Differential and Physical Geometry

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Contents

0.1 Preface

(**note to readers of this preliminary version: You are free to download the book but not to distribute it without the authors permission. What I ask in return is that you take note of mistakes etc. and notify me of such by email (see my website at http://webpages.acs.ttu.edu/jlee for the email address**).

Classical differential geometry is the approach to geometry that takes full advantage of the introduction of numerical coordinates into a geometric space. This use of coordinates in geometry was the essential insight of Rene Descartes that allowed the invention of analytic geometry and paved the way for modern differential geometry. A basic object in differential geometry (and differential topology) is the smooth manifold. This is a topological space on which is defined a sufficiently nice family of coordinate systems or "charts". The charts consist of locally defined n-tuples of functions which are sufficiently independent of each other so as to allow each point in a neighborhood to be specified by the values of these functions in the same way that the polar coordinate functions uniquely specify points in the plane (sans a ray from origin). One may start with a topological space and add charts which are compatible with the topology or the charts themselves can generate the topology. The charts must also be compatible with each other so that changes of coordinates are always smooth maps. Depending on what type of geometry is to be studied, extra structure is assumed which may take the form of a distinguished group of symmetries, or the presence of a distinguished "tensor" such as a metric tensor or symplectic form.

Despite the presence of coordinates, modern differential geometers have learned to present much of the subject without direct reference to a coordinate system. This is called the "invariant" or "coordinate free" approach to differential geometry. The only way to really see exactly what this all means is by diving in and learning the subject.

The relationship between geometry and the physical world is fundamental on many levels. Geometry (especially differential geometry) clarifies, codifies and then generalizes ideas arising from our intuitions about certain aspects of our world. Some of these aspects are those that we think of as forming the spatiotemporal background of our activities while other aspects derive from our experience with objects that have "smooth" surfaces. The earth is both a surface and a "lived in space" and so the prefix "geo" in the word geometry is doubly appropriate. Differential geometry is also an appropriate mathematical setting for the study of what we classically conceive of as continuous physical phenomenon such as fluids and electromagnetic fields.

Manifolds have dimension. The surface of the earth is two dimensional, while the configuration space of a mechanical system is a manifold which may easily have a very high dimension (dimension=degrees of freedom). Stretching the imagination further we can conceive of each possible field configuration for some classical field as being an abstract point in an infinite dimensional manifold.

The physicist is interested in geometry because s/he wants to understand the way the physical world is in "actuality". But there is also a discovered "logical world" of pure geometry that is in some sense a part of reality too. This is the reality which Roger Penrose calls the Platonic world[1]. Thus the mathematician is interested in the way worlds *could be in principal*- they are interested in what might be called "possible geometric worlds". Since the inspiration for what we find interesting has its roots in our experience, even the abstract geometries that we study retain a certain vague physicality. From this point of view, the intuition that guides the pure geometer is fruitfully enhanced by an explicit familiarity with the way geometry plays a role in physical theory.

[1]Penrose seems to take this Platonic world rather literally giving it a great deal of ontological weight as it were.

Knowledge of differential geometry is common among physicists thanks to the success of Einstein's highly geometric theory of gravitation and also because of the discovery of the differential geometric underpinnings of modern gauge theory[2] and string theory. It is interesting to note that the gauge field concept was introduced into physics within just a few years of the time that the notion of a connection on a fiber bundle (of which a gauge field is a special case) was making its appearance in mathematics. Perhaps the most exciting, as well as challenging, piece of mathematical physics to come along in a while is the "string theory" mentioned above. String theory is, at least in part, a highly differential geometric theory. It is too early to tell if string theory will turn out to provide an accurate model of the physical world but the mathematics involved is intrinsically exciting.

The usefulness of differential geometric ideas for physics is also apparent in the conceptual payoff enjoyed when classical mechanics is reformulated in the language of differential geometry. Mathematically, we are led to the subjects of symplectic geometry and Poisson geometry.

The applicability of differential geometry is not limited to pure physics. Differential geometry is also of use in engineering. For example there is the increasingly popular differential geometric approach to control theory.

To be clear, this book is not a physics or engineering book but is a mathematics book which takes inspiration from, and uses examples from physics and engineering. Although there is a great deal of physics that might be included in a book of this sort, the usual constraints of time and space make it possible to include only a small part. A similar statement holds concerning the differential geometry covered in the text. Differential geometry is a huge field and even if we had restricted our attention to just Riemannian geometry, only a small fragment of what could be addressed at this level could possibly be included.

In choosing what to include in this book, I was guided by personal interest and, more importantly, by the limitations of my own understanding. There will most likely be mistakes in the text, some serious. For this I apologize.

[2] The notion of a connection on a fiber bundle and the notion of a gauge field are essentially identical concepts discovered independently by mathematicians and physicists.

Part I

Part I

Chapter 1

Differentiable Manifolds

An undefined problem has an infinite number of solutions.
-Robert A. Humphrey

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For understanding the material to follow it is necessary that the reader have a good background in the following:

1) Linear algebra. In particular, the reader should be familiar with the idea of the dual space to a vector space and also with the notion of a quotient vector space. A bit of the theory of modules is also needed and this is outlined in Appendix D.

2) Multivariable calculus. In particular, the reader should be familiar with the idea that the derivative at a point p of a map between open sets of (normed) vector spaces is a linear transformation between the vector spaces. Usually the normed spaces are assumed to be the Euclidean coordinate spaces such as $\mathbb{R}^n$ with the norm $\|x\| = \sqrt{x \cdot x}$. A reader who felt the need for a review could do no better than to study roughly the first half of the classic book "Calculus on Manifolds" by Michael Spivak. The reader is also asked to look over Appendix C which gives a treatment of differential calculus on Banach spaces.

3) The basics of point set topology- including familiarity with the notions of subspace topologies, compactness and connectedness. The reader should also know the definitions of Hausdorff topological spaces, regular spaces and normal spaces. The reader should also have been exposed to quotient topologies and the way topological spaces may be "glued together" to form new topological spaces. Appendix A reviews some of the basic ideas and definitions.

4) Finally, the reader will need a familiarity with the basics of undergraduate level abstract algebra at least to the level of the basic isomorphism theorems for groups and rings.

Einstein Summation Convention

Summations such as that which occurs in the equation

$$h^i = \sum_{j=1}^{n} \tau^i_j \alpha^j$$

occur often in differential geometry. It is convenient to employ a convention whereby summation over repeated indices is implied even without the $\sum$ symbol. This convention is attributed to Einstein and is called the Einstein summation convention. Using this convention the above equation would be written

$$h^i = \tau^i_j \alpha^j$$

The range of the indices is either determined by context or must be explicitly mentioned. Later we shall see expressions such as

$$\sum_{\substack{i_1 \ldots i_r = 1 \\ j_1 \ldots j_s = 1}}^{n} \tau^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}\, e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s}$$

which is extremely cumbersome. The summation convention alleviates the situation a bit since the above becomes

$$\tau^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}\, e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s}$$

Normally, the repeated indices that are summed over are repeated once as a subscript and once as a superscript. For example, $\tau^{ik}_j \alpha^j v_k$ is shorthand for

$$\sum_{k=1}^{n} \sum_{j=1}^{n} \tau^{ik}_j \alpha^j v_k$$

Of course it is possible that the range of summation may be different for different indices and if this range is not clear from context it must be made explicit. For example, if $A = (a^i_j)$ is an $n \times m$ matrix and $B = (b^i_j)$ and $m \times k$ matrix, where in this case we use upper indices to indicate row and lower for column, then $C = AB$ corresponds to

$$c^i_j = \sum_{l=1}^{m} a^i_l b^l_j$$

which is reduced by the summation convention to $c^i_j = a^i_l b^l_j$. The order of matrix multiplication is correctly reflected in this expression if one arranges that the repeated indices occur down and then up as one reads through the expression from left to right. Thus both $a^i_l b^l_j$ and $b^l_j a^i_l$ indicate the same sum but only the first correctly reflects the order of matrix multiplication.

Remark 1.1 ***From this point onward we will employ the Einstein summation convention freely but we will not adhere to it slavishly. Indeed, there will be times when including the summation symbol $\sum$ makes things clearer.***

1.1 Topological Manifolds

We recall a few concepts from point set topology. A **cover** of a topological space X is a family of sets $\{U_\beta\}_{\beta \in B}$ such that $X = \cup_\beta U_\beta$. If all the sets U_β are open we call it an **open cover.** A **refinement** of a cover $\{U_\beta\}_{\beta \in B}$ of a topological space X is another cover $\{V_i\}_{i \in I}$ such that every set from the second cover is contained in at least one set from the original cover. This means that means that if $\{U_\beta\}_{\beta \in B}$ is the given cover of X, then a refinement may be described as a cover $\{V_i\}_{i \in I}$ together with a set map $i \mapsto \beta(i)$ of the indexing sets $I \to B$ such that $V_i \subset U_{\beta(i)}$ for all i. Two covers $\{U_\alpha\}_{\alpha \in A}$ and $\{U_\beta\}_{\beta \in B}$ have a common refinement. Indeed, we simply let $I = A \times B$ and then let $U_i = U_\alpha \cap U_\beta$ if $i = (\alpha, \beta)$. This common refinement will obviously be open if the two original covers were open. We say that a cover $\{V_i\}_{i \in I}$ of X (by not necessarily open sets) is a **locally finite** cover if every point of X has a neighborhood that intersects only a finite number of sets from the cover. A topological space X is called **paracompact** if every *open* cover of X has a refinement which is a locally finite open cover. Sometimes the notion of paracompactness is defined to include the requirement that the space is Hausdorff. This is not much of an issue for us since we generally deal only with Hausdorff spaces anyway.

A **base** (or **basis**) for the topology of a topological space X is a collection $\mathfrak{B}$ of open sets such that all open sets from the topology $\mathfrak{T}$ are unions of open sets from the family $\mathfrak{B}$. A topological space is called **second countable** if its topology has a countable base. The space $\mathbb{R}^n$ with the usual topology derived from the Euclidean distance function is second countable since we have a base for the topology consisting of open balls with rational radii centered at points with rational coordinates.

Definition 1.1 *An n dimensional* ***topological manifold*** *is a paracompact Hausdorff topological space, say M, such that every point $p \in M$ is contained in some open set U_p that is the domain of a homeomorphism ϕ_p onto an open subset of the Euclidean space $\mathbb{R}^n$. Thus we say that a topological manifold M is "locally Euclidean". The integer n is referred to as the* ***dimension*** *of M.*

Example 1.1 *$\mathbb{R}^n$ is trivially a topological manifold of dimension n.*

Example 1.2 *The unit circle $S^1 := \{e^{i\theta} \in \mathbb{C}\} = \{(x, y) : x^2 + y^2 = 1\}$ is a 1 dimensional topological manifold. Indeed, the map $\mathbb{R} \to S^1$ given by $t \mapsto e^{it}$ has restrictions to small open sets which are homeomorphisms. The boundary of a square in the plane is a topological manifold homeomorphic to the circle and so we say that it is a topological circle. More generally, the n-sphere $S^n := \{(x^1, ..., x^{n+1}) \in \mathbb{R}^{n+1} : \sum (x^i)^2 = 1\}$ is a topological manifold.*

Example 1.3 *If M_1 and M_2 are topological manifolds of dimensions n_1 and n_2 respectively, then $M_1 \times M_2$, with the product topology is a topological manifold of dimension $n_1 + n_2$. Such a manifold is called a product manifold. The required homeomorphisms are constructed in the obvious way from those defined on M_1 and M_2: If $\phi_p : U_p \subset M_1 \to V_{\phi(p)} \subset \mathbb{R}^{n_1}$ and $\psi_q : U_q \subset M_2 \to V_{\psi(q)} \subset \mathbb{R}^{n_2}$*

are homeomorphisms then we have a homeomorphism $\phi_p \times \psi_q : U_p \times U_q \to V_{\phi(p)} \times V_{\psi(q)} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *where* $(p,q) \in M_1 \times M_2$ *and* $(\phi_p \times \psi_q)(x,y) := (\phi_p(x), \psi_q(y))$.

The product manifold construction of the last example can obviously be extended to products of several spaces at a time.

Example 1.4 *The* n*-torus*

$$T^n := S^1 \times S^1 \times \cdots S^1 \text{ (n factors)}$$

is a topological manifold of dimension n.

We shall not give many examples of topological manifolds at this time because our main concern is with *smooth* manifolds defined below. We give plenty examples of smooth manifolds and every smooth manifold is also a topological manifold.

Manifolds are often defined with the requirement of second countability[1] but one of the main reasons that manifolds are traditionally defined to be second countable (and Hausdorff) is that second countability implies paracompactness. Paracompactness is important in connection with the notion of a "partition of unity" discussed later in this book. It is known that for a locally Euclidean Hausdorff space, paracompactness is equivalent to the property that each connected component is second countable and so if there is at most a countable number of components, paracompactness would imply second countability. Now there is at least one theorem that does not work as usually stated if the manifold has an uncountable number of components and that is Sard's theorem 3.1. Our approach will be to add in the requirement of second countability when needed. In such cases we would just be excluding the possibility that the manifold had an uncountably infinite number of connected components.

In defining a topological manifold some authors allow the dimension n of the Euclidean space to depend on the homeomorphism ϕ and so on the point $p \in M$. However, it is a consequence of a result of Brouwer called "invariance of domain" that the n would have to be a well defined locally constant function and therefore constant on connected components of M. This result is rather easy to prove if the manifold has a differentiable structure (defined below) but is more difficult in general. We shall simply record Brouwer's theorem:

Theorem 1.1 (Invariance of Domain) *The image of an open set* $U \subset \mathbb{R}^n$ *by a 1-1 continuous map* $f : U \to \mathbb{R}^n$ *is open. It follows that if* $U \subset \mathbb{R}^n$ *is homeomorphic to* $V \subset \mathbb{R}^m$ *then* $m = n$.

Let us define n dimensional Euclidean half-space to be $\mathbb{H}^n := \mathbb{R}^n_{x^1 \le 0} := \{(x^1, ..., x^n) \in \mathbb{R}^n : x^1 \le 0\}$. The boundary of $\mathbb{H}^n$ is $\partial\mathbb{H}^n = \mathbb{R}^n_0 =: \{(x^1, ..., x^n) : x^1 = 0\}$. The interior of $\mathbb{H}^n$ is $\text{int}(\mathbb{H}^n) = \mathbb{H}^n \backslash \partial\mathbb{H}^n$. The space $\mathbb{H}^n$ is not a

[1] A toplogical space is said to be second countable if it has a countable base for its topology.

manifold because points on the boundary do not have neighborhoods homeomorphic to an open set in a Euclidean space. However, $\mathbb{H}^n$ will be our model for a somewhat different kind of space. There are other half spaces homeomorphic to $\mathbb{H}^n = \mathbb{R}^n_{x^1 \leq 0}$. For instance we could have used the "positive" half space $\mathbb{R}^n_{x^1 \geq 0}$ as our model. However, the special choice we have made does have a rationale that only becomes apparent later[2].

A **topological manifold with boundary** (of dimension n) is a paracompact Hausdorff topological space M such that each point $p \in M$ is contained in some open set U_p that is the domain of a homeomorphism $\psi : U_p \to V_{\psi(p)}$ where the range $V_{\psi(p)}$ is an open subset in a fixed Euclidean half space $\mathbb{H}^n = \mathbb{R}^n_{x^1 \leq 0}$. The reader should recall carefully the meaning of an open set in $\mathbb{H}^n$; these certainly need not be open in the containing $\mathbb{R}^n$. A point $p \in M$ that is mapped to $\partial\mathbb{H}^n$ under some homeomorphism $\psi : U_p \to V_{\psi(p)}$ as above is called a boundary point and the set of all boundary point of M is denoted ∂M. The manifold's interior consists of those points of M that are mapped to points of $\mathrm{int}(\mathbb{H}^n)$. It is a corollary to Brouwer's theorem that these concepts are well defined independent of the homeomorphism used.

Exercise 1.1 *Show that* $\mathrm{int}(\mathbb{H}^n) \cap \partial M = \emptyset$*.*

Exercise 1.2 *The boundary ∂M of an n dimensional topological manifold with boundary is an $n-1$ dimensional topological manifold (without boundary).*

Example 1.5 *Let $[a, b] \subset \mathbb{R}$ be a closed interval. If N is a topological manifold of dimension $n-1$ then $N \times [a, b]$ is an n dimensional topological manifold with boundary.*

Example 1.6 *If N is a topological manifold with boundary ∂N then $N \times \mathbb{R}$ is a topological manifold with boundary and $\partial (N \times \mathbb{R}) = \partial N \times \mathbb{R}$.*

Example 1.7 *The closed cube $\overline{C^n} = \{(x_1, ..., x_n) \in \mathbb{R}^n : \max_{i=1,..,n}\{\left|x^i\right|\} \leq 1\}$ with its subspace topology inherited from $\mathbb{R}^n$ is a topological manifold with boundary. The boundary is (homeomorphic to) an $n-1$ dimensional sphere.*

We shall eventually also define smooth manifolds with boundary. These will all automatically be topological manifolds with boundary and so we shall see more examples later.

Topological manifolds, both with or without boundary, as we have defined them, are paracompact and Hausdorff and hence also **normal**. This means that given any pair of disjoint closed sets $F_1, F_2 \subset M$ there are open sets U_1 and U_2 containing F_1 and F_2 respectively such that U_1 and U_2 are also disjoint. For more on manifold topology consult [Matsu].

[2] This choice is convenient when we define the notion of the induced orientation on the boundary of an oriented smooth manifold with boundary.

1.2 Charts, Atlases and Smooth Structures

The definitions which follow are motivated by the desire to have a well defined notion of what it means for a function on a manifold to be differentiable.

Definition 1.2 *Let M be a set. A* ***chart*** *on M is a bijection of a subset $U \subset M$ onto an open subset some Euclidean space $\mathbb{R}^n$. We say that the chart takes values in $\mathbb{R}^n$ or say simply that the chart is $\mathbb{R}^n$-valued. A chart $\mathtt{x} : U \to \mathtt{x}(U) \subset \mathbb{R}^n$ is traditionally indicated by the pair $(U, \mathtt{x})$ and the pair itself is also called a chart.*

Definition 1.3 *If $p \in M$ and $(U, \mathtt{x})$ is a chart with $p \in U$ then if $\mathtt{x}(p) = 0 \in \mathbb{R}^n$ we say that the chart is* ***centered*** *at p.*

Recall that if U is an open subset of $\mathbb{R}^n$ then a map $h : U \to \mathbb{R}^n$ is said to be of class C^r if it is continuous and all partial derivatives of order less than or equal to r exist and are continuous. Also, h is said to be C^r at p if its restriction to some open neighborhood of p is of class C^r. A map $h : U \to V \subset \mathbb{R}^n$ is a diffeomorphism of class C^r if h is bijective and has a C^r inverse.

Definition 1.4 *Let $\mathcal{A} = \{(U_\alpha, \mathtt{x}_\alpha)\}_{\alpha \in A}$ be a collection of $\mathbb{R}^n$-valued charts on a set M. We call $\mathcal{A}$ an $\mathbb{R}^n$-valued* ***atlas of class*** *C^r $(1 \leq r \leq \infty)$ if the following conditions are satisfied:*

i) $\cup_{\alpha \in A} U_\alpha = M$

ii) the sets of the form $\mathtt{x}_\alpha(U_\alpha \cap U_\beta)$ for $\alpha, \beta \in A$ are all open in $\mathbb{R}^n$,

iii) Whenever $U_\alpha \cap U_\beta$ is not empty then the ***overlap map***

$$\mathtt{x}_\beta \circ \mathtt{x}_\alpha^{-1} : \mathtt{x}_\alpha (U_\alpha \cap U_\beta) \to \mathtt{x}_\beta (U_\alpha \cap U_\beta)$$

is a C^r diffeomorphism.

Remark 1.2 *Some folks might say that $\mathtt{x}_\beta \circ \mathtt{x}_\alpha^{-1}$ is really shorthand for*

$$\left(\mathtt{x}_\beta|_{U_\alpha \cap U_\beta}\right) \circ \left(\mathtt{x}_\alpha|_{\mathtt{x}_\alpha(U_\alpha \cap U_\beta)}\right)^{-1}$$

but this is far too pedantic and cluttered for most people's tastes.

The maps $\mathtt{x}_\beta \circ \mathtt{x}_\alpha^{-1}$ in the definitions are called variously **overlaps map**, **change of coordinate maps** or **transition functions**. It is exactly the way we have required the overlap maps to be diffeomorphisms that will allow for us to have a well defined and useful notion of what it means for a function on M to be differentiable of class C^r. An **atlas of class** C^r is also called a C^r **atlas**.

Definition 1.5 *Two C^r−atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on M are* ***equivalent*** *if $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a C^r-atlas. A C^r* ***differentiable structure*** *on M is an equivalence class of C^r−atlases.*

A C^∞ differentiable structure will also be called a **smooth structure**. The union of all the C^r atlases in an equivalence class is the unique **maximal** C^r **atlas** in the class. The set of equivalence classes of C^r atlases (differentiable structures) is in 1-1 correspondence with the set of maximal C^r atlases. Thus an alternative way to define a C^r differentiable structure is as a maximal C^r atlas. We shall use this alternative quite often without comment. Every atlas is contained in a unique maximal atlas and so as soon as we have a C^r-atlas then we have a determined C^r differentiable structure. We say a chart $(U, \mathtt{x})$ on M is **compatible** with a C^r atlas $\mathcal{A}$ if $\mathcal{A} \cup \{(U, \mathtt{x})\}$ is also an atlas. This just means that if $\mathcal{A} = \{(U_\alpha, \mathtt{x}_\alpha)\}_{\alpha \in A}$ then all the sets of the form $\mathtt{x}_\alpha(U \cap U_\alpha)$ and $\mathtt{x}(U \cap U_\alpha)$ are open and all the maps of the form

$$\mathtt{x}_\alpha \circ \mathtt{x}^{-1} \text{ and } \mathtt{x} \circ \mathtt{x}_\alpha^{-1}$$

for the various α are C^r diffeomorphisms. The maximal C^r atlas determined by $\mathcal{A}$ is easily seen to be exactly composed of all charts compatible with $\mathcal{A}$.

The space $\mathbb{R}^n$ itself has an atlas consisting of the single chart $(\mathrm{id}, \mathbb{R}^n)$ where $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is just the identity map $\mathrm{id}(p) = p$. This atlas consisting of a single global chart determines a maximal atlas. Polar coordinates are defined on the set $U = \mathbb{R}^n \backslash \{0\}$ and provide another chart contained in the maximal atlas.

Lemma 1.1 *Let M is a set with a C^r structure given by an atlas $\mathcal{A} = \{(U_\alpha, \mathtt{x}_\alpha)\}_{\alpha \in A}$. If $(U, \mathtt{x})$ and $(V, \mathtt{y})$ are charts compatible with $\mathcal{A}$ such that $U \cap V \neq \emptyset$, then the charts $(U \cap V, \mathtt{y}|_{U \cap V})$ and $(U \cap V, \mathtt{x}|_{U \cap V})$ are also compatible with $\mathcal{A}$ and hence are in the maximal atlas generated by $\mathcal{A}$. Thus the intersections of compatible chart domains are either empty or are also compatible chart domains. Furthermore, if O is an open subset of $\mathtt{x}(U)$ for some compatible chart $(U, \mathtt{x})$ then taking $V = \mathtt{x}^{-1}(O)$we have that $(V, \mathtt{x}|V)$ is also a compatible chart.*

Proof. This assertions of the lemma are almost obvious: If $\mathtt{x} \circ \mathtt{x}_\alpha^{-1}$, $\mathtt{x}_\alpha \circ \mathtt{x}^{-1}$, $\mathtt{y} \circ \mathtt{x}_\alpha^{-1}$, $\mathtt{x}_\alpha \circ \mathtt{y}^{-1}$ are all C^r diffeomorphisms then certainly the restrictions $\mathtt{x}|_{U \cap V} \circ \mathtt{x}_\alpha^{-1}$, $\mathtt{x}_\alpha \circ \mathtt{x}|_{U \cap V}^{-1}$, $\mathtt{y}|_{U \cap V} \circ \mathtt{x}_\alpha^{-1}$ and $\mathtt{x}_\alpha \circ \mathtt{y}|_{U \cap V}^{-1}$ are also. One might just check that the natural domains of these maps are indeed open in $\mathbb{R}^n$. For example the domain of $\mathtt{x}|_{U \cap V} \circ \mathtt{x}_\alpha^{-1}$ is $\mathtt{x}_\alpha(U_\alpha \cap U \cap V) = \mathtt{x}_\alpha(U_\alpha \cap U) \cap \mathtt{x}_\alpha(U_\alpha \cap V)$ and both $\mathtt{x}_\alpha(U_\alpha \cap U)$ and $\mathtt{x}_\alpha(U_\alpha \cap V)$ are open as part of what it means that $(U, \mathtt{x})$ is compatible with the atlas $\mathcal{A}$. The last assertion of the lemma is equally easy to prove. ■

It is now an easy exercise in point set topology to prove that the family of sets which are the domains of charts compatible with a given atlas provide a base for a topology on M which we call the **topology induced by the** C^r **structure** on M. Thus the open sets are exactly the empty set plus arbitrary unions of chart domains from the maximal atlas. We will also call this the topology induced by the charts or by the atlas.

Notice that if an atlas is contained in another atlas then they both give rise to the same maximal atlas (same C^r-structure) and so both induce the same topology.

Exercise 1.3 *Show that if $\mathcal{A}_1$ is a subatlas of $\mathcal{A}_2$ then they both induce the same topology.*

The proof of the following lemma is trivial but it is used often-sometimes without mention.

Lemma 1.2 *Let M be a set with a C^r structure given by an atlas $\mathcal{A}$. If for every two distinct points $p, q \in M$, we have that either p and q are respectively in disjoint chart domains U_α and U_β from the atlas, or they are both in a common chart domain, then the induced topology is Hausdorff.*

Lemma 1.3 *Let $\mathcal{A}$ be a C^r atlas for M. If $\mathcal{A}$ is countable or has a countable subatlas then the induced topology is second countable.*

Proof. Suppose that the family of chart domains $\{U_\alpha\}$ is countable. For every α, the set $\mathtt{x}_\alpha(U_\alpha)$ is an open set in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is second countable we see that for every α, there is a collection $\{V_{\alpha,i}\}$ such that every open subset of $\mathtt{x}_\alpha(U_\alpha)$ is a union of sets from this family. It follows that every open subset of U_α is a union of sets from the countable collection $\{U_{\alpha,i}\}_{i\in I}$ where $U_{\alpha,i} := \mathtt{x}_\alpha^{-1}(V_{\alpha,i})$. However, the doubly indexed family $\{U_{\alpha,i}\}_{i\in I}$ is also countable. If $U \subset M$ is any open set then we must show that U is a countable union of open sets from the collection $\{U_{\alpha,i}\}_{\alpha\in A, i\in I}$. But $U = \cup_\alpha (U_\alpha \cap U)$ is a countable union and each $U_\alpha \cap U$ is a countable union of sets from $\{U_{\alpha,i}\}_{i\in I}$ which means that U is a countable union of sets from the collection $\{U_{\alpha,i}\}_{\alpha\in A, i\in I}$. ■

We would like to know what condition (or conditions) on an atlas guarantees that the induced topology is paracompact. We recall that paracompactness is equivalent to the condition that each connected component is second countable.

Lemma 1.4 *Let $\mathcal{A}$ be a C^r atlas for M. If the collection of chart domains $\{U_\alpha\}_{\alpha\in A}$ from the atlas $\mathcal{A}$ is such that for every fixed $\alpha_0 \in A$ the set $\{\alpha \in A : U_\alpha \cap U_{\alpha_0} \neq \emptyset\}$ is at most countable, then the induced topology is paracompact. Thus, if this conditions holds and M if is connected then the topology is second countable.*

Proof. Give M the induced topology. If we focus on a connected component then we reduce the problem to showing that M has a countable base. From lemma 1.3 it suffices to show that $\mathcal{A}$ has a countable subatlas. Let U_{α_1} be a particular chart domain from the atlas. We proceed inductively to define a sequence of sets starting with with $X_1 = U_{\alpha_1}$. Now given X_{n-1} let X_n be the union of those chart domains U_α which intersect X_{n-1}. If follows (inductively) that each X_n is a countable union of chart domains and hence the same is true of the union $X = \cup_n X_n$. By construction, if some chart domain U_α meets X then it is actually contained in X since to meet X_{n-1} is to be contained in X_n. All that is left is to show that $M = X$. We have reduced to the case that M is connected and since X is open, it will suffice to show that $M\backslash X$ is also open. Now if $p \in M\backslash X$ then it is in some U_α and as we said, U_α cannot meet X

without being contained in X. Thus it must be the case that $U_\alpha \cap X = \emptyset$ and thus $U_\alpha \subset M$. We see that $M \backslash X$ open as is X so $M = X$. ■

This leads us to a principal definition:

Definition 1.6 *An **dimensional differentiable manifold of class** C^r is a set M together with a specified C^r structure on M such that the topology induced by the C^r structure is Hausdorff and paracompact. If the charts are $\mathbb{R}^n-$valued then we say the manifold has dimension n.*

In other words, we may say that a differentiable manifold of class C^r is a pair $(M, \mathcal{A})$ where $\mathcal{A}$ is a maximal ($\mathbb{R}^n$-valued) C^r atlas and such that the topology induced by the atlas makes M a topological manifold. A differentiable manifold of class C^r is also referred to as a C^r manifold. A differentiable manifold of class C^∞ is also called a **smooth** manifold (a terminology used often in the sequel). Notice that if $0 \leq r_1 < r_2$, then any C^{r_2} atlas is also a C^{r_1} atlas and so any C^{r_2} manifold is also a C^{r_1} manifold ($0 \leq r_1 < r_2$). For every integer $n \geq 0$,the Euclidean space $\mathbb{R}^n$ is a smooth manifold where, as noted above, there is an atlas whose only member is the $(\mathbb{R}^n, id)$ and this atlas determines a maximal atlas providing the usual smooth structure for $\mathbb{R}^n$.

It is important to notice that if $r > 0$ then a C^r manifold is much more than merely a topological manifold. Note also that we have defined manifolds in such a way that they are necessarily paracompact and Hausdorff. This is, in part, a matter of convenience since for some purposes neither assumption is necessary. We could have just defined a C^r manifold to be a set with a C^r structure and the induced topology. Lemmas 1.2 and 1.4 tell us how to determine, from knowledge about a given atlas, whether the induced topology is indeed Hausdorff and/or paracompact. In problem 2 we ask the reader to check that these topological conditions hold for the examples of smooth manifolds that we give in this chapter.

Notation 1.1 *As defined, a C^r manifold is a pair $(M, \mathcal{A})$. However, we follow the tradition of using the single symbol M itself to denote the differentiable manifold.*

Now we come to an important point. Suppose that M already has some natural or previously given topology. For example, perhaps it is already known that M is a topological manifold. If M is given a C^r structure then it is important to know whether this topology is the same as the topology induced by the C^r structure (i.e. the topology induced by the charts). For this consideration we have the following

Lemma 1.5 *If M is a topological space with topology $\mathcal{T}$ which also has a C^r atlas, then if each chart is a homeomorphism with respect to this topology then $\mathcal{T}$ will be the same as the topology induced by the C^r structure.*

A good portion of the examples of C^r manifolds that we provide will be of the type described by this previous lemma. In fact, one often finds the following given as the definitions of atlas and C^r manifold:

Definition 1.7 (Alternative traditional) *An $\mathbb{R}^n$-valued chart on a topological space is a* homeomorphism *from an* open *subset of M to an open subset of $\mathbb{R}^n$. Let $\mathcal{A} = \{(U_\alpha, \mathtt{x}_\alpha)\}_{\alpha\in A}$ be a collection of $\mathbb{R}^n$-valued charts on a topological manifold M. We call $\mathcal{A}$ an $\mathbb{R}^n$-valued* ***atlas of class*** *C^r ($1 \le r \le \infty$) if the following conditions are satisfied:*
i) $\cup_{\alpha\in A} U_\alpha = M$
ii) Whenever $U_\alpha \cap U_\beta$ is not empty then the map

$$\mathtt{x}_\beta \circ \mathtt{x}_\alpha^{-1} : \mathtt{x}_\alpha(U_\alpha \cap U_\beta) \to \mathtt{x}_\beta(U_\alpha \cap U_\beta)$$

is a C^r diffeomorphism.
A maximal atlas is called a C^r differentiable structure on M (here maximal is defined as before).
An n ***dimensional differentiable manifold of class*** *C^r is an n dimensional topological manifold M together with a C^r differentiable structure specified on M.*

In expositions that use the above definitions, the fact that one can start out with a set, provide charts, and then end up with an appropriate topology is presented as a separate lemma (see for example [Lee, John] or [O'Neill]).

Exercise 1.4 *Let M be a smooth manifold of dimension n. Let $p \in M$. Show that for any $r > 0$ there is a chart $(U, \mathtt{x})$ with $p \in U$ and such that $\mathtt{x}(U) = B(0, r) := \{x \in \mathbb{R}^n : |x| < r\}$. Show that for any $p \in U$ we may further arrange that $\mathtt{x}(p) = 0$. Such a chart is both centered and spherical.*

Remark 1.3 *From now on all manifolds in this book will be assumed to be C^∞ manifolds (smooth manifolds) unless otherwise indicated. Also, let us refer to an $n-$dimensional smooth manifold as an "$n-$**manifold**". Note that some authors reserve the term "n-manifold" for connected smooth manifolds.*

As mentioned above, it is certainly possible for there to be two different differentiable structures on the same topological manifold. For example, the chart on $\mathbb{R}^1$ given by the cubing function $x \mapsto x^3$ is not compatible with the identity chart $(\mathrm{id}, \mathbb{R}^1)$ but since the cubing function also has domain all of $\mathbb{R}^1$, it too provides an atlas. But then this atlas cannot be compatible with the atlas $\{(\mathrm{id}, \mathbb{R}^1)\}$ and so they determine different maximal atlases. The problem is that the inverse of $x \mapsto x^3$ is not differentiable (in the usual sense) at the origin. Now we have two differentiable structures on the line $\mathbb{R}^1$. Actually, although the two atlases do give distinct differentiable structures, they are equivalent in another sense that we mentioned above and that we make precise below (they are diffeomorphic; definition 1.12). On the other hand, it is a deep result that there exist infinitely many truly different (non-diffeomorphic) differentiable structures on $\mathbb{R}^4$. The existence of exotic differentiable structures on $\mathbb{R}^4$ follows from the deep results of [Donaldson] and [Freedman]. For $n \neq 4$ all smooth structures on $\mathbb{R}^n$ are diffeomorphic. The reader ought to be wondering what is so special about dimension four.

We have seen that all the Euclidean spaces $\mathbb{R}^n$ are smooth manifolds but so is any open subset of a Euclidean space $\mathbb{R}^n$. If $U \subset \mathbb{R}^n$ is such an open set then an atlas on U is obtained from the standard atlas on $\mathbb{R}^n$ by a process of restriction. These open subsets of Euclidean space might seem to be very uninteresting manifolds but in fact they can be quite interesting and complex. For example, much can be learned about a knot $K \subset \mathbb{R}^3$ by studying its complement $\mathbb{R}^3 \backslash K$ and the latter is an open subset of $\mathbb{R}^3$. More generally, if U is some open subset of a smooth manifold M with atlas $\mathcal{A}_M$, then U is itself a differentiable manifold with an atlas of charts being given by all the restrictions $(\,\mathtt{x}_\alpha|_{U_\alpha \cap U}\,, U_\alpha \cap U\,)$ where $(U_\alpha, \mathtt{x}_\alpha) \in \mathcal{A}_M$. We shall refer to such an open subset $U \subset M$ with this differentiable structure as an **open submanifold** of M.

Here are several more interesting examples of smooth manifolds.

