Review of Selected Topics in Probability

Christoforos Raptopoulos

Lecture 2

The Monty Hall Problem

The Monty Hall Problem

- There are 3 doors A, B, C and one has a prize.
- There are two rounds:
  - In the first round the contestant chooses a door and then the organizers open one of the (remaining) empty ones.
  - In the second round there are two options for the contestant: (a) change his door choice or (b) stick with his first choice.

The Monty Hall Problem

- There are 3 doors A, B, C and one has a prize.
- There are two rounds:
  - In the first round the contestant chooses a door and then the organizers open one of the (remaining) empty ones.
  - In the second round there are two options for the contestant: (a) change his door choice or (b) stick with his first choice. *However, he must declare this choice before the first round.*

The Monty Hall Problem

- There are 3 doors A, B, C and one has a prize.
- There are two rounds:
  - In the first round the contestant chooses a door and then the organizers open one of the (remaining) empty ones.
  - In the second round there are two options for the contestant: (a) change his door choice or (b) stick with his first choice. *However, he must declare this choice before the first round.*

Question: Which strategy should the contestant follow?

How do we Answer a Probability Question?

How do we Answer a Probability Question?

Step 1: Find the Sample Space $\mathcal{S}$, i.e. the set of possible outcomes (or sample points, or atomic events).

How do we Answer a Probability Question?

Step 1: Find the Sample Space $\mathcal{S}$, i.e. the set of possible outcomes (or sample points, or atomic events).

Example: An outcome in the Monty Hall problem is a triple of doors (X, Y, Z): X is the place of the prize, Y is the initial choice of the contestant and Z is the door opened by the organizers.

How do we Answer a Probability Question?

Step 1: Find the Sample Space $\mathcal{S}$, i.e. the set of possible outcomes (or sample points, or atomic events).

Example: An outcome in the Monty Hall problem is a triple of doors (X, Y, Z): X is the place of the prize, Y is the initial choice of the contestant and Z is the door opened by the organizers.

Quiz: How many possible outcomes are there?

How do we Answer a Probability Question?

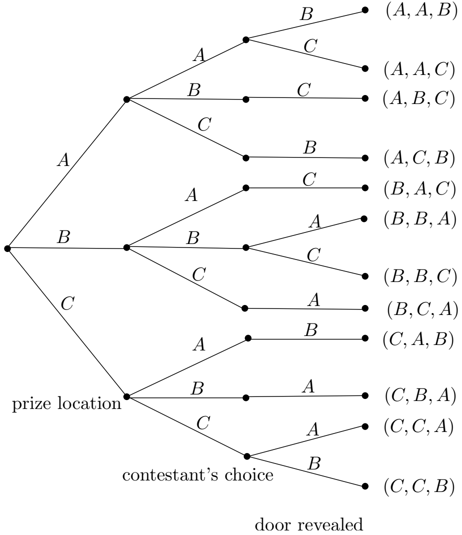
Step 1: Find the Sample Space $\mathcal{S}$, i.e. the set of possible outcomes (or sample points, or atomic events).

Example: An outcome in the Monty Hall problem is a triple of doors (X, Y, Z): X is the place of the prize, Y is the initial choice of the contestant and Z is the door opened by the organizers.

Quiz: How many possible outcomes are there?

(a) 3^3
(b) 12
(c) 15
(d) Something else.

Sample Space for Monty Hall



How do we Answer a Probability Question? (cntd.)

Step 2: Define events of interest, i.e. define subsets of the sample space.

How do we Answer a Probability Question? (cntd.)

Step 2: Define events of interest, i.e. define subsets of the sample space.

Example: Let S_w be the set of outcomes for which the contestant wins if he employs the "stick" strategy. Then

$$S_w = \{(A, A, B), (A, A, C), (B, B, A), (B, B, C), (C, C, A), (C, C, B)\}$$

How do we Answer a Probability Question? (cntd.)

