(ii) *the map $C^0(G)^G \to C^0(T)^W$ given by $f \mapsto f|_T$ is an isomorphism.*

Example 30.1.12. The Weyl group of $U(n)$ is Σ_n, the permutation group on n letters. To see this, we first remark that

$$T = \left\{ \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \mid |\lambda_j| = 1 \right\} \cong \mathbb{T}^n$$

is a maximal torus. Now, two matrices are conjugate if and only if they have the same set of eigenvalues and so the Weylgroup W is a subset of Σ_n. Moreover, if σ_{ij} is the permutation matrix that swaps the ith and jth columns of the identity matrix, and $A \in T$, then $\sigma_{ij}A\sigma_{ij}^{-1}$ transposes the eigenvalues λ_i and λ_j of A. Hence, all two-cycles are in W. But Σ_n is generated by two-cycles and we conclude that $W = \Sigma_n$.

31. Lecture 31

31.1. **The Peter-Weyl Theorems I.** Our final goal will be to prove the Peter-Weyl theorem(s):

Theorem 31.1.1 (Peter-Weyl I)**.** *Any compact Lie group G is isomorphic to a closed subgroup of $U(n)$ for some choice of n.*

Another way to state this is that a compact Lie group G can be realized as a subset of $M_n(\mathbb{C})$, i.e. is a matrix group. Yet another way to look at this is there is a faithful (kernel $= \{1\}$) representation $\rho : G \to Gl(n, \mathbb{C})$.

We will need a few preliminary definitions before we can state version II of this theorem.

Definition 31.1.2. Suppose $\rho : G \to GL(n, \mathbb{C})$ is a representation of a Lie group G. Let $r_{ij} : M_n(\mathbb{C}) \to \mathbb{C}$ denote the standard coordinate functions. The functions

$$r_{ij} \circ \rho : G \to \mathbb{C}$$

are called the *matrix coefficients* of the representation ρ.

More abstractly, the representation coefficients may be realized as

$$(r_{ij} \circ \rho)(g) = \langle e_i^*, \rho(g) e_j \rangle$$

where $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{C}^n$ and $\{e_1^*, \ldots, e_n^*\}$ is the associated dual basis.

Definition 31.1.3. Let G be a Lie group. A function $f : G \to \mathbb{C}$ is a *representative function* (an *abstract matrix coefficient*) if there is a representation $\rho : G \to GL(V)$ such that

$$f(g) = \langle l, \rho(g)\xi \rangle$$

for some $\xi \in V$, $l \in V^*$ and for all $g \in G$. We will usually denote such a function by $f_{V,l,\xi}$.

Lemma 31.1.4. *Representative functions of a Lie group G form a subalgebra of $C^0(G)$.*

Proof. The following equalities are left as an exercise:

$$\begin{aligned} f_{V_1,l_1,\xi_1} + f_{V_2,l_2,\xi_2} &= f_{V_1 \oplus V_2, l_1 \oplus l_2, \xi_1 \oplus \xi_2} \\ f_{V_1,l_1,\xi_1} f_{V_2,l_2,\xi_2} &= f_{V_1 \otimes V_2, l_1 \otimes l_2, \xi_1 \otimes \xi_2} \\ \lambda f_{V,l,\xi} &= f_{V,\lambda l,\xi} \qquad \forall \lambda \in \mathbb{C} \end{aligned}$$

□

Notation We will denote by $C^0_{alg}(G)$ the algebra of representative functions of G.

Theorem 31.1.5 (Peter-Weyl II)**.** *Let G be a compact Lie group. Then, $C^0_{alg}(G)$ is dense in $C^0(G)$ with respect to the supremum norm. That is, for all $f \in C^0(G)$ and for all $\epsilon > 0$, there exists $h \in C^0_{alg}(G)$ such that*

$$\sup_{g \in G} |f(g) - h(g)| < \epsilon$$

We claim that the two versions of the Peter-Weyl Theorem are equivalent. Version I implies Version II by the Stone-Weierstrass Theorem:

Theorem 31.1.6 (Stone-Weierstrass). *Let X be a compact topological space. Suppose $A \subseteq C^0(X)$ is a subalgebra such that*

(1) *A separates points, i.e. for all $x_1, x_2 \in X$ with $x_1 \neq x_2$ there is $f \in A$ so that $f(x_1) \neq f(x_2)$,*
(2) *$1 \in A$ (here, 1 denotes the constant function $1(x) = 1$ for all $x \in X$) and*
(3) *if $f \in A$, then the complex conjugate $\overline{f} \in A$.*

Then, A is dense in $C^0(X)$.