Example 1.8 *The 2-sphere $S^2 \subset \mathbb{R}^3$. Choose a pair of antipodal points such as north and south poles where $z = 1$ and -1 respectively. Then off of these two pole points and off of a single half great circle connecting the poles we have the usual spherical coordinates. We actually have many such systems of spherical coordinates since we can re-choose the poles in many different ways. We can also use projection onto the coordinate planes as charts. For instance let U_z^+ be all $(x, y, z) \in S^2$ such that $z > 0$. Then $(x, y, z) \mapsto (x, y)$ provides a chart $U_z^+ \to \mathbb{R}^2$. The various overlap maps can be computed explicitly and are clearly smooth.*

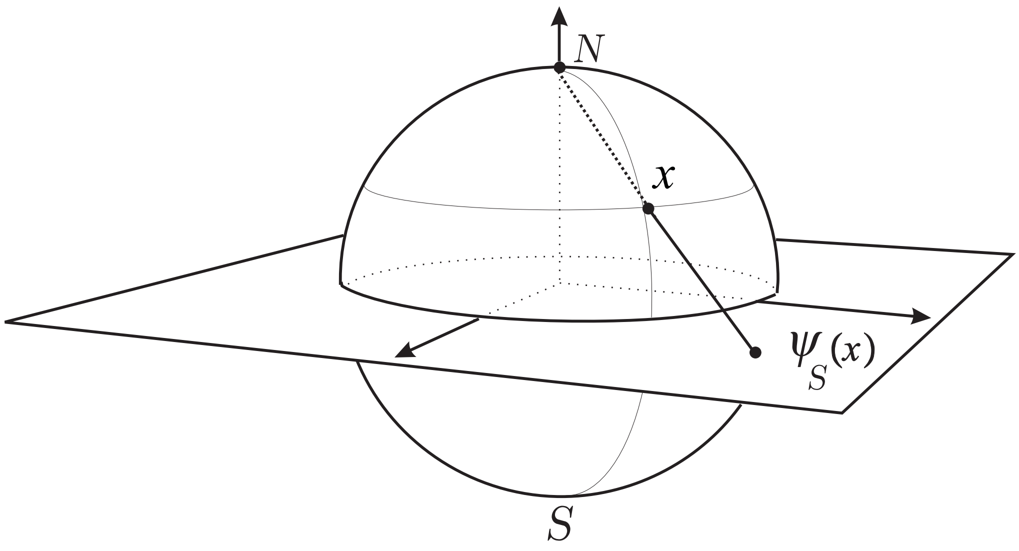
Example 1.9 (Stereographic projection) *We can also use **stereographic projection** to give charts on S^2. More generally, we can provide the $n-$sphere $S^n \subset \mathbb{R}^{n+1}$ with a smooth structure using two charts (U_S, ψ_S) and (U_N, ψ_N). Here,*

$$U_S = \{x = (x_1, ..., x_{n+1}) \in S^n : x_{n+1} \neq 1\}$$
$$U_N = \{x = (x_1, ..., x_{n+1}) \in S^n : x_{n+1} \neq -1\}$$

and $\psi_S : U_S \to \mathbb{R}^n$ (resp. $\psi_N : U_N \to \mathbb{R}^n$) is stereographic projection from the north pole $p_N = (0, 0....0, 1)$ (resp. south pole $p_S = (0, 0, ..., 0, -1)$). Not that ψ_S maps from the southern open set containing p_S. Explicitly we have

$$\psi_S(x) = \frac{1}{(1 - x_{n+1})}(x_1, ..., x_n) \in \mathbb{R}^n$$
$$\psi_N(x) = \frac{1}{(1 + x_{n+1})}(x_1, ..., x_n) \in \mathbb{R}^n$$

***Exercise**: Show that $\psi_S(U_N \cap U_S) = \psi_N(U_N \cap U_S) = \mathbb{R}^n \backslash 0$ and that $\psi_S \circ \psi_N^{-1}(y) = y/\|y\|^2 = \psi_N \circ \psi_S^{-1}(y)$ for all $y \in \mathbb{R}^n \backslash 0$.*
Thus we have an an atlas $\{(U_S, \psi_S), (U_N, \psi_N)\}$. We leave it to the reader to verify that the induced topology is the same as the usual topology on S^n (say as a subspace of $\mathbb{R}^{n+1}$) and that all the maps involved are appropriately smooth.



Let us make an observation that will come in handy later. If we identify $\mathbb{R}^2$ with $\mathbb{C}$, then the overlap maps for charts on S^2 from the last example become

$$\psi_S \circ \psi_N^{-1}(z) = \bar{z}^{-1} = \psi_N \circ \psi_S^{-1}(z)$$

for all $z \in \mathbb{C}\backslash\{0\}$

Example 1.10 (Projective spaces) *The set of all lines through the origin in $\mathbb{R}^3$ is denoted $\mathbb{R}P^2$ and is called the **real projective plane** . Let U_z be the set of all lines $\ell \in \mathbb{R}P^2$ not contained in the x, y plane. Every line $\ell \in U_z$ intersects the plane $z = 1$ at exactly one point of the form $(x(\ell), y(\ell), 1)$. We can define a bijection $\varphi_z : U_z \to \mathbb{R}^2$ by letting $\ell \mapsto (x(\ell), y(\ell))$. This is a chart for $\mathbb{R}P^2$ and there are obviously two other analogous charts (U_x, φ_x) and (U_y, φ_y). These charts cover $\mathbb{R}P^2$ and have smooth overlap maps. Thus we have an atlas for $\mathbb{R}P^2$.*

*More generally, the set of all lines through the origin in $\mathbb{R}^{n+1}$ is called **real projective** $n-$**space** denoted $\mathbb{R}P^n$. We have the surjective map $\pi : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}P^n$ given by letting $p(x)$ be the line through x and the origin and we can give $\mathbb{R}P^n$ the quotient topology where $U \subset P_n(\mathbb{R})$ is open if and only if $\pi^{-1}(U)$ is open. $\mathbb{R}P^n$ can be given an atlas consisting of charts of the form (U_i, φ_i) where*

$$\begin{aligned} U_i &= \{\ell \in \mathbb{R}P^n : \ell \text{ is not contained in the hyperplane } x^i = 0\} \\ \varphi_i(\ell) &= \text{the unique coordinates } (u^1, ..., u^n) \text{ such that } (u^1, ..., 1, ..., u^n) \text{ is} \\ &\quad \text{on the line } \ell. \end{aligned}$$

Once again it can be chacked that the overlap maps are smooth so that we have an atlas. The topology induced by the atlas itself, as described previously, is *exactly the quotient topology and we leave it as an exercise to show that it is both paracompact and Hausdorff.*

It is often useful to view $\mathbb{R}P^n$ as a quotient of the sphere S^n. Consider the map $S^n \to \mathbb{R}P^n$ given by $x \mapsto \ell_x$ where ℓ_x is the unique line through the origin in $\mathbb{R}^{n+1}$ which contains x. Notice that $\ell_x = \ell_{-x}$ for all $x \in S^n$ and if $\ell_x = \ell_y$ for $x, y \in S^n$ then $x = \pm y$. It is not hard to show that this map is open and hence a quotient map. Thus we may equally consider $\mathbb{R}P^n$ to be $S^n / \sim$ where $x \sim y$ if and only if $x = \pm y$ or, in other words, if and only if $x = ya$ for some element of the group $\mathbb{Z}_2 = \{1, -1\}$.

Example 1.11 *In this example we consider a more general way of getting charts for the projective space $\mathbb{R}P^n$. Let $\alpha : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be an affine map whose image does not contain the origin. Thus α has the form $\alpha(x) = Lx + b$ where $L : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is linear and $b \in \mathbb{R}^{n+1}$ is nonzero. The composition $\pi \circ \alpha$ can be easily shown to be a homeomorphism onto its image and we call this type of map an affine parameterization. The inverses of these maps form charts for an atlas. The charts described in the last example are essentially special cases of these charts and give the same smooth structure.*

Example 1.12 *By analogy with the real case we can construct complex projective $n-$space $\mathbb{C}P^n$. As a set $\mathbb{C}P^n$ is the family of all $1-$dimensional complex subspaces of $\mathbb{C}^{n+1}$ (each of these has real dimension 2). In tight analogy with the real case $\mathbb{C}P^n$ can be given an atlas consisting of charts of the form (U_i, φ_i) where*

$$
\begin{aligned}
U_i &= \{\ell \in \mathbb{C}P^n : \ell \text{ is not contained in the complex hyperplane } z^i = 0 \\
\varphi_i(\ell) &= \text{the unique coordinates } (z^1, ..., z^n) \text{ such that } (z^1, ..., 1, ..., z^n) \text{ is} \\
&\quad \text{on the line } \ell.
\end{aligned}
$$

Here $\varphi_i : U_i \to \mathbb{C}^n \cong \mathbb{R}^{2n}$ and so $\mathbb{C}P^n$ is a manifold of (real) dimension $2n$.

Exercise 1.5 *Show that the overlap maps for $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are indeed smooth.*

Notation 1.2 (Homogeneous coordinates) *For $(z_1, ..., z_{n+1}) \in \mathbb{C}^{n+1}$ let $[z_1, ..., z_{n+1}]$ denote the unique $l \in \mathbb{C}P^n$ such that the complex line l contains the point $(z_1, ..., z_{n+1})$. The numbers $(z_1, ..., z_{n+1})$ are said to provide* ***homogeneous*** *coordinates for l. Similarly for $(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1}$ and $[x_1, ..., x_{n+1}] \in \mathbb{R}P^n$.*

In terms of homogeneous coordinates, the chart map $\varphi_i : U_i \to \mathbb{C}^n$ is given by $\varphi_i([z_1, ..., z_{n+1}]) = (z_1 z_i^{-1}, ..., \widehat{1}, ..., z_{n+1} z_i^{-1})$ where the caret symbol ˆ means we have deleted the 1 in the i-th slot to get an element of $\mathbb{C}^n$. A similar statement is true for the real case.

Exercise 1.6 *In reference to the last example, compute the overlap maps $\varphi_i \circ \varphi_j^{-1} : U_i \cap U_j \to \mathbb{C}^n$. For $\mathbb{C}P^1$ show that $U_1 \cap U_2 = \mathbb{C}\backslash 0$ and that $\varphi_2 \circ \varphi_1^{-1}(z) = z^{-1} = \varphi_1 \circ \varphi_2^{-1}(z)$ for $z \in \mathbb{C}\backslash 0$. Show also that if we define $\bar{\varphi}_1 : U_1 \to \mathbb{C}$ by*

$\bar{\varphi}_1(\ell) = \overline{\varphi_1(\ell)}$ then $\{(U_1, \bar{\varphi}_1), (U_2, \varphi_2)\}$ is also an atlas for $\mathbb{C}P^1$ giving the same smooth structure as before. Show that

$$\varphi_2 \circ \bar{\varphi}_1^{-1}(z) = \bar{z}^{-1} = \varphi_1 \circ \varphi_2^{-1}(z) \text{ for } z \in \mathbb{C}\backslash 0$$

Notice that with the atlas $\{(U_1, \bar{\varphi}_1), (U_2, \varphi_2)\}$ for $\mathbb{C}P^1$ from the last example, we have the same overlap maps we saw for S^2 (see equation **??**). This suggests that $\mathbb{C}P^1$ is diffeomorphic to S^2. Let us construct a diffeomorphism $\mathbb{C}P^1 \to S^2$. For each $x = (x_1, x_2, x_3) \in S^2$ with $x_3 \neq 1$ let $z(x) := \frac{x_1}{1+x_3} + \frac{x_2}{1+x_3}i$ and for each $x = (x_1, x_2, x_3) \in S^2$ with $x_3 \neq -1$ let $w(x) := \left(\frac{x_1}{1-x_3} - \frac{x_2}{1-x_3}i\right)$. Now define f by

$$f(x) = \begin{cases} [z(x), 1] \text{ if } x_3 \neq 1 \\ [1, w(x)] \text{ if } x_3 \neq -1 \end{cases}$$

Using the fact that $1 - x_3^2 = x_1^2 + x_2^2$ one finds that for $x = (x_1, x_2, x_3) \in S^2$ with $-1 < x_3 < 1$ we have $w(x) = z(x)^{-1}$. If follows that for such x we have $[z(x), 1] = [1, w(x)]$. This means that f is well defined.

Exercise 1.7 *Show that the map $f : \mathbb{C}P^1 \to S^2$ defined above is a diffeomorphism.*

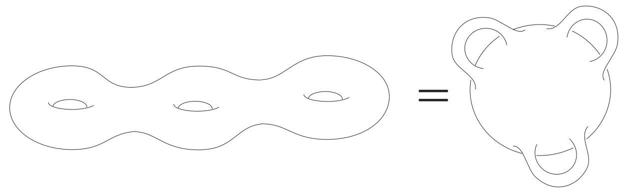
Exercise 1.8 *Show that $\mathbb{R}P^1$ is diffeomorphic to S^1.*

Example 1.13 *The graph of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is the subset of the Cartesian product $\mathbb{R}^n \times \mathbb{R}$ given by $\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}^n\}$. The projection map $\Gamma_f \to \mathbb{R}^n$ is a homeomorphism and provides a global chart on Γ_f making it a smooth manifold.*

Example 1.14 *Let us generalize the last example. Suppose we have a map $f : O \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ where O is open. Let $S = f^{-1}(q_0)$ for some $q_0 \in f(O)$. Suppose further that the Jacobian matrix $\left[\frac{\partial f^i}{\partial x^j}\right]$ has rank $n - k$ everywhere on a neighborhood of S. For every $p \in S$ there is an open neighborhood $U \subset O$ containing p such that on that neighborhood we have*

$$\frac{\partial(f^1, ..., f^{n-k})}{\partial(x_{i_1}, ..., x_{i_{n-k}})} \neq 0$$

on U for some choice of coordinate functions $x_{i_1}, ..., x_{i_{n-k}}$. Thus by the implicit function theorem we deduce that for each $p \in S$ there is at least one such set of coordinates $(x_{i_1}, ..., x_{i_{n-k}})$ from among the x^i with complementary coordinates $(x_{j_1}, ..., x_{j_k})$ such that the projection $\mathbb{R}^n \to \mathbb{R}^k$ given by $(x^1, ..., x^n) \mapsto (x_{j_1}, ..., x_{j_k})$ restricts to a homeomorphism on some (relatively) open subset $U \cap S$ of S with image being an open set in $\mathbb{R}^k$. It is not hard to see that the inverses of these homeomorphisms are smooth as maps into $\mathbb{R}^n$ since up to a permutation of coordinates they have the form of "graph maps": $x \mapsto (x, g(x))$ as in the conclusion of the implicit function theorem (see Appendix C). The set of all such homeomorphisms provide a cover of S by charts. The overlap maps are certainly smooth, being compositions of such graph maps with projections and permutations of coordinates.



Example 1.15 *Let g be a positive integer. Define $f(x) := x(x-1)^2(x-2)^2 \cdots (x-(g-1))^2(x-g)$. For sufficiently smal-l $r > 0$ the set $\{(x,y,z) : (y^2 + f(x))^2 + z^2 = r^2\}$ is a compact surface of genus g. This is an example of subset of $\mathbb{R}^3$ which can be given charts of the form described in the last example. Topologically, a surface of genus g is a sphere with g handles added on or also a $g-$holed torus.*

Example 1.16 *The set of all $m \times n$ real matrices $M_{m\times n}(\mathbb{R})$ is an $mn-$manifold modeled on $\mathbb{R}^{mn}$. We only need one chart again since it is clear that $M_{m\times n}(\mathbb{R})$ is in natural one to one correspondence with $\mathbb{R}^{mn}$ by the map $[a_{ij}] \mapsto (a_{11}, a_{12}, ..., a_{mn})$. Also, the set of all non-singular matrices $Gl(n,\mathbb{R})$ is an open submanifold of $\mathbb{M}_{n\times n} \cong \mathbb{R}^{n^2}$.*

If we have two manifolds M_1 and M_2 of dimensions n_1 and n_2 respectively, we can form the topological Cartesian product $M_1 \times M_2$. We may give $M_1 \times M_2$ a differentiable structure in the following way: Let $\mathcal{A}_{M_1}$ and $\mathcal{A}_{M_2}$ be atlases for M_1 and M_2.Take as charts on $M_1 \times M_2$ the maps of the form

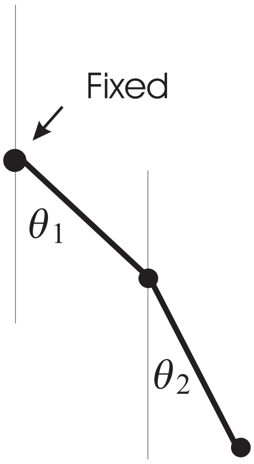
$$\mathtt{x}_\alpha \times \mathtt{y}_\gamma : U_\alpha \times V_\gamma \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$$

where $(U_\alpha, \mathtt{x}_\alpha)$ is a chart from $\mathcal{A}_{M_1}$ and $(\mathtt{y}_\gamma, V_\gamma)$ a chart from $\mathcal{A}_{M_2}$. This gives $M_1 \times M_2$ an atlas called the product atlas which induces a maximal atlas and hence a differentiable structure. With this product differentiable structure, $M_1 \times M_2$ is called a **product manifold**. The product of several manifolds is also possible by an obvious iteration. The induced topology is the product topology and so the underlying topological manifold is the product topological manifold discussed earlier.

Example 1.17 *The circle S^1 is clearly a smooth 1-manifold and hence so is the product $T^2 = S^1 \times S^1$ which is a torus. The set of all configurations of a double pendulum constrained to a plane and where the arms are free to swing past each other can be taken to be modeled by $T^2 = S^1 \times S^1$.*

Example 1.18 *For any manifold M we can construct the "cylinder" $M \times I$ where $I = (a,b)$ is some open interval in $\mathbb{R}$.*

Convention: From now on in this book all topological spaces will be assumed to be Hausdorff unless otherwise stated.



1.3 Smooth Maps and Diffeomorphisms

Definition 1.8 *Let M and N be smooth manifolds with corresponding maximal atlases $\mathcal{A}_M$ and $\mathcal{A}_N$. We say that a map $f : M \to N$ is of class C^r (or $r-$times continuously differentiable) at $p \in M$ if for every chart $(V, \mathtt{y})$ from $\mathcal{A}_N$ with $f(p) \in V$, there exists a chart $(U, \mathtt{x})$ from $\mathcal{A}_M$ with $p \in U$ such that $f(U) \subset V$ and such that $\mathtt{y} \circ f \circ \mathtt{x}^{-1}$ is of class C^r. If f is of class C^r at every point $p \in M$ then we say that f is of class C^r.*

Even though we have restricted our attention to smooth manifolds, that is manifolds with C^∞ structure, we may still be interested in maps which are only of class C^r for some $r < \infty$. This is especially so when one wants to do analysis on smooth manifolds. In fact one could define what it means for a map to be Lebesgue measurable in a similar way. It is obvious from the way we have formulated the definition that the property of being of class C^r is a local property. Also, the above definition does not start out with the assumption that f is continuous but is constructed carefully so as to *imply* that a function which is of class C^r (at a point) according to the definition is automatically continuous (at the point). In fact, since the topologies are induced by the charts we see that in the case $r = 0$ the definition is just a statement of the definition of continuity. However, we might know in advance that $f : M \to N$ is continuous. In this case the condition that f be C^r at $p \in M$ for $r > 0$ can be seen to be equivalent to the condition that for some (and hence every) choice of charts $(U, \mathtt{x})$ from $\mathcal{A}_M$ and $(V, \mathtt{y})$ from $\mathcal{A}_N$ such that $p \in U$ and $f(p) \in V$, the composite map

$$\mathtt{y} \circ f \circ \mathtt{x}^{-1} : \mathtt{x}(U) \to \mathtt{y}(V)$$

is C^r. Note carefully, the use of the phrase "and hence every" above. The point is that that if we choose another pair of charts $(\mathtt{x}', U')$ and $(\mathtt{y}', V')$ with $p \in U'$ and $f(p) \in V'$ then $\mathtt{y}' \circ f \circ \mathtt{x}'^{-1}$ must be C^r on some neighborhood of $\mathtt{x}'(p)$ if and only if $\mathtt{y} \circ f \circ \mathtt{x}^{-1}$ is C^r on some neighborhood of $\mathtt{x}(p)$. This is true because the overlap maps $\mathtt{x}' \circ \mathtt{x}^{-1}$ and $\mathtt{y}' \circ \mathtt{y}^{-1}$ are diffeomorphisms (The chain rule is at work here of course). Without worrying about domains, the point is that

$$\begin{aligned}
&\mathtt{y}' \circ f \circ \mathtt{x}'^{-1} \\
&= \mathtt{y}' \circ \left(\mathtt{y}^{-1} \circ \mathtt{y}\right) \circ f \circ \left(\mathtt{x}^{-1} \circ \mathtt{x}\right) \circ \mathtt{x}'^{-1} \\
&= \left(\mathtt{y}' \circ \mathtt{y}^{-1}\right) \circ \left(\mathtt{y} \circ f \circ \mathtt{x}^{-1}\right) \circ \left(\mathtt{x}' \circ \mathtt{x}^{-1}\right)^{-1}.
\end{aligned}$$

Now the reader should be able to see quite clearly why we required overlap maps to be diffeomorphisms. The functions of the form $\mathtt{y} \circ f \circ \mathtt{x}^{-1}$ are called representative functions. Thus once we know that a map f is continuous at p then f is of class C^r at p exactly if some representative function whose domain contains p is of class C^r. Also, to know that f is (globally) of class C^r on M, it is enough to know that for some, not necessarily maximal, atlases for M and N, all of the representative functions corresponding to charts from these atlases are C^r whenever the have nonempty domains. The set of all C^r maps $M \to N$ is denoted $C^r(M, N)$.

As a special case, we note that a function $f : M \to \mathbb{R}$ (resp.$\mathbb{C}$) is C^r differentiable at $p \in M$ if and only if it is continuous and

$$f \circ \mathtt{x}^{-1} : \mathtt{x}(U) \to \mathbb{R} \quad (\text{resp. } \mathbb{C})$$

is C^r-differentiable for some admissible chart $(U, \mathtt{x})$ with $p \in U$ and f is of class C^r if it is of class C^r at every p. The set of all C^r functions defined on all of M is denoted $C^r(M)$. A map f which is defined only on some proper open subset of a manifold is defined to be C^r if it is smooth as a map of the corresponding open submanifold but this is again just to say that it is C^r at each point in the open set. We shall often need to consider maps that are defined on subsets $S \subset M$ that are not necessarily open.

Definition 1.9 *Let S be an arbitrary subset of a smooth manifold M. Let $f : S \to N$ be a continuous map where N is a smooth manifold. The map f is said to be **smooth** (resp. C^r) if for every $s \in S$ there is an open set O containing s and map $\widetilde{f}$ that is smooth (resp. C^r) on O and such that $\left.\widetilde{f}\right|_{S \cap O} = f$.*

It is easy to show that if S has compact closure then a function f with domain S is smooth if and only if it has a smooth extension to some open set containing all of S. In particular a curve defined on a closed interval $[a, b]$ is smooth if it has a smooth extension to an open interval containing $[a, b]$. We will occasionally need the following simple concept:

Definition 1.10 *A continuous curve $c : [a, b] \to M$ into a smooth manifold is called **piecewise smooth** if there exists a partition $a = t_0 < t_1 < \cdots < t_k = b$ such that c restricted to $[t_i, t_{i+1}]$ is smooth for $0 \leq i \leq k - 1$.*

We already have the notion of a diffeomorphism between open sets of some Euclidean space $\mathbb{R}^n$. We are now in a position to extend this notion to the realm of smooth manifolds.

Definition 1.11 *Let M and N be smooth (or C^r) manifolds. A homeomorphism $f : M \to N$ such that f and f^{-1} are C^r differentiable with $r \geq 1$ is called a C^r-**diffeomorphism**. In case $r = \infty$ we shorten C^∞-diffeomorphism to just **diffeomorphism**. With composition of maps, the set of all C^r diffeomorphisms of a manifold M onto itself is a group denoted $\mathrm{Diff}^r(M)$. In case $r = \infty$ we simply write $\mathrm{Diff}(M)$.*

We will use the convention that $\mathrm{Diff}^0(M)$ denotes the group of homeomorphisms of M onto itself. Also, it should be pointed out that if we refer to a map between open subsets of manifolds as being a diffeomorphism, we mean that the map is a C^r-diffeomorphism of the corresponding open submanifolds. This just means that the map on the open sets is a homeomorphism which is appropriately differentiable (C^r) and whose inverse is also differentiable.

Example 1.19 *The map $r_\theta : S^2 \to S^2$ given by $r_\theta(x, y, z) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta, z)$ for $x^2 + y^2 + z^2 = 1$ is a diffeomorphism (and also an isometry!).*

Example 1.20 *Let $0 < \theta < 2\pi$ The map $f : S^2 \to S^2$ given by $f_\theta(x, y, z) = (x \cos((1 - z^2)\theta) - y \sin((1 - z^2)\theta), x \sin((1 - z^2)\theta) + y \cos((1 - z^2)\theta), z)$ is also a diffeomorphism (but not an isometry). Try to picture this map.*

Definition 1.12 *C^r manifolds M and N will be called (C^r) diffeomorphic and then said to be in the same diffeomorphism class if and only if there is a C^r diffeomorphism $f : M \to N$.*

We will be almost exclusively concerned with the smooth (C^∞) case. In the definition of diffeomorphism we have suppressed explicit reference to the maximal atlases but note that whether or not a map is differentiable (C^r or smooth) essentially involves the choice of differentiable structures on the manifolds. Recall that we have pointed out that we can put more than one differentiable structure on $\mathbb{R}$ by using the function $x^{1/3}$ as a chart. This generalizes in the obvious way: The map $\varepsilon : (x^1, x^2, ..., x^n) \mapsto ((x^1)^{1/3}, x^2, ..., x^n)$ is a chart for $\mathbb{R}^n$ but not compatible with the standard (identity) chart. It induces the usual topology again but the resulting maximal atlas is different! Thus we seem to have two smooth manifolds $(\mathbb{R}^n, \mathcal{A}_1)$ and $(\mathbb{R}^n, \mathcal{A}_2)$. This is true. Technically, they are different. But they are equivalent and therefore the same in another sense. Namely, they are diffeomorphic via the map ε. So it may be that the same underlying topological space M carries two different differentiable structures and so we really have two differentiable manifolds with the same underlying set. It remains to ask whether they are nevertheless diffeomorphic. It is an interesting question whether a given topological manifold can carry differentiable structures that are not diffeomorphic. We have mentioned that $\mathbb{R}^4$ carries non-diffeomorphic structures. In fact, it turns out that $\mathbb{R}^4$ carries infinitely many pairwise non-diffeomorphic structures all having the same underlying topology. Each $\mathbb{R}^k$ for $k \neq 4$ has only one diffeomorphism class compatible with the usual topology on $\mathbb{R}^k$.

Definition 1.13 *Let N and M be smooth manifolds of the same dimension. A map $f : M \to N$ is called a local diffeomorphism if and only if every point $p \in M$ is contained in an open subset $U_p \subset M$ such that $f|_{U_p} : U_p \to f(U)$ is a diffeomorphism onto an open subset of N. For C^r manifolds, a C^r local diffeomorphism is defined similarly.*

Example 1.21 *The map $\pi : S^2 \to P(\mathbb{R}^2)$ given by taking the point (x, y, z) to the line through this point and the origin is a local diffeomorphism but is not a diffeomorphism since it is 2-1 rather than 1-1.*

Example 1.22 *The map $(x, y) \mapsto (\frac{1}{1-z(x,y)}x, \frac{1}{1-z(x,y)}y)$ where $z(x, y) = \sqrt{1 - x^2 - y^2}$ is a diffeomorphism from the open disk $B(0, 1) = \{(x, y) : x^2 + y^2 < 1\}$ onto the whole plane. Thus $B(0, 1)$ and $\mathbb{R}^2$ are diffeomorphic and in this sense are the "same" differentiable manifold.*

The following terminology will be used often in the sequel.

Definition 1.14 *If $\pi : M \to N$ is a smooth surjection then a smooth* ***section*** *of π is a smooth map $\sigma : N \to M$ such that $\pi \circ \sigma = id$. If σ is defined only on an open subset U and $\pi \circ \sigma = id_U$ then we call σ a local section.*

Sometimes it is only important how maps behave near a certain point. Let M and N be smooth manifolds and consider the set $S(p, M, N)$ of all smooth maps which are defined on some open neighborhood of a fixed point $p \in M$. In other words,

$$S(p, M, N) := \bigcup_{U \in \mathcal{N}_p} C^\infty(U, N)$$

where $\mathcal{N}_p$ denotes the set of all open neighborhoods of $p \in M$. On this set we define the equivalence relation where f and g are equivalent at p if and only if there is an open set U containing p and with U contained in the domains of f and g, such that

$$f|_U = g|_U$$

In other word, f and g are equivalent in this sense if they agree on a neighborhood of p. The equivalence class of f is denoted $[f]$ or by $[f]_p$ if the point in question needs to be made clear. The set of equivalence classes $S(p, M, N)/\sim$ is denoted $C_p^\infty(M, N)$.

Definition 1.15 *Elements of $C_p^\infty(M, N)$ are called* ***germs*** *and if f and g are in the same equivalence class we write $f \underset{p}{\sim} g$ and we say that f and g have the same* ***germ at*** *p.*

The value of a germ at p is well defined $[f](p) = f(p)$. The evaluation map ev_p is defined by

$$ev_p : [f] \longmapsto f(p)$$

Taking $N = \mathbb{R}$ (resp. $\mathbb{C}$) we see that $C_p^\infty(M, \mathbb{R})$ (resp. $C_p^\infty(M, \mathbb{C})$) is a real (resp. complex) commutative algebra (and hence a ring) if we make the definitions

$$\begin{aligned} a[f] + b[g] &:= [af + bg] \text{ for } a, b \in \mathbb{R}(\text{ resp. } \mathbb{C}) \\ [f][g] &:= [fg] \end{aligned}$$

1.4 Cut-off functions and partitions of unity

There is a special and extremely useful kind of function called a bump function or cut-off function which we now take the opportunity to introduce. Recall that given a topological space X, the **support,** $\mathrm{supp}(f)$, of a function $f : X \to \mathbb{R}$ is the closure of the subset on which it takes nonzero values. The same definition applies for vector space valued functions $f : X \to \mathrm{V}$.

Lemma 1.6 (Existence of cut-off functions) *Let M be a smooth manifold. Let K be a compact subset of M and O an open set containing K. There exists a smooth function β on M that is identically equal to 1 on K, has compact support in O and $0 \leq \beta \leq 1$.*

Proof. Special case 1: Assume that $O = B(0, R)$ and $K = \overline{B}(0, r)$ for $0 < r < R$. In this case we may take

$$\phi(x) = \frac{\int_{|x|}^{R} g(t)dt}{\int_{r}^{R} g(t)dt}$$

where

$$g(t) = \begin{cases} e^{-(t-r)^{-1}} e^{(t-R)^{-1}} & \text{if } r < t < R \\ 0 & \text{otherwise.} \end{cases}$$

It is an exercise in calculus to show that g is a smooth function and thus that ϕ is smooth.

Special case 2: Assume that $M = \mathbb{R}^n$. Let $K \subset O$ be as in the hypotheses. Let $K_i \subset U_i$ be concentric balls as in the special case above but now with various choices of radii and such that $K \subset \cup K_i$. The U_i are chosen small enough that $U_i \subset O$. Let ϕ_i be the corresponding functions provided in the proof of the special case 1. By compactness there are only a finite number of pairs $K_i \subset U_i$ needed so we may assume that a reduction to a finite cover has been made. Examination of the following function will convince the reader that it is well defined and provides the needed cut-off function;

$$\beta(x) = 1 - \prod_i (1 - \phi_i(x)).$$

General case: It is clear that if K is contained in the domain U of a chart $(U, \mathtt{x})$ then by composing with $\mathtt{x}$ and extending to zero outside of U we obtain the result from the case $M = \mathbb{R}^n$ proved above. If K is not contained in such a chart then we may take a finite number of charts $(\mathtt{x}_1, U_1), ..., (\mathtt{x}_k, U_k)$ and compact sets $K_1, ..., K_k$ with

$$
\begin{aligned}
K &\subset \cup_{i=1}^{k} K_i \\
K_i &\subset U_i \\
\cup U_i &\subset O
\end{aligned}
$$

(we need the normality of M here). Now let ϕ_i be identically 1 on C_i and identically 0 on $U_i^c = M \backslash U_i$. Then the function β we are looking for is given by

$$\beta = 1 - \prod_{i=1}^{k} (1 - \phi_i).$$

■

Let $[f] \in C_p^\infty(M, \mathbb{R})$ (or $\in C_p^\infty(M, \mathbb{C})$) and let f be a representative of the equivalence class $[f]$. We can find an open set U such that $\overline{U}$ contains p and is contained in the domain of f. Now if β is a cut-off function that is identically equal to 1 on $\overline{U}$ and has support inside the domain of f then βf is smooth and it can be extended to a globally defined smooth function which is zero outside of the domain of f. Denote this extended function by $(\beta f)_{ext}$. Then $(\beta f)_{ext} \in [f]$ (usually, the extended function is just written as βf). Thus every element of $C_p^\infty(M, \mathbb{R})$ has a representative in $C^\infty(M, \mathbb{R})$. In short, each germ has a global representative.

A partition of unity is a technical tool that can help one piece together locally defined smooth objects with some desirable properties to obtain a globally defined object that also has the desired properties. For example, we will use this tool to show that on any (paracompact) smooth manifold there exists a Riemannian metric tensor. As we shall see, the metric tensor is the basic object whose existence allows the introduction of metric notions such as length and volume.

Definition 1.16 *A* ***partition of unity*** *on a smooth manifold M is a collection $\{\varphi_\alpha\}_{\alpha \in A}$ of* nonnegative *smooth functions on M such that*

(i) The collection of supports $\{supp(\varphi_\alpha)\}_{\alpha \in A}$ is locally finite, that is each point p of M has a neighborhood W_p such that $W_p \cap supp(\varphi_\alpha) = \emptyset$ for all but a finite number of $\alpha \in A$.

(ii) $\sum_{\alpha \in A} \varphi_\alpha(p) = 1$ for all $p \in M$ (this sum is always finite by (i)).

If $\mathcal{O} = \{O_\alpha\}_{\alpha \in A}$ is an open cover of M and $supp(\varphi_\alpha) \subset O_\alpha$ for each $\alpha \in A$ the we say that $\{\varphi_\alpha\}_{\alpha \in A}$ is a partition of unity ***subordinate to*** *$\mathcal{O} = \{O_\alpha\}_{\alpha \in A}$.*

Notice that every partition of unity is subordinate to some cover since we may take $O_\alpha = \{\varphi_\alpha > 0\}$. That $\{O_\alpha\}$ is a cover follows from (ii). (By definition $\varphi_\alpha \geq 0$ for all α).

Remark 1.4 *Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be a cover of M. Suppose that $\mathcal{W} = \{W_\beta\}_{\beta\in B}$ is a cover which is a refinement of $\mathcal{U}$. Let $\{\psi_\beta\}_{\beta\in B}$ is a partition of unity subordinate to a cover $\mathcal{W}$. We may obtain a partition of unity subordinate to $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$. Indeed, let $f : B \to A$ be such that for $W_\beta \subset U_{f(\beta)}$ for every $\beta \in B$. Then we let $\varphi_\alpha := \sum_{\beta\in f^{-1}(\alpha)} \psi_\beta$.*

Our definition of smooth manifold M includes the requirement that M be paracompact (and Hausdorff). Paracompact Hausdorff spaces are normal spaces but the following theorem would be true for a normal locally Euclidean space with smooth structure even without the assumption of paracompactness. The reason is that we *explicitly assume* the local finiteness of the cover. For this reason we put the word "normal" in parentheses as a pedagogical device.

Theorem 1.2 *Let M be a (normal) smooth manifold and $\{U_\alpha\}_{\alpha\in A}$ be a locally finite cover of M. If each U_α has compact closure then there is a partition of unity $\{\varphi_\alpha\}_{\alpha\in A}$ subordinate to $\{U_\alpha\}_{\alpha\in A}$.*

Proof. We shall use a well known result about normal spaces. Namely, if $\{U_\alpha\}_{\alpha\in A}$ is a locally finite cover of a normal space M then there exists another cover $\{V_\alpha\}_{\alpha\in A}$ of M such that $\overline{V_\alpha} \subset U_\alpha$.

We do this to our cover and then notice that since each U_α has compact closure, each $\overline{V_\alpha}$ is compact. We apply lemma 1.6 to obtain nonnegative smooth functions ψ_α such that supp$\psi_\alpha \subset U_\alpha$ and $\psi_\alpha|_{\overline{V_\alpha}} \equiv 1$. Let $\psi := \sum_{\alpha\in A} \psi_\alpha$ and notice that for each $p \in M$ the sum $\sum_{\alpha\in A} \psi_\alpha(p)$ is finite and $\psi(p) > 0$. Now let $\varphi_\alpha := \psi_\alpha/\psi$. It is now easy to check that $\{\varphi_\alpha\}_{\alpha\in A}$ is the desired partition of unity. ■

If we use the paracompactness assumption then we can show that a partition of unity exists which is subordinate to *any* given cover.

Theorem 1.3 *Let M be a (paracompact) smooth manifold and $\{U_\alpha\}_{\alpha\in A}$ be a cover of M. Then there is a partition of unity $\{\varphi_\alpha\}_{\alpha\in A}$ subordinate to $\{U_\alpha\}_{\alpha\in A}$.*

Proof. By remark 1.4 and the fact that M is locally compact we may assume without loss of generality that each U_α has compact closure. Then since M is paracompact we may find a locally finite refinement of $\{U_\alpha\}_{\alpha\in A}$ which we denote by $\{V_i\}_{i\in I}$. Now use the previous theorem to get a partition of unity subordinate to $\{V_i\}_{i\in I}$. Finally use remark 1.4 one more time to get a partition of unity subordinate to $\{U_\alpha\}_{\alpha\in A}$. ■

1.5 Coverings and Discrete groups

1.5.1 Covering spaces and the fundamental group

In this section and later when we study fiber bundles many of the results are interesting and true in either the purely topological category or in the smooth category. In order not to have to do things twice let us agree that a C^0-manifold is simply a topological manifold or more generally any paracompact Hausdorff topological space that satisfies the technical condition of being "semi-locally simply connected" (SLSC) All manifolds are SLSC.. Thus all relevant maps in this section are to be C^r where if $r = 0$ we just mean continuous and then only require that the spaces be sufficiently nice topological spaces. Also, "C^0 diffeomorphism" just means homeomorphism.

"C^0-diffeomorphism"	=	C^0-isomorphism	=	homeomorphism
"C^0-manifold"	=	topological space		
C^0-group	=	topological group		

We may define a simple equivalence relation on a topological space M by declaring

$$p \sim q \Leftrightarrow \text{ there is a continuous curve connecting } p \text{ to } q.$$

The equivalence classes are called **path components** and if there is only one path component then we say that M is **path connected**. The following exercise will be used whenever needed without explicit mention:

Exercise 1.9 *The path components of a manifold M are exactly the connected components of M. Thus, a manifold is connected if and only if it is path connected.*

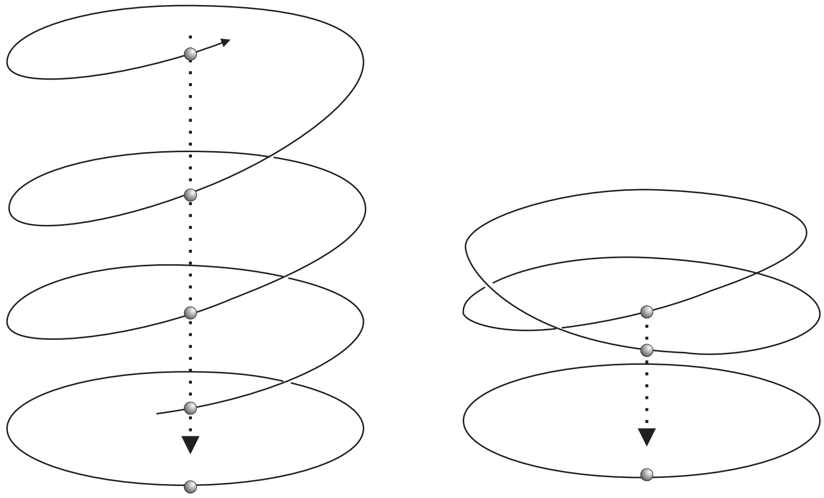
In the definition of path component given above we used continuous paths but it is not hard to show that if two points on a smooth manifold can be connected by a continuous path then they can be connected by smooth path and so the notion of path component remains unchanged by the use of smooth paths.