Step 3: Compute outcome probabilities, i.e. determine the probability space.

How do we Answer a Probability Question? (cntd.)

Step 3: Compute outcome probabilities, i.e. determine the probability space.

Definition (Probability Space)

The probability space of an experiment consists of the sample space $\mathcal{S}$ and a function $\Pr : \mathcal{S} \to \mathbb{R}$ such that

How do we Answer a Probability Question? (cntd.)

Step 3: Compute outcome probabilities, i.e. determine the probability space.

Definition (Probability Space)

The probability space of an experiment consists of the sample space $\mathcal{S}$ and a function $\Pr : \mathcal{S} \to \mathbb{R}$ such that

1. $0 \leq \Pr(w) \leq 1$, for all atomic events $w \in \mathcal{S}$.
2. $\sum_{w \in \mathcal{S}} \Pr(w) = 1$.

Additional Assumptions for Step 3

Additional Assumptions for Step 3

1. The prize is behind a door with probability $\frac{1}{3}$.
2. The contestant chooses a door equiprobably.
3. If the organizers have a choice for which door to open, they choose one equiprobably.

Additional Assumptions for Step 3

1. The prize is behind a door with probability $\frac{1}{3}$.
2. The contestant chooses a door equiprobably.
3. If the organizers have a choice for which door to open, they choose one equiprobably.

Example: In the Monty Hall problem $\Pr(A, A, C) = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{18}$ and $\Pr(A, B, C) = \frac{1}{3} \cdot \frac{1}{3} \cdot 1 = \frac{1}{9}$.

How do we Answer a Probability Question? (cntd.)

Step 4: Compute event probabilities; for any event $S' \in \mathcal{S}$,

$$\Pr(S') = \sum_{w \in S'} \Pr(w). \tag{1}$$

How do we Answer a Probability Question? (cntd.)

Step 4: Compute event probabilities; for any event $S' \in \mathcal{S}$,

$$\Pr(S') = \sum_{w \in S'} \Pr(w). \tag{1}$$

Example: Let S_w and D_w be the events that the contestant wins if he employs the "stick" strategy or the "change" strategy respectively. Then

$$\Pr(S_w) = 6 \cdot \frac{1}{18} = \frac{1}{3}$$

and

$$\Pr(D_w) = 6 \cdot \frac{1}{9} = \frac{2}{3}.$$

Probability and Counting

Theorem (Equiprobable atomic events)

Let $\mathcal{S}$ be a sample space and for any atomic events w, w', $\Pr(w) = \Pr(w')$, then

$$\Pr(A) = \frac{\#\ \textit{atomic events in } A}{\#\ \textit{atomic events in } \mathcal{S}}. \tag{2}$$

□

Probability and Counting

Theorem (Equiprobable atomic events)

Let $\mathcal{S}$ *be a sample space and for any atomic events* w, w'*,* $\Pr(w) = \Pr(w')$*, then*

$$\Pr(A) = \frac{\#\ \textit{atomic events in}\ A}{\#\ \textit{atomic events in}\ \mathcal{S}}. \quad (2)$$

□

Use with caution!! Makes computing probabilities easier ONLY when atomic events are equiprobable; extra care is needed when defining the sample space.

Quiz

Quiz

Consider a random 3-length DNA sequence (using A, C, T, G) in which each of the 4^3 possible (ordered) outcomes are equally likely to appear. What is the probability that we get the sequence TGA when the order does matter (i.e. $TGA \neq TAG$) and when not (i.e. $TGA = TAG$)?

Quiz

Consider a random 3-length DNA sequence (using A, C, T, G) in which each of the 4^3 possible (ordered) outcomes are equally likely to appear. What is the probability that we get the sequence TGA when the order does matter (i.e. $TGA \neq TAG$) and when not (i.e. $TGA = TAG$)?

(a) $\frac{1}{4^3}$ and $\frac{1}{4^3}$.