Assuming the Stone-Weierstrass Theorem, let's see why version I of the Peter-Weyl Theorem implies the second version:

Proof. Suppose $G \in GL(n, \mathbb{C})$ is a compact Lie subgroup. Consider the subalgebra A of $C^0(G)$ generated by $r_{ij}|_G, \overline{r}_{ij}|_G$ and 1. The functions r_{ij} separate points. Also, $r_{ij}, \overline{r}_{ij}, 1 \in C^0_{alg}(G)$ so $A \subseteq C^0_{alg}(G)$. By Stone-Weierstrass, A is dense in $C^0(G)$ and so $C^0_{alg}(G)$ is dense in $C^0(G)$. □

We now endeavor to prove that the second version of the Peter-Weyl Theorem implies the first. We will need a considerable amount of background material to complete the proof.

Proposition 31.1.7. *Any descending chain*

$$G_1 \not\supseteq G_2 \supsetneq \ldots \supsetneq G_k \supsetneq \ldots$$

of compact Lie groups is finite.

Proof. Suppose A and B are two compact Lie groups with $A \subsetneq B$. We claim that either $\dim A < \dim B$ or the connected components A_0 and B_0 of the identities are equal and $A/A_0 \subsetneq B/B_0$.

To prove this claim, first suppose that $\dim A = \dim B$. Then, since $\dim Lie(A) = \dim Lie(B)$ and $Lie(A) \subseteq Lie(B)$ they must be equal. Hence, $A_0 = B_0$ and $A/A_0 \subseteq B/B_0$. Since $A \neq B$ $A/A_0 \neq B/B_0$.

Now that the claim has been established, note that at eash step of the chain, say $G_i \supsetneq G_{i+1}$, either $\dim G_{i+1} \lneq \dim G_i$ or $|(G_i/G_{i+1})| < |G_i/(G_i)_0|$. So, eventually $\dim G_i = 0$ and then $|G_{i+k}/(G_{i+k})_0| = 1$. □

32. Lecture 32

32.1. **A Bit of Analysis.** We will now review some properties of normed vector spaces that we will need to prove the Peter-Weyl Theorems via a little bit of functional analysis.

Definition 32.1.1. A *norm* $\|\cdot\|$ on a vector space V is a map

$$\|\cdot\| : V \to [0,\infty)$$

such that

(1) $\|f\| = 0$ if and only if $f = 0$ for all $f \in V$,
(2) $\|\alpha f\| = |\alpha|\|f\|$ for all $\alpha \in \mathbb{C}$, $f \in V$,
(3) $\|f+g\| \leq \|f\| + \|g\|$ for all $f, g \in V$.

A vector space V with a norm is called a *normed vector space.*

Recall that a normed vector space V is a *Banach* space if it is complete, i.e. every Cauchy sequence is a convergent sequence.

Fact: $(C^0(G), \|\cdot\|_{C^0})$ is a Banach space, where G is a compact Lie group and

$$\|f\|_{C^0} = \sup_{g\in G} |f(g)|$$

A Banach space is a *Hilbert space* if there is a Hermitian inner product $(\cdot,\cdot) : V \times V \to \mathbb{C}$ such that

$$\|f\| = (f,f)^{1/2}$$

for all $f \in V$.

Example 32.1.2. Let G be a compact Lie group. Define

$$L^2(G) = \left\{ f : G \to \mathbb{C} \mid \int_G |f(x)|^2\, dx < \infty \right\}$$

Then, $L^2(G)$ is a Hilbert space with

$$\|f\|_{L^2} = \left(\int_G |f(x)|^2\, dx \right)^{\frac{1}{2}}$$

and

$$(f,g) = \int_G \overline{f(x)} g(x)\, dx.$$

Suppose now that $(V_1, \|\cdot\|_1)$ and $V_2, \|\cdot\|_2)$ are Banach spaces. A linear map $T : V_1 \to V_2$ is continuous if there exists $A > 0$ so that $\|T(v)\|_2 \leq A\|v\|_1$.

Example 32.1.3. If $f \in C^0(G)$, then

$$\sup_{x\in G} |f(x)|^2 = \left(\sup_{x\in G} |f(x)| \right)^2 = \|f\|_{C^0}^2$$

and so

$$\begin{aligned}
\int \|f\|_{L^2}^2 &= \int_G |f(x)|^2\,dx \\
&\leq \int_G \sup_{x\in G} |f(x)|^2\,dx \\
&\leq \int_G \|f\|_{C^0}^2\,dx \\
&= \|f\|_{C^0}^2 \int_G 1\,dx \\
&= \|f\|_{C^0}^2.
\end{aligned}$$

That is, $\|f\|_{L^2} \leq \|f\|_{C^0}$. Hence, the identity map $\mathrm{id} : C^0(G) \to L^2(G)$ (with the above indicated norms) is continuous.