Definition 1.17 *Two C^r maps on C^r manifolds, say $f_0 : X \to Y$ and $f_1 : X \to Y$ are said to be C^r homotopic if there exists a C^r map $H : X \times [0,1] \to Y$ such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$ for all x. We then say that f_0 is C^r homotopic to f_1 and write $f_0 \overset{C^r}{\simeq} f_1$. If $A \subset X$ is a closed subset and if $H(a,s) = f_0(a) = f_1(a)$ for all $a \in A$ and all $s \in [0,1]$, then we say that H is a homotopy relative to A and we write $f_0 \overset{C^r}{\simeq} f_1$ (rel A).*

At first it may seem that there might be a big difference between C^∞ and C^0 homotopy but if all the spaces involved are smooth manifolds then the difference is not big at all. In fact, we have the following theorems which we merely state but proofs may be found in [Lee, John].

Theorem 1.4 *If $f : M \to N$ is a continuous map on smooth manifolds then f is homotopic to a smooth map $f_0 : M \to N$. If the continuous map $f : M \to N$ is smooth on a closed subset A then it can be arranged that $f \overset{C^\infty}{\simeq} f_0$ (rel A).*

Theorem 1.5 *If $f_0 : M \to N$ and $f_1 : M \to N$ are smooth maps which are homotopic then they are smoothly homotopic. If f_0 is homotopic to f_1 relative to a closed subset A then f_0 is smoothly homotopic to f_1 relative to A.*



Because of these last two theorems we will usually simply write $f \simeq f_0$ instead of $f \overset{C^r}{\simeq} f_0$ the value of or r being of little significance in this setting.

Definition 1.18 *Let $\widetilde{M}$ and M be C^r-spaces. A surjective C^r map $\wp : \widetilde{M} \to M$ is called a C^r* ***covering map*** *if every point $p \in M$ has an open connected neighborhood U such that each connected component $\widetilde{U}_i$ of $\wp^{-1}(U)$ is C^r diffeomorphic to U via the restrictions $\wp|_{\widetilde{U}_i} : \widetilde{U}_i \to U$. In this case we say that U is* ***evenly covered*** *by $\wp$ (or by the sets $\widetilde{U}_i$). The triple $(\widetilde{M}, \wp, M)$ is called a covering space. We also refer to the space $\widetilde{M}$ (somewhat informally) as a* ***covering space*** *for M.*

Example 1.23 *The map $\mathbb{R} \to S^1$ given by $t \mapsto e^{it}$ is a covering. The set of points $\{e^{it} : \theta - \pi < t < \theta + \pi\}$ is an open set evenly covered by the intervals I_n in the real line given by $I_n := (\theta - \pi + n2\pi, \theta + \pi + n2\pi)$.*

Exercise 1.10 *Explain why the map $(-2\pi, 2\pi) \to S^1$ given by $t \mapsto e^{it}$ is* not *a covering map.*

Definition 1.19 *A continuous map f is said to be proper if $f^{-1}(K)$ is compact whenever K is compact.*

Exercise 1.11 *Show that a C^r* proper *map between connected smooth manifolds is a smooth covering map if and only if it is a local C^r diffeomorphism.*

The set of all C^r covering spaces are the objects of a category. A morphism between C^r-covering spaces, say $(\widetilde{M}_1, \wp_1, M_1)$ and $(\widetilde{M}_2, \wp_2, M_2)$ is a pair of

(C^r) maps $(\widetilde{f}, f)$ that give a commutative diagram

$$\begin{array}{ccc} \widetilde{M}_1 & \xrightarrow{\widetilde{f}} & \widetilde{M}_2 \\ \downarrow & & \downarrow \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

which means that $f \circ \wp_1 = \wp_2 \circ \widetilde{f}$. Similarly the set of coverings of a fixed space M are the objects of a category where the morphisms are maps $\Phi : \widetilde{M}_1 \to \widetilde{M}_2$ required to make the following diagram commute:

$$\begin{tikzcd} \widetilde{M}_1 \arrow[rr, "\Phi"] \arrow[dr] & & \widetilde{M}_2 \arrow[dl] \\ & M & \end{tikzcd}$$

so that $\wp_1 = \wp_2 \circ \Phi$. Now let $(\widetilde{M}, \wp, M)$ be a C^r covering space. The set of all C^r-diffeomorphisms Φ that are automorphisms in the above category, that is, diffeomorphisms for which $\wp_1 = \wp_2 \circ \Phi$, are called **deck transformations** or **covering transformations**. This set of deck transformations is a group of C^r diffeomorphisms of $\widetilde{M}$ called the deck transformation group which we denote by $Deck(\wp)$ or sometimes by $Deck(\widetilde{M})$. A deck transformation permutes the elements of each fiber $\wp^{-1}(p)$. In fact, it is not to hard to see that if $U \subset M$ is evenly covered then Φ permutes the connected components of $\wp^{-1}(U)$.

Proposition 1.1 *If $\wp : \widetilde{M} \to M$ is a C^r covering map with M being connected then the cardinality of $\wp^{-1}(p)$ is either infinite or is independent of p. In the latter case the cardinality of $\wp^{-1}(p)$ is called the* ***multiplicity*** *of the covering.*

Proof. Fix $k < \infty$. Let U_k be the set of all points such that $\wp^{-1}(p)$ has cardinality k. It is easy to show that U_k is both open and closed and so, since M is connected, U_k is either empty or all of M. ∎

Since we are mainly interested in the smooth case the following theorem is quite useful:

Theorem 1.6 *Let M be a C^r manifold with $r > 0$ and suppose that $\wp : \widetilde{M} \to M$ is a C^0 covering map. Then there exists a (unique) C^r structure on $\widetilde{M}$ making $\wp$ a C^r covering map.*

Proof. Choose an atlas $\{(U_\alpha, \mathtt{x}_\alpha)\}_{\alpha \in A}$ such that each domain U_α is small enough to be evenly covered by $\wp$. Thus we have that $\wp^{-1}(U_\alpha)$ is a disjoint union of open set U_α^i with each restriction $\wp|_{U_\alpha^i}$ a homeomorphism. We now construct charts on $\widetilde{M}$ using the maps $\mathtt{x}_\alpha \circ \wp|_{U_\alpha^i}$ defined on the sets U_α^i (which cover $\widetilde{M}$). The overlap maps are smooth since for example

$$\begin{aligned}
&\left(\mathtt{x}_\alpha \circ \wp|_{U_\alpha^i}\right) \circ \left(\mathtt{x}_\beta \circ \wp|_{U_\beta^j}\right)^{-1} \\
&= \mathtt{x}_\alpha \circ \wp|_{U_\alpha^i} \circ \left(\wp|_{U_\beta^j}\right)^{-1} \circ \mathtt{x}_\beta^{-1} \\
&= \mathtt{x}_\alpha \circ \mathtt{x}_\beta^{-1}
\end{aligned}$$

We leave it to the reader to show, or trust that $\widetilde{M}$ must be paracompact and Hausdorff if M is. ■

The following is a special case of definition 1.17.

Definition 1.20 *Let $\alpha : [0,1] \to M$ and $\beta : [0,1] \to M$ be two C^r maps (paths) both starting at $p \in M$ and both ending at q. A C^r* ***fixed end point homotopy*** *from α to β is a family of C^r maps $H_s : [0,1] \to M$ parameterized by $s \in [0,1]$ such that*
1) $H : [0,1] \times [0,1] \to M$ defined by $H(t,s) := H_s(t)$ is C^r ,
2) $H_0 = \alpha$ and $H_1 = \beta$,
3) $H_s(0) = p$ and $H_s(1) = q$ for all $s \in [0,1]$.

Definition 1.21 *If there is a C^r homotopy from α to β then we say that α is C^r* ***homotopic*** *to β and write $\alpha \simeq \beta$ (C^r). If $r = 0$ we speak of paths being continuously homotopic.*

Remark 1.5 *By Theorems 1.5 and 1.4 above we know that in the case of smooth manifolds we have that $\alpha \simeq \beta$ (C^0) if and only if $\alpha \simeq \beta$ (C^r) for $r > 0$. Thus we can just say that α is homotopic to β and write $\alpha \simeq \beta$.*

It is easily checked that homotopy is an equivalence relation. Let $P(p,q)$ denote the set of all continuous (or smooth) paths from p to q defined on $[0,1]$. Every $\alpha \in P(p,q)$ has a unique inverse (or reverse) path $\alpha^{\leftarrow}$ defined by

$$\alpha^{\leftarrow}(t) := \alpha(1-t).$$

If p_1, p_2 and p_3 are three points in M then for $\alpha \in P(p_1,p_2)$ and $\beta \in P(p_2,p_3)$ we can "multiply" the paths to get a path $\alpha * \beta \in P(p_1,p_3)$ defined by

$$\alpha * \beta(t) := \begin{cases} \alpha(2t) & \text{for } 0 \le t < 1/2 \\ \beta(2t-1) & \text{for } 1/2 \le t < 1 \end{cases}.$$

An important observation is that if $\alpha_1 \simeq \alpha_2$ and $\beta_1 \simeq \beta_2$ then

$\alpha_1 * \beta_1 \simeq \alpha_2 * \beta_2$ where the homotopy between $\alpha_1 * \beta_1$ and $\alpha_2 * \beta_2$ is given in terms of the homotopies $H_\alpha : \alpha_1 \simeq \alpha_2$ and $H_\beta : \beta_1 \simeq \beta_2$ by

$$H(t,s) := \begin{cases} H_\alpha(2t,s) & \text{for } 0 \le t < 1/2 \\ H_\beta(2t-1,s) & \text{for } 1/2 \le t < 1 \end{cases} \quad \text{and } 0 \le s < 1$$

Similarly, if $\alpha_1 \simeq \alpha_2$ then $\alpha_1^{\leftarrow} \simeq \alpha_2^{\leftarrow}$. Using this information we can define a group structure on the set of homotopy equivalence classes of loops, that is, of paths in $P(p,p)$ for some fixed $p \in M$. First of all, we can always form $\alpha * \beta$ for any $\alpha, \beta \in P(p,p)$ since we are always starting and stopping at the same point p. Secondly we have the following

Proposition 1.2 *Let $\pi_1(M,p)$ denote the set of fixed end point homotopy classes of paths from $P(p,p)$. For $[\alpha], [\beta] \in \pi_1(M,p)$ define $[\alpha] \cdot [\beta] := [\alpha * \beta]$. This is a well defined multiplication and with this multiplication $\pi_1(M,p)$ is a group. The identity element of the group is the homotopy class 1 of the constant map $1_p : t \to p$, the inverse of a class $[\alpha]$ is $[\alpha^{\leftarrow}]$.*

Proof. We have already shown that $[\alpha] \cdot [\beta] := [\alpha * \beta]$ is well defined. One must also show that

1) For any α, the paths $\alpha \circ \alpha^{\leftarrow}$ and $\alpha^{\leftarrow} \circ \alpha$ are both homotopic to the constant map 1_p.

2) For any $\alpha \in P(p,p)$ we have $1_p * \alpha \simeq \alpha$ and $\alpha * 1_p \simeq \alpha$.

3) For any $\alpha, \beta, \gamma \in P(p,p)$ we have $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$.

Proof of (1): 1_p is homotopic to $\alpha \circ \alpha^{\leftarrow}$ via

$$H(s,t) = \begin{cases} \alpha(2t) & \text{for } 0 \le 2t \le s \\ \alpha(s) & \text{for } s \le 2t \le 2-s \\ \alpha^{\leftarrow}(2t-1) & \text{for } 2-s \le 2t \le 2 \end{cases}$$

where $0 \le s \le 1$. Interchanging the roles of α and $\alpha^{\leftarrow}$ we also get that 1_p is homotopic to $\alpha^{\leftarrow} \circ \alpha$.

Proof of (2): Use the homotopy

$$H(s,t) = \begin{cases} \alpha(\frac{2}{1+s}t) & \text{for } 0 \le t \le 1/2 + s/2 \\ \\ p & \text{for } 1/2 + s/2 \le t \le 1 \end{cases}$$

Proof of (3): Use the homotopy

$$H(s,t) = \begin{cases} \alpha(\frac{4}{1+s}t) & \text{for } 0 \le t \le \frac{1+s}{4} \\ \beta(4(t - \frac{1+s}{4})) & \frac{1+s}{4} \le t \le \frac{2+s}{4} \\ \gamma(\frac{4}{2-s}(t - \frac{2+s}{4})) & \text{for } \frac{2+s}{4} \le t \le 1 \end{cases}$$

■

The group $\pi_1(M,p)$ is called the **fundamental group** of M at p. If desired one can take the equivalence classes in $\pi_1(M,p)$ to be represented by smooth maps. If $\gamma : [0,1] \to M$ is a path from p to q then we have a group isomorphism $\pi_1(M,q) \to \pi_1(M,p)$ given by

$$[\alpha] \mapsto [\gamma * \alpha * \gamma^{\leftarrow}].$$

Exercise 1.12 *Show that the previous prescription is well defined and that the map is really a group isomorphism.*

As a result we have

Proposition 1.3 *For any two points p, q in the same path component of M, the groups $\pi_1(M, p)$ and $\pi_1(M, q)$ are isomorphic (by the map described above).*

Corollary 1.1 *If M is connected then the fundamental groups based at different points are all isomorphic.*

Because of this last proposition, if M is connected we may simply refer to *the* fundamental group of M which we write as $\pi_1(M)$.

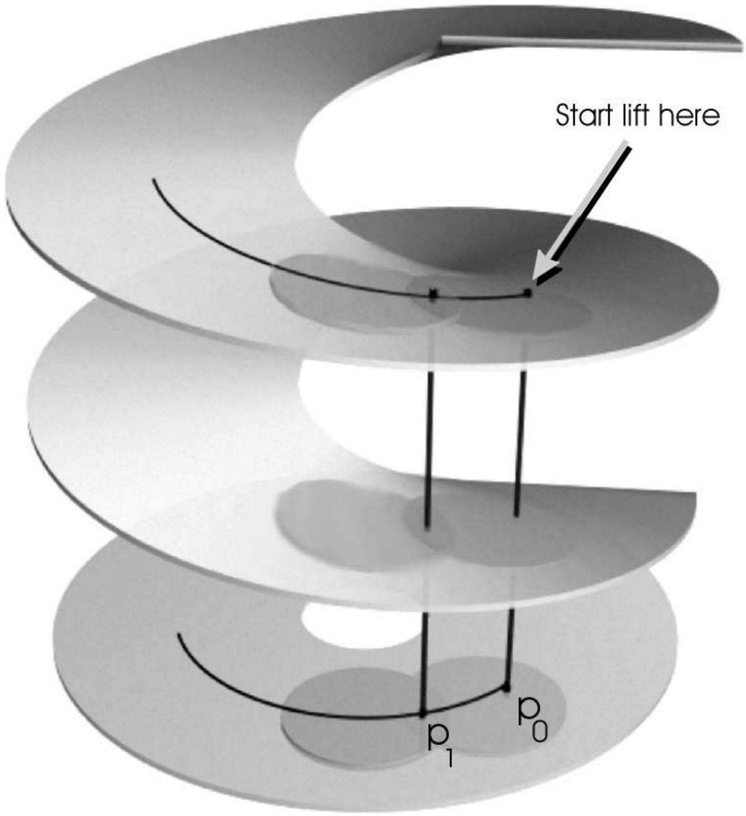
Definition 1.22 *A path connected topological space is called* ***simply connected*** *if $\pi_1(M) = \{1\}$.*

The fundamental group is actually the result of applying a functor (see Appendix B). Consider the category whose objects are pairs (M, p) where M is a C^r manifolds and p a distinguished point (base point) and whose morphisms $f : (M, p) \to (N, q)$ are C^r maps $f : M \to N$ such that $f(p) = q$. The pairs are called pointed C^r spaces and the morphisms are called pointed C^r maps (or base point preserving maps). To every pointed space (M, p) we assign the fundamental group $\pi_1(M, p)$ and to every pointed map $f : (M, p) \to (N, f(p))$ we may assign a group homomorphism $\pi_1(f) : \pi_1(M, p) \to \pi_1(N, f(p))$ by

$$\pi_1(f)([\alpha]) = [f \circ \alpha].$$

It is easy to check that this is a covariant functor and so for pointed maps f and g that can be composed $(M, x) \xrightarrow{f} (N, y) \xrightarrow{g} (P, z)$ we have $\pi_1(g \circ f) = \pi_1(g)\pi_1(f)$.

Notation 1.3 *To avoid notational clutter we will often denote $\pi_1(f)$ by $f_\#$.*



Definition 1.23 *Let $\wp : \widetilde{M} \to M$ be a C^r covering and let $f : P \to M$ be a C^r map. A map $\widetilde{f} : P \to \widetilde{M}$ is said to be a **lift** of the map f if $\wp \circ \widetilde{f} = f$.*

Theorem 1.7 *Let $\wp : \widetilde{M} \to M$ be a C^r covering, let $\gamma : [a,b] \to M$ a C^r curve and pick a point y in $\wp^{-1}(\gamma(a))$. Then there exists a unique C^r lift $\widetilde{\gamma} : [a,b] \to \widetilde{M}$ of γ such that $\widetilde{\gamma}(a) = y$. Thus the following diagram commutes.*

$$\begin{tikzcd} & \widetilde{M} \arrow[d, "\wp"] \\ {[a,b]} \arrow[ur, "\widetilde{\gamma}"] \arrow[r, "\gamma"] & M \end{tikzcd}$$

If two paths α and β with $\alpha(a) = \beta(a)$ are fixed end point homotopic via an homotopy h, then for a given point y in $\wp^{-1}(\gamma(a))$, we have the corresponding lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ starting at y. In this case the homotopy h lifts to a fixed endpoint homotopy $\widetilde{h}$ between $\widetilde{\alpha}$ and $\widetilde{\beta}$. In short, homotopic paths lift to homotopic paths.

Proof. We just give the basic idea and refer the reader to the extensive literature for details (see [**?**, **?**]). Figure 1.5.1 shows the way. Decompose the curve γ into segments that lie in evenly covered open sets. Lift inductively starting by using the inverse of $\wp$ in the first evenly covered open set. It is clear that in order to connect up continuously, each step is forced and so the lifted curve is unique. The proof of the second half is just slightly more complicated but the idea is the same and the proof is left to the curious reader. A tiny technicality in either case is the fact that for $r > 0$ a C^r-map on a closed set is defined to mean that there is a C^r-map on a slightly larger open set. For instance, for the curve γ we must lift an extension $\gamma_{ext} : (a - \varepsilon, b + \varepsilon) \to M$ but considering how the proof went we see that the procedure is basically the same and gives a C^r- extension $\widetilde{\gamma}_{ext}$ of the lift $\widetilde{\gamma}$.

A similar argument shows how to lift the homotopy h. ∎

There are several important corollaries to this result. One is simply that if $\alpha : [0, 1] \to M$ is a path starting at a base point $p \in M$, then since there is one and only one lift $\widetilde{\alpha}$ starting at a given p' in the fiber $\wp^{-1}(p)$, the endpoint $\widetilde{\alpha}(1)$ is completely determined by the path α and by the point p' from which we want the lifted path to start. In fact, the endpoint only depends on the homotopy class of α (and the choice of starting point p'). To see this note that if $\alpha, \beta : [0, 1] \to M$ are fixed end point homotopic paths in M beginning at p and if $\widetilde{\alpha}$ and $\widetilde{\beta}$ are the corresponding lifts with $\widetilde{\alpha}(0) = \widetilde{\beta}(0) = p'$ then by the second part of the theorem, any homotopy $h_t : \alpha \simeq \beta$ lifts to a unique fixed endpoint homotopy $\widetilde{h}_t : \widetilde{\alpha} \simeq \widetilde{\beta}$. This then implies that $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$. Applying these ideas to loops based at $p \in M$ we will next see that the fundamental group $\pi_1(M, p)$ acts on the fiber $\wp^{-1}(p)$ as a group of permutations. (This is a "right action" as we will see). In case the covering space $\widetilde{M}$ is simply connected we will also obtain an isomorphism of the group $\pi_1(M, p)$ with the deck transformation group (which acts from the *left* on $\widetilde{M}$). Before we delve into these matters, we state, without proof, two more more standard results (see [Gr-Harp]):

Theorem 1.8 *Let $\wp : \widetilde{M} \to M$ be a C^r covering. Fix a point $q \in Q$ and a point $\widetilde{p} \in \widetilde{M}$. Let $\phi : Q \to M$ be a C^r map with $\phi(q) = \wp(\widetilde{p})$. If Q is* connected *then there is at most one lift $\widetilde{\phi} : Q \to \widetilde{M}$ of ϕ such that $\widetilde{\phi}(p) = \widetilde{p}$. If $\phi_\#(\pi_1(Q, q)) \subset \wp_\#(\pi_1(\widetilde{M}, \widetilde{p}))$ then ϕ has such a lift. In particular, if Q is simply connected then the lift exists.*

Theorem 1.9 *Every connected topological manifold M has a C^0 simply connected covering space which is unique up to isomorphism of coverings. This is called the **universal cover**. Furthermore, if H is any subgroup of $\pi_1(M, p)$, then there is a connected covering $\wp : \widetilde{M} \to M$ and a point $\widetilde{p} \in \widetilde{M}$ such that $\wp_\#(\pi_1(\widetilde{M}, \widetilde{p})) = H$.*

If follows from this and Theorem 1.6 that if M is a C^r manifold then there is a unique C^r structure on the universal covering space $\widetilde{M}$ so that $\wp : \widetilde{M} \to M$ is a C^r covering.

Since a deck transformation is a lift we have the following

Corollary 1.2 *If $\wp : \widetilde{M} \to M$ is a C^r covering map and we choose a base point $p \in M$ then if $\widetilde{M}$ is connected there is at most one deck transformation ϕ that maps a given $p_1 \in \wp^{-1}(p)$ to a given $p_2 \in \wp^{-1}(p)$. If $\widetilde{M}$ is simply connected then such a deck transformation exists and is unique.*

Theorem 1.10 *If $\widetilde{M}$ is the universal cover of M and $\wp : \widetilde{M} \to M$ the corresponding universal covering map then for any base point $p_0 \in M$, there is an isomorphism of $\pi_1(M, p_0)$ with the deck transformation group $Deck(\wp)$.*

Proof. Fix a point $\widetilde{p} \in \wp^{-1}(p_0)$. Let $a \in \pi_1(M, p_0)$ and let α be a loop representing a. Lift to a path $\widetilde{\alpha}$ starting at $\widetilde{p}$. As we have seen the point $\widetilde{\alpha}(1)$ depends only on the choice of $\widetilde{p}$ and $a = [\alpha]$. Let ϕ_a be the unique deck transformation such that $\phi_a(\widetilde{p}) = \widetilde{\alpha}(1)$. The assignment $a \mapsto \phi_a$ gives a map $\pi_1(M, p_0) \to Deck(\wp)$. Now for $a = [\alpha]$ and $b = [\beta]$ chosen from $\pi_1(M, p_0)$, we have the lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ and we see that $\phi_a \circ \widetilde{\beta}$ is a path from $\phi_a(\widetilde{p})$ to $\phi_a(\widetilde{\beta}(1)) = \phi_a(\phi_b(\widetilde{p}))$. Thus the path $\widetilde{\gamma} := \widetilde{\alpha} * \left(\phi_a \circ \widetilde{\beta}\right)$ is defined. Now

$$
\begin{aligned}
\wp \circ \widetilde{\gamma} &= \wp \circ \left[\widetilde{\alpha} * \left(\phi_a \circ \widetilde{\beta}\right)\right] \\
&= (\wp \circ \widetilde{\alpha}) * \left(\wp \circ \left(\phi_a \circ \widetilde{\beta}\right)\right) \\
&= (\wp \circ \widetilde{\alpha}) * \left((\wp \circ \phi_a) \circ \widetilde{\beta}\right) \\
&= (\wp \circ \widetilde{\alpha}) * \left(\wp \circ \widetilde{\beta}\right) = \alpha * \beta
\end{aligned}
$$

Since $\alpha * \beta$ represents the element $ab \in \pi_1(M, p_0)$ we have $\phi_{ab}(\widetilde{p}) = \widetilde{\gamma}(1) = \phi_a(\phi_b(\widetilde{p}))$. Now since $\widetilde{M}$ is connected this forces $\phi_{ab} = \phi_a \circ \phi_b$. Thus the map is a group homomorphism

Next we show that the map $a \mapsto \phi_a$ is onto. Given $\varphi \in Deck(\wp)$ we simply take a curve $\widetilde{\gamma}$ from $\widetilde{p}$ to $f(\widetilde{p})$ and then $f = \phi_g$ where $g = [\wp \circ \widetilde{\gamma}] \in \pi_1(M, p_0)$.

Finally, if $\phi_a = id$ then we conclude that any loop $\alpha \in [\alpha] = a$ lifts to a loop $\widetilde{\alpha}$ based at $\widetilde{p}$. But $\widetilde{M}$ is simply connected and so $\widetilde{\alpha}$ is homotopic to a constant map to $\widetilde{p}$ and its projection α is therefore homotopic to a constant map to p. Thus $a = [\alpha] = 0$ and so the homomorphism is 1-1. ■

1.5.2 Discrete Group Actions

Definition 1.24 *Let G be a group and M a set. A* ***left group action*** *is a map $l : G \times M \to M$ such that for every $g \in G$ the partial map $l_g(.) := l(g, .)$ satisfies:*

1) $l(g_2, l(g_1, x)) = l(g_2 g_1, x)$ for all $g_1, g_2 \in G$ and all $x \in M$.

2) $l(e, x) = x$ for all $x \in M$.

We often write $g \cdot x$ or just gx in place of the more pedantic notation $l(g, x)$. Using this notation we have $g_2(g_1 x) = (g_2 g_1)x$ and $ex = x$. Similarly, we have

Definition 1.25 *Let G be a group and M a set. A **right group action** is a map $r : M \times G \to M$ such that for every $g \in G$ the partial map $r_g(.) := r(., g)$ satisfies:*

1) $r(r(x, g_1), g_2) = r(x, g_1 g_2)$ for all $g_1, g_2 \in G$ and all $x \in M$.

2) $r(x, e) = x$ for all $x \in M$.

In the case of right actions we write $r(x, g)$ as $x \cdot g$ or xg. For every result for left actions there is an analogous result for right actions. However, mathematical conventions are such that while $g \mapsto l_g$ is a group homomorphism, the map $g \mapsto r_g$ is a a group *anti-homomorphism* which means that $r_{g_1} \circ r_{g_2} = r_{g_2 g_1}$ for all $g_1, g_2 \in G$ (notice the order reversal).

Given a left action, the sets of the form $Gx = \{gx : g \in G\}$ are called **orbits** or cosets. In particular Gx is called the orbit of x. Two points x and y are in the same orbit if there is a group element g such that $gx = y$. The orbits are equivalence classes and so they partition M. Let $G\backslash M$ be the set of orbits (cosets) and let $\wp : M \to G\backslash M$ be the projection taking each x to Gx.

Definition 1.26 *Suppose the G acts on a set M by $l : G \times M \to M$. We say that G acts **transitively** if for any $x, y \in M$ there is a g such that $l_g(x) = y$. Equivalently, the action is **transitive** if the action has only one orbit. If $l_g = id_M$ implies that $g = e$ we say that the action is an **effective action** and if $l_g(x) = x$ for some $x \in M$ implies that $g = e$ the we say that G acts **freely** (or that the action is free).*

Similar statements and definitions apply for right actions except that now the orbits have the form xG and the quotient space (space of orbits) will then be denoted by M/G.

Warning: *The notational distinction between $G\backslash M$ and M/G is not universal and often M/G is used to denote $G\backslash M$. In situations where left-right distinctions are not relevant we find that the forward slash "/" is often used to denote quotients of either kind.*

Example 1.24 *Let $\wp : \widetilde{M} \to M$ be a covering map. Fix a base point $p_0 \in M$ and a base point $\widetilde{p}_0 \in \wp^{-1}(p_0)$. If $a \in \pi_1(M, p_0)$ then for each $x \in \wp^{-1}(p_0)$ we define $r_a(x) := xa := \widetilde{\alpha}(1)$ where $\widetilde{\alpha}$ is the lift of any loop α representing a. The reader may check that r_a is a right action on the set $\wp^{-1}(p_0)$.*

Example 1.25 *Recall that if $\wp : \widetilde{M} \to M$ is a universal C^r covering map (so that $\widetilde{M}$ is simply connected) we have an isomorphism $\pi_1(M, p_0) \to Deck(\wp)$ which we denoted by $a \mapsto \phi_a$. This means that $l(a, x) = \phi_a(x)$ defines a left action of $\pi_1(M, p_0)$ on $\widetilde{M}$.*

Let G be a group and endow G with the discrete topology so that, in particular, every point is an open set. In this case we call G a discrete group. If M is a topological space then so is $G \times M$ with the product topology. What does it mean for a map $\alpha : G \times M \to M$ to be continuous? The topology of $G \times M$ is clearly generated by sets of the form $S \times U$ where S is an arbitrary subset of G and U is open in M. The map $\alpha : G \times M \to M$ will be continuous if for any point $(g_0, x_0) \in G \times M$ and any open set $U \subset M$ containing $\alpha(g_0, x_0)$ we can find an open set $S \times V$ containing (g_0, x_0) such that $\alpha(S \times V) \subset U$. Since the topology of G is discrete, it is necessary and sufficient that there is an open V such that $\alpha(g_0 \times V) \subset U$. It is easy to see that a necessary and sufficient condition for α to be continuous on all of $G \times M$ is that the partial maps $\alpha_g(.) := \alpha(g, .)$ are continuous for every $g \in G$.

Definition 1.27 *Let G be a discrete group and M a manifold. A left* ***discrete group action*** *is a group action $l : G \times M \to M$ such that for every $g \in G$ the partial map $l_g(.) := l(g, .)$ is continuous. A right discrete group action is defined similarly.*

It follows that if $l : G \times M \to M$ is a discrete action then each partial map $l_g(.)$ is a homeomorphism with $l_g^{-1}(.) = l_{g^{-1}}(.)$.

Definition 1.28 *A discrete group action is C^r if M is a C^r manifold and each $l_g(.)$ (or $r_g(.)$ for a right actions) is a C^r map.*

Example 1.26 *Let $\phi : M \to M$ be a diffeomorphism and let $\mathbb{Z}$ act on M by $n \cdot x := \phi^n(x)$ where*

$$\begin{aligned} \phi^0 &:= \mathrm{id}_M, \\ \phi^n &:= \phi \circ \cdots \circ \phi \text{ for } n > 0 \\ \phi^{-n} &:= (\phi^{-1})^n \text{ for } n > 0. \end{aligned}$$

This gives a discrete action of $\mathbb{Z}$ on M.

Definition 1.29 *A discrete group action $\alpha : G \times M \to M$ is said to act* ***properly*** *if every two points $x, y \in M$ have open neighborhoods U_x and U_y respectively such that the set $\{g \in G : gU_x \cap U_y \neq \emptyset\}$ is finite.*

There is a more general notion of proper action which we shall meet later. For free *and* proper discrete actions we have the following useful characterization.

Proposition 1.4 *A discrete group action $\alpha : G \times M \to M$ acts* ***properly and freely*** *if and only if the following two conditions hold:*

i) Each $x \in M$ has a neighborhood U such that $gU \cap U = \emptyset$ for all g except the identity e. We shall call such open sets ***self avoiding****.*

ii) If $x, y \in M$ are not in the same orbit then they have self avoiding neighborhoods U_x and U_y such that $gU_x \cap U_y = \emptyset$ for all $g \in G$.

Proof. Suppose that the action α is proper and free. Let x be given. We then know that there is an open V containing x such that $gV \cap V = \emptyset$ except for a finite number of g, say, $g_1, ..., g_k$ which are distinct. One of these, say

g_1, must be e. Since the action is free we know that for each fixed $i > 1$ we have $g_i x \in M \backslash \{x\}$. By choosing V smaller if necessary we can use continuity to obtain that $g_i V \subset M \backslash \{x\}$ for all $i = 2, ..., k$ or, in other words, that $x \notin g_2 V \cup \cdots \cup g_k V$. Let $U = V \backslash (g_2 V \cup \cdots \cup g_k V)$. Notice that U is open and we have arranged that U is also nonempty. We show that $U \cap gU$ is empty unless $g = e$. So suppose $g \neq e$. Now $U \cap gU \subset V \cap gV$ so we know that this is empty for sure in all cases except maybe where $g = g_i$ for $i = 2, ..., k$. If $x \in U \cap g_i U$ then $x \in U$ and so $x \notin g_i V$ by the definition of U. But we must also have $x \in g_i U \subset g_i V-$ a contradiction. We conclude that (i) holds. Now suppose that $x, y \in M$ are not in the same orbit. Again we know that there exist U_x and U_y open with $gU_x \cap U_y$ empty except possibly for some finite set of g which we again denote by $g_1, ..., g_k$. Since the action is free $g_1 x, ..., g_k x$ are distinct. We also know that y is not equal to any of $g_1 x, ..., g_k x$ and so since M is a normal space there exists disjoint open sets $O_1, ..., O_k$, O_y with $g_i x \in O_i$ and $y \in O_y$. By continuity, we may shrink U_x so that $g_i U_x \subset O_i$ for all $i = 1, ..., k$ and then we also replace U_y with $O_y \cap U_y$ (renaming this U_y again). As a result we now see that $gU_x \cap U_y = \emptyset$ for $g = g_1, ..., g_k$ and hence for all g. By shrinking the sets U_x and U_y further we may make them self avoiding.

Now we suppose that (i) and (ii) always hold for some discrete action α. First we show that α is free. Suppose that $y = gx$. Then for every neighborhood U of x the set $gU \cap U$ is empty which by (i) means that $g = e$. Thus the action is free.

Pick $x, y \in M$. If x, y are not in the same orbit then by (ii) we may pick U_x and U_y so that $\{g \in G : gU_x \cap U_y \neq \emptyset\}$ is empty and so certainly a finite set. If x, y are in the same orbit then $y = g_0 x$ for a unique g_0 since we now know the action is free. Choose a neighborhood U of x so that $gU \cap U = \emptyset$ for $g \neq e$. Let $U_x = U$ and $U_y = g_0 U$ then $gU_x \cap U_y = gU \cap g_0 U$. If $gU \cap g_0 U \neq \emptyset$ then $g_0^{-1} gU \cap U \neq \emptyset$ and so $g_0^{-1} g = e$ and $g = g_0$. Thus the only way that $gU_x \cap U_y$ is nonempty is if $g = g_0$ and so the set $\{g \in G : gU_x \cap U_y \neq \emptyset\}$ has cardinality one. In either case we may choose U_x and U_y so that the set is finite which is what we wanted to show. ■

It is easy to see that if $U \subset M$ is self avoiding then any open subset $V \subset U$ is also self avoiding. Thus every point $x \in M$ has a self avoiding neighborhood that is a connected chart domain.

Example 1.27 *Fix a basis* $(\mathbf{f}_1, \mathbf{f}_2)$ *of* $\mathbb{R}^2$. *Let* $\mathbb{Z}^2$ *act on* $\mathbb{R}^2$ *by* $(m, n) \cdot (x, y) := (x, y) + m\mathbf{f}_1 + n\mathbf{f}_2$. *This action is easily seen to be properly discontinuous.*

Proposition 1.5 *Let* $l : G \times M \to M$ *be a smooth action which is proper and free (*M *is an* n *dimensional smooth manifold). Then the quotient space* $G \backslash M$ *has a natural smooth structure such that the quotient map is a smooth covering map.*

Proof. Giving $G \backslash M$ the quotient topology makes $\wp : M \to G \backslash M$ both an open map and continuous. Using (ii) of proposition 1.4 it is easy to show that the quotient topology on $G \backslash M$ is Hausdorff.