(b) $\frac{1}{4^3}$ and $\frac{1}{\binom{4+3-1}{3}}$

(c) $\frac{1}{4^3}$ and $\frac{1}{\binom{4}{3}}$

(d) $\frac{1}{4^3}$ and $\frac{6}{4^3}$

Quiz

Consider a random 3-length DNA sequence (using A, C, T, G) in which each of the 4^3 possible (ordered) outcomes are equally likely to appear. What is the probability that we get the sequence TGA when the order does matter (i.e. $TGA \neq TAG$) and when not (i.e. $TGA = TAG$)?

(a) $\frac{1}{4^3}$ and $\frac{1}{4^3}$.

(b) $\frac{1}{4^3}$ and $\frac{1}{\binom{4+3-1}{3}}$

(c) $\frac{1}{4^3}$ and $\frac{1}{\binom{4}{3}}$

(d) $\frac{1}{4^3}$ and $\frac{6}{4^3}$

Answer: (d)

Another Quiz

Suppose we have 2 white balls and 2 black balls and we randomly choose 2 of them. What is the probability that the pair that we choose is of different color when (i) balls of the same color are identical and (ii) when all 4 balls are distinct and the order matters?

Another Quiz

Suppose we have 2 white balls and 2 black balls and we randomly choose 2 of them. What is the probability that the pair that we choose is of different color when (i) balls of the same color are identical and (ii) when all 4 balls are distinct and the order matters?

(a) $\frac{1}{2}$ in both cases.

(b) $\frac{1}{2}$ and $\frac{2}{3}$

(c) $\frac{2}{3}$ in both cases

(d) $\frac{2}{3}$ and $\frac{1}{2}$

Another Quiz

Suppose we have 2 white balls and 2 black balls and we randomly choose 2 of them. What is the probability that the pair that we choose is of different color when (i) balls of the same color are identical and (ii) when all 4 balls are distinct and the order matters?

(a) $\frac{1}{2}$ in both cases.

(b) $\frac{1}{2}$ and $\frac{2}{3}$

(c) $\frac{2}{3}$ in both cases

(d) $\frac{2}{3}$ and $\frac{1}{2}$

Answer: Homework! (Do not use conditional probability)

Yet Another Quiz - Balls in Bins

Suppose we have m distinct bins and n balls, which we place randomly inside the bins. What is the probability that the first bin remains empty?

Yet Another Quiz - Balls in Bins

Suppose we have m distinct bins and n balls, which we place randomly inside the bins. What is the probability that the first bin remains empty?

(a) $\left(\frac{m-1}{m}\right)^n$

(b) $\left(\frac{n-1}{n}\right)^m$

(c) The problem is not well defined; we need to know if the balls are distinct or not

Yet Another Quiz - Balls in Bins

Suppose we have m distinct bins and n balls, which we place randomly inside the bins. What is the probability that the first bin remains empty?

(a) $\left(\frac{m-1}{m}\right)^n$

(b) $\left(\frac{n-1}{n}\right)^m$

(c) The problem is not well defined; we need to know if the balls are distinct or not

Answer: (a)

Axiomatic Definition of Probability

Definition (Probability - alternative definition)

Probability is a function $\Pr : 2^{\mathcal{S}} \rightarrow \mathbb{R}$ such that

1. $\Pr(A) \geq 0, \forall A \subseteq \mathcal{S}$.
2. $\Pr(\mathcal{S}) = 1$.
3. If $A \cap B = \emptyset$, $\Pr(A \cup B) = \Pr(A) + \Pr(B)$ (Sum Rule, special case).

Some Properties of Probability

For any given sample space $\mathcal{S}$ and any events $A, B, A_1, \ldots, A_n$:

- $\Pr(A) \in [0, 1]$.
- $\Pr(\bar{A}) = 1 - \Pr(A)$.
- $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.
- $\Pr(A_1 \cup \cdots \cup A_n) \leq \Pr(A_1) + \cdots + \Pr(A_n)$ (Boole's inequality).