Theorem 32.1.4 (Cauchy Schwarz Inequality)**.** *Suppose (X, dx) is a measure space. Then, for all $f, g \in L^2(X)$ we have*

$$|(f,g)| \leq \|f\|_2 \|g\|_2$$

In particular, we see that if $X = G$ is a compact Lie group, then

$$\left|\int_X \overline{f(x)} g(x)\,dx\right| \leq \left(\int_X |f(x)|^2\,dx\right)^{\frac{1}{2}} \left(\int_X |g(x)|^2\,dx\right)^{\frac{1}{2}}$$

and

$$\begin{aligned}
\left|\int_G |f(x)|\,dx\right| &= \left|\int_G |f(x)| \cdot 1\,dx\right| \\
&\leq \left(\int_X |f(x)|^2\,dx\right)^{\frac{1}{2}} \left(\int_X |1^2\,dx\right)^{\frac{1}{2}} \\
&\leq \|f\|_{L^2}.
\end{aligned}$$

From this we conclude that if $f \in L^2(G)$, then $f \in L^1(G)$ and the inclusion map $L^2(G) \hookrightarrow L^1(G)$ is continuous.

Example 32.1.5. A function $k \in C^0(G \times G)$ defines a linear map

$$\begin{aligned}
K : C^0(G) &\to \text{functions on } G \\
(Kf)(x) &:= \int_G k(x,y) f(y)\,dy.
\end{aligned}$$

It is left as an exercise to the reader to prove that $Kf \in C^0(G)$. Moreover,

$$\begin{aligned}
|(Kf)(x)| &\leq \int_G |k(x,y)||f(y)|\,dy \\
&\leq \int_G (\sup|k|)|f(y)|\,dy \\
&= \|k\|_{C^0} \int_G |f(y)|\,dy \\
&\leq \|k\|_{C^0}\|f\|_{L^2}.
\end{aligned}$$

So, $K : (C^0(G), \|\cdot\|_{L^2}) \to (C^0(G), \|\cdot\|_{C^0})$ is continuous.

Definition 32.1.6. A linear map $K : V_1 \to V_2$ of normed vector spaces is *compact* if it maps bounded sets $B \subseteq V_1$ to precompact sets $K(B) \subseteq V_2$.

Definition 32.1.7. Let X be a topological space. A subset $L \subset C^0(X)$ is *equicontinuous at* $x_0 \in X$ if for all $\epsilon > 0$ there is a neighborhood U of x_0 such that

$$|f(x) - f(x_0)| < \epsilon \qquad \forall x \in X, \forall f \in L.$$

L is *equicontinuous* if it is equicontinuous at every point of X.

Theorem 32.1.8 (Ascoli). *A subset $L \subseteq (C^0(G), \|\cdot\|_{C^0})$ is precompact if and only if it is bounded and equicontinuous.*

We now present an application of the Ascoli Theorem:

Example 32.1.9. Let $k \in C^0(G \times G)$ and $K : (C^0(G), \|\cdot\|_{L^2}) \to (C^0(G), \|\cdot\|_{C^0})$ be as above. We claim that K, and hence $K : (C^0(G), \|\cdot\|_{L^2}) \to (C^0(G), \|\cdot\|_{L^2})$, is compact.

Proof. Suppose $B \subseteq C^0(G)$ is L^2-bounded by $C > 0$, i.e. for all $f \in B$, $\|f\|_{L^2} < C$. We want to show that $K(B)$ is precompact in the C^0-norm. By Ascoli, it is enough to show that $K(B)$ is C^0-bounded and equicontinuous. Since $|(Kf)(x)| \leq \|k\|_{C^0}\|f\|_{L^2}$,

$$\|Kf\|_{C^0} = \sup_{x \in G} |(Kf)(x)| \leq \|k\|_{C^0} \cdot C$$

To continue, we'll need the following lemma:

Lemma 32.1.10. *If $h \in C^0(G)$, then for all $\epsilon > 0$ there is a neighborhood U of 1 in G so that*

$$xy^{-1} \in U \implies |h(x) - h(y)| < \epsilon$$

In light of this lemma, given $\epsilon > 0$ we can find a neighborhood V of 1 in G such that

$$|k(x,y) - k(x',y)| < \epsilon C^{-1}$$

as long as $x'x^{-1} \in V$. Then,

$$\begin{aligned}
|(Kf)(x) - (Kf)(x')| &= \left| \int_G (k(x,y) - k(x'y)) f(y)\, dy \right| \\
&\leq \epsilon C^{-1} \int_G |f(w)|\, dy \\
&\leq \epsilon C^{-1} \|f\|_{L^2} \\
&= \epsilon
\end{aligned}$$

$\square$

33. Lecture 33

33.1. The Peter-Weyl Theorems II.

We continue to apply analysis to our study of compact Lie groups. In the last lecture, we used a lemma which we will now prove.