By proposition 1.4 we may cover M by charts whose domains are self avoiding and connected. Let $(U, \mathtt{x})$ be one such chart and consider the restriction $\wp|\,U$. If $x, y \in U$ and $\wp(x) = \wp(y)$ then x and y are in the same orbit and so $y = gx$ for some g. Therefore $y \in gU \cap U$ which means that $gU \cap U$ is not empty and so $g = e$ since U is self avoiding. Thus $x = y$ and we conclude that $\wp|\,U$ is injective. Since it is also surjective we see that it is a bijection. Since $\wp|\,U$ is also open it has a continuous inverse and so it is a homeomorphism. Since U is connected, $\wp(U)$ is evenly covered by $\wp$ and now we see that $\wp$ is a covering map. For every such chart $(U, \mathtt{x})$ we have a map

$$\mathtt{x} \circ (\wp|\,U)^{-1} : \wp(U) \to \mathtt{x}(U)$$

which is a chart on $G\backslash M$. Given any other map constructed in this way, say $\mathtt{y} \circ (\wp|\,V)^{-1}$, the domains $\wp(U)$ and $\wp(V)$ only meet if there is a $g \in G$ such that gU meets V and α_g maps an open subset of U diffeomorphically onto a subset of V. In fact, by exercise 1.13 below $(\wp|\,V)^{-1} \circ \wp|\,U$ is a restriction of α_g. Thus for the overlap map we have

$$\begin{aligned}
&\mathtt{y} \circ (\wp|\,V)^{-1} \circ \left(\mathtt{x} \circ (\wp|\,U)^{-1}\right)^{-1} \\
&= \mathtt{y} \circ (\wp|\,V)^{-1} \circ \wp|\,U \circ \mathtt{x}^{-1} \\
&= \mathtt{y} \circ \alpha_g \circ \mathtt{x}^{-1}
\end{aligned}$$

which is smooth. ■

Exercise 1.13 *In the context of the proof above, show that $\wp|_V \circ (\wp|_U)^{-1}$ is defined on an open set $O = (\wp|_U)^{-1}(\wp(V) \cap \wp(V))$ and coincides with a restriction of the map $x \mapsto gx$ for some fixed g and so is a C^r map. Hint: For $x \in O$ we must have $\wp|_V \circ (\wp|_U)^{-1}(x) = gx$ for some g. But what of other points x' also in O? If $\wp|_V \circ (\wp|_U)^{-1}(x') = g'x'$ then is it true that $g = g'$? Think about the unique path lifting property and the fate of the path $t \mapsto g\gamma(t)$ where γ connects x and x'.*

Example 1.28 *We have seen the torus previously presented as $T^2 = S^1 \times S^1$. Another presentation that uses group action is given as follows: Let the group $\mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ act on $\mathbb{R}^2$ by*

$$(m, n) \times (x, y) \mapsto (x + m, y + n).$$

It is easy to check that proposition 1.5 applies to give a manifold $\mathbb{R}^2/\mathbb{Z}^2$. This is actually the torus in another guise and we have the diffeomorphism $\phi : \mathbb{R}^2/\mathbb{Z}^2 \to S^1 \times S^1 = T^2$ given by $[(x, y)] \mapsto (e^{ix}, e^{iy})$. The following diagram commutes:

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r] \arrow[d] & S^1 \times S^1 \\
\mathbb{R}^2/\mathbb{Z}^2 \arrow[ur] &
\end{tikzcd}$$

Exercise 1.14 *Show that if $\wp : M \to G\backslash M$ is the covering arising from a free and proper discrete action of G on M then G is exactly the deck transformations $Deck(\wp)$.*

Covering spaces $\wp : \widetilde{M} \to M$ that arise from a proper and free discrete group action are special in that if M is connected then the covering is a normal covering which means that the group $Deck\,(\wp)$ acts transitively on each fiber $\wp^{-1}(p)$.

Example 1.29 *Recall the abelian group $\mathbb{Z}_2$ of two elements has both a multiplicative presentation and an additive presentation. In this example we take the multiplicative version. Let $\mathbb{Z}_2 := \{1, -1\}$ act on the sphere $S^n \subset \mathbb{R}^{n+1}$ by $(\pm 1) \cdot x := \pm x$. Thus the action is generated by letting -1 send a point on the sphere to its antipode. This action is also easily seen to be free and proper. The quotient space is the real projective space $\mathbb{R}P^n$. (See Example 1.10)*

$$\mathbb{R}P^n = S^n/\mathbb{Z}_2$$

1.6 Grassmann Manifolds

Grassmann manifolds generalize the projective spaces. Let $G_{n,k}$ denote the set of k-dimensional subspaces of $\mathbb{R}^n$. We will exhibit a natural differentiable structure on this set. The idea is the following: An alternative way of defining the points of projective space is as equivalence classes of $n-$tuples $(v^1, ..., v^n) \in \mathbb{R}^n - \{0\}$ where $(v^1, ..., v^n) \sim (\lambda v^1, ..., \lambda v^n)$ for any nonzero λ. This is clearly just a way of specifying a line through the origin. Generalizing, we shall represent a $k-$plane as an $n \times k$ matrix whose column vectors span the $k-$ plane. Thus we are putting an equivalence relation on the set of $n \times k$ matrices where $A \sim Ag$ for any nonsingular $k \times k$ matrix g. Let $\mathbb{M}_{n \times k}$ be the set of $n \times k$ matrices with rank $k < n$ (maximal rank). Two matrices from $\mathbb{M}_{n \times k}$ are equivalent exactly if their columns span the same k-dimensional subspace. Thus the set $G(k, n)$ of equivalence classes is in one to one correspondence with the set of real k dimensional subspaces of $\mathbb{R}^n$.

Now let U be the set of all $[A] \in G(k, n)$ such that A has its first k rows linearly independent. This property is independent of the representative A of the equivalence class $[A]$ and so U is a well defined set. This last fact is easily proven by a Gaussian column reduction argument. Now every element $[A] \in U \subset G(k, n)$ is an equivalence class that has a unique member A_0 of the form

$$\begin{pmatrix} I_{k \times k} \\ Z \end{pmatrix}.$$

Thus we have a map on U defined by $\Psi : [A] \mapsto Z \in \mathbb{M}_{(n-k)\times k} \cong \mathbb{R}^{k(n-k)}$. We wish to cover $G(k, n)$ with sets U_σ similar to U and define similar maps. Let $\sigma_{i_1 \ldots i_k}$ be the shuffle permutation that puts the k columns indexed by $i_1, ..., i_k$ into the positions $1, ..., k$ without changing the relative order of the remaining columns. Now consider the set $U_{i_1 \ldots i_k}$ of all $[A] \in G(k,n)$ such that any representative A has the property that the k rows indexed by $i_1, ..., i_k$ are linearly independent. This characterization of $[A]$ is independent of the representative A. The the permutation induces an obvious 1-1 onto map $\widetilde{\sigma_{i_1 \ldots i_k}}$ from $U_{i_1 \ldots i_k}$ onto $U = U_{1 \ldots k}$. We now have maps $\Psi_{i_1 \ldots i_k} : U_{i_1 \ldots i_k} \to \mathbb{M}_{(n-k)\times k} \cong \mathbb{R}^{k(n-k)}$ given by composition $\Psi_{i_1 \ldots i_k} := \Psi \circ \widetilde{\sigma_{i_1 \ldots i_k}}$. These maps form an atlas $\{\Psi_{i_1 \ldots i_k}, U_{i_1 \ldots i_k}\}$ for $G(k,n)$ and gives it the structure of a smooth manifold called the **Grassmann manifold** of real k-planes in $\mathbb{R}^n$. The topology induced by the charts is the same as the quotient topology and one can check that this topology is Hausdorff and paracompact.

1.7 Regular Submanifolds

A subset S of a smooth manifold M of dimension $n = l + k$ is called a **regular submanifold** of dimension l if every point $p \in S$ is in the domain of a chart $(U, \mathtt{x})$ that has the following **submanifold property**:

$$\mathtt{x}(U \cap S) = \mathtt{x}(U) \cap (\mathbb{R}^l \times \{0\})$$

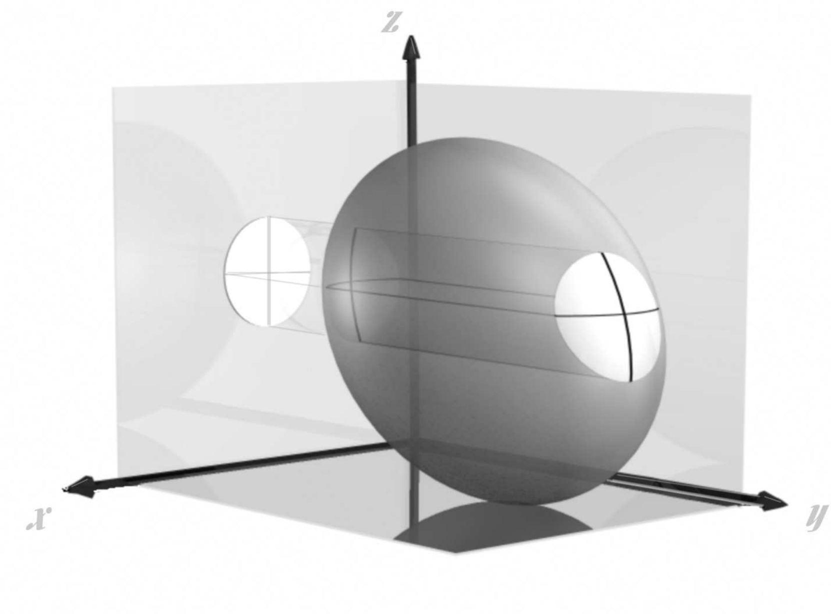
We will refer to such charts as **regular submanifold charts** and as being **adapted** (to S). The restrictions $\mathtt{x}|_{U \cap S}$ of regular submanifold charts provide an atlas for S (called an induced submanifold atlas) making it a smooth manifold in its own right. Indeed, one checks that the overlap maps for adapted charts are smooth.

Exercise 1.15 *Prove this last statement.*

We will see more general types of submanifolds in the sequel. An important aspect of regular submanifolds is that the induced topology is the same as the relative topology. The integer k is called the **codimension** of S (in M) and we say that S is a regular submanifold of codimension k.

Exercise 1.16 *Show that S is a smooth manifold and that a continuous map $f : N \to M$ that has its image contained in a regular submanifold S is differentiable with respect to the submanifold atlas, if and only if it is differentiable as a map into M.*

When S is a regular submanifold of M then the tangent space T_pS at $p \in S \subset M$ is intuitively a subspace of T_pM. In fact, this is true as long as one is not bent on distinguishing a curve in S through p from the "same" curve thought of as a map into M. If one wants to be pedantic then we have the inclusion map $\iota : S \hookrightarrow M$ and if $c : I \to S$ is a curve into S then $\iota \circ c : I \to M$ is a map into M as such. At the tangent level this means that $c'(0) \in T_pS$ while $(\iota \circ c)'(0) \in T_pM$. Thus from this more pedantic point of view we have to explicitly declare $T_p\iota : T_pS \to T_p\iota(T_pS) \subset T_pM$ to be an identifying map. We will avoid the use of inclusion maps when possible and simply write $T_pS \subset T_pM$ and trust the intuitive notion that T_pS is indeed a subspace of T_pM.



Exercise 1.17 *Convince yourself that* $Sym_{n\times n}(\mathbb{R})$ *is a regular submanifold of* $\mathbb{M}_{n\times n}(\mathbb{R})$. *Under the canonical identification of* $T_S\mathbb{M}_{n\times n}(\mathbb{R})$ *with* $\mathbb{M}_{n\times n}(\mathbb{R})$ *the tangent space of* $Sym_{n\times n}(\mathbb{R})$ *at the symmetric matrix* S *becomes what subspace of* $\mathbb{M}_{n\times n}(\mathbb{R})$*?*

In example 1.14 we saw how certain subsets of $\mathbb{R}^n$ can be given a smooth structure where the charts are restrictions of projections onto coordinate planes. One can show that such subsets of $\mathbb{R}^n$ which are locally graphs over coordinate planes are regular submanifolds in accordance with definition given above. In the following exercise we ask the reader to show the converse.

Exercise 1.18 *Show that if* M *is an* l*-dimensional regular submanifold of* $\mathbb{R}^n$ *then for every* $p \in M$ *there exists at least one* l*-dimensional coordinate plane* P *such that linear projection* $\mathbb{R}^n \to P \cong \mathbb{R}^l$ *restricts to a coordinate system for* M *defined on some neighborhood of* p.

1.8 Manifolds with boundary.

For the general Stokes theorem where the notion of flux has its natural setting we will need to have the concept of a smooth manifold with boundary . We have already introduced the notion of a topological manifold with boundary but now we want to see how to handle the issue of the smooth structures. A basic example to keep in mind is the closed hemisphere S^2_+ which is the set of all $(x, y, z) \in S^2$ with $z \geq 0$. Recall that we defined the n dimensional (left) Euclidean half-space to be $\mathbb{H}^n := \mathbb{R}^n_{x^1 \leq 0} := \{(x^1, ..., x^n) \in \mathbb{R}^n : x^1 \leq 0\}$. n-dimensional differentiable manifolds are modeled on $\mathbb{H}^n$.

Remark 1.6 *We have chosen the space $\mathbb{R}^n_{x^1 \leq 0}$ rather than $\mathbb{R}^n_{x^1 \geq 0}$ on purpose. $\frac{\partial}{\partial x^1}$ is outward pointing for $\mathbb{R}^n_{x^1 \leq 0}$ but not for $\mathbb{R}^n_{x^1 \geq 0}$. This turns out to be convenient with regard to defining the induced orientation on the boundary of a manifold with boundary. $\left(\frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}\right)$ will be positively oriented on $0 \times \mathbb{R}^{n-1}$ whenever $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}\right)$ is positively oriented on $\mathbb{R}^n$.*

Give $\mathbb{H}^n$ the relative topology as a subset of $\mathbb{R}^n$. Since $\mathbb{H}^n \subset \mathbb{R}^n$ we already have a notion of differentiability on open subsets of $\mathbb{H}^n$ via definition 1.3. For convenience let us introduce for an open set $U \subset \mathbb{H}^n$ (always relatively open) the following notations: Let ∂U denote $\partial \mathbb{H}^n \cap U$ and $\text{int}(U)$ denote $U \setminus \partial U$.

We have the following three facts:

1. First, it is an easy exercise to show that if $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ is C^r differentiable (with $r \geq 1$) and g is another such map with the same domain, then if $f = g$ on $\mathbb{H}^n \cap U$ then $Df(x) = Dg(x)$ for all $x \in \mathbb{H}^n \cap U$.

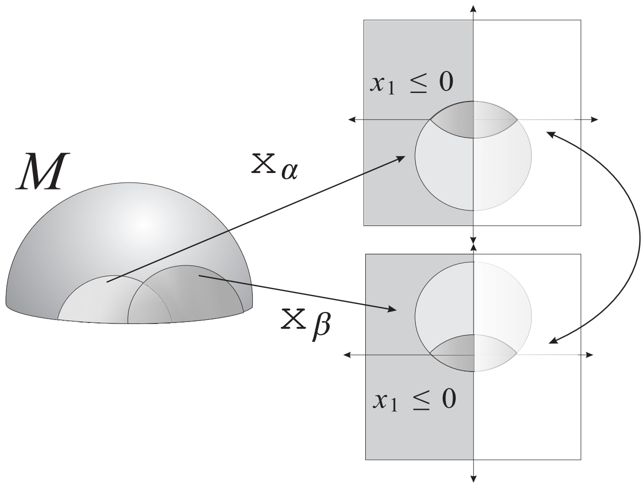
2. If $f : U \subset \mathbb{R}^n \to \mathbb{H}^n$ is C^r differentiable (with $r \geq 1$) and $f(x) \in \partial \mathbb{H}^n$ for all $x \in U$ then $Df\ (x) : \mathbb{R}^n \to \mathbb{R}^n$ must have its image in $\partial \mathbb{H}^n$.

3. Let $f : U_1 \subset \mathbb{H}^n \to U_2 \subset \mathbb{H}^n$ be a diffeomorphism (in our new extended sense). Assume that $\mathbb{H}^n \cap U_1$ and $\mathbb{H}^n \cap U_2$ are not empty. Then f induces diffeomorphisms $\partial U_1 \to \partial U_2$ and $\text{int}(U_1) \to \text{int}(U_2)$.

These three claims are not exactly obvious but there are very intuitive. On the other hand, none of them are difficult to prove and we will leave these as problems.

We can now form a definition of smooth manifold with boundary in a fashion completely analogous to the definition of a smooth manifold without boundary. A **half space chart** $\mathtt{x}_\alpha$ for a set M is a bijection of some subset U_α of M onto an open subset of $\mathbb{H}^n$. A C^r **half space atlas** is a collection $(U_\alpha, \mathtt{x}_\alpha)$ of such half space charts such that for any two, say $(U_\alpha, \mathtt{x}_\alpha)$ and $(U_\beta, \mathtt{x}_\beta)$, the map $\mathtt{x}_\alpha \circ \mathtt{x}_\beta^{-1}$ is a C^r diffeomorphism on its natural domain (if non-empty). Note: "Diffeomorphism" means diffeomorphism in the extended sense of being a homeomorphism and such that both $\mathtt{x}_\alpha \circ \mathtt{x}_\beta^{-1}$ and its inverse are C^r in the sense of Definition 1.3.

Definition 1.30 *A C^r-**manifold with boundary** $(M, \mathcal{A})$ is a pair consisting of a set M together with a maximal atlas of half space charts $\mathcal{A}$. The manifold topology is that generated by the domains of all such charts. The boundary of M is denoted by ∂M and is the set of points that have their image in the boundary $\partial \mathbb{H}^n$ of $\mathbb{H}^n$ under some and hence every chart.*



The three facts listed above show that the notion of a boundary is well defined concept and is a natural notion in the context of smooth manifolds; it is a "differentiable invariant".

Colloquially, one usually just refers to M as a manifold with boundary and forgoes the explicit reference to the atlas. The interior of a manifold with boundary is $M\backslash\partial M$. It is a manifold without boundary and is denoted $\overset{\circ}{M}$.

Exercise 1.19 *Show that $M \cup \partial M$ is closed and that $\overset{\circ}{M} = M\backslash\partial M$ is open.*

In the present context, a manifold without boundary that is compact (and hence closed in the usual topological sense if M is Hausdorff) is often referred to as a **closed manifold**. If no component of a manifold without boundary is compact, it is called an **open manifold**. For example, the "interior" $\overset{\circ}{M}$ of a connected manifold M with nonempty boundary is never compact and is an open manifold in the above sense. So $\overset{\circ}{M}$ will be an open manifold if every component of M contains part of the boundary.

Remark 1.7 *The phrase "closed manifold" is a bit problematic since the word closed is acting as an adjective and so conflicts with the notion of closed in the ordinary topological sense. For this reason we will try to avoid this terminology and use instead the phrase "**compact manifold without boundary**".*

Remark 1.8 (Warning) *Some authors let M denote the interior, so that $M \cup \partial M$ is the closure and is the smooth manifold with boundary in our sense.*

Theorem 1.11 *If M is a C^r manifold with boundary then M is a C^r manifold (without boundary) with an atlas being given by all maps of the form $\mathtt{x}_\alpha| , U_\alpha \cap \partial M$. The manifold ∂M is called the **boundary of** M.*

Idea of Proof. The truth of this theorem becomes obvious once we recall what it means for a chart overlap map $\mathtt{y}\circ\mathtt{x}^{-1} : U \to V$ to be a diffeomorphism in a neighborhood a point $x \in U\cap\mathbb{H}^n$. First of all there must be a set U' containing U that is open in $\mathbb{R}^n$ and an extension of $\mathtt{y}\circ\mathtt{x}^{-1}$ to a differentiable map on U'. But the same is true for $(\mathtt{y}\circ\mathtt{x}^{-1})^{-1} = \mathtt{x}\circ\mathtt{y}^{-1}$. The extensions are inverses of each other on U and V. But we must also have that the derivatives of the chart overlap maps are isomorphisms at all points up to and including ∂U and ∂V. But then the inverse function theorem says that there are neighborhoods of points in ∂U in $\mathbb{R}^n$ and ∂V in $\mathbb{R}^n$ such that these extensions are actually diffeomorphisms and inverses of each other. Now it follows that the restrictions $\mathtt{y}\circ\mathtt{x}^{-1}\big|_{\partial U} : \partial U \to \partial V$ are diffeomorphisms. In fact, this is the main point of the comment (3) above and we have now seen the idea of its proof also. ∎

Example 1.30 *The closed ball $\overline{B}(p,R)$ in $\mathbb{R}^n$ is a smooth manifold with boundary $\partial\overline{B}(p,R) = S^{n-1}$.*

Example 1.31 *The hemisphere $S^n_+ = \{x \in \mathbb{R}^{n+1} : x^{n+1} \geq 0\}$ is a smooth manifold with boundary.*

Exercise 1.20 *Is the Cartesian product of two smooth manifolds with boundary necessarily a smooth manifold with boundary?*

1.9 Local expressions

Many authors seem to be over zealous and overly pedantic when it comes to the notation used for calculations in a coordinate chart. We will often make some simplifying conventions that are exactly what every student at the level of advanced calculus is already using anyway. Consider an arbitrary pair of charts $\mathtt{x}$ and $\mathtt{y}$ and the overlap maps $\mathtt{y}\circ\mathtt{x}^{-1} : \mathtt{x}(U\cap V) \to \mathtt{y}(U\cap V)$. We write

$$\mathtt{y}(p) = \mathtt{y}\circ\mathtt{x}^{-1}(\mathtt{x}(p))$$

for $p \in U\cap V$. For finite dimensional manifolds we see this written as

$$y^i(p) = y^i(x^1(p), ..., x^n(p)) \tag{1.1}$$

which makes sense but in the literature we also see

$$y^i = y^i(x^1, ..., x^n). \tag{1.2}$$

In this last expression one might wonder if the x^i are functions or numbers. But this ambiguity is sort of purposeful for if 1.1 is true for all $p \in U\cap V$ then 1.2 is true for all $(x^1, ..., x^n) \in \mathtt{x}(U\cap V)$ and so we are unlikely to be led into error.

If $f : M \to N$ is a map of smooth manifolds then for every pair of charts $(U,\mathtt{x}) \in \mathcal{A}_M$ and $(V,\mathtt{y}) \in \mathcal{A}_N$ such that $f(U)\cap V \neq \emptyset$ we have a map $\bar{f} = \mathtt{y}\circ f\circ\mathtt{x}^{-1}$ defined on an open subset of $\mathbb{R}^n$ where $n = \dim(M)$. Now $\bar{f}$ is called the local representative of with respect to the chosen charts. If $\dim(N) = k$ then

$\bar{f} = (\bar{f}^1, ..., \bar{f}^k)$ and each $\bar{f}^i$ is a function of n variables. If we denote generic points in $\mathbb{R}^n$ as $(u^1, ..., u^n)$ and those in $\mathbb{R}^k$ as $(v^1, ..., v^k)$ then we may write $v^i = \bar{f}^i(u^1, ..., u^n)$, $1 \leq i \leq k$. However, in the spirit of the last paragraph, it is also common and sometimes psychologically helpful to write $y^i = \bar{f}^i(x^1, ..., x^n)$. The bars over the f's are also sometimes dropped. Another common way to indicate $\mathtt{y} \circ f \circ \mathtt{x}^{-1}$ is with the notation f_{VU} which is very suggestive and tempting but it has a slight logical defect since there may be many charts with domain U and many charts with domain V. How would one deal with the situation where $U = V$ but $\mathtt{x} \neq \mathtt{y}$?

1.10 Applications

At this point in our explorations we haven't covered enough ground to do justice to any real applications so we just make a few general remarks. The notion of differentiable manifold which we have introduced serves several purposes in physics. The set of all configurations of a classical mechanical system or system of particles is usually a manifold. For a single particle the manifold would represent the set of all possible positions that the particle could take. A set of constraints may fix the particle to a submanifold of Euclidean space for example. If we include the set of all possible momenta then we have the phase space which is most naturally taken to be an associated manifold called the cotangent bundle T^*M which we introduce in the next chapter along with the tangent bundle TM. Given initial conditions, the evolution of a particle or more general system will be a smooth path in T^*M. In relativity theory a manifold will represent the set of all possible idealized events in spacetime.

Much more machinery must be developed in order to appreciate how physics is done with manifolds. Later we will study fields and they will be special kinds of maps between smooth manifolds. Such fields provide a mathematical description of physical concepts such as the (classical) electromagnetic fields, velocity fields of fluid mechanics, first quantized matter fields and gauge fields from modern particle physics and more. Engineering concepts are also sometimes amplified and clarified by the use of the language of manifolds and smooth maps. For example, there is much activity in what is called geometric control theory wherein the tools of manifold theory are used with full force.

Symmetry is an important topic for physics and engineering. Many types of problems cannot be solved in detail unless sufficient symmetry is present and properly understood. The groups of matrices that describe many symmetries are usually manifolds themselves.

It should also be mentioned that many of the basic laws of physics are formulated as ordinary or partial differential equations. As more machinery is introduced we will also see that highly geometric versions of these ODE's and PDE's will appear naturally in the manifold setting. In fact, the vector fields and flows that we study will correspond to the ODE's while the heat, wave, and Laplace type equations also have natural geometric generalizations. On the other hand, the pure geometry itself gives rise to its own differential equations that describe integrability conditions for geometric fields and encode various facts about curvature, parallel transport etc.

1.11 Problems

1. Prove Lemma 1.2.

2. Check that each of the manifolds given as examples are indeed paracompact and Hausdorff.

3. Let M and N be smooth manifolds, and $f : M \to N$ a C^∞ map. Suppose that M is compact, N is connected. Suppose further that f is injective and that $T_x f$ is an isomorphism for each $x \in M$. Show that f is a diffeomorphism.

4. Let M_1 , M_2 and M_3 be smooth manifolds.

   (a) Show that $(M_1 \times M_2) \times M_3$ is diffeomorphic to $M_1 \times (M_2 \times M_3)$ in a natural way.

   (b) Show that $f : M \to M_1 \times M_2$ is C^∞ if and only if the composite maps $pr_1 \circ f : M \to M_1$ and $pr_2 \circ f : M \to M_2$ are both C^∞.

5. Show that a C^r a manifold M is connected as a topological space if and only it is C^r-path connected in the sense that for any two points $p_1, p_2 \in M$ there is a C^r map $c : [0, 1] \to M$ such that $c(0) = p_1$ and $c(1) = p_2$.

6. An affine map between two vector spaces is a map of the form $x \mapsto Lx + b$ for a fixed vector b. An affine space has the structure of a vector space except that only the difference of vectors rather than the sum is defined. For example,let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let $A_c = \{x \in \mathbb{R}^n : Lx = c\}$. Now A_c is not a vector subspace of $\mathbb{R}^n$; it is an affine space with "difference space" $\ker L$. This means that for any $p_1, p_2 \in A_c$ the difference $p_2 - p_1$ is an element of $\ker L$. Furthermore, for any fixed $p_0 \in A_c$ the map $x \mapsto p_0 + x$ is a bijection $\ker L \cong A_c$. Show that the set of all such bijections forms an atlas for A_c such that for any two charts from this atlas the overlap map is an affine isomorphism from $\ker L$ to itself. Develop a more abstract definition of topological affine space with a Banach space as difference space. Show show that this affine structure is enough to make sense out of the derivative via the difference quotient.

7. A k-frame in $\mathbb{R}^n$ is a linearly independent ordered set of vectors $(v_1, ..., v_k)$. Show that the set of all $k-$frames in $\mathbb{R}^n$ can be given the structure of a smooth manifold. This kind of manifold is called a **Stiefel manifold**.

8. Embed the Stiefel manifold of k-frames in $\mathbb{R}^n$ into a Euclidean space $\mathbb{R}^N$ for some large N.

9. Show in detail that the subsets of $\mathbb{R}^3$ described as compact surfaces of genus g in example 1.15 are indeed regular submanifolds of $\mathbb{R}^3$.

10. If $M \times N$ is a product manifold we have the two projection maps $pr_1 : M \times N \to M$ and $pr_2 : M \times N \to N$ defined by $(x, y) \longmapsto x$ and $(x, y) \longmapsto y$ respectively. Show that if we have smooth maps $f_1 : P \to M$ and $f_2 : P \to N$ then the map $(f, g) : P \to M \times N$ given by $(f, g)(p) = (f(p), g(p))$ is the unique smooth map such that $pr_1 \circ (f, g) = f$ and $pr_2 \circ (f, g) = g$.

11. Let M be a topological manifold. Give necessary conditions such that the topology induced by an atlas is the same as the original topology.

12. Show that the atlas natural projection charts of a regular submanifold induce the relative topology inherited from the ambient manifold.

13. The purpose of this problem is to make it clear that even though a set may carry an atlas, it is not necessarily true that the induced topology as described in problem **??** is Hausdorff. Let S be the subset of $\mathbb{R}^2$ given by the union $(\mathbb{R} \times 0) \cup \{\ (0, 1)\}$. Let U be $\mathbb{R} \times 0$ and let V be the set obtained from U by replacing the point $(0, 0)$ by $(0, 1)$. Define a chart map $\mathtt{x}$ on U by $\mathtt{x}(x, 0) = x$ and a chart $\mathtt{y}$ on V by

$$\mathtt{y}(x, 0) = \begin{cases} x \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Show that these two charts provide a C^∞ atlas on S but that the induced topology is not Hausdorff.

14. Because we have not required a manifold to be second countable but only Hausdorff and paracompact, we have the possibility of having an uncountable number of connected components. Consider the set $\mathbb{R}^2$ without its usual topology. For each $a \in \mathbb{R}$ define a bijection $\phi_a : \mathbb{R} \times \{a\} \to \mathbb{R}$ by $\phi_a(x, a) = x$. Show that the family of sets of the form $U \times \{a\}$ for U open in $\mathbb{R}$ and $a \in \mathbb{R}$ provide a base for a paracompact topology on $\mathbb{R}^2$. Show that the maps ϕ_a are charts and together provide an atlas for $\mathbb{R}^2$ with this unusual topology. Show that the resulting smooth manifold has an uncountable number of connected components.

15. Show that every connected manifold has a countable atlas consisting of charts whose domains have compact closure and are simply connected. Hint: We are assuming that our manifolds are paracompact so each connected component is second countable.

16. Show that each every second countable manifold has a countable fundamental group (see [Lee, **?**] page 10 if you get stuck).

17. If $\mathbb{C} \times \mathbb{C}$ is identified with $\mathbb{R}^4$ in the obvious way then S^3 is exactly the subset of $\mathbb{C} \times \mathbb{C}$ given by $\{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\}$. Let p, q be coprime integers and $p > 0$. Now let ω be a primitive $n-$th root of unity so that $\mathbb{Z}_p = \{1, \omega, ..., \omega^{p-1}\}$. For $(z_1, z_2) \in S^3$ let $\omega \cdot (z_1, z_2) := (\omega z_1, \omega^q z_2)$ and extend this to an action of $\mathbb{Z}_p$ on S^3 so that $\omega^k \cdot (z_1, z_2) = (\omega^k z_1, \omega^{qk} z_2)$. Show that this action is free and proper. The quotient space $\mathbb{Z}_p \backslash S^3$ is called a lens space and is denoted by $L(p; q)$.

18. Let S^1 be realized as the set of complex numbers of modulus one. Define a map $\theta : S^1 \times S^1 \to S^1 \times S^1$ by $\theta(z, w) = (-z, \overline{w})$ and note that $\theta \circ \theta = id$. Let G be the group $\{id, \theta\}$. Show that $M := \left(S^1 \times S^1\right)/G$ is a smooth 2-manifold. Is M orientable?

19. Show that if S is a regular submanifold of M then we may cover S by special adapted charts from the atlas of M which are of the form $\mathtt{x} : U \to V_1 \times V_2 \subset \mathbb{R}^l \times \mathbb{R}^k = \mathbb{R}^n$ with

$$\mathtt{x}(U \cap S) = V_1 \times \{0\}$$

for some open sets $V_1 \subset \mathbb{R}^l, V_2 \subset \mathbb{R}^k$.

20. Prove the three properties about $\mathbb{H}^n$ and $\partial\mathbb{H}^n$ listed in the section on manifolds with boundary.

Chapter 2

The Tangent Structure

2.1 The Tangent Space

If $c : (-\epsilon, \epsilon) \to \mathbb{R}^N$ is a smooth curve then it is common to visualize the "velocity vector" $\dot{c}(0)$ as being based at the point $p = c(0)$. It is often desirable to explicitly form a separate $N-$dimensional vector space whose elements are to be thought of as being based at a point $p \in \mathbb{R}^N$. One way to do this is to use $\{p\} \times \mathbb{R}^N$ so that a tangent vector based at p is taken to be a pair (p, v) where $v \in \mathbb{R}^N$. The set $\{p\} \times \mathbb{R}^N$ inherits a vector space structure from $\mathbb{R}^N$ in the obvious way and is one version of what is called the tangent space to $\mathbb{R}^N$ at p. In this context, we denote $\{p\} \times \mathbb{R}^N$ by $T_p\mathbb{R}^N$. If we write $c(t) = (x^1(t), ..., x^N(t))$, then the velocity vector of a curve c at $t = 0$ and based at $p = c(0)$ is $(p, \frac{dx^1}{dt}(0), ..., \frac{dx^N}{dt}(0))$. Ambiguously, both $(p, \frac{dx^1}{dt}(0), ..., \frac{dx^N}{dt}(0))$ and $(\frac{dx^1}{dt}(0), ..., \frac{dx^N}{dt}(0))$ are often denoted by $\dot{c}(0)$. A bit more generally, if V is a finite dimensional vector space then V is a manifold and the tangent space at $p \in \mathrm{V}$ can be taken to be the set $\{p\} \times \mathrm{V}$.

Definition 2.1 *If $v_p := (p, v)$ is a tangent vector at p then v is called the* ***principal part*** *of v_p.*

The existence of the obvious and natural isomorphism between $\mathbb{R}^N$ and $T_p\mathbb{R}^N = \{p\} \times \mathbb{R}^N$ for any p is the reason that in the context of calculus on $\mathbb{R}^N$ this explicit construction of vectors based at a point is often deemed unnecessary. All the tangent spaces $T_p\mathbb{R}^N$ are canonically isomorphic to $\mathbb{R}^N$ and hence to each other. However, from the point of view of manifold theory, the tangent space at a point is a fundamental construction. We will define the notion of a tangent space at a point of a differentiable manifold and it will be seen that there is in general no way to canonically identify tangent spaces at different points.

Actually, we shall give several (ultimately equivalent) definitions of tangent space. Let us start with the special case of a submanifold of $\mathbb{R}^N$. A tangent vector at p can be thought of as the velocity of a curve, as a direction for a directional derivative and also as geometric object which has components which depend on what coordinates are being used. Let us explore these aspects in the case of a submanifold. If M is an n-dimensional regular submanifold of $\mathbb{R}^N$ then a smooth curve $c : (-\epsilon, \epsilon) \to M$ is also a smooth curve into $\mathbb{R}^N$ and $\dot{c}(0)$ is normally thought of as based at the point $p = c(0)$ and is tangent to M according to any reasonable definition of what it means to be tangent. The set of all vectors obtained in this way from curves into M with $c(0) = p$ is an n dimensional subspace of the tangent space of $\mathbb{R}^N$ at p (described above) and in this special case this space could play the role of the tangent space of M at p. Let us tentatively accept this definition of the tangent space at p and denote it by T_pM. Let $v_p := (p, v) \in T_pM$. There are three things we should notice about v_p. The first thing to notice is that there are many different curves $c : (-\epsilon, \epsilon) \to M$ with $c(0) = p$ which all give the same tangent vector v_p and there is an obvious equivalence relation among these curves: two curves passing through p at time $t = 0$ are equivalent if they have the same tangent vector. Already one can see that perhaps this could be turned around so that we can think of a tangent vector as equivalence class of curves. Curves would be equivalent if they agree infinitesimally in some appropriate sense.

The second thing that we wish to bring out is that a tangent vector can be used to construct a directional derivative operator. Thus from $v_p = (p, v)$ we get a linear map

$$C^\infty(M) \to \mathbb{R}$$

given by

$$f \mapsto \left.\frac{d}{dt}\right|_0 f \circ c$$

where $c : I \to M$ is any curve whose velocity at time $t = 0$ is v_p. This is the idea which we will exploit later when we use the abstract properties of such a directional derivative to actually *define* the notion of a tangent vector.

The final aspect we wish to bring out is how v_p relates to charts for the submanifold. If $(\mathtt{y}, U)$ is a chart on M with $p \in M$ then by inverting we obtain a map $\mathtt{y}^{-1} : V \to M$ which we may then think of as a map into the ambient space $\mathbb{R}^N$. The map $\mathtt{y}^{-1}$ parameterizes a portion of M. For convenience let us suppose that $\mathtt{y}^{-1}(0) = p$. Then we have the "coordinate curves" $y^i \mapsto \mathtt{y}^{-1}(0, ..., y^i, ..., 0)$ for $i = 1, ..., n$. The resulting tangent vectors at p have principal parts given by the partial derivatives so that

$$E_i := (p, \frac{\partial \mathtt{y}^{-1}}{\partial y^i}(0)).$$

It can be shown that $(E_1, ..., E_n)$ is a basis for T_pM. For another coordinate system $\bar{\mathtt{y}}$ with $\bar{\mathtt{y}}^{-1}(0) = p$ we similarly define a basis $(\bar{E}_1, ..., \bar{E}_n)$. Now if $v_p = \sum_{i=1}^n a^i E_i = \sum_{i=1}^n \bar{a}^i \bar{E}_i$ then letting $a = (a^1, ..., a^n)$ and $\bar{a} = (\bar{a}^1, ..., \bar{a}^n)$, the chain rule can be used to show that

$$\bar{a} = D(\bar{\mathtt{y}} \circ \mathtt{y}^{-1})\big|_{\mathtt{y}(p)}\, a$$

which is classically written as

$$\bar{a}^i = \frac{\partial \bar{y}^i}{\partial y^j} a^j \quad \text{(summation convention)}.$$

Both $(a^1, ..., a^n)$ and $(\bar{a}^1, ..., \bar{a}^n)$ represent the tangent vector v_p but with respect to different charts. This is a simple example of a transformation law.

The various definitions for the notion of a tangent vector given below in the general setting, will be based in turn on the following three ideas:

1. Equivalence classes of curves through a point. Curves are equivalent if they are "tangent" at the point.

2. The use of charts and the idea of the components a tangent vector with respect to the charts. The transformation law for the components of a tangent vector with respect to various charts, plays a central role here.

3. The use of the idea of a "derivation", a kind of abstract directional derivative.

Of course we will also have to show how to relate these various definitions to see how they are really equivalent. We start with the idea from (1) above to get a definition that will be our main and default definition.

