Review of Selected Topics in Probability
Probability Distributions

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Lecture 6

Bernoulli Distribution - Indicator Random Variable

X is an indicator random variable iff $X \in \{0, 1\}$ and

$$\Pr(X = 1) = p = 1 - \Pr(X = 0) \quad (1)$$

for some $p \in [0, 1]$.

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Parameters:

1. (Expectation) $\mathbb{E}[X] = p$.
2. (Variance) $Var(X) = p(1 - p)$.
3. (PGF) $\mathbb{E}[z^X] = 1 - p + pz$.
4. (MGF) $\mathbb{E}[e^{tX}] = 1 - p + pe^t$.

Binomial Distribution

X follows the Binomial distribution iff $X \in \{0, 1, \ldots, n\}$ and

$$\Pr(X = k) = \binom{n}{k} p^k (1 - p)^{n-k} \tag{2}$$

for some $p \in [0, 1]$ and integer $n > 0$.

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Parameters:

1. (Expectation) $\mathbb{E}[X] = np$.
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Parameters:

1. (Expectation) $\mathbb{E}[X] = \frac{1}{p}$.
2. (Variance) $Var(X) = \frac{1-p}{p^2}$.
3. (MGF) $\mathbb{E}[e^{tX}] = \frac{pe^t}{1-(1-p)e^t}$, for $t < -\ln(1 - p)$.
4. (PGF) $\mathbb{E}[z^X] =?$

Poisson Distribution

X follows the Poisson distribution with parameter λ iff $X \in \{0, 1, \ldots\}$ and

$$\Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}. \quad (4)$$

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Parameters:

1. (Expectation) $\mathbb{E}[X] = \lambda$.
2. (Variance) $Var(X) = \lambda$.
3. (PGF) $\mathbb{E}[z^X] = e^{\lambda(z-1)}$.

Convergence of Binomial to Poisson

Let $X \sim \mathcal{B}(n, p)$ and $Y \sim Poisson(\lambda)$. Assume (a) $\lambda = np$ is bounded and (a) $n \to \infty$. Then

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$$\begin{aligned}
\mathbb{E}[z^X] &= (1 + p(z-1))^n = \left(1 + \frac{\lambda(z-1)}{n}\right)^n \\
&= \left(\left(1 + \frac{\lambda(z-1)}{n}\right)^{\frac{n}{\lambda(z-1)}}\right)^{\lambda(z-1)} \\
&\to e^{\lambda(z-1)} = \mathbb{E}[z^Y].
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Theorem (Poisson Paradigm)

Consider n Bernoulli trials X_i with success probability $p_i, i = 1, \ldots, n$. If p_i are "small" and the trials are either independent or "weakly dependent", then $Y = \sum_i X_i$ follows "approximately" the Poisson distribution with parameter $\sum_i p_i$.

Uniform Distribution (Continuous case)

X follows the Uniform distribution in $[a, b]$ iff $X \in [a, b]$ and

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{,for } a < x < b \\ 0 & \text{elsewhere.} \end{cases} \tag{5}$$

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Parameters:

1. (Expectation) $\mathbb{E}[X] = \frac{a+b}{2}$.
2. (Variance) $Var(X) = \frac{(b-a)^2}{12}$.
3. (MGF) $\mathbb{E}[e^{tX}] = \frac{e^{tb}-e^{ta}}{t(b-a)}$.

Exponential Distribution

X follows the Exponential distribution with parameter λ iff $X \in [0, \infty)$ and

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Note: Expresses interarrival times (more on this in Poisson process lecture). Also has the memoryless property (Homework!).

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Normal (or Gaussian) Distribution

X follows the Normal distribution with mean value μ and typical deviation σ iff $X \in (-\infty, \infty)$ and

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \tag{7}$$

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3. (MGF) $\mathbb{E}[e^{tX}] = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.

The Central Limit Theorem

Theorem (Central Limit Theorem)

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $\mathbb{E}[X_i] = \mu_i$ and $Var(X_i) = \sigma_i^2$. Under "mild conditions", for any $\alpha \in \mathbb{R}$,

$$\Pr\left(\frac{\sum_{i=1}^{n}(X_i - \mu_i)}{\sqrt{\sum_{i=1}^{n}\sigma_i^2}} \leq \alpha\right) \to \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-x^2} dx. \tag{8}$$

i.e. as $n \to \infty$, $\sum_{i=1}^{n} X_i$ is distributed according to $\mathcal{N}\left(\sum_{i=1}^{n}\mu_i, \sum_{i=1}^{n}\sigma_i^2\right)$.

Another well known Limit Theorem

Theorem (Strong law of large numbers)

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables $\mathbb{E}[X_i] = \mu$, for all i. Then, with probability 1, as $n \to \infty$

$$\frac{X_1 + X_2 + \cdots + X_n}{n} \to \mu. \tag{9}$$

Further reading

S. Ross. A first course in probability:
Chapter 4, “Random Variables”
Chapter 5, “Continuous Random Variables”
Chapter 8, “Limit Theorems”