Lemma 33.1.1. *Let H be a compact Lie group and $f \in C^0(H)$. Then, for any $\epsilon > 0$ there is a neighborhood V of 1 in H so that*

$$yx^{-1} \in V \implies |f(x) - f(y)| < \epsilon$$

Proof. Since f is continuous, for any $x \in H$ there is a neighborhood U_x of 1 such that for any $y \in U_x x$

$$|f(y) - f(x)| < \frac{\epsilon}{2}$$

Choose $V_x \subseteq H$ so that $1 \in V_x$ and $V_x^2 \subseteq U_x$. Then, $\{V_x x\}$ is an open cover of H. Since H is compact, there is a finite subcover: there exist $x_1, \ldots, x_n$ so that

$$\bigcup_{i=1}^{n} V_{x_i} x_i$$

Set $V = \cap_i V_{x_i}$ and suppose $yx^{-1} \in V$. Then, $x \in V_{x_i} x_i$ for some i. So, $y = (yx^{-1})x \in V V_{x_i} x_i \subseteq U_{x_i} x_i$. We then have

$$|f(y) - f(x_i)| < \frac{\epsilon}{2}, |f(x) - f(x_i)| < \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned} |f(y) - f(x)| &\leq |(f(y) - f(x_i)) + (f(x_i) - f(x))| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Remark 33.1.2. We may further assume that $V = V^{-1}$ by replacing V with $V \cap V^{-1}$.

Corollary 33.1.3. *If G is a compact Lie group and $k \in C^0(G \times G)$, then for all $\epsilon > 0$ there exists a neighborhood U of 1 in G such that if $xy^{-1} \in U$,*

$$|k(x, h) - k(y, h)| < \epsilon$$

for all $h \in G$.

Proof. Let $H = G \times G$. We may assume that $U \subseteq G \times G$ is of the form $U \times U$. Then, $xy^{-1} \in U$ implies that $(x, h)(y, h)^{-1} \in U \times U$ for any $h \in G$. □

Lemma 33.1.4. *Let G be a compact Lie group, $k \in C^0(G \times G)$. Then, for any $f \in L^2(G)$,*

$$(Kf)(x) := \int_G k(x, y) f(y)\, dy$$

is a continuous function of x. In fact, for all $\epsilon > 0$ there exists a neighborhood U of 1 in G so that

$$xy^{-1} \in U \implies |(Kf)(x) - (Kf)(y)| < \epsilon$$

Proof. By Corollary 33.3, there exists a neighborhood U of 1 such that if $xy^{-1} \in U$, then

$$|k(x,h) - k(y,h)| < \frac{\epsilon}{C}$$

where C will be chosen later. Then,

$$\begin{aligned}
|(Kf)(x) - (Kf)(y)| &\leq \left| \int_G (k(x,h) - k(y,h)) f(h)\, dh \right| \\
&\leq \int_G |k(x,h) - k(y,h)||f(h)|\, dh \\
&\leq \int_G \frac{\epsilon}{C} |f(h)|\, dh \\
&= \frac{\epsilon}{C} \int_G |f(h)|\, dh \\
&\leq \frac{\epsilon}{C} \|f\|_{L^2}.
\end{aligned}$$

Now, simply take $C = \|f\|_{L^2}$. □

Lemma 33.1.5. *let G, k and K be as above. Then, for all $f \in L^2(G)$, $\|Kf\|_{C^0} \leq \|k\|_{C^0} \|f\|_{L^2}$. Hence, $K : (L^2(G), \|\cdot\|_{L^1}) \to (C^0(G), \|\cdot\|_{C^0})$ is bounded.*

Proof. For any $x \in G$,

$$\begin{aligned}
|(Kf)(x)| &= \left| \int_G k(x,y) f(y)\, dy \right| \\
&\leq \int_G |k(x,y||f(y)|\, dy \\
&\leq \int_G \sup_{x,y \in G} |k(x,y)||f(y)|\, dy \\
&= \|k\|_{C^0} \int_G |f(y)|\, dy \\
&\leq \|k\|_{C^0} \|f\|_{L^2}.
\end{aligned}$$

Therefore, $|Kf|_{C^0} = \sup_{x \in G} |(Kf)(x)| \leq \|k\|_{C^0} \|f\|_{L^2}$ □

It is a well-known fact, and left as an exercise for the reader, that bounded linear maps between normed vector spaces are continuous, and we have seen that $K : L^1(G) \to L^2(G)$ is compact. The question is: how does this help us with the Peter-Weyl Theorem: the density of $C^0_{alg}(G)$ in $C^0(G)$?

