Review of Selected Topics in Counting

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Lecture 1

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  Example: For $t = 0, 1, \ldots$, let S_t be the number of individuals in a population at time t. Let also $S_0 = 2$ and $S_{t+1} = S_t + t$, for all $t \geq 0$. How much is S_n?

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- *But there is more...!* - General Counting Rules: (a) The Sum Rule, (b) The Product Rule and (c) The Division Rule

Union of Sets

Theorem (The Sum Rule)

If $A_1, \ldots, A_n$ are disjoint, then

$$|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|. \tag{1}$$

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If the sets are not disjoint - Inclusion-Exclusion Principle

Example: Let A the set of people in this class with cyan hair and B those with purple skin. Then $|A \cup B| = |A| + |B| - |A \cap B|$.

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(a) $|A| + |B| + |C|$

(b) $|A|+|B|+|C|-2|A\cap B|-2|B\cap C|-2|B\cap C|+3|A\cap B\cap C|$

(c) $|A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

(d) $|A| + |B| + |C| - |A \cap B| - |B \cap C| - |B \cap C| + 2|A \cap B \cap C|$

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Answer: (c)

Union of Sets - General (ugly) Case

Theorem (The Inclusion-Exclusion Principle)

Let $A_1, \ldots, A_n$ not necessarily disjoint, then

$$
\begin{aligned}
|A_1 \cup \cdots \cup A_n| = & \sum_{1 \leq i \leq n} |A_i| \\
& - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| \\
& + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| \\
& \vdots \\
& (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|
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Products of Sets

- Product of two sets $A \times B \overset{def}{=} \{(a,b) | a \in A, b \in B\}$.

  Example: Let $A = \{x, y, z\}$ and $B = \{1, 2\}$, then $A \times B = \{(x,1), (x,2), (y,1), (y,2), (z,1), (z,2)\}$.

Theorem (The Product Rule)

For sets $A_1, A_2, \ldots, A_n$,

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_n|. \tag{3}$$

Proof. By induction. □

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Answer: (a); using a combination of the Sum and Product Rules.

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- r-Permutations with Repetition: The number of ways we present r out of n items in a sorted order when repetitions are allowed is n^r

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Example 2: (r-Combinations of a Set) How many subsets of size r does an n-element set have? Answer: $\binom{n}{r} = \frac{n!}{(n-r)! \cdot r!}$.

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  Example: Assuming the order of letters A, C, T and G in a DNA sequence does not matter, how many different 3-length sequences can we have? Answer: $\binom{4+3-1}{3} = 20$ (compare this to $4^3 = 64$ when the order does matter).

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- *r*-Combinations with Repetition, with at least one of each item: This number is equal to $\binom{r-1}{n-1}$.

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Answer: (b); use the Division Rule.

More Refined Ways for Counting (cntd.)

- Permutations with Limited Repetition: The number of ways to arrange in a line n items such that item i is repeated exactly r_i times is $\frac{(r_1+\cdots+r_n)!}{r_1!\cdots r_n!} = \binom{r_1+\cdots+r_n}{r_1,\ldots,r_n}$

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  Example: The Multinomial Theorem.

$$(x_1+x_2+\cdots+x_n)^r = \sum_{r_1+r_2+\cdots+r_n=r} \binom{r}{r_1, r_2, \ldots, r_n} x_1^{r_1} \cdot x_2^{r_2} \cdots x_n^{r_n}$$

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FAQ: Are there more ways to count objects?

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FAQ: Are there more ways to count objects? Yes...but...

Further reading

C. Liu: Elements of Discrete Mathematics.