2.1.1 Tangent space via curves

Let p be a point in a smooth manifold M of dimension n. Suppose that we have smooth curves c_1 and c_2 mapping into M, each with open interval domains containing $0 \in \mathbb{R}$ and with $c_1(0) = c_2(0) = p$. We say that c_1 is tangent to c_2 at p if for all smooth functions $f : M \to \mathbb{R}$ we have $\frac{d}{dt}\big|_{t=0} f \circ c_1 = \frac{d}{dt}\big|_{t=0} f \circ c_2$. This is an equivalence relation on the set of all such curves. Define a **tangent vector at** p to be an equivalence class $X_p = [c]$ under this relation. In this case we will also write $\dot{c}(0) = X_p$. The **tangent space** T_pM is defined to be the set of all tangent vectors at $p \in M$.

The definition of tangent space just given is very geometric but it has one disadvantage. Namely, it is not immediately obvious that T_pM is a vector space in a natural way. The following principle is used to obtain a vector space structure:

Proposition 2.1 (Consistent transfer of linear structure) *Suppose that S is a set and $\{V_\alpha\}_{\alpha \in A}$ is a family of n dimensional vector spaces. Suppose that for each α we have a bijection $b_\alpha : V_\alpha \to S$. If for every $\alpha, \beta \in A$ the map $b_\beta^{-1} \circ b_\alpha : V_\alpha \to V_\beta$ is a linear isomorphism then there is a unique vector space structure on the set S such that each b_α is a linear isomorphism.*

The proof of this proposition is straightforward.

Exercise 2.1 *Use the above proposition to show that there is a natural vector space structure on* T_pM *as defined above. Hint: For every chart* $(\mathtt{x}_\alpha, U)$ *with* $p \in U$ *we have a map* $b_\alpha : \mathbb{R}^n \to T_pM$ *given by* $v \mapsto [\gamma_v]$ *where* $\gamma_v : t \mapsto \mathtt{x}_\alpha^{-1}(\mathtt{x}_\alpha(p) + tv)$. *Show that* b_α *is a bijection and let* $V_\alpha = \mathbb{R}^n$ *for all* α.

We will give two more definitions of tangent space and although we will eventually come to see all these as versions of the same object let us temporarily called the tangent space defined above the **kinematic tangent space** and denote it also by $(T_pM)_{kin}$. To summarize, if $\mathcal{C}_p$ is the set of smooth curves defined on some interval containing 0, then

$$(T_pM)_{kin} = \mathcal{C}_p / \sim$$

where the equivalence is a described above.

Exercise 2.2 *Let* c_1 *and* c_2 *be smooth curves mapping into a smooth manifold* M, *each with open domains containing* $0 \in \mathbb{R}$ *and with* $c_1(0) = c_2(0) = p$. *Show that*

$$\left.\frac{d}{dt}\right|_{t=0} f \circ c_1 = \left.\frac{d}{dt}\right|_{t=0} f \circ c_2$$

for all smooth f *if and only if the curves* $\mathtt{x} \circ c_1$ *and* $\mathtt{x} \circ c_2$ *have the same tangent vector in* $\mathbb{R}^n$ *for any chart* $(U, \mathtt{x})$.

2.1.2 Tangents space via charts

Let $\mathcal{A}$ be the maximal atlas for a smooth manifold M of dimension n. For fixed $p \in M$, consider the set Γ_p of all triples $(p, v, (U, \mathtt{x})) \in \{p\} \times \mathbb{R}^n \times \mathcal{A}$ such that $p \in U$. Define an equivalence relation on Γ_p be requiring that $(p, v, (U, \mathtt{x})) \sim (p, w, (U, \mathtt{y}))$ if and only if

$$D(\mathtt{y} \circ \mathtt{x}^{-1})\big|_{\mathtt{x}(p)} \cdot v = w.$$

In other words, the derivative at $\mathtt{x}(p)$ of the coordinate change $\mathtt{y} \circ \mathtt{x}^{-1}$ "identifies" v with w. The set $\Gamma_p / \sim$ of equivalence classes can be given a vector space structure as follows: For each chart $(U, \mathtt{x})$ containing p we have a map $b_{(U,\mathtt{x})} : \mathbb{R}^n \to \Gamma_p / \sim$ given by $v \mapsto [p, v, (U, \mathtt{x})]$ where $[p, v, (U, \mathtt{x})]$ denotes the equivalence class of $(p, v, (U, \mathtt{x}))$. To see that this map is a bijection we just notice that if $[p, v, (U, \mathtt{x})] = [p, w, (U, \mathtt{x})]$ then $v = D(\mathtt{x} \circ \mathtt{x}^{-1})\big|_{\mathtt{x}(p)} \cdot v = w$ by definition. By proposition 2.1 we obtain a vector space structure on $\Gamma_p / \sim$. This is another version of the tangent space at p and we shall (temporarily) denote this by $(T_pM)_{phys}$. The subscript "phys" refers to the fact that this version of the tangent space is based on a "transformation law" and corresponds to a way of looking at things that has traditionally been popular among physicists. If $v_p = [p, v, (U, \mathtt{x})] \in (T_pM)_{phys}$ then we say that $v \in \mathbb{R}^n$ represents v_p with respect to the chart $(U, \mathtt{x})$.

This viewpoint takes on a more familiar appearance if we use a more classical notation; Let $(U, \mathtt{x})$ and $(V, \mathtt{y})$ be two charts containing p in their domains. If an $n-$tuple $(v^1, ..., v^n)$ represents a tangent vector at p from the point of view of $(U, \mathtt{x})$ and if the $n-$tuple $(w^1, ..., w^n)$ represents the same vector from the point of view of $(V, \mathtt{y})$ then

$$w^i = \left.\frac{\partial y^i}{\partial x^j}\right|_{\mathtt{x}(p)} v^j \quad \text{(sum over } j\text{)}$$

where we write the change of coordinates as $y^i = y^i(x^1, ..., x^n)$ with $1 \leq i \leq n$.

Notation 2.1 *It is sometimes convenient to index the maximal atlas: $\mathcal{A} = \{(U_\alpha, \mathtt{x}_\alpha)\}_{\alpha \in A}$. Then we would consider triples of the form (p, v, α) and let the equivalence relation be $(p, v, \alpha) \sim (q, w, \beta)$ if and only if $p = q$ and $D(\mathtt{x}_\beta \circ \mathtt{x}_\alpha^{-1})\big|_{\mathtt{x}_\alpha(p)} \cdot v = w$.*

2.1.3 Tangent space via derivations

We abstract the notion of directional derivative for our next approach to the tangent space. There are actually at least two common versions of this and we explain both. Let M be a smooth manifold of dimension n. A **tangent vector** X_p at p is a linear map $X_p : C^\infty(M) \to \mathbb{R}$ with the property that for $f, g \in C^\infty(M)$

$$X_p(fg) = g(p)X_pf + f(p)X_pg.$$

This is called the Liebniz law. We may say that a tangent vector at p is a derivation of the algebra $C^\infty(M)$ with respect to the evaluation map ev_p at p defined by $ev_p(f) := f(p)$. Alternatively, we say that X_p is a derivation at p. The set of such derivations at p is easily seen to be a vector space which is called the **tangent space** at p and is denoted T_pM. We temporarily distinguish this version of the tangent space from $(T_pM)_{\text{kin}}$ and $(T_pM)_{\text{phys}}$ defined previously by denoting it by $(T_pM)_{\text{alg}}$ and referring to it as the algebraic tangent space. We could also consider the vector space of derivations of $C^r(M)$ at a point for $r < \infty$ but this would not give a finite dimensional vector space and so is not a good candidate definition for the tangent space (see problem 14).

Definition 2.2 *Let $(U, \mathtt{x})$ be a chart on a smooth n dimensional manifold M with $p \in U$. We write $\mathtt{x}= (x^1, ..., x^n)$ as usual and denote standard coordinates on $\mathbb{R}^n$ by $(u^1, ..., u^n)$. For $f \in C^\infty(M)$ define*

$$\frac{\partial f}{\partial x^i}(p) := \frac{\partial \left(f \circ \mathtt{x}^{-1}\right)}{\partial u^i}(\mathtt{x}(p)).$$

Also, define the operator $\left.\frac{\partial}{\partial x^i}\right|_p : C^\infty(M) \to \mathbb{R}$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \frac{\partial f}{\partial x^i}(p)$$

From the usual product rule it follows that $\frac{\partial}{\partial x^i}\big|_p$ is a derivation with respect to ev_p and so is an element of $(T_pM)_{\text{alg}}$. We will show that $(\frac{\partial}{\partial x^1}\big|_p, ..., \frac{\partial}{\partial x^n}\big|_p)$ is a basis for the vector space $(T_pM)_{\text{alg}}$.

Lemma 2.1 *Let $v_p \in (T_pM)_{alg}$ (so v_p is a derivation as explained above). Then (i) if $f, g \in C^\infty(M)$ are equal on some neighborhood of p then $v_pf = v_pg$,*

(ii) if $h \in C^\infty(M)$ is constant on some neighborhood of p then $v_ph = 0$.

Proof. (i) Since v_p is a linear map it is clear that it suffices to show that if $f = 0$ on a neighborhood U of p then $v_pf = 0$. Of course $v_p(0) = 0$. Let β be a cut-off function with support in U and $\beta(p) = 1$. Then we have that βf is identically zero and so

$$\begin{aligned}0 &= v_p(\beta f) = f(p)v_p\beta + \beta(p)v_pf \\ &= v_pf \quad (\text{since } \beta(p) = 1 \text{ and } f(p) = 0)\end{aligned}$$

(ii) From what we have just shown, it suffices to assume that h is equal to a constant c globally on M.

In the special case $c = 1$ we have

$$\begin{aligned}v_p1 &= v_p(1 \cdot 1) = 1 \cdot v_p1 + 1 \cdot v_p1 \\ &2v_p1\end{aligned}$$

so that $v_p1 = 0$. Finally we have $v_pc = v_p\,(1c) = c\,(v_p1) = 0$. ■

Let $p \in U \subset M$ with U open. We construct a rather obvious map $\Phi : (T_pU)_{\text{alg}} \to (T_pM)_{\text{alg}}$ by using the restriction map $C^\infty(M) \to C^\infty(U)$. We for each $w_p \in T_pU$ we define $\widetilde{w_p} : C^\infty(M) \to \mathbb{R}$ by $\widetilde{w_p}(f) := w_p(f|U)$. It is simple to show that $\widetilde{w_p}$ is a derivation of the appropriate type and so $\widetilde{w_p} \in (T_pM)_{\text{alg}}$. Thus we get a map $\Phi : (T_pU)_{\text{alg}} \to (T_pM)_{\text{alg}}$ which is manifestly linear. We want to show that this map is an isomorphism but notice that we have not established the finite dimensionality of either $(T_pU)_{\text{alg}}$ or $(T_pM)_{\text{alg}}$. First we show that $\Phi : w_p \mapsto \widetilde{w_p}$ has trivial kernel. So suppose that $\widetilde{w_p} = 0$, i.e. $\widetilde{w_p}(f) = 0$ for all $f \in C^\infty(M)$. Now let $h \in C^\infty(U)$. Pick a cut-off function β with support in U so that βh extends by zero to a smooth function f on all of M and agreeing with h on a neighborhood of p. Then by the above lemma $w_p(h) = w_p(\beta h) = \widetilde{w_p}(f) = 0$. Thus, since h was arbitrary, we see that $w_p = 0$ and so Φ has trivial kernel. Next we show that Φ is *onto*. Let $v_p \in (T_pM)_{\text{alg}}$. We wish to define $w_p \in (T_pU)_{\text{alg}}$ by $w_p(h) := v_p(\beta h)$ where β is as above and βh extended by zero to a function in $C^\infty(M)$. Another similar choice of cut-off function, say β_1, would make βh and $\beta_1 h$ (both extended to all of M) functions agreeing on a neighborhood of p and so by Lemma 2.1, $v_p(\beta h) = v_p(\beta_1 h)$. Thus w_p is well defined. Now $\widetilde{w_p}(f) := w_p(f|U) = v_p(\beta\, f|U) = v_p\,(f)$ since $\beta f|U$ and f agree on a neighborhood of p. Thus $\Phi : (T_pU)_{\text{alg}} \to (T_pM)_{\text{alg}}$ is an isomorphism.

Because of this isomorphism we tend to identify $(T_pU)_{\text{alg}}$ with $(T_pM)_{\text{alg}}$ and in particular, if $(U, \mathtt{x})$ is a chart we think of the derivations $\frac{\partial}{\partial x^i}\big|_p$, $1 \leq i \leq n$ as being simultaneously elements of both $(T_pU)_{\text{alg}}$ and $(T_pM)_{\text{alg}}$. In either case the formula is the same: $\frac{\partial}{\partial x^i}\big|_p f = \frac{\partial(f \circ \mathtt{x}^{-1})}{\partial u^i}(\mathtt{x}(p))$.

Notice that agreeing on a neighborhood of a point is an important thing here and this provides motivation for employing the notion of a germ of a function which was defined at the end of section 1.3. We will do this but first we establish the basis theorem.

Theorem 2.1 *Let M be a smooth $n-$dimensional manifold and $(U, \mathtt{x})$ a chart with $p \in U$. Then the ordered $n-$tuple of vectors (derivations) $(\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p)$ is a basis for $(T_pM)_{alg}$. Furthermore, for each $v_p \in (T_pM)_{alg}$ we have*

$$v_p = v_p(x^i) \left.\frac{\partial}{\partial x^i}\right|_p \quad \textit{(sum over i)}$$

Proof. From our discussion we may assume without loss that U has been shrunk in such a way that $\mathtt{x}(U)$ is a convex set such as a ball of radius ε in $\mathbb{R}^n$. By composing with a translation we assume that $\mathtt{x}(p) = 0$. This make no difference for what we wish to prove since v_p applied to a constant is 0. Now for any smooth function g defined on the convex set $\mathtt{x}(U)$ we define

$$g_i(u) := \int_0^1 \frac{\partial g}{\partial u^i}(tu)dt \text{ for all } u \in \mathtt{x}(U).$$

The fundamental theorem of calculus can be used to show that $g = g(0) + g_i u^i$. Applying $\frac{\partial}{\partial u^i}\big|_0$ we see that $g_i(0) = \left.\frac{\partial g}{\partial u^i}\right|_0$. Now for a function $f \in C^\infty(U)$ we let $g := f \circ \mathtt{x}^{-1}$ and then using the above, we arrive at the expression $f = f(p) + f_i x^i$ and applying $\frac{\partial}{\partial x^i}\big|_p$ we get $f_i(p) = \left.\frac{\partial f}{\partial x^i}\right|_p$. Now apply the derivation v_p to $f = f(p) + f_i x^i$ we get

$$\begin{aligned} v_p f &= 0 + \sum v_p(f_i x^i) \\ &= 0 + \sum v_p(x^i) f_i(p) + \sum 0 v_p f_i \\ &= \sum v_p(x^i) \left.\frac{\partial f}{\partial x^i}\right|_p \end{aligned}$$

which shows that $v_p = \sum v_p(x^i) \frac{\partial}{\partial x^i}\big|_p$ so that we have a spanning set. To see that $(\frac{\partial}{\partial x^i}\big|_p, \ldots, \frac{\partial}{\partial x^i}\big|_p)$ is a linearly independent set, let us assume that $\sum a^i \frac{\partial}{\partial x^i}\big|_p = 0$ (the zero derivation). Applying this to x^j we get $0 = \sum a^i \left.\frac{\partial x^j}{\partial x^i}\right|_p = \sum a^i \delta_i^j = a^j$ and since j was arbitrary we get the result. ■

There is another version of this definition of a tangent vector as a derivation that emphasizes the local character. Using this method allows us to worry a bit less about the relation between $(T_pU)_{\text{alg}}$ and $(T_pM)_{\text{alg}}$. Let $\mathcal{F}_p = C_p^\infty(M, \mathbb{R})$ be the algebra of germs of functions defined near p. Recall that if f is a representative for the equivalence class $[f] \in \mathcal{F}_p$ then we can unambiguously define the value of $[f]$ at p by $[f](p) = f(p)$. Thus we have an evaluation map $ev_p : \mathcal{F}_p \to \mathbb{R}$.

Remark 2.1 *We will sometimes abuse notation and write f instead of $[f]$ to denote the germ represented by a function f.*

Definition 2.3 *A **derivation** (with respect to the evaluation map ev_p) of the algebra $\mathcal{F}_p$ is a map $\mathcal{D} : \mathcal{F}_p \to \mathbb{R}$ such that $\mathcal{D}([f][g]) = f(p)\mathcal{D}[g] + g(p)\mathcal{D}[f]$ for all $[f], [g] \in \mathcal{F}_p$.*

The set of all these derivations on $\mathcal{F}_p$ is easily seen to be a real vector space and is sometimes denoted by $\mathsf{Der}(\mathcal{F}_p)$.

Let M be a smooth manifold of dimension n. Consider the set of all germs of C^∞ functions $\mathcal{F}_p$ at $p \in M$. The set $\mathsf{Der}(\mathcal{F}_p)$ of derivations of $\mathcal{F}_p$ with respect to the evaluation map ev_p is again a vector space which could be taken as the definition of the tangent space at p. This would be a slight variation of what we have called the algebraic tangent space.

2.2 Interpretations

We will now show how to move from one definition of tangent vector to the next. For simplicity let us assume that M is a smooth (C^∞) n-manifold.

Suppose that we think of a tangent vector X_p as an equivalence class of curves represented by $c : I \to M$ with $c(0) = p$. We obtain a derivation by defining

$$X_p f := \left.\frac{d}{dt}\right|_{t=0} f \circ c$$

This gives a map $(T_pM)_{\text{kin}} \to (T_pM)_{\text{alg}}$ which can be shown to be an isomorphism.

We also have a natural isomorphism $(T_pM)_{\text{kin}} \to (T_pM)_{\text{phys}}$. Given $[c] \in (T_pM)_{\text{kin}}$ we obtain an element $X_p \in (T_pM)_{\text{phys}}$ by letting X_p be associated to the triple $(p, v, (U, \mathtt{x}))$ where $v^i := \left.\frac{d}{dt}\right|_{t=0} x^i \circ c$ for a chart $(U, \mathtt{x})$ with $p \in U$.

If X_p is a derivation at p and $(U, \mathtt{x})$ an admissible chart with domain containing p, then X_p, as a tangent vector a la definition 2.1.2, is represented by the triple $(p, v, (U, \mathtt{x}))$ where $v = (v^1, ...v^n)$ is given by

$$v^i = X_p x^i \quad (\text{ } X_p \text{ is acting as a derivation})$$

Next we show how to get an isomorphism $(T_pM)_{\text{phys}} \to (T_pM)_{\text{alg}}$. Suppose that $[(p, v, (U, \mathtt{x}))] \in (T_pM)_{\text{phys}}$ where $v \in \mathbb{R}^n$. We obtain a derivation by defining

$$X_p f = D(f \circ \mathtt{x}^{-1})\big|_{\mathtt{x}(p)} \cdot v$$

We have the more traditional notation

$$X_p f = v^i \left. \frac{\partial}{\partial x^i} \right|_p f.$$

for $v = (v^1, ...v^n)$. If is an easy exercise that X_p defined in this way is independent of the representative triple $(p, v, (U, \mathtt{x}))$.

We now adopt the explicitly flexible attitude of interpreting a tangent vector in any of the ways we have described above depending on the situation. Thus we effectively identify the spaces $(T_pM)_{\mathrm{kin}}, (T_pM)_{\mathrm{phys}}$ and $(T_pM)_{\mathrm{alg}}$. Henceforth we use the notation T_pM for the tangent space of a manifold M at a point p.

Definition 2.4 *The dual space to a tangent space T_pM is called the **cotangent space** and is denoted by T_p^*M.*

The basis for T_p^*M that is dual to the coordinate basis $\{ \left.\frac{\partial}{\partial x^1}\right|_p, ..., \left.\frac{\partial}{\partial x^n}\right|_p \}$ described above is denoted $\{ \left.dx^1\right|_p, ..., \left.dx^n\right|_p \}$. By definition $\left.dx^i\right|_p \left(\left.\frac{\partial}{\partial x^j}\right|_p \right) = \delta^i_j$. The reason for the differential notation dx^i will be explained below. Sometimes one abbreviates $\left.\frac{\partial}{\partial x^j}\right|_p$ and $\left.dx^i\right|_p$ to $\frac{\partial}{\partial x^j}$ and dx^i but there is some risk of confusion since later $\frac{\partial}{\partial x^j}$ and dx^i will more properly denote not elements of the vector spaces T_pM and T_p^*M, but rather *fields* defined over a chart domain. More on this shortly.

2.3 Tangent spaces on manifolds with boundary

Recall, that a manifold with boundary is modeled on the half space $\mathbb{H}^n := \{x \in \mathbb{R}^n : x^1 \leq 0\}$. If M is a manifold with boundary, the tangent space T_pM is defined as before. For instance, even if $p \in \partial M$ the fiber T_pM may still be thought of as consisting of equivalence classes where $(p, v, \alpha) \sim (p, w, \beta)$ if and only if $\left. D(\mathtt{x}_\beta \circ \mathtt{x}_\alpha^{-1})\right|_{\mathtt{x}_\alpha(p)} \cdot v = w$. Notice that for a given chart $(U_\alpha, \mathtt{x}_\alpha)$, the vectors v in (p, v, α) still run through all of $\mathbb{R}^n$ and so T_pM still has tangent vectors "pointing in all directions" as it were. On the other hand, if $p \in \partial M$ then for any half-space chart $\mathtt{x}_\alpha : U_\alpha \to \mathbb{H}^n$ with p in its domain, $T\mathtt{x}_\alpha^{-1}(\partial \mathbb{H}^n)$ is a subspace of T_pM. This is the subspace of vectors tangent to the boundary and is identified with $T_p\partial M$ the tangent space to ∂M (also a manifold as well shall see).

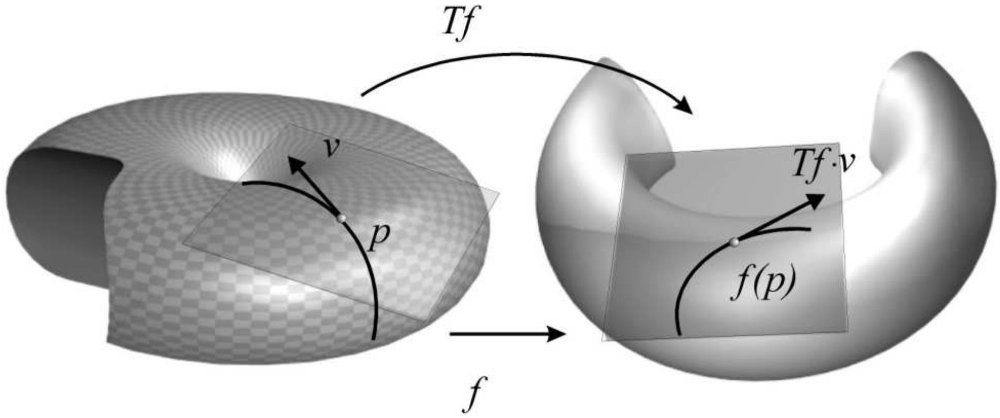
Exercise 2.3 *Show that this subspace does not depend on the choice of $\mathtt{x}_\alpha$.*

Definition 2.5 *Let M be a manifold with boundary and suppose that $p \in \partial M$. A tangent vector $v_p = [(p, v, \alpha)] \in T_pM$ is said to be **outward pointing** if $v^1 > 0$ and **inward pointing** if $v^1 < 0$. Here $\alpha \in A$ indexes charts as before.*

Exercise 2.4 *Show that the above definition is independent of the choice of the half-space chart $\mathtt{x}_\alpha : U_\alpha \to \mathbb{H}$.*

2.4 The Tangent Map

The first definition given below of the tangent map at $p \in M$ of a smooth map $f : M \to N$ will be considered our main definition but the others are actually equivalent. Given f and p as above wish to define a linear map $T_p f : T_p M \to T_{f(p)} N$



Definition 2.6 (Tangent map I) *If we have a smooth function between manifolds*

$$f : M \to N$$

and we consider a point $p \in M$ and its image $q = f(p) \in N$ then we define the ***tangent map*** *at p*

$$T_p f : T_p M \to T_q N$$

in the following way: Suppose that $v_p \in T_p M$ and we pick a curve c with $c(0) = p$ so that $v_p = [c]$, then by definition

$$T_p f \cdot v_p = [f \circ c] \in T_q N$$

where $[f \circ c] \in T_q N$ is the vector represented by the curve $f \circ c$.

Since we have several definitions of tangent vector we expect to see several equivalent definitions of the tangent map. Here is another:

Definition 2.7 (Tangent map II) *Let $f : M \to N$ be a smooth map and consider a point $p \in M$ with image $q = f(p) \in N$. Choose any chart $(U, \mathtt{x})$ containing p and a chart $(V, \mathtt{y})$ containing $q = f(p)$ so that for any $v_p \in T_p M$ we have the representative $(p, v, (U, \mathtt{x}))$. Then the* ***tangent map*** *$T_p f : T_p M \to T_{f(p)} N$ is defined by letting the representative of $T_p f \cdot v_p$ in the chart $(V, \mathtt{y})$ be given by $(q, w, (V, \mathtt{y}))$ where*

$$v = D(\mathtt{y} \circ f \circ \mathtt{x}^{-1}) \cdot w.$$

This uniquely determines $T_p f \cdot v$ and the chain rule guarantees that this is well defined (independent of the choice of charts).

Another alternative definition of tangent map is given in terms of derivations:

Definition 2.8 (Tangent Map III) *Let M be a smooth n-manifold. View tangent vectors as derivations as explained above. Then continuing our set up above we define $T_p f \cdot v_p$ as a derivation by*

$$(T_p f \cdot v_p) g = v_p (f \circ g)$$

*for g a smooth function . It is easy to check that this defines a derivation so is also a tangent vector in $T_q M$. This map is yet another version of the **tangent map** $T_p f$.*

In the above definition one could take g to be the germ of a smooth function defined on a neighborhood of $f(p)$ and then $T_p f \cdot v_p$ would act as a derivation of such germs.

Notation 2.2 *Another popular way to denote the tangent map $T_p f$ is f_{p*}.*

Now we introduce the differential of a function.

Definition 2.9 *Let M be a smooth manifold let $p \in M$. For $f \in C^\infty(M)$ we define the **differential** of f at p as the linear map $df(p) : T_p M \to \mathbb{R}$ defined by*

$$df(p) \cdot v_p = v_p f$$

for all $v_p \in T_p M$.

One may view $df_p(v_p)$ as the "infinitesimal" version of the composition $f \circ \gamma$ where $\gamma'(0) = v_p$. The notation df_p or $df|_p$ is also used in place of $df(p)$. It is easy to show that $df(p) = pr_1 \circ T_p f$ where we take $T_{f(p)}\mathbb{R} = \{f(p)\} \times \mathbb{R}$ and $pr_2 : \{f(p)\} \times \mathbb{R} \to \mathbb{R}$ is projection onto the second factor which in this context gives that natural identification of $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$. So, in a way, $df(p)$ is just a version of the tangent map that takes advantage of the identification of $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$.

Exercise 2.5 *Let $(U_\alpha, \mathtt{x}_\alpha)$ be a chart with $\mathtt{x}_\alpha = (x^1, ..., x^n)$ and let $p \in U_\alpha$. Show that the definition of $dx^1\big|_p$ given previously is consistent with the last general definition.*

The differential can be generalized:

Definition 2.10 *Let V be a vector space. For a smooth $f : M \to \mathrm{V}$ with $p \in M$ as above, the differential $df(p) : T_p M \to \mathrm{V}$ is the composition of the tangent map $T_p f$ and the canonical map $T_y \mathrm{V} \to \mathrm{V}$ where $y = f(p)$. Diagrammatically we have*

$$df(p) : T_p M \overset{Tf}{\to} T_y \mathrm{V} = \{y\} \times \mathrm{V} \overset{pr_2}{\to} \mathrm{V}.$$

The notational distinction between $T_p f$ and df_p is not universal and df_p is itself often used to denote $T_p f$.

We now consider the inclusion map $\iota : U \hookrightarrow M$ where U is open. For $p \in U$ we get the tangent map $T_p\iota : T_pU \to T_pM$. Let us look at this map from several of the points of view corresponding to the various ways one can define a tangent space. First, consider tangent spaces from the derivation point of view. From this point of view the map $T_p\iota$ is defined for $v_p \in T_pU$ as acting on $C^\infty(M)$ as follows $T_p\iota(v_p)f = v_p\,(f \circ \iota) = v_p\,(f|\,U)$. We have seen this map before where we called it $\Phi : (T_pU)_{\text{alg}} \to (T_pM)_{\text{alg}}$ and it was observed to be an isomorphism and we decided to identify $(T_pU)_{\text{alg}}$ with $(T_pM)_{\text{alg}}$. From the point of view of equivalence classes of curves, the map $T_p\iota$ sends $[\gamma]$ to $[\iota \circ \gamma]$. But while γ is a curve into U the map $\iota \circ \gamma$ is simply the same curve but thought of as mapping into M. We leave it to the reader to verify the expected fact that $T_p\iota$ is a linear isomorphism. Thus it makes sense to identify $[\gamma]$ with $[\iota \circ \gamma]$ and so again to identify T_pU with T_pM via this isomorphism. Next consider $v_p \in T_pU$ to be represented by a triple $(p, v, (U_\alpha, \mathtt{x}_\alpha))$ where $(U_\alpha, \mathtt{x}_\alpha)$ is a chart on the open manifold U. Now since $(U_\alpha, \mathtt{x}_\alpha)$ is also a chart on M the triple also represents an element of T_pM which is none other than $T_p\iota \cdot v_p$. The map $T_p\iota$ looks more natural and trivial than ever and we once again see the motivation for identifying T_pU and T_pM.

2.5 Tangents of Products.

Suppose that $f : M_1 \times M_2 \to N$ is a smooth map. Define the partial maps by

$$\begin{aligned} f_{,y}(.) &:= f(., y) \text{ for fixed } y \in M_2 \\ f_x(.) &:= f(x, .) \text{ for fixed } x \in M_1 \end{aligned}$$

Notice the comma in $f_{,y}$. The reason for this comma is to avoid confusion in case $M_1 = M_2$ since we then need to distinguish $f(., x)$ from $f(x, .)$.

Definition 2.11 (Partial Tangential) *Let $f : M_1 \times M_2 \to N$ be as above. Define the* ***partial tangent maps*** *$\partial_1 f$ and $\partial_2 f$ by*

$$\begin{aligned} (\partial_1 f)(x, y) &:= T_x f_{.y} : T_x M_1 \to T_{f(x,y)} N \\ (\partial_2 f)(x, y) &:= T_y f_x : T_y M_2 \to T_{f(x,y)} N \end{aligned}$$

Next we introduce another natural identification. It is obvious that a curve $c : I \to M_1 \times M_2$ is equivalent to a pair of curves

$$\begin{aligned} c_1 &: I \to M_1 \\ c_2 &: I \to M_2 \end{aligned}$$

The infinitesimal version of this fact gives rise to a natural identification

$$T_{(x,y)}(M_1 \times M_2) \cong T_x M_1 \times T_y M_2$$

This is perhaps easiest to see if we view tangent vectors as equivalence classes of curves (tangency classes). If $c(t) = (c_1(t), c_2(t))$ and $c(0) = (x, y)$ then the map $[c] \mapsto ([c_1], [c_2])$ is a natural isomorphism which we use to simply identify $[c] \in T_{(x,y)}(M_1 \times M_2)$ with $([c_1], [c_2]) \in T_x M_1 \times T_y M_2$. For another view, consider the insertion maps $\iota_x : y \mapsto (x, y)$ and $\iota^y : x \mapsto (x, y)$.

$$\begin{tikzcd}
(M_1, x) \arrow[r, shift right, "\iota^y"'] & (M_1 \times M_2, (x, y)) \arrow[l, shift right, "pr_1"'] \arrow[r, shift left, "pr_2"] & (M_2, x) \arrow[l, shift left, "\iota_x"]
\end{tikzcd}$$

$$\begin{tikzcd}
T_x M_1 \arrow[r, shift right, "T_x \iota^y"'] & T_{(x,y)} M_1 \times M_2 \arrow[l, shift right, "T_{(x,y)} pr_1"'] \arrow[r, shift left, "T_{(x,y)} pr_2"] & T_x M_2 \arrow[l, shift left, "T_y \iota_x"]
\end{tikzcd}$$

We have linear monomorphisms $T\iota^y(x) : T_x M_1 \to T_{(x,y)}(M_1 \times M_2)$ and $T\iota_x(y) : T_y M_2 \to T_{(x,y)}(M_1 \times M_2)$. Let us temporarily denote the isomorphic images of $T_x M_1$ and $T_y M_2$ in $T_{(x,y)}(M_1 \times M_2)$ under these two maps by the symbols $(T_x M)_1$ and $(T_y M)_2$. We then have the internal direct sum decomposition $(T_x M)_1 \oplus (T_y M)_2 = T_{(x,y)}(M_1 \times M_2)$ and the isomorphism

$$T\iota^y \times T\iota_x : T_x M_1 \times T_y M_2 \to (T_x M)_1 \oplus (T_y M)_2 = T_{(x,y)}(M_1 \times M_2).$$

The inverse of this isomorphism is

$$T_{(x,y)} pr_1 \times T_{(x,y)} pr_2 : T_{(x,y)}(M_1 \times M_2) \to T_x M_1 \times T_y M_2$$

which is then taken as an identification and, in fact, this is none other than the map $[c] \mapsto ([c_1], [c_2])$. Let us say a bit about the naturalness of the identification of $[c] \in T_{(x,y)}(M_1 \times M_2)$ with $([c_1], [c_2]) \in T_x M_1 \times T_y M_2$. In the smooth category there is a product operation. The essential point is that for any two manifolds M_1 and M_2 the manifold $M_1 \times M_2$ together with the two projection maps serves as the product in the technical sense that for any smooth maps $f : N \longrightarrow M_1$ and $g : N \longrightarrow M_2$ we always have the unique map $f \times g$ which makes the following diagram commute:

$$\begin{tikzcd}
 & N \arrow[dl, "f"'] \arrow[d, "f \times g"] \arrow[dr, "g"] & \\
M_1 & M_1 \times M_2 \arrow[l, "pr_1"'] \arrow[r, "pr_2"] & M_2
\end{tikzcd}$$

Now for a point $x \in N$ write $p = f(x)$ and $q = g(x)$. On the tangent level we have

$$\begin{tikzcd}
 & T_x N \arrow[dl, "T_x f"'] \arrow[d, "T_x(f \times g)"] \arrow[dr, "T_x g"] & \\
T_p M_1 & T_{(p,q)}(M_1 \times M_2) \arrow[l, "T_{(p,q)} pr_1"'] \arrow[r, "T_{(p,q)} pr_2"] & T_q M_2
\end{tikzcd}$$

which is a diagram in the vector space category. In the category of vector spaces the product of T_pM_1 and T_pM_2 is $T_pM_1 \times T_pM_2$ (outer direct sum) together with the projections onto the two factors. It is then quite reassuring to notice that under the identification introduced above this diagram corresponds to

$$\begin{tikzcd} & T_xN \arrow[dl, "T_xf"'] \arrow[d, "T_xf \times T_xg"] \arrow[dr, "T_xg"] & \\ T_pM_1 & T_pM_1 \times T_qM_2 \arrow[l, "Pr_1"'] \arrow[r, "Pr_2"] & T_qM_2 \end{tikzcd}$$

Notice that we have $f \circ \iota^y = f_{,y}$ and $f \circ \iota_x = f_x$.

Looking again at the definition of partial tangential one arrives at

Lemma 2.2 (partials lemma) *For a map $f : M_1 \times M_2 \to N$ we have*

$$T_{(x,y)}f \cdot (v, w) = (\partial_1 f)_{(x,y)} \cdot v + (\partial_2 f)_{(x,y)} \cdot w.$$

where we have used the aforementioned identification $T_{(x,y)}(M_1 \times M_2) = T_xM_1 \times T_yM_2$

Proving this last lemma is much easier and more instructive than reading the proof so we leave it to the reader in good conscience.

The following diagram commutes:

$$\begin{array}{ccc} & T_{(x,y)}(M_1 \times M_2) & \\ & & \searrow \\ {\scriptstyle T_{(x,y)}pr_1 \times T_{(x,y)}pr_2} & \downarrow & & T_{f(x,y)}N \\ & & \nearrow \\ & T_xM_1 \times T_yM_2 & \end{array}$$

Essentially, both diagonal maps refer to $T_{(x,y)}f$ because of our identification.

2.6 The Tangent and Cotangent Bundles

2.6.1 Tangent Bundle

We define the **tangent bundle** of a manifold M as the (disjoint) union of the tangent spaces; $TM = \bigcup_{p \in M} T_pM$. Similarly the cotangent bundle is the (disjoint) union of the cotangent spaces; $T^*M = \bigcup_{p \in M} T_p^*M$. Also, if U is an open set in a finite dimensional vector space V then the tangent space at $x \in$ V can be viewed at the set $\{x\} \times$ V and so the tangent bundle of U can be viewed as the product $U \times$ V. We show in proposition 2.2 below that both TM and T^*M are themselves smooth manifolds but first we introduce a couple of definitions.

Definition 2.12 *Given a smooth map $f : M \to N$ as above then the tangent maps on the individual tangent spaces combine to give a map $Tf : TM \to TN$ on the tangent bundles that is linear on each fiber called the* ***tangent map*** *or sometimes the* ***tangent lift*** *of f .*

Definition 2.13 *If $f : M \to \mathrm{V}$ where V is a finite dimensional vector space (most usually $\mathbb{R}$ or $\mathbb{C}$), then we have the differential $df(p) : T_pM \to \mathrm{V}$ for each p. These maps can be combined to give a single map $df : TM \to \mathrm{V}$ (also called the differential) which is defined by $df(v) = df(p)(v)$ when $v \in T_pM$.*

Looking over the definitions one can see immediately that $df = pr_2 \circ Tf$ where $pr_2 : T\mathrm{V} = \mathrm{V} \times \mathrm{V} \to \mathrm{V}$ is projection onto the second factor.