Two Ideas:

(1) G acts on functions on G,

$$(a \cdot f)(x) = f(a^{-1}x)$$

and we will prove that

Lemma 33.1.6. *$f \in C^0(G)$ is a representative function if and only if there is a finitw dimensional subspace $W \subseteq C^0(G)$ with $f \in W$.*

(2) Let $f \in C^0(G)$ be a function. Fix $\epsilon > 0$. We want to find $h \in C^0_{alg}(G)$ such that

$$\|f - h\|_{C^0} < \epsilon$$

Since $f \in C^0(G)$ there exists a neighborhood U of 1 in G such that

$$xy^{-1} \in U \implies |f(x) - f(y) < \frac{\epsilon}{2}.$$

Pick $\rho \in C^0(G)$ with $\rho \geq 0$ such that $\operatorname{supp}\rho \subseteq U$,

(i) $\rho(x) = \rho(x^{-1})$ and

(ii) $\displaystyle\int_G \rho(x)\,dx = 1.$

To get (ii), take any $\tilde{\rho}$ with $\operatorname{supp}\tilde{\rho} \subseteq U$ and let

$$\rho(x) = \frac{1}{\int_G \tilde{\rho}(x)\,dx}\tilde{\rho}(x)$$

To get (ii), take any $\tilde{\rho}$ and let

$$\rho(x) = \frac{1}{2}(\tilde{\rho}(x) + \tilde{\rho}(x^{-1}))$$

Now, let $k(x,y) = \rho(x^{-1}y) = \rho((x^{-1}y)^{-1}) = \rho(y^{-1}x)$ and consider

$$\begin{array}{rcl} K : L^2(G) & \to & C^0(G) \hookrightarrow L^2(G) \\ (Kf)(x) & = & \displaystyle\int_G k(x,y)f(y)\,dy \end{array}$$

Then, we have

$$\|Kf - f\|_{C^0} < \frac{epsilon}{2}$$

On the other hand, since we know K is compact, and we also see that $g \cdot (Kf) = K(g \cdot f)$ (that is, K is G-equivariant). Hence, the eigenspaces of K are G-invariant and finite dimensional. Furthermore, the span of eigenfunctions is dense in $(L^2(G), \|\cdot\|_{L^2})$. That is, there exist $v_1, \ldots, v_k \in L^2$ so that $KV_i = \lambda_i v_i$ and

$$\|\sum v_i - f\|_{L^2} < \frac{\epsilon}{2}\|k\|_{C^0}.$$

Hence,

$$\begin{array}{rcl} \|\sum \lambda_i v_i - Kf\|_{C^0} & = & \|K(\sum v_i - f)\|_{C^0} \\ & \leq & \|k\|_{C^0}\|\sum v_i - f\|_{L^2} \end{array}$$

34. Lecture 34

34.1. The Peter-Weyl Theorems III.

Proposition 34.1.1. *Let $\mathcal{H}$ be a Hilbert space and $K : \mathcal{H} \to \mathcal{H}$ be a symmetric, compact operator. Then,*

$$\|K\| = \sup_{\|f\|=1} |(Kf, f)|$$

Proof. For $f \in \mathcal{H}$ with $\|f\| = 1$, we have by Cauchy-Schwarz

$$|(Kf, f)| \leq \|Kf\|\|f\| \leq \|K\|1^2$$

and so

$$M := \sup_{\|f\|=1} |(Kf, f)| \leq \|K\|$$

We now argue that $\|K\| \leq M$. For this, it's enough to show that $\|Kf\| \leq M$ for all $f \in \mathcal{H}$ and all $\|f\| = 1$. Note that if $Kf = 0$, we're done. So, assume $x \in \mathcal{H}$, $\|x\| = 1$ and $Kx \neq 0$. Let $y = Kx/\|Kx\|$. We then see that $\|y\| = 1$ as well. Also,

$$\begin{aligned} (x, Ky) &= (Kx, y) \\ &= (Kx, Kx/\|Kx\|) \\ &= \frac{1}{\|Kx\|}\|Kx\|^2 \\ &= \|Kx\| \end{aligned}$$

It is also easy to verify that

$$(K(x+y), x+y) - (K(x-y), x-y) = 2(Kx, y) + 2(Ky, x) = 4\|Kx\|$$

On the other hand, for any $z \in \mathcal{H}$, $z \neq 0$ we have

$$|(Kz, z)| = (|(K(z/\|z\|), z/\|z\||)\|z\|^2 \leq M\|z\|^2$$

and so

$$\begin{aligned} (K(x+y), x+y) &\leq M\|x+y\|^2 \\ -(K(x-y), x-y) &\leq M\|x-y\|^2. \end{aligned}$$

We therefore conclude that

$$\begin{aligned} 4\|Kx\| &= (K(x+y), x+y) - (K(x-y), x-y) \\ &\leq M(\|x+y\|^2 + \|x-y\|^2) \\ &= M(2\|x\|^2 + 2\|y\|^2 \\ &= 4M \end{aligned}$$

and so $\|Kx\| \leq M$. $\square$

Proposition 34.1.2. *Let $K : \mathcal{H} \to \mathcal{H}$ be a compact symmetric operator. Then, either $\|K\|$ or $-\|K\|$ is an eigenvalue of K.*

Proof. By the last proposition, there is a sequence $\{x_n\} \subseteq \mathcal{H}$ with $\|x_n\| = 1$ such that

$$\lim_{n\to\infty} |(Kx_n, x_n)| = \|K\|$$

By passing to a subsequence. if necessary, we may assume

$$\lim_{n\to\infty} (Kx_n, x_n) = \alpha$$

where α is either $\|K\|$ or $-\|K\|$. Note, if $\alpha = 0$ there is nothing to prove. So, assume $\alpha \neq 0$. Then,

$$\begin{aligned}
0 \leq \|Kx_n - \alpha x_n\|^2 &= (Kx_n - \alpha x_n, Kx_n - \alpha x_n)\\
&= \|Kx_n\|^2 - (\alpha x_n, Kx_n) - (Kx_n, \alpha x_n) + |\alpha|^2 \|x_n\|^2\\
&\leq \|K\|^2 \|x_n\|^2 - 2\alpha(x_n, Kx_n) + \alpha^2 \|x_n\|^2\\
&= 2\alpha(\alpha - (x_n, Kx_n))
\end{aligned}$$

Since $(Kx_n, x_n) \to \alpha$ as $n \to \infty$. $(\alpha - (x_n, Kx_n)) \to 0$. Thus,

$$\|Kx_n - \alpha x_n\|^2 \to 0$$

Since K is compact and x_n is bounded, by passing to a subsequence we may assume that $Kx_n \to y$. But, $\|Kx_n - \alpha x_n\|^2 \to 0$ implies that $\alpha x_n \to y$ as $n \to \infty$. Now,

$$\begin{aligned}
Ky &= K(\lim_{n\to\infty} \alpha x_n)\\
&= \lim_{n\to\infty} K(\alpha x_n)\\
&= \alpha \lim_{n\to\infty} (Kx_n)\\
&= \alpha y.
\end{aligned}$$

So, y is an eigenvector of K with eigenvalue α, provided $y \neq 0$. But y cannot be zero since

$$\|y\| = \|\lim_{n\to\infty} \alpha x_n\| = \lim_{n\to\infty} \|\alpha x_n\| = |\alpha|$$

□

Suppose that $\mathcal{H}$ is a Hilbert space and $K : \mathcal{H} \to \mathcal{H}$ is compact and symmetric. If v is an eigenvector with eigenvalue α, then

(1) $(Kv, v) = (\lambda v, v) = \overline{\lambda}(v, v) = \overline{\lambda}\|v\|^2$ and
(2) $(v, Kv) = (v, \lambda v) = \lambda(v, v) = \lambda\|v\|^2$.

We conclude that all eigenvalues of K are <u>real</u>.

Also, if $Kv = \lambda v$, $Kw - \mu w$ and $\mu \neq \lambda$, then

$$\lambda(v, w) = (Kv, w) = (v, Kw) = \mu(v, w).$$

So, $(v, w) = 0$ and eigenspaces associated to distinct eigenvalues are mutually orthogonal.

Theorem 34.1.3. *Let $\mathcal{H}$ be a Hilbert space, $K : \mathcal{H} \to \mathcal{H}$ a compact symmetric operator. Then,*

(1) *For each $\epsilon > 0$, the direct sum of eigenspaces $\bigoplus_{|\lambda|>\epsilon} \mathcal{H}_\lambda$ is finite dimensional and*
(2) *$\bigoplus_\lambda \mathcal{H}_\lambda$ is dense in $\mathcal{H}$.*

Proof. (1) Suppose that $\bigoplus_{|\lambda|>\epsilon} \mathcal{H}_\lambda$ is infinite dimensional. Then, there is a sequence $\{x_n\}$ of orthornormal eigenvectors with eigenvalues λ_n such that $|\lambda_n| > \epsilon$ for each n. Since $\{x_n\}$ is bounded and K is compact, $\{Kx_n\}$ contains a convergent subsequence.

On the other hand, for all positive integers m, n we have

$$\begin{aligned} \|Kx_n - Kx_m\|^2 &= \|\lambda_n x_n - \lambda_m x_m\|^2 \\ &= |\lambda_n|^2\|x_n\|^2 + |\lambda_m|^2\|x_m\|^2 \\ &= \lambda_n^2 + \lambda_n^2 > 2\epsilon. \end{aligned}$$

To prove (2), let $E = \overline{\bigoplus_\lambda \mathcal{H}_\lambda}$, the closure of $\bigoplus_\lambda \mathcal{H}_\lambda$ in $\mathcal{H}$ Since $K\left(\bigoplus_\lambda \mathcal{H}_\lambda\right) \subseteq \mathcal{H}_\lambda$, $K\left(\bigoplus_\lambda \mathcal{H}_\lambda\right) \subseteq \bigoplus_\lambda \mathcal{H}_\lambda$. Since K is continuous, $K(E) \subseteq E$. Let $F = E^\perp$. Since $K(E) \subseteq E$, $K(F) \subseteq F$. So, for any $e \in E$, $f \in F$,

$$(K(f), e) = (f, K(e)) = 0$$

$K(e) \in E$. Moreover, F is closed. So, F is a Hilbert space.