Remark 2.2 (Warning) *The notation "df" is subject to interpretation. Beside the map $df : TM \to \mathrm{V}$ described above it could also refer to the map $df : p \mapsto df(p)$ or to another map on vector fields which we describe later in this chapter.*

Definition 2.14 *The map $\pi_{TM} : TM \to M$ defined by $\pi_{TM}(v) = p$ if $v \in T_pM$ is called the* ***tangent bundle projection map****. (The set TM together with the map $\pi_{TM} : TM \to M$ is an example of a vector bundle which is defined in the sequel.)*

For every chart $(U, \mathtt{x})$ on M we obtain a chart $\left(\widetilde{U}, \widetilde{\mathtt{x}}\right)$ on TM by letting

$$\widetilde{U} := TU = \pi^{-1}(U) \subset TM$$

and defining $\widetilde{\mathtt{x}}$ on $\widetilde{U}$ by the prescription

$$\widetilde{\mathtt{x}}(v_p) = (x^1(p), ..., x^1(p), v^1, ..., v^n) \text{ where } v_p \in T_pM$$

and where $v^1, ..., v^n$ are the (unique) coefficients in the coordinate expression $v = v^i \left.\frac{\partial}{\partial x^i}\right|_p$. Thus $\widetilde{\mathtt{x}}^{-1}(u^1, ..., u^n, v^1, ..., v^n) = v^i \left.\frac{\partial}{\partial x^i}\right|_{\mathtt{x}^{-1}(u)}$. Now recall that if $v_p = v^i \left.\frac{\partial}{\partial x^i}\right|_p$ then $v^i = dx^i(v_p)$. From this we see that $\widetilde{\mathtt{x}} = (x^1 \circ \pi, ..., x^n \circ \pi,\ dx^1, ..., dx^n)$. By creative abuse of notation one can also write $\widetilde{\mathtt{x}} = (x^1, ..., x^n, v^1, ..., v^n)$ which has the advantage of being relatively uncluttered.

For any $(U, \mathtt{x})$ we have the tangent lift $T\mathtt{x} : TU \to TV$ where $V = \mathtt{x}(U)$. This is defined fiberwise as described above $T\mathtt{x} : T_pU \to T_{\mathtt{x}(p)}V$. Recall that since $V \subset \mathbb{R}^n$ we can identify $T_{\mathtt{x}(p)}V$ with $\{\mathtt{x}(p)\} \times \mathbb{R}^n$. Let us invoke this identification. Now let $v_p \in T_pU$ and let γ be a curve that represents v_p so that $\gamma'(0) = v_p$. Then $T_p\mathtt{x} \cdot v_p = \left(\mathtt{x}(p), \left.\frac{d}{dt}\right|_{t=0} (\mathtt{x} \circ \gamma)\right)$. Now if $v_p = \left.\frac{\partial}{\partial x^i}\right|_p$ then we can take $\gamma(t) := \mathtt{x}^{-1}(\mathtt{x}(p) + t\mathbf{e}_i)$ where $\mathbf{e}_i$ is the $i-$th member of the standard basis of $\mathbb{R}^n$. Thus $T_p\mathtt{x} \cdot \left.\frac{\partial}{\partial x^i}\right|_p = \left(\mathtt{x}(p), \left.\frac{d}{dt}\right|_{t=0} (\mathtt{x}(p) + t\mathbf{e}_i)\right) = (\mathtt{x}(p), \mathbf{e}_i)$. Now suppose that $v_p = v^i \left.\frac{\partial}{\partial x^i}\right|_p$. Then

$$\begin{aligned}
&T_p\mathtt{x} \cdot v_p \\
&= T_p\mathtt{x} \cdot \left(v^i \left.\frac{\partial}{\partial x^i}\right|_p \right) \\
&= (\mathtt{x}(p), v^i \mathbf{e}_i) = (x^1(p), ..., x^n(p), v^1, ..., v^n)
\end{aligned}$$

From this we see that $T\mathtt{x}$ is none other than $\widetilde{\mathtt{x}}$ defined above and since $\widetilde{U} = TU$ we see that an alternative and suggestive notation for $\left(\widetilde{U}, \widetilde{\mathtt{x}}\right)$ is $(TU, T\mathtt{x})$. This notation reminds one that these charts we have constructed are not just any charts on TM but are each associated naturally with a chart on M. They are called standard or natural charts.

Proposition 2.2 *For any smooth M, the set TM is a smooth manifold in a natural way and $\pi : TM \to M$ is a smooth map. Furthermore, for a smooth map $f : M \to N$ the tangent lift Tf is smooth and the following diagram commutes.*

$$\begin{array}{cccccc}
 & TM & \overset{Tf}{\to} & TN & \\
\pi_{TM} & \downarrow & & \downarrow & \pi_{TN} \\
 & M & \overset{f}{\to} & N &
\end{array}$$

Proof. For every chart $(U, \mathtt{x})$ let $TU = \pi^{-1}(U)$ and let $T\mathtt{x}$ be the map $T\mathtt{x} : TU \to \mathtt{x}(U) \times \mathbb{R}^n$. The pair $(TU, T\mathtt{x})$ is a chart on TM. Suppose that $(TU, T\mathtt{x})$ and $(TV, T\mathtt{y})$ are two such charts constructed as above from two charts $(U, \mathtt{x})$ and $(V, \mathtt{y})$ and that $U \cap V \neq \emptyset$. Then $TU \cap TV \neq \emptyset$ and on the overlap we have the coordinate transitions $T\mathtt{y} \circ T\mathtt{x}^{-1}$:$(x, v) \mapsto (y, w)$ where

$$\begin{aligned}
y &= \mathtt{y} \circ \mathtt{x}^{-1}(x) \\
w &= D(\mathtt{y} \circ \mathtt{x}^{-1})\big|_{\mathtt{x}(p)}\, v
\end{aligned}$$

and so we see that the overlap maps are smooth. It follows from Lemmas 1.2 and 1.4 that TM is Hausdorff and paracompact.

To test for the smoothness of π we now look at maps of the form $\mathtt{x} \circ \pi \circ (T\mathtt{x})^{-1}$. We have

$$\begin{aligned}
&\mathtt{x} \circ \pi \circ (T\mathtt{x})^{-1}(x, v) \\
&= \mathtt{x} \circ \pi \left(v^i \left.\frac{\partial}{\partial x^i}\right|_{\mathtt{x}^{-1}(x)} \right) \\
&= \mathtt{x} \circ \mathtt{x}^{-1}(x) = x
\end{aligned}$$

which is just a projection and so clearly smooth. ∎

If $p \in \mathtt{x}(U \cap V)$ and $\mathtt{x}(p) = (x^1(p), ..., x^n(p))$ then as in the proof above $T\mathtt{y} \circ T\mathtt{x}^{-1}$ sends $(x^1(p), ..., x^n(p), v^1, ..., v^n)$ to $(y^1(p), ..., y^n(p), w^1, ..., w^n)$ where

$$w^i = \frac{\partial(\mathtt{y} \circ \mathtt{x}^{-1})^i}{\partial x^k} v^k$$

If we abbreviate the i-th component of $\mathtt{y} \circ \mathtt{x}^{-1}(x^1(p), ..., x^n(p))$ to

$$y^i = y^i(x^1(p), ..., x^n(p))$$

then we could express the tangent bundle overlap map by the relations

$$\begin{aligned} y^i &= y^i(x^1(p), ..., x^n(p)) \\ w^i &= \frac{\partial y^i}{\partial x^k} v^k \end{aligned}$$

Since this is true for all $p \in \mathtt{x}(U \cap V)$ we can write the very classical looking expression

$$\begin{aligned} y^i &= y^i(x^1, ..., x^n) \\ w^i &= \frac{\partial y^i}{\partial x^k} v^k \end{aligned}$$

where we now can interpret $(x^1, ..., x^n)$ as an $n-$tuple of numbers. This is in the spirit of section 1.9 and equation 2.1.2. It is often the case that local expression can either be interpreted as living on the manifold in the chart domain or equally, in Euclidean space on the image of the chart domain. This should not be upsetting since, after all, one could argue that the charts are there to *identify* chart domains in the manifold with open sets in Euclidean space.

Now that TM has a smooth structure we can inquire about smoothness of maps which have TM as either domain or codomain. This becomes important in section 2.7 below. Also we have

Exercise 2.6 *For a smooth map $f : M \to N$, the map*

$$Tf : TM \to TN$$

is itself a smooth map.

We have seen that if U is an open set in a vector space V then the tangent bundle is $U \times \mathrm{V}$. Suppose that for some smooth manifold M the is a diffeomorphism $F : TM \to M \times \mathrm{V}$ for some vector space V which is such that the diagram

$$\begin{array}{ccc} TM & \to & M \times \mathrm{V} \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

commutes and such that the restriction of F to each tangent space is a linear map $T_pM \to \{p\} \times \mathrm{V}$. Then for some purposes, we can identify TM with $M \times \mathrm{V}$.

Definition 2.15 *A diffeomorphism $F : TM \to M \times \mathrm{V}$ with the properties described above is called a (global) **trivialization** of TM. If a trivialization exist then we say that TM is trivial. For an open set $U \subset M$ a trivialization of TU is called a **local trivialization** of TM over U.*

For most manifolds, there does not exist a global trivialization of the tangent bundle. On the other hand, every point p in a manifold M is contained in an open set U so that TM has a local trivialization over U. The existence of these local trivializations is quickly deduced from the existence of the special charts which we constructed for a tangent bundle.

2.6.2 The Cotangent Bundle

Recall that for each $p \in M$, T_pM has a dual space T_p^*M called the cotangent space at p.

Definition 2.16 *Define the* ***cotangent bundle*** *of a manifold M to be the set*

$$T^*M := \bigcup_{p \in M} T_p^*M$$

and define the map $\pi_{T^*M} : T^*M \to M$ to be the obvious projection taking elements in each space T_p^*M to the corresponding point p.

Remark 2.3 *We will denote both the tangent bundle projection and the cotangent bundle projection by simply π whenever no confusion is likely.*

We now see that T^*M is also a smooth manifold. Let $\mathcal{A}$ be an atlas of admissible charts on M. For each chart $(U, \mathtt{x}) \in \mathcal{A}$ we obtain a chart $(T^*U, T^*\mathtt{x})$ for T^*M which we now describe. First, $T^*U = \bigcup_{p \in U} T_p^*M = \bigcup_{p \in U} T_p^*U$. Secondly, $T^*\mathtt{x}$ is a map which we now define directly and then show that in some sense it is dual to the map $T\mathtt{x}$. For convenience consider that map $p_i : \theta_p \mapsto \xi_i$ which just peals off the coefficients in the expansion of any $\theta_p \in T_p^*M$ in the basis $\{\, dx^i\big|_p\}$:

$$p_i\,(\theta_p) = p_i\left(\xi_j\ dx^i\big|_p\right) = \xi_i$$

Notice that we have

$$\theta_p(\left.\frac{\partial}{\partial x^i}\right|_p) = \xi_j\ dx^i\big|_p\,(\left.\frac{\partial}{\partial x^i}\right|_p) = \xi_j \delta_i^j = p_i(\theta_p).$$

and so

$$p_i(\theta_p) = \theta_p(\left.\frac{\partial}{\partial x^i}\right|_p).$$

Now with the definition of the p_i in hand we can define

$$T^*\mathtt{x} = (x^1 \circ \pi, ..., x^n \circ \pi, p_1, ..., p_n).$$

on T^*U. We call $(T^*U, T^*\mathtt{x})$ a natural chart. If $\mathtt{x} = (x^1, ..., x^n)$ then for the natural chart $(T^*U, T^*\mathtt{x})$ we could use the abbreviation $T^*\mathtt{x} = (x^1, ..., x^n, p_1, ..., p_n)$. Another common choice of notation is $(q^1, ..., q^n, p_1, ..., p_n)$ where $q^i = x^i \circ \pi$. This notation is very popular in applications to mechanics. We will reserve the option of using ξ_i instead of p_i which is also a popular choice.

We now claim that if we take advantage of the identifications of $T_x\mathbb{R}^n = \mathbb{R}^n = (\mathbb{R}^n)^* = T^*\mathbb{R}^n$ where $(\mathbb{R}^n)^*$ is the dual space of $\mathbb{R}^n$, then $T^*\mathtt{x}$ acts on each fiber T_p^*M as the inverse of the dual to the map $T_p\mathtt{x}$, i.e. the contragradient of $T_p\mathtt{x}$:

$$T_p\mathtt{x}^{-1^*}(\theta_p) \cdot (v) = \theta_p \left(T_p\mathtt{x}^{-1} \cdot v\right)$$

Let us unravel this contragradient map. If $\theta_p \in T_p^*M$ for some $p \in U$ then can write

$$\theta_p = \xi_i \; dx^i\big|_p$$

for some numbers ξ_i depending on θ_p which are what we have called $p_i(\theta_p)$. We have

$$\begin{aligned} T_p\mathtt{x}^{-1^*}(\theta_p) \cdot (v) &= \theta_p \left(T_p\mathtt{x}^{-1} \cdot v\right) \\ &= \xi_i \; dx^i\big|_p \cdot \left(T_p\mathtt{x}^{-1} \cdot v\right) \\ &= \xi_i \; dx^i\big|_p \left(v^k \; \frac{\partial}{\partial x^k}\bigg|_p\right) \\ &= \xi_i v^i \end{aligned}$$

Thus, under the usual identification of $\mathbb{R}^n$ with its dual we see that $T_p\mathtt{x}^{-1^*}(\theta_p)$ is just $(\xi_1, ..., \xi_n)$. But recall that $T^*\mathtt{x}\,(\theta_p) = (x^1(p), ..., x^n(p), \xi_1, ..., \xi_n)$. Thus for $\theta_p \in T_p^*M$ we have

$$T^*\mathtt{x}\,(\theta_p) = (\mathtt{x}(p), T_p\mathtt{x}^{-1^*}(\theta_p))$$

Suppose that $(T^*U, T^*\mathtt{x})$ and $(T^*V, T^*\mathtt{y})$ are the coordinates constructed as above from two charts $(U, \mathtt{x})$ and $(V, \mathtt{y})$ respectively with $U \cap V \neq \emptyset$. Then on the overlap $T^*U \cap T^*V$ we have

$$T^*\mathtt{y} \circ T^*\mathtt{x}^{-1} : \mathtt{x}(U \cap V) \times \mathbb{R}^{n*} \to \mathtt{y}(U \cap V) \times \mathbb{R}^{n*}.$$

This last map will send something of the form $(x, \xi) \in U \times \mathbb{R}^{n*}$ to $(\bar{x}, \bar{\xi}) = (\mathtt{y} \circ \mathtt{x}^{-1}(x), D(\mathtt{y} \circ \mathtt{x}^{-1})^* \cdot \xi)$ where $D(\mathtt{y} \circ \mathtt{x}^{-1})^*$ is the dual map to the inverse of $D(\mathtt{y} \circ \mathtt{x}^{-1})$, i.e., the contragradient of $D(\mathtt{y} \circ \mathtt{x}^{-1})$. If we identify $\mathbb{R}^{n*}$ with $\mathbb{R}^n$ and write $\xi = (\xi_1, .., \xi_n)$ and $\bar{\xi} = (\bar{\xi}_1, .., \bar{\xi}_n)$ then in classical style we have:

$$\begin{aligned} y^i &= y^i(x^1, ..., x^n) \\ \bar{\xi}_i &= \xi_k \frac{\partial x^k}{\partial y^i}. \text{ (sum)} \end{aligned}$$

This should be compared to the expression 2.1.2.

Exercise 2.7 *The topology of T^*M is paracompact and Hausdorff .*

2.7 Vector fields

Definition 2.17 *A smooth* ***vector field*** *is a smooth map $X : M \to TM$ such that $X(p) \in T_pM$ for all $p \in M$. In other words, a vector field on M is a* ***smooth section*** *of the tangent bundle $\pi : TM \to M$. We often write $X_p = X(p)$.*

If $(U, \mathtt{x})$ is a chart on a smooth n-manifold then, writing $\mathtt{x} = (x^1, ..., x^n)$, we have vector fields defined on U by

$$\frac{\partial}{\partial x^i} : p \mapsto \left.\frac{\partial}{\partial x^i}\right|_p$$

The set of fields $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\}$ is called a **coordinate frame field** (or also "holonomic frame field"). If X is a vector field defined on some set including this chart domain U then for some smooth functions X^i defined on U we have

$$X(p) = X^i(p) \left.\frac{\partial}{\partial x^i}\right|_p$$

or in other words

$$X|_U = X^i \frac{\partial}{\partial x^i}.$$

Notation 2.3 *In this context, we will not usually bother to distinguish X from its restrictions to chart domains and so we just write $X = X^i \frac{\partial}{\partial x^i}$.*

A vector field is a smooth section of the tangent bundle. Let us unravel what the smoothness condition means. Let $(TU, T\mathtt{x})$ be one of the natural charts that we constructed for TM from a corresponding chart $(U, \mathtt{x})$ on M. If X is smooth then restriction to U is also smooth and takes values in TU. Thus to test the smoothness of X we look at the composition $T\mathtt{x} \circ X \circ \mathtt{x}^{-1}$. For $x \in \mathtt{x}(U)$ we have

$$\begin{aligned}
&T\mathtt{x} \circ X \circ \mathtt{x}^{-1}(x) \\
&= T\mathtt{x} \circ \left(\sum X^i \frac{\partial}{\partial x^i}\right) \circ \mathtt{x}^{-1}(x) \\
&= T\mathtt{x} \left(\sum X^i(\mathtt{x}^{-1}(x)) \left.\frac{\partial}{\partial x^i}\right|_{\mathtt{x}^{-1}(x)}\right) \\
&= \left(x, T_{\mathtt{x}^{-1}(x)}\mathtt{x} \left(\sum X^i(\mathtt{x}^{-1}(x)) \left.\frac{\partial}{\partial x^i}\right|_{\mathtt{x}^{-1}(x)}\right)\right) \\
&= \big(x,\, X^1 \circ \mathtt{x}^{-1}(x), ..., X^n \circ \mathtt{x}^{-1}(x)\big)
\end{aligned}$$

Our chart was arbitrary and so we see that the smoothness of X is equivalent to the smoothness of the component functions X^i in every chart.

Exercise 2.8 *Show that if $X : M \to TM$ is continuous and $\pi \circ X = id$ then X is smooth if and only if $Xf : p \mapsto X_p f$ is a smooth function for every locally defined smooth function f on M. Show that it is enough to consider globally defined smooth functions.*

Notation 2.4 *The set of all smooth vector fields on M is denoted by $\Gamma(M, TM)$ or by the common notation $\mathfrak{X}(M)$. Smooth vector fields may at times be defined only on some open set so we also have the notation $\mathfrak{X}(U) = \mathfrak{X}_M(U)$ for these fields.*

Vector fields as derivations

We have seen how individual tangent vectors in T_pM can be identified as derivations with at p. The derivation idea can be globalized

Definition 2.18 *Let M be a smooth manifold. A (global)* ***derivation*** *on $C^\infty(M)$ is a linear map $\mathcal{D} : C^\infty(M) \to C^\infty(M)$ such that*

$$\mathcal{D}(fg) = \mathcal{D}(f)g + f\mathcal{D}(g).$$

We denote the set of all such derivations of $C^\infty(M)$ by $\mathrm{Der}(C^\infty(M))$.

Notice the subtle difference between a derivation in this sense and a derivation at a point.

Definition 2.19 *To a vector field X on M we associate the map $\mathcal{L}_X : C^\infty(M) \to \mathfrak{X}(M)$ defined by*

$$(\mathcal{L}_X f)(p) := X_p \cdot f$$

$\mathcal{L}_X$ is called the ***Lie derivative*** *on functions.*

Remark 2.4 *It is important to notice that $(\mathcal{L}_X f)(p) = X_p \cdot f = df\,(X_p)$ for any p and so $\mathcal{L}_X f = df \circ X$.*

Lemma 2.3 *Let $U \subset M$ be an open set. If $\mathcal{L}_X f = 0$ for all $f \in C^\infty(U)$ then $X|_U = 0$.*

Proof. Working locally in a chart $(U, \mathtt{x})$, let X_U be the local representative of X (defined in section 2.7). Suppose $X_U(p) \neq 0$ and that $\ell : \mathbb{R}^n \to \mathbb{R}$ is a continuous linear map such that $\ell(X_U(p)) \neq 0$. Let $f := \ell \circ \mathtt{x}$. Then the local representative of $\mathcal{L}_X f(p)$ is $d(\ell \circ \mathtt{x})(X(p)) = D\ell|_p \cdot X_U = \ell(X_U(p)) \neq 0$ i.e. $\mathcal{L}_X f(p) \neq 0$. We have used the fact that $D\ell|_p\, \ell = \ell$ (since ℓ is linear). ■

Theorem 2.2 *For $X \in \mathfrak{X}(M)$ we have $\mathcal{L}_X \in \mathrm{Der}(C^\infty(M))$ and if $\mathcal{D} \in \mathrm{Der}(C^\infty(M))$ then $\mathcal{D} = \mathcal{L}_X$ for a uniquely determined $X \in \mathfrak{X}(M)$.*

Proof. That $\mathcal{L}_X$ is in $\mathrm{Der}(C^\infty(M))$ follows from the Liebniz law, in other words, from the fact each X_p is a derivation at p. Now if we are given a derivation $\mathcal{D}$ we define a derivation X_p at p (i.e. at tangent vector) by the rule $X_p f := (\mathcal{D}f)(p)$. We need to show that the assignment $p \mapsto X_p$ is smooth.

Recall that any locally defined function can be extended to a global one by using a cutoff function. Because of this it suffices to show that $X_p f$ is smooth for any $f \in C^\infty(M)$. But this is clear since $X_p f := (\mathcal{D}f)(p)$ and $\mathcal{D}f \in C^\infty(M)$. Suppose now that $\mathcal{D}f = \mathcal{L}_{X_1} = \mathcal{L}_{X_2}$. Notice that $\mathcal{L}_{X_1} - \mathcal{L}_{X_2} = \mathcal{L}_{X_1 - X_2}$ and so $\mathcal{L}_{X_1 - X_2}$ is the zero derivation. By Lemma 2.3 we have $X_1 - X_2 = 0$. ■

Because of this theorem we can identify $\mathrm{Der}(C^\infty(M))$ with $\mathfrak{X}(M)$ and we can write Xf in place of $\mathcal{L}_X f$ if we choose. The derivation law (also called the Liebniz law) $\mathcal{L}_X(fg) = g\mathcal{L}_X\, f + f\mathcal{L}_X\, g$ becomes simply $X(fg) = gXf + fXg$. Another thing worth noting is that if we have a derivation of $C^\infty(M)$ then that corresponds to a vector field which is a field of vectors which can be restricted to any open set $U \subset M$ and thus we get a derivation of $C^\infty(U)$. The up shot of this is that if $X \in \mathfrak{X}(M)$ then X also acts as a derivation on $C^\infty(U)$ and if $f \in C^\infty(U)$ we write Xf instead of the more pedantic $\left.X\right|_U f$.

While it makes sense to talk of vector fields on M of differentiability r where $0 < r < \infty$ and these do act as derivations on $C^r(M)$, it is only in the smooth case ($r = \infty$) that we can say that vector fields account for *all* derivations of $C^r(M)$.

Theorem 2.3 *If* $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{Der}(C^\infty(M))$ *then* $[\mathcal{D}_1, \mathcal{D}_2] := \mathcal{D}_1 \circ \mathcal{D}_2 - \mathcal{D}_2 \circ \mathcal{D}_1 \in \mathrm{Der}(C^\infty(M))$.

Proof. We compute

$$\begin{aligned}
\mathcal{D}_1\left(\mathcal{D}_2\left(fg\right)\right) &= \mathcal{D}_1\left(\mathcal{D}_2(f)g + f\mathcal{D}_2(g)\right) \\
&= \left(\mathcal{D}_1\mathcal{D}_2 f\right)g + \mathcal{D}_2 f\mathcal{D}_1 g \\
&+ \mathcal{D}_1 f\mathcal{D}_2 g + f\mathcal{D}_1\mathcal{D}_2 g
\end{aligned}$$

Write out the similar expression for $\mathcal{D}_2\left(\mathcal{D}_1\left(fg\right)\right)$ and subtracting we obtain after fortuitous cancellation

$$\begin{aligned}
\left[\mathcal{D}_1, \mathcal{D}_2\right](fg) &= \left(\mathcal{D}_1\mathcal{D}_2 f\right)g + f\mathcal{D}_1\mathcal{D}_2 g \\
&- \left(\left(\mathcal{D}_2\mathcal{D}_1 f\right)g + f\mathcal{D}_2\mathcal{D}_1 g\right) \\
&= \left(\left[\mathcal{D}_1, \mathcal{D}_2\right]f\right)g + f[\mathcal{D}_1, \mathcal{D}_2]g
\end{aligned}$$

■

Corollary 2.1 *If* $X, Y \in \mathfrak{X}(M)$ *then there is a unique vector field* $[X, Y]$ *such that* $\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X$.

We ought to see what the local formula for the Lie derivative looks like in more conventional "index" notation. Suppose we have $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^i \frac{\partial}{\partial x^i}$. Then using the summation convention we have

$$[X, Y] = \sum_{ij}\left(\frac{\partial Y^i}{\partial x^j}X^j - \frac{\partial X^i}{\partial x^j}Y^j\right)\frac{\partial}{\partial x^i}$$

Exercise 2.9 *Verify this last formula.*

Definition 2.20 *The vector field* $[X,Y]$ *from the previous corollary is called the* ***Lie bracket*** *of* X *and* Y.

Proposition 2.3 *For any* $X, Y, Z \in \mathfrak{X}(M)$,

1. The map $(X,Y) \mapsto [X,Y]$ is bilinear over $\mathbb{R}$.
2. $[X,Y] = -[Y,X]$
3. $[X,[Y,Z]]+ [Y,[Z,X]] + [Z,[X,Y]] = 0$
4. $[fX, gY] = fg[X,Y] + f\,(Xg)\,Y - g\,(Yf)\,X$ for all $f, g \in C^\infty(M)$.

Proof. These results follows from direct (albeit tedious) calculation and the easily checked fact that $\mathcal{L}_{aX+bY} = a\mathcal{L}_X + b\mathcal{L}_Y$ for $a, b \in \mathbb{R}$ and $X, Y \in \mathfrak{X}(M)$. ∎

The map $(X,Y) \mapsto [X,Y]$ is linear over $\mathbb{R}$ but by number 4 it is not linear over $C^\infty(M)$.

The $\mathbb{R}-$vector space $\mathfrak{X}(M)$ together with the $\mathbb{R}-$bilinear map $(X,Y) \mapsto [X,Y]$ is an example of an extremely important abstract algebraic structure:

Definition 2.21 (Lie Algebra) *A vector space* $\mathfrak{a}$ *(over a field* $\mathbb{F}$*) is called a* ***Lie algebra*** *if it is equipped with a bilinear map* $\mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$ *(a multiplication) denoted* $(v,w) \mapsto [v,w]$ *such that*

$$[v,w] = -[w,v]$$

and such that we have the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

for all $x, y, z \in \mathfrak{a}$.

Notice that the Jacobian identity may be restated as $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ which just says that for fixed x the map $y \mapsto [x,y]$ is a derivation of the Lie algebra $\mathfrak{a}$. This is a possibly easier way to remember the identity besides being significant mathematically. The Lie algebra $\mathfrak{X}(M)$ is infinite dimensional (unless M is zero dimensional) but we will later be very interested in certain finite dimensional Lie algebras.

Given a diffeomorphism $\phi : M \to N$ we define the **pull-back** $\phi^*Y \in \mathfrak{X}(M)$ for $Y \in \mathfrak{X}(N)$ and the **push-forward** $\phi_*X \in \mathfrak{X}(N)$ of $X \in \mathfrak{X}(M)$ via ϕ by defining

$$\phi^*Y = T\phi^{-1} \circ Y \circ \phi \text{ and}$$
$$\phi_*X = T\phi \circ X \circ \phi^{-1}.$$

In other words, $(\phi^*Y)(p) = T\phi^{-1} \cdot Y_{\phi(p)}$ and $(\phi_*X)(p) = T\phi \cdot X_{\phi^{-1}(p)}$. Notice that ϕ^*Y and ϕ_*X are both smooth vector fields.

Let $\phi : M \to N$ be a smooth map of manifolds. The following commutative diagrams summarize some of mapping concepts introduced so far:

Pointwise:

$$\begin{array}{ccc} T_pM & \xrightarrow{T_p\phi} & T_{\phi p}N \\ \downarrow & & \downarrow \\ (M,p) & \xrightarrow{\phi} & (N,\phi(p)) \end{array}$$

Global:

$$\begin{array}{ccc} TM & \xrightarrow{T\phi} & TN \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & N \end{array}$$

If ϕ is a diffeomorphism then

$$\begin{array}{ccc} \mathfrak{X}(M) & \xrightarrow{\phi_*} & \mathfrak{X}(N) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & N \end{array}$$

and also

$$\begin{array}{ccc} \mathfrak{X}(M) & \xleftarrow{\phi^*} & \mathfrak{X}(N) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & N \end{array} .$$

Notice the arrow reversal. If $M = N$, this gives a right and left pair of actions of the diffeomorphism group $\mathrm{Diff}(M)$ on the space of vector fields: $\mathfrak{X}(M) = \Gamma(M, TM)$.

$$\begin{aligned} \mathrm{Diff}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (\phi_*, X) &\mapsto \phi_* X \end{aligned}$$

and

$$\begin{aligned} \mathfrak{X}(M) \times \mathrm{Diff}(M) &\to \mathfrak{X}(M) \\ (X, \phi) &\mapsto \phi^* X \end{aligned}$$

On functions the pull-back is defined by $\phi^* g := g \circ \phi$ but if ϕ is a diffeomorphism then we can also define a push-forward $\phi_* = (\phi^{-1})^*$. With this notation we have

Proposition 2.4 *The Lie derivative on functions is natural with respect to pull-back and push-forward by diffeomorphisms. In other words, if $\phi : M \to N$ is a diffeomorphism and $f \in C^\infty(M)$, $g \in C^\infty(N)$, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ then*

$$\mathcal{L}_{\phi^* Y} \phi^* g = \phi^* \mathcal{L}_Y g$$

and

$$\mathcal{L}_{\phi_* X} \phi_* f = \phi_* \mathcal{L}_X f$$

Proof.

$$\begin{aligned}(\mathcal{L}_{\phi^* Y}\phi^* g)(p) &= d(\phi^* g)\circ(\phi^* Y)(p)\\ &= (\phi^* dg)(T\phi^{-1}Y_{\phi p}) = dg(T\phi T\phi^{-1}Y_{\phi p})\\ &= dg(Y_{\phi p}) = \phi^*\left(dg\circ Y\right)(p) = (\phi^*\mathcal{L}_Y g)(p)\end{aligned}$$

The second statement follows from the first since $\phi_* = (\phi^{-1})^*$. ■

Now even if $f : M \to N$ is *not* a diffeomorphism it *may* still be that there is a vector field $Y \in \mathfrak{X}(N)$ such that

$$Tf \circ X = Y \circ f.$$

Or in other words, $Tf \cdot X_p = Y_{\phi(p)}$ for all p in M. In this case we say that Y is f**-related** to X and write $X \sim_f Y$.

Lemma 2.4 *Suppose that $f : M \to N$ a smooth map and $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Then X and Z are f-related if and only if $X(g \circ f) = (Yg)\circ f$ for all $g \in C^\infty(N)$.*

Proof. Let $p \in M$ and let $g \in C^\infty(N)$. Then

$$X(g \circ f)(p) = X_p(g \circ f) = (T_p f \cdot X_p)\, g$$

and

$$(Yg \circ f)\,(p) = Y_{f(p)}g$$

so that $X(g \circ f) = (Yg) \circ f$ for all such g if and only if $T_p f \cdot X_p = Y_{f(p)}$ ■

Proposition 2.5 *If $f : M \to N$ is a smooth map and X_i is f-related to Y_i for $i = 1, 2$ then $[X_1, X_2]$ is f-related to $[Y_1, Y_2]$. In particular, if ϕ is a diffeomorphism then $[\phi_* X_1, \phi_* X_2] = \phi_*[X_1, X_2]$ for all $X_1, X_2 \in \mathfrak{X}(M)$.*

Proof. We use the previous lemma: Let $g \in C^\infty(N)$. The $X_1X_2(g \circ f) = X_1((Y_2 g) \circ f) = (Y_1Y_2 g) \circ f$. In the same way $X_2X_1(g \circ f) = (Y_2Y_1 g) \circ f$ and subtracting we obtain

$$\begin{aligned}[X_1, X_2]\,(g \circ f) &= X_1X_2(g \circ f) - X_2X_1(g \circ f)\\ &= (Y_1Y_2 g) \circ f - (Y_2Y_1 g) \circ f\\ &= ([X_1, X_2]g) \circ f\end{aligned}$$

and so using the lemma one more time we have the result. ■

In case the map $f : M \to N$ is not a diffeomorphism we still have a result when two vector fields are f**-related**.

Theorem 2.4 *Let $f : M \to N$ be a smooth map and suppose that $X \sim_f Y$. Then we have for any $g \in C^\infty(N)$ $\mathcal{L}_X\,(f^* g) = f^*\mathcal{L}_Y g$.*

The proof is similar to the previous theorem and is left to the reader.

2.7.1 Integral curves and Flows

All flows of vector fields near points where the field doesn't vanish look the same.

A family of diffeomorphisms $\Phi_t : M \to M$ is called a **(global) flow** if $t \mapsto \Phi_t$ is a group homomorphism from the additive group $\mathbb{R}$ to the diffeomorphism group of M and such that $\Phi_t(x) = \Phi(t, x)$ gives a smooth map $\mathbb{R}\times M \to M$. A **local flow** is defined similarly except that $\Phi(t, x)$ may not be defined on all of $\mathbb{R}\times M$ but rather on some open neighborhood of $\{0\} \times M \subset \mathbb{R}\times M$ and so we explicitly require that

1. $\Phi_t \circ \Phi_s = \Phi_{t+s}$ and
2. $\Phi_t^{-1} = \Phi_{-t}$
   for all t and s such that both sides of these equations are defined.

Using a smooth local flow we can define a vector field X^Φ by

$$X^\Phi(p) = \left.\frac{d}{dt}\right|_0 \Phi(t, p) \in T_pM.$$

If one computes the velocity vector $\dot{c}(0)$ of the curve $c : t \mapsto \Phi(t, x)$ one gets $X^\Phi(x)$. On the other hand, if we are given a smooth vector field X in open set $U \subset M$ then we say that $c : (a, b) \to M$ is an **integral curve** for X if $\dot{c}(t) = X(c(t))$ for $t \in (a, b)$.

Our study begins with a quick recounting of a basic existence and uniqueness theorem for differential equations stated here in the setting of real Banach spaces. If desired, the reader may take the Banach space to be a finite dimensional normed space such as $\mathbb{R}^n$. The proof may be found in Appendix H.

Theorem 2.5 *Let E be a Banach space and let $F : U \subset \mathsf{E} \to \mathsf{E}$ be a smooth map. Given any $x_0 \in U$ there is a smooth curve $c : (-\epsilon, \epsilon) \to U$ with $c(0) = x_0$ such that $c'(t) = F(c(t))$ for all $t \in (-\epsilon, \epsilon)$. If $c_1 : (-\epsilon_1, \epsilon_1) \to U$ is another such curve with $c_1(0) = x_0$ and $c_1'(t) = F(c(t))$ for all $t \in (-\epsilon_1, \epsilon_1)$ then $c = c_1$ on the intersection $(-\epsilon_1, \epsilon_1) \cap (-\epsilon, \epsilon)$. Furthermore, there is an open set V with $x_0 \in V \subset U$ and a smooth map $\Phi : V \times (-a, a) \to U$ such that $t \mapsto c_x(t) := \Phi(x, t)$ is a curve satisfying $c'(t) = F(c(t))$ for all $t \in (-a, a)$.*

Now let $X \in \mathfrak{X}(M)$ and consider a point p in the domain of a chart $(U, \mathtt{x})$. The local expression for the integral curve equation $\dot{c}(t) = X(c(t))$ is of the form treated in the the last theorem and so we see that there certainly exists an integral curve for X through p defined on at least some small interval $(-\epsilon, \epsilon)$. We will now use this theorem to obtain similar but more global results on smooth manifolds. First of all we can get a more global version of uniqueness:

Lemma 2.5 *If $c_1 : (-\epsilon_1, \epsilon_1) \to M$ and $c_2 : (-\epsilon_2, \epsilon_2) \to M$ are integral curves of a vector field X with $c_1(0) = c_2(0)$ then $c_1 = c_2$ on the intersection of their domains.*

Proof. Let $K = \{t \in (-\epsilon_1, \epsilon_1) \cap (-\epsilon_2, \epsilon_2) : c_1(t) = c_2(t)\}$. The set K is closed since M is Hausdorff. If follows from the local theorem 2.5 that K contains a (small) open interval $(-\epsilon, \epsilon)$. Now let t_0 be any point in K and consider the translated curves $c_1^{t_0}(t) = c_1(t_0 + t)$ and $c_2^{t_0}(t) = c_2(t_0 + t)$. These are also integral curves of X and agree at $t = 0$ and by 2.5 again we see that $c_1^{t_0} = c_2^{t_0}$ on some open neighborhood of 0. But this means that c_1 and c_2 agree in this neighborhood so in fact this neighborhood is contained in K implying K is also open since t_0 was an arbitrary point in K. Thus, since $I = (-\epsilon_1, \epsilon_1) \cap (-\epsilon_2, \epsilon_2)$ is connected, it must be that $I = K$ and so c_1 and c_2 agree on $I = (-\epsilon_1, \epsilon_1) \cap (-\epsilon_2, \epsilon_2)$. ∎

Let X be a C^r vector field on M with $r \geq 1$. A **flow box** for X at a point $p_0 \in M$ is a triple (U, a, φ^X) where

1. U is an open set in M containing p.
2. $\varphi^X : U \times (-a, a) \to M$ is a C^r map and $0 < a \leq \infty$.
3. For each $p \in M$ the curve $t \mapsto c_p(t) = \varphi^X(p, t)$ is an integral curve of X with $c_p(0) = p$.
4. The map $\varphi_t^X : U \to M$ given by $\varphi_t^X(p) = \varphi^X(p, t)$ is a diffeomorphism onto its image for all $t \in (-a, a)$.