The inclusion map $F \hookrightarrow \mathcal{H}$ is continuous and so $K|_F : F \to F$ is compact, and $K|_F$ is still symmetric. By Proposition 34.2, $K|_F$ has an eigenvector $v \neq 0$. So, $v \in \mathcal{H}_\lambda$ for some λ. But then, $v \in E$ which is a contradiction. $\square$

35. Lecture 35

35.1. The Peter-Weyl Theorems IV. Let's first recall a few facts we have discussed earlier:

(1) A function $f \in C^0(G)$ is a representative function if and only if there exists a representation $\rho : G \to Gl(V)$, $v \in V$ and $l \in V^*$ such that

$$f(g) = \langle l, \rho(g)c \rangle.$$

(2) G acts on $C^0(G)$ by

$$(g \cdot f)(x) = f(g^{-1}x).$$

We also owe the reader a proof of the following lemma:

Lemma 35.1.1. *The function $f \in C^0(G)$ is a representative function of and only if there exists a finite dimensional subspace $W \subseteq C^0(G)$ which is a subrepresentation and $f \in W$.*

Proof. Suppose $\rho : G \to GL(V)$ is a finite dimensional representation. Take a basis $\{v_1, \dots, v_n\}$ of V and let $\{v_1^*, \dots, v_n^*\}$ be the associated dual basis. Then,

$$a_{ij}(x) := \sum_j v_j^*(\rho(x)v_i)$$

are representative functions.

Since $\rho(gx) = \rho(g)\rho(x)$, $(a_{ij}(gx)) = (a_{ij}(g))(a_{kj}(x))$ and $(g \cdot a_{ij})(x) = a_{ij}(g^{-1}x) = \sum_k a_{ik}(g^{-1})a_{kj}(x)$ That is, $g \cdot a_{ij} \in \operatorname{span}_{\mathbb{C}}\{a_{kj}$ for all $g \in G$ and a_{ij}.

Also, for any $l \in V^*$,$v \in V$

$$f(g) = l(\rho(g)v) \in \operatorname{span}_{\mathbb{C}}\{a_{ij}(g)\}$$

Therefore, $W = \operatorname{span}_{\mathbb{C}}\{a_{ij}\}$.

Conversely, suppose that $W \subseteq C^0(G)$ is finite dimensional and for any $f \in W$, $g \cdot f \in W$. Consider $l : W \to \mathbb{C}$ given by $l(f) = f(1)$. Then, $l \in W^*$. Also,

$$l(g^{-1} \cdot f) = f(g(x))|_{x=1} = f(g).$$

Let $V = W^*$ and $\rho : G \to GL(W^*)$ be given by

$$\rho = (g \cdot f)|_W)^*$$

Then, $f \in V = (W^*)^*$ and $f(g) = \langle f, \rho(g)l \rangle$. □

We now finally begin the proof of

Theorem 35.1.2 (Peter-Weyl Version II)**.** *Let G be a compact Lie group. Then, representative functions are dense in $C^0(G)$.*

Proof. Fix $f_0 \in C^0(G)$. For any $\epsilon > 0$, we want to find a representative function h so that

$$\|f_0 - h\|_{C^0} < \epsilon.$$

Since $f_0 \in C^0(G)$, there exists a neighborhood U of 1 in G such that

$$xy^{-1} \in U \implies |f_0(x) - f_0(y)| < \frac{\epsilon}{2}$$

Pick $\tau_1 \in C^0(G)$ such that $\tau_1 \geq 0$ and $\operatorname{supp} \tau_1 \subseteq U$ and

$$\int_G \tau_1(x)\, dx = 1$$

Let $\tau(x) = (1/2)(\tau_1(x) + \tau_1(x^{-1}))$. Then, $\displaystyle\int \tau(x)\, dx = 1$, $\operatorname{supp} \tau \subseteq U$ and $\tau(x) = \tau(x^{-1})$.

Now, let

$$\begin{aligned} k(x,y) &= \tau(x^{-1}y), \\ K : L^2(G) &\to C^0(G) \hookrightarrow L^2(G) \\ (Kf)(x) = \int_G \tau(x^{-1}y) f(y)\, dy \end{aligned}$$

Claim 1: K is a compact, symmetric operator.
Claim 2: $\|Kf_0 - f_0\|_{C^0} < \epsilon/2$.
Claim 3: $g \cdot (Kf) = K(g \cdot f)$ for all $f \in L^2(G)$.

Let's see, assuming these claims, how to finish the proof. Since k is compact, the nonzero eigenspaces of K are finite dimensional. So, if $Kf = \lambda f$ for some nonzero λ, then f is a representative function.