Now before we prove that flow boxes actually exist, we make the following observation: If we have a triple that satisfies 1-3 above then both $c_1 : t \mapsto \varphi_{t+s}^X(p)$ and $c_2 : t \mapsto \varphi_t^X(\varphi_s^X(p))$ are integral curves of X with $c_1(0) = c_2(0) = \varphi_s^X(p)$ so by uniqueness (Lemma 2.5) we conclude that $\varphi_t^X(\varphi_s^X(p)) = \varphi_{t+s}^X(p)$ as long as both sides are defined. This also shows that

$$\varphi_s^X \circ \varphi_t^X = \varphi_{t+s}^X = \varphi_t^X \circ \varphi_s^X$$

whenever defined. This is the local group property, so called because if φ_t^X were defined for all $t \in \mathbb{R}$ (and X a global vector field) then $t \mapsto \varphi_t^X$ would be a group homomorphism from $\mathbb{R}$ into $\mathrm{Diff}(M)$. Whenever this happens, that is, whenever φ_t^X is defined for all t, we say that X is a **complete vector field**. The group property also implies that $\varphi_t^X \circ \varphi_{-t}^X = \mathrm{id}$ and so in general φ_t^X must at least be a locally defined diffeomorphism with inverse φ_{-t}^X.

Exercise 2.10 *Show that on $\mathbb{R}^2$ the vector fields $y^2 \frac{\partial}{\partial x}$ and $x^2 \frac{\partial}{\partial y}$ are complete but $y^2 \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y}$ is not complete.*

Theorem 2.6 (Flow Box) *Let X be a C^r vector field on an n-manifold M with $r \geq 1$. Then for every point $p_0 \in M$ there exists a flow box for X at p_0. If (U_1, a_1, φ_1^X) and (U_2, a_2, φ_2^X) are two flow boxes for X at p_0, then $\varphi_1^X = \varphi_2^X$ on $(-a_1, a_1) \cap (-a_2, a_2) \times U_1 \cap U_2$.*

Proof. First of all notice that the U in the triple (U, a, φ^X) does not have to be contained in a chart or even homeomorphic to an open set in $\mathbb{R}^n$. However, to prove that there are flow boxes at any point we can work in the domain of a chart $(U, \mathtt{x})$ and so we might as well assume that the vector field is defined on an open set in $\mathbb{R}^n$. Of course, we may have to choose a to be smaller so that the flow stays within the range of the chart map $\mathtt{x}$. Now a vector field in this setting can be taken to be a map $U \to \mathbb{R}^n$ so the theorem 2.5 provides us with the flow box data (V, a, Φ) where we have taken $a > 0$ small enough that $V_t = \Phi(t, V) \subset U$ for all $t \in (-a, a)$. Now the flow box is transferred back to the manifold via $\mathtt{x}$

$$U = \mathtt{x}^{-1}(V)$$
$$\varphi^X(t, p) = \Phi(t, \mathtt{x}(p)).$$

Now if we have two such flow boxes (U_1, a_1, φ_1^X) and (U_2, a_2, φ_2^X) then by lemma 2.5 we have for any $x \in U_1 \cap U_2$ we must have $\varphi_1^X(t, x) = \varphi_2^X(t, x)$ for all $t \in (-a_1, a_1) \cap (-a_2, a_2)$.

Finally, since both $\varphi_t^X = \varphi^X(t, .)$ and $\varphi_{-t}^X = \varphi^X(-t, .)$ are both smooth and inverse of each other we see that φ_t^X is a diffeomorphism onto its image $U_t = \mathtt{x}^{-1}(V_t)$. ■

Now if $c_p(t)$ is an integral curve of X defined on some interval (a, b) with $c_p(0) = p$ then we may consider the limit

$$\lim_{t \to b-} c_p(t)$$

If this limit exists as a point $p_1 \in M$ then we may consider the integral curve c_{p_1} begining at p_1. One may now use Lemma 2.5 to combine $t \mapsto c_p(t)$ with $t \mapsto c_{p_1}(t - b)$ to produce an extended integral curve begining at p. We may repeat this process until the limit fails to exist and so extend the integral curve as far as possible. Similarly we may extend in the negative direction and so produce a maximal integral curve defined on a maximal interval $J_p := (T_{p,X}^-, T_{p,X}^+)$ where $T_{p,X}^-$ might be $-\infty$ and $T_{p,X}^+$ might be $+\infty$.

Definition 2.22 *Let X be a C^r vector field on M with $r \geq 1$. For any given $p \in M$ let $J_p^{\max} := (T_{p,X}^-, T_{p,X}^+) \subset \mathbb{R}$ be the largest interval (as above) such that there is an integral curve $c : J_p^{\max} \to M$ of X with $c(0) = p$. The* ***maximal flow*** *φ^X is defined on the set (called the maximal flow domain)*

$$\mathcal{D}_X = \bigcup_{p \in M} J_p^{\max} \times \{p\}.$$

by the prescription that $t \mapsto \varphi^X(t, p)$ is the maximal integral curve of X such that $\varphi^X(0, p) = p$

Theorem 2.7 *For $X \in \mathfrak{X}(M)$, the set $\mathcal{D}_X$ is an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$ and the map $\varphi^X : \mathcal{D}_X \to M$ is smooth. Furthermore,*

$$\varphi^X(t + s, p) = \varphi^X(t, \varphi^X(s, p)) \tag{2.1}$$

whenever both sides are defined. If the left hand side is defined then the right hand side is defined. If $t, s \geq 0$ or $t, s \leq 0$ then if the right hand side is defined so is the left hand side.

Proof. If the left hand side of the equation 2.1 is defined then it is defined for $t + s$ in some small interval and we may differentiate

$$\begin{aligned} \frac{d}{dt}\varphi^X(t+s,p) &= \left.\frac{d}{du}\right|_{u=s+t} \varphi^X(u,p) \\ &= X(\varphi^X(t+s,p)) \end{aligned}$$

we also have $\left.\varphi^X(t+s,p)\right|_{t=0} = \varphi^X(s,p)$ so $t \mapsto \varphi^X(t+s,p)$ is an integral curve starting at $q = \varphi^X(s,p)$ which means that $\varphi^X(t, \varphi^X(s,p))$ is defined and $\varphi^X(t+s,p) = \varphi^X(t, \varphi^X(s,p))$.

Now let us assume that $t, s \geq 0$ and that $\varphi^X(s,p)$ is defined and that $\varphi^X(t, \varphi^X(s,p))$ is defined. Then $\varphi^X(u,p)$ is defined for $u \in (0, s+\epsilon_1)$ for some small $\epsilon_1 > 0$ and $\varphi^X(u-s, \varphi^X(s,p))$ is defined for $u - s \in (0, t+\epsilon_2)$ for some small $\epsilon_2 > 0$. Now define

$$c_p(u) := \begin{cases} \varphi^X(u,p) \text{ for } u \leq s \\ \varphi^X(u-s, \varphi^X(s,p)) \text{ for } s \leq u \leq s+t \end{cases}$$

For $u \leq s$ we have

$$\frac{d}{du}c_p(u) = \frac{d}{du}\varphi^X(u,p) = X(\varphi^X(u,p)) = X(c_p(u))$$

On the other hand, for $s \leq u \leq s+t$ we also have

$$\frac{d}{du}c_p(u) = \frac{d}{du}\varphi^X(u-s, \varphi^X(s,p)) = X(c_p(u))$$

and since $c_p(0) = p$ we conclude that $\varphi^X(u,p)$ exist for $0 \leq u \leq s+t$ and that $c_p(u) = \varphi^X(u,p)$ for all such u. This means that $\varphi^X(u-s, \varphi^X(s,p)) = \varphi^X(u,p)$ for $0 \leq u \leq s+t$ and so $\varphi^X(s+t,p)$ exists and

$$\varphi^X(t, \varphi^X(s,p)) = \varphi^X(s+t,p)$$

The fact that $\mathcal{D}_X$ is open follows from Theorem 2.6 since the domains of flow boxes must be contained in $\mathcal{D}_X$.

Now let I_p be the set of all $t \in \mathbb{R}$ such that φ^X is defined and smooth on some open neighborhood of $I_t \times \{p\}$ where $I_t = [0,t]$ if $t \geq 0$ and $I_t = [t,0]$ if $t < 0$. We will be done if we can show that I_p is equal to $J_p^{\max}$. We will show that I_p is nonempty and both open and closed in $J_p^{\max}$. Notice that by uniqueness the restriction of φ^X to the domain of any local flow box must coincide with that local flow which is smooth and so I_p is not empty since it therefore contains some neighborhood of 0. Let $t_0 \in J_p^{\max} \cap (\overline{I_p} \backslash I_p)$. Clearly, $t_0 \neq 0$. Let us assume for concreteness that $t_0 > 0$ (the $t_0 < 0$ case is similar). Again since we have coincidence with the local flows, φ^X must be smooth on a neighborhood O of $[-\epsilon, \epsilon] \times \{\varphi^X(t_0, p)\}$ for some small $\epsilon > 0$. Also, by our choice of t_0, the flow φ^X must be smooth on a neighborhood of $[0, t_0 - \epsilon] \times \{p\}$. We now show that φ^X is smooth on some neighborhood of $[0, t_0] \times \{p\}$. Indeed, for (t, x) sufficiently close to $[0, t_0] \times \{p\}$ we may write $(t, x) = (t' + t'', x)$ in such a way that (t'', x) is near$[0, t_0 - \epsilon] \times \{p\}$ and (t', x) is near $[-\epsilon, \epsilon] \times \{\varphi^X(t_0, p)\}$. Then we have

$$\varphi^X(t' + t'', x) = \varphi^X(t', \varphi^X(t'', x))$$

which shows that φ^X is smooth at (t, x). But this means that $t_0 \in I_p$ after all which is a contradiction. Thus I_p is closed and thus equal to $J_p^{\max}$. ■

Remark 2.5 *Up until now we have used the notation φ^X ambiguously to refer to any (local or global) flow of X and now we have used the same notation for the* unique *maximal flow defined on $\mathcal{D}_X$. We could have introduced notation such as $\varphi^X_{\max}$ but prefer not to clutter up the notation to that extent unless necessary. We hope that the reader will be able to tell from context what we are referring to when we write φ^X.*

If φ^X is the maximal flow of X then we would like to write φ^X_t for the map $p \mapsto \varphi^X_t(p)$. The domain of this map is $\mathcal{D}^t_X = \{p : t \in (T^-_{p,X}, T^+_{p,X})\}$. Note well that, in general, the domain of φ^X_t depends on t.

Exercise 2.11 *Let s and t be real numbers. Show that the domain of $\varphi^X_s \circ \varphi^X_t$ is contained in $\mathcal{D}^{s+t}_X$ and show that for each t, $\mathcal{D}^t_X$ is open.*

X is a **complete** vector field if and only if $\mathcal{D}_X = \mathbb{R} \times M$.

Definition 2.23 *The **support** of a vector field X is the* closure *of the set $\{p : X(p) \neq 0\}$ and is denoted* $\mathrm{supp}(X)$.

Lemma 2.6 *Every vector field that has compact support is a complete vector field. In particular if M is compact then every vector field is complete.*

Proof. Let c^X_p be the maximal integral curve through p and $J_p^{\max} = (T^-_{p,X}, T^+_{p,X})$ its domain. It is clear that for any $t \in (T^-_{p,X}, T^+_{p,X})$ the image point $c^X_p(t)$ must always lie in the support of X. But we show that if $T^+_{p,X} < \infty$ then given any compact set $K \subset M$, for example the support of X, there is an $\epsilon > 0$ such that for all $t \in (T^+_{p,X} - \epsilon, T^+_{p,X})$ the image $c^X_p(t)$ is outside K. If not then we may take a sequence t_i converging to $T^+_{p,X}$ such that $c^X_p(t_i) \in K$. But then going to a subsequence if necessary we have $x_i := c^X_p(t_i) \to x \in K$. Now there must be a flow box (U, a, x) so for large enough k, we have that t_k is within a of $T^+_{p,X}$ and $x_i = c^X_p(t_i)$ is inside U. We then a guaranteed to have an integral curve $c^X_{x_i}(t)$ of X that continues beyond $T^+_{p,X}$ and thus can be used to extend c^X_p a contradiction of the maximality of $T^+_{p,X}$. Hence we must have $T^+_{p,X} = \infty$. A similar argument give the result that $T^-_{p,X} = -\infty$. ■

Exercise 2.12 *Let $a > 0$ be any positive real number. Show that if for a given vector field X the flow φ^X is defined on $(-a, a) \times M$ then in fact the (maximal) flow is defined on $\mathbb{R} \times M$ and so X is a complete vector field.*

2.8 Lie Derivative

Let X be a vector field on M and let $\varphi^X(p,t) = \varphi_p^X(t) = \varphi_t^X(p)$ be the flow so that

$$\left.\frac{d}{dt}\right|_0 \varphi_p^X(t) = T_0\varphi_p^X \left.\frac{\partial}{\partial t}\right|_0 = X_p$$

Recall our definition of the Lie derivative of a function (2.19). The following is an alternative definition.

Definition 2.24 *For a smooth function $f : M \to \mathbb{R}$ and a smooth vector field $X \in \mathfrak{X}(M)$ define the Lie derivative $\mathcal{L}_X$ of f with respect to X by*

$$\begin{aligned}\mathcal{L}_X f(p) &= \left.\frac{d}{dt}\right|_0 f \circ \varphi^X(p,t) \\ &= X_p f\end{aligned}$$

Exercise 2.13 *Show that this definition is compatible with definition 2.19.*

We now introduce the important concept of the Lie derivative of a vector field extending the previous definition. The Lie derivative will be extend further to tensor fields.

Definition 2.25 *Given a vector field X we define a map $\mathcal{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by*

$$\mathcal{L}_X Y := [X,Y]$$

this map is called the ***Lie derivative****.*

We will further characterized the Lie derivative but we need a technical lemma:

Lemma 2.7 *Let $X \in \mathfrak{X}(M)$ and $f \in C^\infty(U)$ with U open and $p \in U$. There is an interval $I_\delta := [-\delta, \delta]$ and an open set V containing p such that $\varphi^X(I_\delta \times V) \subset U$ and a function $g \in C^\infty(I_\delta \times V)$ such that*

$$f(\varphi^X(t,q)) = f(q) + tg(t,q)$$

for all $(t,q) \in I_\delta \times V$ and such that $g(0,q) = X_q f$ for all $q \in V$.

Proof. The existence of the set $I_\delta \times V$ with $\varphi^X(I_\delta \times V) \subset U$ follows from our study of flows. The function $r(\tau, q) := f(\varphi^X(\tau,q)) - f(q)$ is smooth on $I_\delta \times V$ and $r(0,q) = 0$. Let

$$g(t,q) := \int_0^1 \frac{\partial r}{\partial \tau}(st,q)ds$$

so that

$$tg(t,q) = \int_0^1 \frac{\partial r}{\partial \tau}(st,q)tds = \int_0^1 \frac{\partial}{\partial t} r(st,q)ds = r(t,q)$$

Then $f(\varphi^X(t,q)) = f(q) + tg(t,q)$. Also

$$g(0,q) = \lim_{t\to 0} \frac{1}{t} r(t,q) = \lim_{t\to 0} \frac{f(\varphi^X(t,q)) - f(q)}{t} = X_q$$

■

Proposition 2.6 *Let X and Y be smooth vector fields on M. Let $\varphi = \varphi^X$ be the flow. The function $t \mapsto T\varphi_{-t} \cdot Y_{\varphi_t(p)}$ is differentiable at $t = 0$ and*

$$\left.\frac{d}{dt}\right|_{t=0} T\varphi_{-t} \cdot Y_{\varphi_t(p)} = [X,Y]_p = \left(\mathcal{L}_X Y\right)(p) \tag{2.2}$$

Proof. Let $f \in C^\infty(U)$ with $p \in U$ as in the previous lemma.

$$\frac{T\varphi_{-t} \cdot Y_{\varphi_t(p)} - Y_p}{t} f = \frac{Y_p f - \left(T\varphi \cdot Y_{\varphi_{-t}(p)}\right) f}{t} =$$
$$\frac{Y_p f - Y_{\varphi_{-t}(p)}\left(f \circ \varphi\right)}{t} = \frac{Y_p f - Y_{\varphi_{-t}(p)}\left(f + t g_t\right)}{t}$$

where g is as in the lemma and $g_t(q) = g(t,q)$. Continuing, we have

$$\frac{Y_p f - Y_{\varphi_{-t}(p)}\left(f + t g_t\right)}{t} = \frac{(Yf)(\varphi_t(p)) - (Yf)(p)}{t} - Y_{\varphi_t(p)} g_t$$

Taking the limit as $t \to 0$ and recalling that $g_0 = Xf$ on V we obtain the result that

$$\left.\frac{d}{dt}\right|_{t=0} \left(T\varphi_{-t} \cdot Y_{\varphi_t(p)}\right) f = [X,Y]_p f$$

for all $f \in C^\infty(U)$. Now if we let U be the domain of a chart $(U, \mathtt{x})$ then letting f be each of the coordinate functions we see that each component T_pM-valued function $t \to T\varphi_{-t} \cdot Y_{\varphi_t(p)}$ is differentiable at $t = 0$ and so the function is also. Then we can conclude that $\left.\frac{d}{dt}\right|_{t=0} \left(T\varphi_{-t} \cdot Y_{\varphi_t(p)}\right) = [X,Y]_p$. ■

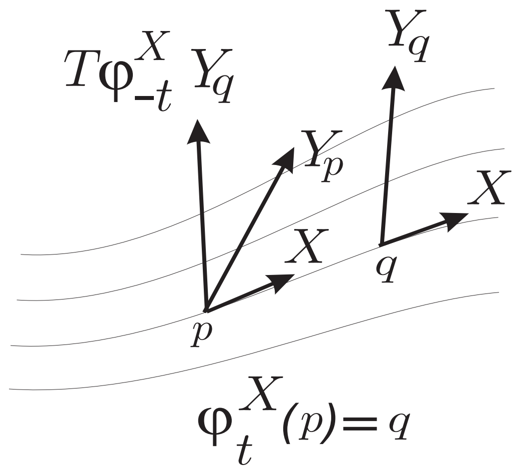
Discussion: Notice that if X is a complete vector field then for each $t \in \mathbb{R}$ the map φ_t^X is a diffeomorphism $M \to M$ and we may define $(\varphi_t^{X*} Y)(p) := (T\varphi_t^X)^{-1} Y(\varphi_t^X(p))$ or

$$\varphi_t^{X*} Y = (T\varphi_t^X)^{-1} \circ Y \circ \varphi_t. \tag{2.3}$$

One may write

$$\mathcal{L}_X Y = \left.\frac{d}{dt}\right|_0 (\varphi_t^{X*} Y)$$

On the other hand, if X is not complete then there exist no t such that φ_t^X is a diffeomorphism of M since for any specific t there might be points of M for which φ_t^X is not even defined! For an X which is not necessarily complete it is best to consider the map $\varphi^X : (t,x) \longmapsto \varphi^X(t,x)$ which is defined on some open



neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$ which just doesn't need to contain any set of form $(0, \epsilon] \times M$ unless $\epsilon = 0$. In fact, suppose that the domain of φ^X contains such an set with $\epsilon > 0$. It follow that for all $0 \leq t \leq \epsilon$ the map φ_t^X is defined on all of M and $\varphi_t^X(p)$ exists for $0 \leq t \leq \epsilon$ independent of p. But now a standard argument shows that $t \mapsto \varphi_t^X(p)$ is defined for *all* t which means that X is a complete vector field. If X is not complete we really have no business writing φ_t^X without some qualification. Despite this it has become common to write this expression anyway especially when we are taking a derivative with respect to t. Whether or not this is just a mistake or liberal use of notation is not clear. Here is what we can say. Given any relatively compact open set $U \subset M$, the map φ_t^X will be defined at least for all $t \in (-\varepsilon, \varepsilon)$ for some ε depending only on X and the choice of U. Because of this, the expression $\varphi_t^{X*} Y = (T_p \varphi_t^X)^{-1} \circ Y \circ \varphi_t$ is a well defined map on U for all $t \in (-\varepsilon, \varepsilon)$. Now if our manifold has a cover by relatively compact open sets $M = \bigcup U_i$ then we can make sense of $\varphi_t^{X*} Y$ on as large a relatively compact set we like as long as t is small enough. Furthermore, if $\varphi_t^{X*} Y\big|_{U_i}$ and $\varphi_t^{X*} Y\big|_{U_j}$ are both defined for the same t then they both restrict to $\varphi_t^{X*} Y\big|_{U_i \cap U_j}$. So $\varphi_t^{X*} Y$ makes sense point by point for small enough t. At any rate $t \mapsto (\varphi_t^{X*} Y)(p)$ has a well defined germ at $t = 0$. With this in mind we might still write $\mathcal{L}_X Y = \frac{d}{dt}\big|_0 (\varphi_t^{X*} Y)$ even for vector fields that are not complete as long as we take a loose interpretation of $\varphi_t^{X*} Y$.

Theorem 2.8 *Let X, Y be vector fields on a smooth manifold M. Then*

$$\frac{d}{dt} \varphi_{\mathbf{t}}^{\mathbf{X}*} Y = \varphi_t^{X*} (\mathcal{L}_X Y)$$

Proof.

$$\begin{aligned} \left.\frac{d}{dt}\right|_t \varphi_t^{X*}Y &= \left.\frac{d}{ds}\right|_0 \varphi_{t+s}^{X*}Y \\ &= \left.\frac{d}{ds}\right|_0 \varphi_t^{X*}(\varphi_s^{X*}Y) \\ &= \varphi_t^{X*}\left.\frac{d}{ds}\right|_0 (\varphi_s^{X*}Y) \\ &= \varphi_t^{X*}\mathcal{L}_X Y \end{aligned}$$

■

Now we can see that the infinitesimal version of the action

$$\begin{aligned} \mathfrak{X}(M) \times \mathrm{Diff}(M) &\to \mathfrak{X}(M) \\ (X, \varphi) &\mapsto \varphi^* X \end{aligned}$$

is just the Lie derivative. As for the infinitesimal version of the left action of $\mathrm{Diff}(M)$ on $\mathfrak{X}(M)$ we have for $X, Y \in \mathfrak{X}(M)$

$$\begin{aligned} \left.\frac{d}{dt}\right|_0 (\varphi_{t*}^X Y)(p) &= \left.\frac{d}{dt}\right|_0 T\varphi_t^X(Y(\varphi_t^{-1}(p))) \\ &= -\left.\frac{d}{dt}\right|_0 (T\varphi_{-t}^X)^{-1} Y(\varphi_{-t}(p)) \\ &= -(L_X Y) = -[X,Y] \end{aligned}$$

Proposition 2.7 *Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ be ϕ-related vector fields for a smooth map $\phi : M \to N$. Then*

$$\phi \circ \varphi_t^X = \varphi_t^Y \circ \phi$$

whenever both sides are defined. Suppose that $\phi : M \to M$ is a diffeomorphism and $X \in \mathfrak{X}(M)$. Then the flow of $\phi_* X = (\phi^{-1})^* X$ is $\phi \circ \varphi_t^X \circ \phi^{-1}$ and the flow of $\phi^* X$ is $\phi^{-1} \circ \varphi_t^X \circ \phi$.

Proof. Differentiating we have $\frac{d}{dt}(\phi \circ \varphi_t^X) = T\phi \circ \frac{d}{dt}\varphi_t^X = T\phi \circ X \circ \varphi_t^X = Y \circ \phi \circ \varphi_t^X$. But $\phi \circ \varphi_0^X(x) = \phi(x)$ and so $t \mapsto \phi \circ \varphi_t^X(x)$ is an integral curve of Y starting at $\phi(x)$. By uniqueness we have $\phi \circ \varphi_t^X(x) = \varphi_t^Y(\phi(x))$. ■

Theorem 2.9 *For $X, Y \in \mathfrak{X}(M)$ each of the following are equivalent:*

1. $\mathcal{L}_X Y = [X, Y] = 0$
2. $(\varphi_t^X)^* Y = Y$
3. *The flows of X and Y commute:*

$$\varphi_t^X \circ \varphi_s^Y = \varphi_s^Y \circ \varphi_t^X \text{ whenever defined.}$$

Proof. The equivalence of 1 and 2 is follows easily from Proposition 2.6 and Theorem 2.8. The equivalence of 2 and 3 can be seen by noticing that $\varphi_t^X \circ \varphi_s^Y = \varphi_s^Y \circ \varphi_t^X$ is true and defined exactly when $\varphi_s^Y = \varphi_t^X \circ \varphi_s^Y \circ \varphi_t^X$ which happens exactly when

$$\varphi_s^Y = \varphi_s^{(\varphi_t^X)^* Y}$$

and in turn exactly when $Y = (\varphi_t^X)^* Y$. ∎

The Lie derivative and the Lie bracket are essentially the same object and are defined for local sections $X \in \mathfrak{X}_M(U)$ as well as global sections. This is obvious anyway since open subsets are themselves manifolds. As is so often the case for operators in differential geometry, the Lie derivative is natural with respect to restriction so we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{X}(U) & \stackrel{\mathcal{L}_{X|_U}}{\longrightarrow} & \mathfrak{X}(U) \\ r_V^U \downarrow & & \downarrow r_V^U \\ \mathfrak{X}(V) & \stackrel{\mathcal{L}_{X|_U}}{\longrightarrow} & \mathfrak{X}(V) \end{array}$$

where $X|_U$ denotes the restriction of $X \in \mathfrak{X}(M)$ to the open set U and r_V^U is the map that restricts from U to $V \subset U$.

2.9 Time Dependent Fields

Definition 2.26 *A C^∞* ***time dependent vector field*** *on M is a C^∞ map $X : (a,b) \times M \to TM$ such that for each fixed $t \in (a,b) \subset \mathbb{R}$ the map $X_t : M \to TM$ given by $X_t(x) := X(t,x)$ is a C^∞ vector field.*

Definition 2.27 *Let X be a time dependent vector field. A curve $c : (a,b) \to M$ is called an* ***integral curve*** *of X if and only if*

$$\dot{c}(t) = X(t, c(t)) \text{ for all } t \in (a,b).$$

One can study time dependent vector fields by studying their so called suspensions. Let $pr_1 : (a,b) \times M \to (a,b)$ and $pr_2 : (a,b) \times M \to M$ be the projection maps. Let $\widetilde{X} \in \mathfrak{X}((a,b) \times M)$ be defined by $\widetilde{X}(t,p) = (\frac{\partial}{\partial t}, X(t,p)) \in T_t(a,b) \times T_pM = T_{(t,p)}((a,b) \times M)$. The vector field $\widetilde{X}$ is called the suspension of X. It can be checked quite easily that if $\widetilde{c}$ is an integral curve of $\widetilde{X}$ then $c := pr_2 \circ \widetilde{c}$ is an integral curve of the time dependent field X. This allows us to use what we know about integral curves to the time dependent case.

Definition 2.28 *The* ***evolution operator*** *$\Phi_{t,s}^X$ for X is defined by the requirement that*

$$\frac{d}{dt}\Phi_{t,s}^X(x) = X(t, \Phi_{t,s}^X(x)) \text{ and } \Phi_{s,s}^X(x) = x.$$

In other words, $t \mapsto \Phi_{t,s}^X(x)$ is the integral curve that goes through x at time s.

We have chosen to use the term "evolution operator" as opposed to "flow" in order to emphasize that the local group property does not hold in general. Instead we have the following

Theorem 2.10 *Let X be a time dependent vector field. Suppose that $X_t \in \mathfrak{X}(M)$ for each t and that $X : (a,b) \times M \to TM$ is continuous. Then $\Phi^X_{t,s}$ is C^∞ and we have $\Phi^X_{s,a} \circ \Phi^X_{a,t} = \Phi^X_{s,t}$ whenever defined.*

Exercise 2.14 *If $\Phi^X_{t,s}$ is the evolution operator of X then the flow of the supsension $\widetilde{X}$ is given by*

$$\Phi(t,(s,p)) := (t+s, \Phi^X_{t+s,s}(p))$$

Let $\phi_t(p) := \Phi_{0,t}(p)$. Is it true that $\phi_s \circ \phi_t(p) = \phi_{s+t}(p)$? The answer is that in general this equality does *not* hold. The evolution of a time dependent vector field does *not* give rise to is local 1-parameter group diffeomorphism. On the other hand, we do have

$$\Phi_{s,r} \circ \Phi_{r,t} = \Phi_{s,t}$$

which is called the Chapman-Kolmogorov law. If in a special case $\Phi_{r,t}$ depends only on $s-t$ then setting $\phi_t := \Phi_{0,t}$ we recover a flow corresponding to a time-independent vector field. We need a lemma. In the following, note the reversal of the order of s and t across the equal sign:

Theorem 2.11 *Let X and Y be smooth time dependent vector fields and let $f : \mathbb{R}\times M \to \mathbb{R}$ be smooth. We have the following formulas:*

$$\frac{d}{dt}(\Phi^X_{t,s})^* f = (\Phi^X_{t,s})^*(X_t f + \frac{\partial f}{\partial t})$$

and

$$\frac{d}{dt}(\Phi^X_{t,s})^* Y_t = (\Phi^X_{t,s})^*([X_t, Y_t] + \frac{\partial Y}{\partial t}).$$

Proof. Let f_t denote the function $f(t,.)$. Consider the map $(u,v) \mapsto \Phi^X_{u,s} f_v$.

If we let $u(t) = t, v(t) = t$ and compose, then by the chain rule

$$\begin{aligned}
\frac{d}{dt}\left(\left(\Phi^X_{u,s}\right)^* f_v\right)(p) &= \left.\frac{\partial}{\partial u}\right|_{(u,v)=(t,t)} \left(\left(\Phi^X_{u,s}\right)^* f_v\right)(p) + \left.\frac{\partial}{\partial v}\right|_{(u,v)=(t,t)} \left(\left(\Phi^X_{u,s}\right)^* f_v\right)(p) \\
&= \left.\frac{d}{du}\right|_{u=t} \left(\left(\Phi^X_{u,s}\right)^* f_t\right)(p) + \left.\frac{d}{dv}\right|_{v=t} \left(\left(\Phi^X_{t,s}\right)^* f_v\right)(p) \\
&= \left.\frac{d}{du}\right|_{u=t} \left(f_t \circ \Phi^X_{u,s}\right)(p) + \left(\left(\Phi^X_{t,s}\right)^* \frac{\partial f}{\partial t}\right)(p) \\
&= df_t|_{\Phi^X_{u,s}(p)} \cdot \left.\frac{d}{du}\right|_{u=t} \Phi^X_{u,s}(p) + \left(\left(\Phi^X_{t,s}\right)^* \frac{\partial f}{\partial t}\right)(p) \\
&= df_t|_{\Phi^X_{u,s}(p)} \cdot X_t(\Phi^X_{t,s}(p)) + \left(\left(\Phi^X_{t,s}\right)^* \frac{\partial f}{\partial t}\right)(p) \\
&= (X_t f)\left(\Phi^X_{t,s}(p)\right) + \left(\left(\Phi^X_{t,s}\right)^* \frac{\partial f}{\partial t}\right)(p) \\
&= \left(\Phi^X_{t,s}\right)^* (X_t f)(p) + \left(\left(\Phi^X_{t,s}\right)^* \frac{\partial f}{\partial t}\right)(p) = (\Phi^X_{t,s})^*(X_t f + \frac{\partial f}{\partial t})
\end{aligned}$$

Note that a similar but simpler proof shows that if $f \in C^\infty(M)$ then

$$\frac{d}{dt}(\Phi^X_{t,s})^* f = (\Phi^X_{t,s})^*(X_t f) \qquad (*)$$

Claim: $\frac{d}{dt}(\Phi^X_{s,t})^* f = -X_t\{\left(\Phi^X_{s,t}\right)^* f\}$. Proof of claim: Let $g := \left(\Phi^X_{s,t}\right)^* f$. Fix p and consider the map $(u,v) \mapsto \left((\Phi^X_{s,u})^* \left(\Phi^X_{v,s}\right)^* g\right)(p)$. If we let $u(t) = t, v(t) = t$ then the composed map $t \mapsto \left((\Phi^X_{s,t})^* \left(\Phi^X_{t,s}\right)^* g\right)(p) = p$ is constant. Thus by the chain rule

$$\begin{aligned}
0 &= \frac{d}{dt}\left((\Phi^X_{s,t})^* \left(\Phi^X_{t,s}\right)^* g\right)(p) \\
&= \left.\frac{\partial}{\partial u}\right|_{(u,v)=(t,t)} \left[\left((\Phi^X_{s,u})^* \left(\Phi^X_{v,s}\right)^* g\right)(p)\right] + \left.\frac{\partial}{\partial v}\right|_{(u,v)=(t,t)} \left[\left((\Phi^X_{s,u})^* \left(\Phi^X_{v,s}\right)^* g\right)(p)\right] \\
&= \left.\frac{d}{du}\right|_{u=t} \left[\left((\Phi^X_{s,u})^* \left(\Phi^X_{t,s}\right)^* g\right)(p)\right] + \left.\frac{d}{dv}\right|_{v=t} \left[\left((\Phi^X_{s,t})^* \left(\Phi^X_{v,s}\right)^* g\right)(p)\right] \\
&= \left.\frac{d}{du}\right|_{u=t} \left[\left((\Phi^X_{s,u})^* f\right)(p)\right] + (\Phi^X_{s,t})^*(\Phi^X_{t,s})^* X_t g \quad \text{(using (*))} \\
&= \frac{d}{dt}\left[\left((\Phi^X_{s,t})^* f\right)(p)\right] + X_t g \\
&= \frac{d}{dt}\left[\left((\Phi^X_{s,t})^* f\right)(p)\right] + X_t[\left(\Phi^X_{s,t}\right)^* f]
\end{aligned}$$

This proves the claim. Next, note that by Proposition 2.4we have that since $\Phi^X_{s,t} = \left(\Phi^X_{t,s}\right)^{-1}$

$$\left(\left(\Phi^X_{t,s}\right)^* Y\right) f = \left(\Phi^X_{t,s}\right)^* \left(Y \left(\Phi^X_{s,t}\right)^* f\right) \qquad (**)$$

for $Y \in \mathfrak{X}(M)$ and smooth f. This last equation still holds for $Y_t = Y(t, .)$ for a time dependent Y.

Next consider a time dependent vector field Y. We wish to compute $\frac{d}{dt}\left(\left(\Phi^X_{t,s}\right)^* Y_t\right)$ using the chain rule as before we have

$$\begin{aligned}
\frac{d}{dt}\left(\left(\Phi^X_{t,s}\right)^* Y_t\right) f &= \left.\frac{d}{du}\right|_{u=t}\left(\left(\Phi^X_{u,s}\right)^* Y_t\right) f + \left.\frac{d}{dv}\right|_{v=t}\left(\left(\Phi^X_{t,s}\right)^* Y_v\right) f \\
&= \left.\frac{d}{du}\right|_{u=t}\left(\Phi^X_{u,s}\right)^*\left(Y_t\left(\Phi^X_{s,u}\right)^* f\right) + \left.\frac{d}{dv}\right|_{v=t}\left(\left(\Phi^X_{t,s}\right)^* Y_v\right) \text{ (using (**))} \\
&= \frac{d}{dt}\left(\Phi^X_{t,s}\right)^*\left(Y_t\left(\Phi^X_{s,t}\right)^* f\right) + \left(\left(\Phi^X_{t,s}\right)^* \frac{\partial Y}{\partial t}\right) f \\
&\frac{d}{dt}\left(\Phi^X_{t,s}\right)^*\left(Y_t\left(\Phi^X_{s,t}\right)^* f\right) + \left(\left(\Phi^X_{t,s}\right)^* \frac{\partial Y}{\partial t}\right) f \\
&= \left.\frac{d}{du}\right|_{u=t}\left(\Phi^X_{u,s}\right)^*\left(Y_t\left(\Phi^X_{s,t}\right)^* f\right) + \left.\frac{d}{dv}\right|_{v=t}\left(\Phi^X_{t,s}\right)^*\left(Y\left(\Phi^X_{s,v}\right)^* f\right) + \left(\left(\Phi^X_{t,s}\right)^* \frac{\partial Y}{\partial t}\right) f \\
&= \left(\Phi^X_{t,s}\right)^* X_t\left(Y_t\left(\Phi^X_{s,t}\right)^* f\right) - \left(\Phi^X_{t,s}\right)^* Y_t\left(X_t\left(\Phi^X_{s,t}\right)^* f\right) + \left(\left(\Phi^X_{t,s}\right)^* \frac{\partial Y}{\partial t}\right) f \\
&= \left(\Phi^X_{t,s}\right)^*\left([X_t, Y_t]\left(\Phi^X_{s,t}\right)^* f\right) + \left(\Phi^X_{s,t}\right)^* \frac{\partial Y}{\partial t} f \\
&= \left(\left(\Phi^X_{t,s}\right)^*\left[[X_t, Y_t] + \frac{\partial Y}{\partial t}\right]\right) f \text{ (using (**) again on } [X_t, Y_t])
\end{aligned}$$

■

2.10 Covector Fields

Recall the notation $\pi_{T^*M} : T^*M \to M$ for the projection map for the tangent bundle. Recall that a map $\alpha : M \to T^*M$ such that $\pi_{T^*M} \circ \alpha = id$ is a called a section of the cotangent bundle.