On the other hand, eigenfunctions of K are dense in $L^2(G)$. Thus, there exist $h-1, \ldots, h_n \in L^2(G)$ such that $Kh_i = \lambda_i h_i$ and

$$\|\sum_i h_i - f_0\|_{L^2} \leq (\|k\|_{C^0})^{-1}(\frac{\epsilon}{2})$$

So, we get

$$\begin{aligned} \|\sum_i \lambda_i h_i - Kf_0\|_{C^0} &= \|K(\sum_i h_i - f_0\|_{C^0} \\ &\leq \|k\|_{C^0}\|\sum_i h_i - f_0\|_{L^2} \\ &= \|k\|_{C^0}(\|k\|_{C^0})^{-1}(\frac{\epsilon}{2}) = \frac{\epsilon}{2} \end{aligned}$$

Now, let $h = \sum_i h_i$. Then,

$$\begin{aligned} \|f_0 - h\|_{C^0} &\leq \|f_0 - Kf_0\|_{C^0} + \|Kf_0 - h\|_{C^0} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Modulo the proof of the above three claims, this completes the proof of the theorem. $\square$

36. Lecture 36

36.1. **Finale.** We still need to prove the claims from the last lecture.

Proof of Claim 1: Since $\tau(z) = \tau(z^{-1})$, $\tau(x^{-1}y) = \tau(y^{-1}x)$. So, $k(x, y) = \tau(x^{-1}y) = k(y, x)$. Therefore,

$$\begin{aligned}
(Kf_1, f_2) &= \int_G \overline{Kf_1}(x) f_2(x)\, dx \\
&= \int_G \int_G \overline{k(x, y) f_1(y)}\, dy f_2(x)\, dx \\
&= \int_G \int_G \overline{f_1(y)} k(x, y) f_2(x)\, dy\, dx \\
&= \int_G \overline{f_1}(y) \left(\int_G k(y, x) f_2(x)\, dx \right) dy \\
&= \int_G \overline{f_1}(y) K f_2(y)\, dy \\
&= (f_1, Kf_2)
\end{aligned}$$

and it follows that K is symmetric.

Proof of Claim 2: Note first that if we write $y' = x^{-1}y$, we have

$$Kf_0(x) = \int_G \tau(y') f_0(y)\, dy = \int_G \tau(y') f_0(xy')\, dy.$$

This implies

$$\begin{aligned}
|Kf_0(x) - f_0(x)| &= \left| \int_G \tau(y) f_0(xy')\, dy - f_0(x) \int_G \tau(y)\, dy \right| \\
&\leq \int_G |\tau(y)| |f_0(xy) - f_0(x)|\, dy.
\end{aligned}$$

If $\tau(y) \neq 0$, then $y \in U$ is so that $y = x^{-1}xy$ and hence $|f_0(xy) - f_0(x)|, \epsilon/2$. Then,

$$\begin{aligned}
\|Kf_0 - f_0\| &= \sup_x |Kf_0(x) - f_0(x)| \\
&\leq \int_G \tau(y) (\frac{\epsilon}{2})\, dy = \frac{\epsilon}{2}
\end{aligned}$$

Proof of Claim 3: Claim 3 follows from a quick calculation:

$$\begin{aligned}(g\cdot Kf)(x) &= Kf(g^{-1}x)\\ &= \int_G \tau((g^{-1}x)^{-1}y)f(y)\,dy\\ &= \int_G \tau(x^{-1}y')f(g^{-1}y')\,dy'\\ &= \int_G \tau(x^{-1}y)(g\cdot f)(y)\,dy\\ &= (K(g\cdot f))(x)\end{aligned}$$

Finally, we present an application of this theorem:

Example 36.1.1. Let $G = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Any representation of $\mathbb{T}^n$ is a direct sum of irreducibles. A matrix coefficient of an irreducible representation of $\mathbb{T}^n$ is of the form

$$a_m(x \mod \mathbb{Z}^n) = e^{2\pi i(\sum_{j=1}^n m_j x_j)}$$

where $n = (m_1, \ldots, m_n) \in \mathbb{Z}^n$.

We can think of $f \in C^0(\mathbb{T}^n)$ as $f \in C^0(\mathbb{R}^n)$ with $f(x_1, \ldots, x_m) = f(x_1 + k_1, \ldots, x_n + k_n)$ for $k_i \in \mathbb{Z}$. Then,

$$C^0_{alg}(\mathbb{T}^n) = \left\{ \sum_{m\in\mathbb{Z}^n} c_m e^{2\pi i(\sum_{j=1}^n m_j x_j)} \mid c_m \in \mathbb{C} \right\},$$

the space of trigonometric polynomials. In this context, the Peter-Weyl Theorem can be stated as follows:

Theorem 36.1.2. *Any continuous $\mathbb{Z}^n$-periodic function on $\mathbb{R}^n$ can be approximated by a trigonometric polynomial.*