Definition 2.29 *A smooth (resp. C^r) section of the cotangent bundle is called a smooth (resp. C^r)* ***covector field*** *or also a smooth (resp. C^r)* ***1-form*** *. The set of all C^r 1-forms is denoted by $\mathfrak{X}^{r*}(M)$ and the smooth 1-forms are denoted by $\mathfrak{X}^*(M)$.*

Later we will have reason to denote $\mathfrak{X}^*(M)$ also by $\Omega^1(M)$.

Definition 2.30 *Let $f : M \to \mathbb{R}$ be a C^r function with $r \geq 1$. The map $df : M \to T^*M$ defined by $p \mapsto df(p)$ where $df(p)$ is the differential at p as defined in definitions 2.9 and 2.10. df is a 1-form called the differential of f.*

Three views on a 1-form: The novice may easily become confused about what should be the argument of a 1-form or covector field. The reason for this is that one can view a 1-form in at least three different ways. If α is a smooth one form then

1. We may view α as map $\alpha : M \to T^*M$ (as in the definition) so that $\alpha(.)$ takes points as arguments; $\alpha(p) \in T_p^*M$. We also sometimes need to write $\alpha(p) = \alpha_p$ just as for a vector field X we sometimes write $X(p)$ as X_p.

2. We may view α as a smooth map $\alpha : TM \to \mathbb{R}$ so that for $v \in T_pM$ we make sense of $\alpha(v)$ by $\alpha(v) = \alpha_p(v)$.

3. We may view α as a map $\alpha : \mathfrak{X}(M) \to C^\infty(M)$ where for $X \in \mathfrak{X}(M)$ we interpret $\alpha(X)$ as the smooth function $p \mapsto \alpha_p(X_p)$.

The second and third interpretation are dependent on the first.

Notice that given a chart $(U, \mathtt{x})$ with $\mathtt{x} = (x^1, ..., x^n)$ then $dx^i : p \mapsto dx^i\big|_p$ defines covector fields on U such that $dx^1\big|_p, ..., dx^n\big|_p$ forms a basis of T_p^*M for each $p \in U$. If α is any smooth $1-$form (covector field) defined at least on U then

$$\alpha = a_i dx^i$$

for uniquely determined functions a_i. In fact $\alpha(\frac{\partial}{\partial x^i}) = a_i$. In particular $df(\frac{\partial}{\partial x^i}) = \frac{\partial f}{\partial x^i}$ and so we have the following familiar looking formula

$$df = \frac{\partial f}{\partial x^i} dx^i$$

which is interpreted to mean that at each $p \in U_\alpha$ we have

$$df(p) = \left.\frac{\partial f}{\partial x^i}\right|_p dx^i\big|_p .$$

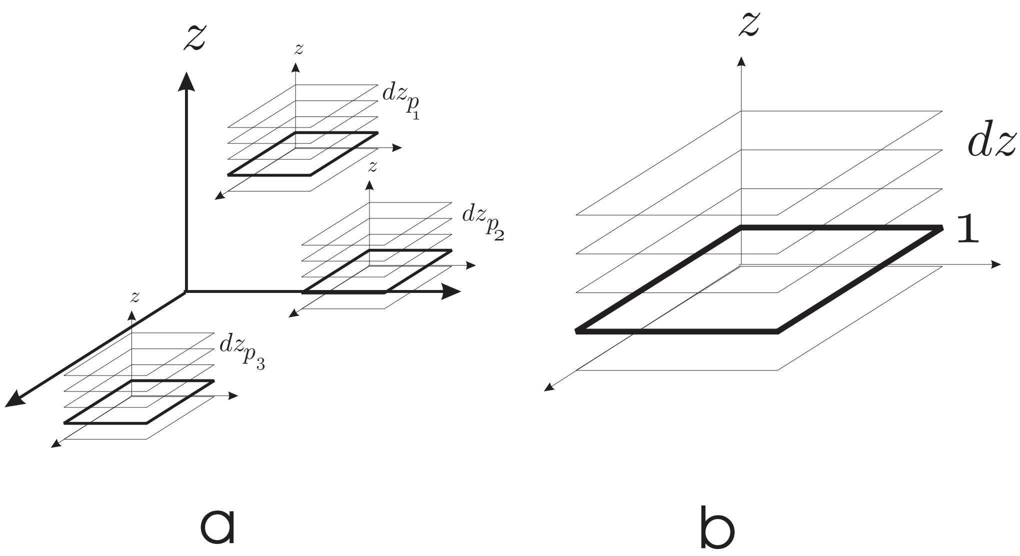
The covector fields dx^i form what is called a **coordinate coframe field** or **holonomic coframe field** [1] over U. Note that the component functions X^i of a vector field with resect to the chart above are given by $X^i = dx^i(X)$ where by definition $dx^i(X)$ is the function $p \mapsto dx^i\big|_p (X_p)$. Thus

$$X|_U = dx^i(X) \frac{\partial}{\partial x^i}.$$

Note. If α is a 1-form on M and $p \in M$ then one can always find many functions f such that $df(p) = \alpha(p)$ but there may not be a single function f so that this is true for all points in a neighborhood let alone all points on M. If in fact $df = \alpha$ for some $f \in C^\infty(M)$ then we say that α is **exact**. More on this later.

Let try to picture one covectors and 1-forms. As a warm up lets review how we might picture a tangent vector v_p at a point $p \in \mathbb{R}^n$. If γ is a curve with $\gamma'(0) = v_p$. If we zoom in on the curve near p then it appear to straighten out and so begins to look like a curve $t \mapsto p + tv_p$. So one might say that a tangent vector is the infinitesimal representation of a (parameterized) curve.

[1] The word holonomic comes from mechanics and just means that the frame field derives from a chart. A related fact is that $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$.



At each point a 1−form gives a linear functional in that tangent space and as we know, the level sets of a linear functional are parallel affine subspaces or hyperplanes. Of course we must have the the zero level set and one of the positive level sets labeled so we can see, for example, which way is the direction of increase. In a strong sense, a covector puts a ruling in the tangents space that measures tangents vectors stretching across this ruling. For example, the 1−form dz in $\mathbb{R}^3$ gives a ruling in each tangents space as suggested by figure 2.10 a. Now the fact that the individual dz_p's at each point somehow coalesce into the level sets of the global function z as shown in 2.10 b, is due to the fact that the 1-form dz is the differential of z. Now consider a more general smooth function f defined on an open set in $\mathbb{R}^n$ containing a point p. Consider the level sets (level curves or hypersurfaces) of f near p. If we zoom in again these look straight and so will in the limit become the straight level sets of an affine linear function of the form $x \mapsto f(p) + df_p(x - p)$ for some linear function df_p. If we picture $x - p$ to be based at p (and so living in $T_p\mathbb{R}^n$) then df_p will be interpreted as a linear functional acting in $T_p\mathbb{R}^n$ and so a member of $T_p^*\mathbb{R}^n$. Thus since df_p is a linear functional its level sets in $T_p\mathbb{R}^n$ will be parallel (affine) hyperplanes. If we let p vary then we obtain df and we have a similar picture at each point. Since df comes from a function these straight level sets which live in each tangent space coalesce into the level sets of f. Now a more general 1−form α is still pictured as straight parallel hyperplanes in each tangent space. Because these level sets live in the tangent space we might call them infinitesimal level sets. It is important to remember that these levels sets must be labeled or parameterized if they are to represent a covector at the point. In particular, we must be told which way is uphill so to speak. Now these level sets live in each tangent space but they may not coalesce into the level sets of any smooth function on the manifold. In other words, the 1-form may not be exact. There are various, increasingly severe ways coalescing may fail to happen. The least severe situation is when α is not the differential of a global function but is still locally a differential near each point. For example, if $M = \mathbb{R}^2 \backslash \{0\}$ then the familiar $1-$form $\alpha = (x^2 + y^2)^{-1}(-ydx + xdy)$ is locally equal to $d\theta$ for some angle θ function measured from some fixed ray such as the positive x axis. But there is no such single smooth angle functions defined on all of $\mathbb{R}^2 \backslash \{0\}$. Thus, globally speaking, α is not the differential of any function. In figure 2.1 we see the coalesced result of "integrating" the infinitesimal level sets which live in the tangent spaces. While these suggest an angular function we see that if we try to picture rising as we travel around the origin we find that we do not return to the same level in one full circulation but rather we keep rising. Locally, however, we really do have level sets of a smooth function

Now the second more severe way that a 1-form may fail to be the differential of a function is where there is not even a *local* function that does the job near a point. The infinitesimal level sets do not coalesce to the level sets of a smooth function even in small neighborhoods. This is much harder to represent but figure 2.2 is meant to at least be suggestive. Nearby curves cross inconsistent numbers of level sets. As an example consider the $1-$form

$$\beta = ydx - xdy$$

The astute read may object that surely radial rays do match up with the directions described by this $1-$form but the point is that a covector in a tangent space is not completely described by the level sets as such, but rather the level sets to be though of as labeled according to the values the represent. Here we have a case where the the level sets coalesce but the values assigned to them do not; they are 1-dimensional submanifolds that fit the 1-form but they are not level sets of a smooth function. This brings us the most severe case which only happens in dimension 3 or above. It can be the case that there are no nice family of $n-1$ dimensional submanifolds that line up with the $1-$form either global or locally. This is the topic of the Frobenius integrability theory for distributions that we study in the sequel and we shall forgo any further discussion of this until then.

If $\phi : M \to N$ is a C^∞ map and $f : N \to \mathbb{R}$ a C^∞ function we define the **pull-back** of f by ϕ as

$$\phi^* f = f \circ \phi$$

and the **pull-back** of a 1-form $\alpha \in \mathfrak{X}^*(N)$ by $\phi^*\alpha = \alpha \circ T\phi$. To get a clearer picture of what is going on we could view things at a point and then we have $\phi^*\alpha|_p \cdot v = \alpha|_{\phi(p)} \cdot (T_p\phi \cdot v)$.

Exercise 2.15 *The pull-back is contravariant in the sense that if* $\phi_1 : M_1 \to M_2$ *and* $\phi_2 : M_2 \to N$ *then for* $\alpha \in \mathfrak{X}^*(N)$ *we have* $(\phi_2 \circ \phi_1)^* = \phi_1^* \circ \phi_2^*$.

Next we describe the local expression for the pull-back of a 1-form. Let $(U, \mathtt{x})$ be a chart on M and $(V, \mathtt{y})$ be a coordinate chart on N with $\phi(U) \subset V$. A typical

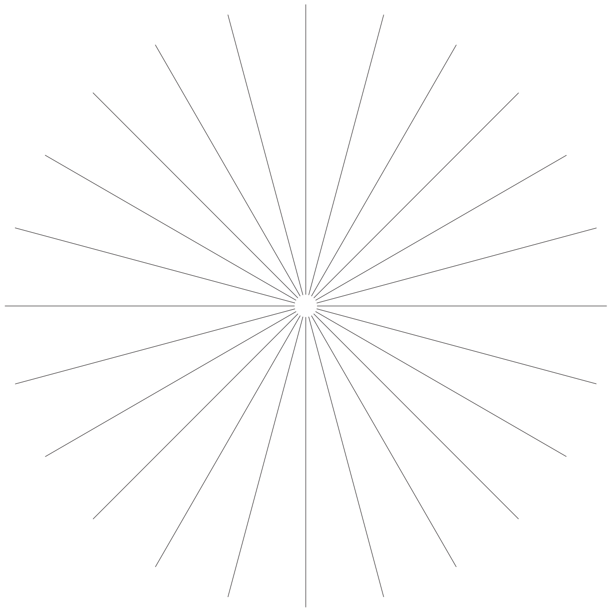


Figure 2.1: Level sets of overlapping angle functions. No single smooth function has these as level sets.

1-form has a local expression on V of the form $\alpha = \sum a_i dy^i$ for $a_i \in C^\infty(V)$. The local expression for $\phi^*\alpha$ on U is $\phi^*\alpha = \sum a_i \circ \phi \, d\left(y^i \circ \phi\right) = \sum a_i \circ \phi \frac{\partial\left(y^i \circ \phi\right)}{\partial x^j} dx^j$. Thus we get a local pull-back formula[2] convenient for computations:

$$\phi^*\left(\sum a_i dy^i\right) = \sum a_i \circ \phi \frac{\partial\left(y^i \circ \phi\right)}{\partial x^j} dx^j$$

The pull-back of a function or 1-form is defined whether $\phi : M \to N$ happens to be a diffeomorphism or not. On the other hand, when we define the pull-back of a vector field in a later section we will only be able to do this if the map that we are using is a diffeomorphism. Push-forward is another matter.

Definition 2.31 *Let $\phi : M \to N$ be a C^∞ diffeomorphism with $r \geq 1$. The* ***push-forward*** *of a function $f \in C^\infty(M)$ is denoted $\phi_* f$ and defined by $\phi_* f(p) := f(\phi^{-1}(p))$. We can also define the push-forward of a 1-form as $\phi_*\alpha = \alpha \circ T\phi^{-1}$.*

Exercise 2.16 *Find the local expression for $\phi_* f$ and $\phi_*\alpha$. Explain why we need ϕ to be a diffeomorphism.*

It should be clear that the pull-back is the more natural of the two when it comes to forms and functions but in the case of vector fields this is not true.

[2]To ensure clarity we have not use the Einstein summation convention here.

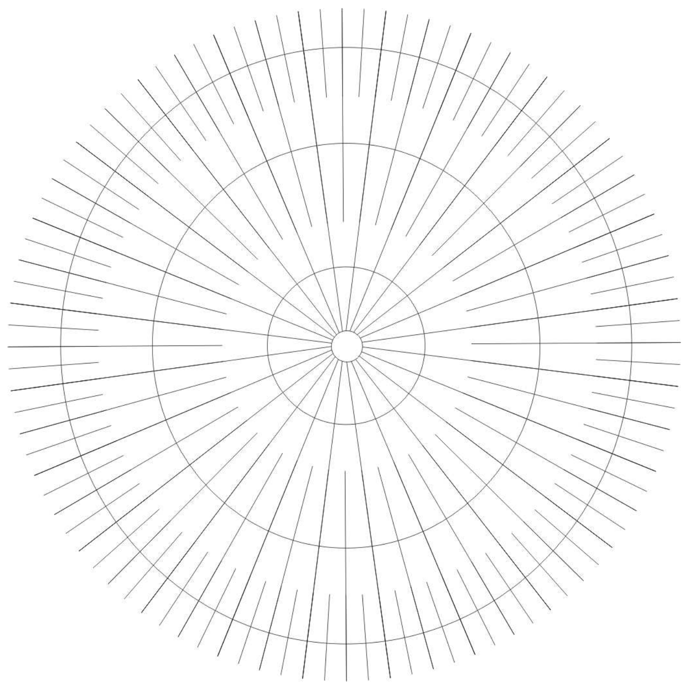


Figure 2.2: Suggestive representation of a form which is not closed.

Lemma 2.8 *The differential is natural with respect to pull-back. In other words, if $\phi : N \to M$ is a C^∞ map and $f : M \to \mathbb{R}$ a C^∞ function with $r \geq 1$ then $d(\phi^* f) = \phi^* df$. Consequently, the differential is also natural with respect to restrictions.*

Proof. Let v be a curve such that $\dot{c}(0) = v$. Then

$$\begin{aligned} d(\phi^* f)(v) &= \left.\frac{d}{dt}\right|_0 \phi^* f(c(t)) = \left.\frac{d}{dt}\right|_0 f(\phi(c(t))) \\ &= df \left.\frac{d}{dt}\right|_0 \phi(c(t)) = df(T\phi \cdot v) \end{aligned}$$

As for the second statement (besides being obvious from local coordinate expressions) notice that if U is open in M and $\iota : U \hookrightarrow M$ is the inclusion map (i.e. identity map id_M restricted to U) then $f|_U = \iota^* f$ and $df|_U = \iota^* df$ so the statement about restrictions is just a special case. ■

The tangent and cotangent bundles TM and T^*M are themselves manifolds and so have their own tangent and cotangent bundles. Among other things, this means that there exist 1−forms and vector fields on these manifolds. Here we introduce the canonical 1−form on T^*M. This is a form we denote by θ_{can} and it is a section of $T^*(T^*M)$. Let $a \in T_p^*M$ and consider a vector $u_a \in T_a^*(T^*M)$. We define

$$\theta_{can}(u_a) = a\,(T_a\pi \cdot u_a)$$

where $\pi : T^*M \to M$ is the usual projection and thus $T_a\pi : T_a^*(T^*M) \to T_pM$. To see that our definition makes sense note that $T_a\pi \cdot u_a$ is a vector in T_pM while $a \in T_p^*M$ which means that $a\,(T_a\pi \cdot u_a) \in \mathbb{R}$. Let $(U, \mathtt{x})$ be a chart containing p and let $(x^1 \circ \pi, ..., x^n \circ \pi_M, p_1, ..., p_n) = (q^1, ..., q^n, p_1, ..., p_n)$ be the associated natural coordinates for T^*M. We wish to show that locally $\theta_{can} = p_i dq^i$. It will suffice to show that $\theta_{can}(\left.\frac{\partial}{\partial q^i}\right|_a) = p_i(a)$ and $\theta_{can}(\left.\frac{\partial}{\partial p^i}\right|_a) = 0$ for all i. We have

$$\begin{aligned} &\theta_{can}(\left.\frac{\partial}{\partial p^i}\right|_a) \\ &= a\left(T_a\pi \cdot \left.\frac{\partial}{\partial p^i}\right|_a\right) = 0 \end{aligned}$$

since in fact $T_a\pi \cdot \left.\frac{\partial}{\partial p^i}\right|_a = 0$. Also we have

$$\begin{aligned} &\theta_{can}(\left.\frac{\partial}{\partial q^i}\right|_a) \\ &= a\left(T_a\pi \cdot \left.\frac{\partial}{\partial q^i}\right|_a\right) \\ &= a\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = a^i = p_i(a) \end{aligned}$$

where we have use the fact that $T_a\pi \cdot \frac{\partial}{\partial q^i}\Big|_a = \frac{\partial}{\partial x^i}\big|_p$ which follows from the definition $q^i = x^i \circ \pi$. Indeed, we know that $T_a\pi \cdot \frac{\partial}{\partial q^i}\Big|_a = c_i^k \frac{\partial}{\partial x^k}\big|_p$ for some constants c_i^k, but we have

$$\begin{aligned} c_i^k &= dx^k \left(T_a\pi \cdot \frac{\partial}{\partial q^i}\bigg|_a \right) \\ &= \pi^* dx^k \left(\frac{\partial}{\partial q^i}\bigg|_a \right) \\ &= d\left(x^k \circ \pi\right) \left(\frac{\partial}{\partial q^i}\bigg|_a \right) \\ &= dq^k \left(\frac{\partial}{\partial q^i}\bigg|_a \right) = \delta_i^k \end{aligned}$$

We will eventually see that this form plays a role in classical mechanics.

2.11 Line Integrals and Conservative Fields

Just as in calculus on Euclidean space we can consider line integrals on manifolds and it is exactly the 1−forms that are the appropriate objects to integrate. First notice that all 1−forms on open sets in $\mathbb{R}^1$ must be of the form fdt for some smooth function f and where t is the coordinate function on $\mathbb{R}^1$. We begin by defining the line integral of a 1−form defined and smooth on an interval $[a, b] \subset \mathbb{R}^1$. If $\beta = fdt$ is such a 1−form then

$$\int_{[a,b]} \beta := \int_a^b f(t)dt$$

Any smooth map $\gamma : [a, b] \to M$ is the restriction of a smooth map on some larger open interval $(a - \varepsilon, b + \varepsilon)$ and so there is no problem defining the pull-back $\gamma^*\alpha$. If $\gamma : [a, b] \to M$ is a smooth curve then we define the **line integral** of a 1−form α along γ to be

$$\int_\gamma \alpha := \int_{[a,b]} \gamma^*\alpha = \int_a^b f(t)dt$$

where $\gamma^*\alpha = fdt$. Now if $t = \phi(s)$ is a smooth increasing function then we obtain a positive reparameterization $\widetilde{\gamma} = \gamma \circ \phi : [c, d] \to M$ where $\phi(c) = a$ and $\phi(d) = b$. With such a reparametrization we have

$$\begin{aligned} \int_{[c,d]} \widetilde{\gamma}^*\alpha &= \int_{[c,d]} \phi^*\gamma^*\alpha = \int_{[c,d]} \phi^*\gamma^*\alpha \\ &= \int_{[c,d]} \phi^*(fdt) = \int_{[c,d]} f \circ \phi \frac{d\phi}{ds} ds \\ &= \int_c^d f(\phi(s))\phi^{'}(s)\, ds = \int_a^b f(t)dt \end{aligned}$$

where the last line is the standard change of variable formula and where we have used $\phi^*(fdt) = \frac{d(f\circ\phi)}{ds}ds$ which is a special case of the pull-back formula mentioned above. We see now that we get the same result as before. This is just as in ordinary multivariable calculus. We have just transferred the usual calculus ideas to the manifold setting.

It is convenient to extend the definitions a bit to include integration along piecewise smooth curves. Thus if $\gamma : [a,b] \to M$ is such a curve and we have $a = t_0 < t_1 < \cdots < t_k = b$ where γ is smooth on each interval $[t_i, t_{i+1}]$ then we define for a 1-form α

$$\int_\gamma \alpha = \sum_{i=0}^{k-1} \int_{[t_i,t_{i+1}]} \gamma_i^* \alpha$$

where γ_i is the restriction of γ to the interval $[t_i, t_{i+1}]$.

Just as in ordinary multivariable calculus we have the following:

Proposition 2.8 *If $\gamma : [a,b] \to M$ is a piecewise smooth curve with $\gamma(a) = p_1$ and $\gamma(b) = p_2$. If $\alpha = df$ then*

$$\int_\gamma \alpha = \int_\gamma df = f(p_2) - f(p_1)$$

In particular, $\int_\gamma \alpha$ is path independent in the sense that it is equal to $\int_c \alpha$ for any other piecewise smooth path c that also begins at p_1 and ends at p_2.

Definition 2.32 *If α is a 1-form on a smooth manifold M such that $\int_c \alpha = 0$ for all closed piecewise smooth curves c then we say that α is* ***conservative****.*

We will need a lemma on differentiability.

Lemma 2.9 *Suppose f is a function defined on a smooth manifold M and let α be smooth $1-$form on M. Suppose that for any $p \in M$, $v_p \in TM$ and smooth curve c with $c'(0) = v_p$ the derivative $\frac{d}{dt}\big|_0 f(c(t))$ exists and*

$$\frac{d}{dt}\bigg|_0 f(c(t)) = \alpha_p(v_p)$$

Then f is smooth and $df = \alpha$.

Proof. We work in a chart $(U, \mathtt{x})$. If we take $c(t) := \mathtt{x}^{-1}(\mathtt{x}(p) + t\mathbf{e}_i)$ then the hypotheses lead to the conclusion that all the first order partial derivatives of $f \circ \mathtt{x}^{-1}$ exist and are continuous. Thus f is C^1. But then also $df_p \cdot v_p = \frac{d}{dt}\big|_0 f(c(t)) = \alpha_p(v_p)$ for all v_p, it follows that $df = \alpha$ and this also implies that f is actually smooth. ■

Proposition 2.9 *If α is a 1-form on a smooth manifold M then α is conservative if and only if it is exact.*

Proof. We know already that if $\alpha = df$ then α is a conservative. Now suppose α is conservative. Fix $p_0 \in M$. Then we can define $f(p) = \int_\gamma \alpha$ where γ is any curve beginning at p_0 and ending at p. Given any $v_p \in T_pM$ we pick a curve $c : [-1, \varepsilon)$ with $\varepsilon > 0$ such that $c(-1) = p_0$, $c(0) = p$ and $c'(0) = v_p$. Then

$$\begin{aligned}
&\left.\frac{d}{d\tau}\right|_0 f(c(\tau)) \\
&= \left.\frac{d}{d\tau}\right|_0 \int_{c|[-1,\tau]} \alpha \\
&= \left.\frac{d}{d\tau}\right|_0 \int_{c|[-1,0]} \alpha + \left.\frac{d}{d\tau}\right|_0 \int_{c|[0,\tau]} \alpha \\
&= 0 + \left.\frac{d}{d\tau}\right|_1 \int_0^\tau c^*\alpha \\
&= \left.\frac{d}{d\tau}\right|_s \int_0^\tau g(t)dt \\
&= g(0)
\end{aligned}$$

where $c^*\alpha = gdt$. On the other hand,

$$\begin{aligned}
\alpha(v_p) &= \alpha(c'(0)) = \alpha(T_0c \cdot \left.\frac{d}{dt}\right|_0) \\
&= c^*\alpha(\left.\frac{d}{dt}\right|_0) = g(0)\ dt|_0 \left(\left.\frac{d}{dt}\right|_0\right) \\
&= g(0)
\end{aligned}$$

Thus $\left.\frac{d}{d\tau}\right|_0 f(c(\tau)) = \alpha_p(v_p)$ for any $v_p \in T_pM$ and any $p \in M$. Now the result follows from the previous lemma. ■

It is important to realize that when we say that a form is conservative in this context, we mean that it is *globally* conservative. It may also be the case that a form is *locally* conservative. This would mean that all restrictions of the 1−form to small contractible open sets are conservative. The following examples explore in simple terms these differences.

Example 2.1 *Let* $\alpha = \left(x^2 + y^2\right)(-ydx + xdy)$. *Let us consider the small circular path* c *given by* $(x, y) = (x_0 + \varepsilon \cos t, y_0 + \varepsilon \sin t)$ *with* $0 \le t \le 2\pi$ *and* ε. *If* $(x_0, y_0) = (0, 0)$ *we get*

$$\begin{aligned}
\int_c \alpha &= \int_0^{2\pi} \left(-y(t)\frac{dx}{dt} + x(t)\frac{dy}{dt}\right) dt \\
&= \int_0^{2\pi} \frac{1}{\varepsilon^2} \left(-\left(\varepsilon \sin t\right)\left(-\varepsilon \sin t\right) + \left(\varepsilon \cos t\right)\left(\varepsilon \cos t\right)\right) dt = 2\pi
\end{aligned}$$

Thus α *is not conservative and hence not exact. On the other hand if* $(x_0, y_0) \neq (0, 0)$ *then we pick a ray* R_0 *that does* not *pass through* (x_0, y_0) *and smooth* *function $\theta(x,y)$ which gives the angle of the the ray R passing through (x,y) measured counterclockwise from R_0. This angle function is smooth and defined on $U = \mathbb{R}^2 \backslash R_0$. If $\varepsilon < \frac{1}{2}\sqrt{x_0^2 + y_0^2}$ then c has image inside the domain of θ and we have that $\alpha| U = d\theta$. Thus $\int_c \alpha = \theta(c(0)) - \theta(c(2\pi)) = 0$. We see that α is locally conservative.*

Example 2.2 *Consider $\beta = ydx - xdy$ on $\mathbb{R}^2 \backslash \{0\}$. If it were that case that for some small open set $U \subset \mathbb{R}^2 \backslash \{0\}$ we had $\beta|_U = df$ then for a closed path c with image in that set we would expect that $\int_c \beta = f(c(2\pi)) - f(c(0)) = 0$. However,*

$$\begin{aligned}
\int_c \beta &= \int_c \left(y(t)\frac{dx}{dt} - x(t)\frac{dy}{dt} \right) dt \\
&= \int_0^{2\pi} \left((x_0 + \varepsilon \sin t)\left(-\varepsilon \sin t\right) - (y_0 + \varepsilon \cos t)\left(\varepsilon \cos t\right) \right) dt \\
&= -2\varepsilon^2 \pi
\end{aligned}$$

so we do not get zero no matter what the point (x_0, y_0) and no matter how small ε. We conclude that β is not even locally conservative.

The distinction between (globally) conservative and locally conservative is often not made sufficiently clear in the physics and engineering literature.

Example 2.3 *In classical physics, the static electric field set up by a fixed point charge of magnitude q can be described, with an appropriate choice of units, by the $1-$form*

$$\frac{q}{r^3} xdx + \frac{q}{r^3} ydy + \frac{q}{r^3} zdz$$

where we have imposed Cartesian coordinates centered at the point charge and where $r = \sqrt{x^2 + y^2 + z^2}$. Notice that the domain of the form is the punctured space $\mathbb{R}^3 \backslash \{0\}$. In spherical coordinates (r, θ, ϕ) this same form is

$$\frac{q}{r^2} d\rho = d\left(\frac{-q}{r} \right)$$

so we see that the form is exact and the field conservative.

2.12 Moving frames

It is important to realize it is possible to get a family of locally defined vector (resp. covector) fields that are linearly independent at each point in their mutual domain and yet are not necessarily of the form $\frac{\partial}{\partial x^i}$ (resp. dx^i) for any coordinate chart. In fact, this may be achieved by carefully choosing n^2 smooth functions f_k^i (resp. a_i^k) and then letting $E_k := f_k^i \frac{\partial}{\partial x^i}$ (resp. $\theta^k := a_i^k dx^i$).

Definition 2.33 *Let $E_1, E_2, ..., E_n$ be smooth vector fields defined on some open subset U of a smooth n-manifold M. If $E_1(p), E_2(p), ..., E_n(p)$ form a basis for T_pM for each $p \in U$ then we say that $E_1, E_2, ..., E_n$ is a (non-holonomic)* ***moving frame*** *or a* ***frame field*** *over U.*

If $E_1, E_2, ..., E_n$ is moving frame over $U \subset M$ and X is a vector field defined on U then we may write

$$X = X^i E_i \text{ on } U \text{ (summation over } i)$$

for some functions X^i defined on U. If the moving frame $(E_1, ..., E_n)$ is not identical to some frame field $(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^1})$ arising from a coordinate chart on U then we say that the moving frame is non-holonomic. It is often possible to find such moving frame fields with domains which can never be the domain of any chart.

Definition 2.34 *If $E_1, E_2, ..., E_n$ is a frame field with domain equal to the whole manifold M then we call it a **global frame field**.*

Most manifolds do not have global frame fields (see problem

Taking the basis dual to $(E_1(p), ..., E_n(p))$ in T_p^*M for each $p \in U$ we get a moving coframe field $(\theta^1, ..., \theta^n)$. The θ^i are 1-forms defined on U. Any $1-$form α can be expanded in terms of these basic $1-$forms as $\alpha = \sum a_i \theta^i$. Actually it is the restriction of α to U that is being expressed in terms of the θ^i but we shall not be so pedantic as to indicate this in the notation. In a manner similar to the case of a coordinate frame we have that for a vector field X defined at least on U, the components with respect to $(E_1, ..., E_n)$ are given by $\theta^i(X)$:

$$X = \theta^i(X) E_i \text{ on } U.$$

Let us now consider an important special situation. If $M \times N$ is a product manifold and $(U, \mathtt{x})$ is a chart on M and $(V, \mathtt{y})$ is a chart on N then we have a chart $(U \times V, \mathtt{x} \times \mathtt{y})$ on $M \times N$ where the individual coordinate functions are $x^1 \circ pr_1, ..., x^m \circ pr_1$ $y^1 \circ pr_2, ..., y^n \circ pr_2$ which we temporarily denote by $\widetilde{x}^1, ..., \widetilde{x}^m, \widetilde{y}^1, ..., \widetilde{y}^n$. Now we consider what is the relation between the coordinate frame fields $(\frac{\partial}{\partial x^1}, ... \frac{\partial}{\partial x^1}), (\frac{\partial}{\partial y^1}, ... \frac{\partial}{\partial y^n})$ and the frame field

$$\left(\frac{\partial}{\partial \widetilde{x}^1}, ..., \frac{\partial}{\partial \widetilde{y}^n} \right).$$

The latter set of $n + m$ vector fields is certainly a linearly independent set at each point $(p, q) \in U \times V$. The crucial relations are $\frac{\partial}{\partial \widetilde{x}^i} f = \frac{\partial}{\partial x^i}(f \circ pr_1)$ and $\frac{\partial}{\partial \widetilde{y}^i} = \frac{\partial}{\partial y^i}(f \circ pr_2)$.

Exercise 2.17 *Show that $\left.\frac{\partial}{\partial \widetilde{x}^i}\right|_{(p,q)} = T(pr_1) \left.\frac{\partial}{\partial x^i}\right|_p$ for all q and that $\frac{\partial}{\partial \widetilde{y}^i} = T(pr_2) \left.\frac{\partial}{\partial y^i}\right|_q$.*

Remark 2.6 *In some circumstances it is safe to abuse notation and denote $x^i \circ pr_1$ by x^i and $y^i \circ pr_2$ by y^i. Of course we then are denoting $\frac{\partial}{\partial \widetilde{x}^i}$ by $\frac{\partial}{\partial x^i}$ and so on.*

A warning (The fundamental confusion of multidimensional calculus). For a chart $(U, \mathtt{x})$ with $\mathtt{x} = (x^1, ..., x^n)$ we have defined $\frac{\partial f}{\partial x^i}$ for any appropriately defined smooth (or C^1) function f. However, this notation can be ambiguous. For example, the meaning of $\frac{\partial f}{\partial x^1}$ is not determined by the coordinate function x^1 alone but implicitly depends of the rest of the coordinate functions. For example, in thermodynamics we see the following situation. We have three functions P, V and T which are not independent but may be interpreted as functions on some $2-$dimensional manifold. Then it may be the case that any two of the three functions my serve as a coordinate system. Now the meaning of $\frac{\partial f}{\partial P}$ depends on whether we are using the coordinate functions (P, V) or alternatively (P, T). We must know not only which function we are allowing to vary but also which other functions are held fixed. To get rid of the ambiguity one can use the notations $\left(\frac{\partial f}{\partial P}\right)_V$ and $\left(\frac{\partial f}{\partial P}\right)_T$. In the first case the coordinates are (P, V) and V is held fixed while in the second case we use coordinates (P, T) and T is help fixed. Another way to avoid ambiguity would be to use different names for the same functions depending on the chart of which they are considered coordinate functions. For example, consider the following change of coordinates:

$$\begin{aligned} y^1 &= x^1 + x^2 \\ y^2 &= x^1 - x^2 + x^3 \\ y^3 &= x^3 \end{aligned}$$

Here $y^3 = x^3$ as functions on the underlying manifold but we use different symbols. Thus $\frac{\partial}{\partial x^3}$ may not be the same as $\frac{\partial}{\partial y^3}$. The chain rule shows that in fact $\frac{\partial}{\partial x^3} = \frac{\partial}{\partial y^2} + \frac{\partial}{\partial y^3}$. This latter method of destroying ambiguity is not very helpful in our thermodynamic example since the letters P, V and T are chosen to stand for the physical quantities of pressure, volume and temperature. Giving these functions more than one name would only be confusing.

2.13 Problems

1. Find the integral curves in $\mathbb{R}^2$ of the vector field $X = e^{-x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and determine if X is complete or not.

2. Find a concrete description of the tangent bundle for each of the following manifolds:
(a) Projective space $P(\mathbb{R}^n)$
(b) The Grassmann manifold $Gr(k, n)$

3. Recall that we have charts on $\mathbb{R}P^2$ given by

$$\begin{aligned} [x, y, z] &\mapsto (u_1, u_2) = (x/z, y/z) \text{ on } U_3 = \{z \neq 0\} \\ [x, y, z] &\mapsto (v_1, v_2) = (x/y, z/y) \text{ on } U_2 = \{y \neq 0\} \\ [x, y, z] &\mapsto (w_1, w_2) = (y/x, z/x) \text{ on } U_1 = \{x \neq 0\} \end{aligned}$$

Show that there is a vector field on $\mathbb{R}P^2$ which in the last coordinate chart above has the following coordinate expression

$$w_1 \frac{\partial}{\partial w_1} - w_2 \frac{\partial}{\partial w_2}$$

What are the expressions for this vector field in the other two charts? (Caution: your guess may be wrong!).

4. Show that if $f : M \to N$ is a diffeomorphism then for each $p \in M$ the tangent map $T_p f : T_p M \to T_{f(p)} N$ is a vector space isomorphism

5. Show that the graph $\Gamma(f) = \{(p, f(p)) \in M \times N : p \in M\}$ of a smooth map $f : M \to N$ is a smooth manifold and that we have an isomorphism $T_{(p,f(p))}(M \times N) \cong T_{(p,f(p))}\Gamma(f) \oplus T_{f(p)} N$.

6. Show that a manifold supports a frame field defined on the whole of M exactly when there is a trivialization of TM (see definitions 2.34 and 2.15.

7. Find natural coordinates for the double tangent bundle TTM. Show that there is a nice map $s : TTM \to TTM$ such that $s \circ s = id_{TTM}$ and such that $T\pi \circ s = T\pi_{TM}$ and $T\pi_{TM} \circ s = T\pi$. Here $\pi : TM \to M$ and $\pi_{TM} : TTM \to TM$ are the appropriate tangent bundle projection maps.

8. Let N be the subspace of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ defined by $N = \{(x, y) : \|x\| = 1$ and $x \cdot y = 0\}$ is a smooth manifold that is diffeomorphic to TS^n.

9. (**Hessian**)Suppose that $f \in C^\infty(M)$ and that $df_p = 0$ for some $p \in M$. In this case show that for any smooth vector fields X and Y on M we have that $Y_p(Xf) = X_p(Yf)$. Let $H_{f,p}(v, w) := X_p(Yf)$ where X and Y are such that $X_p = v$ and $Y_p = w$. Show that $H_{f,p}(v, w)$ is independent of extension vector fields X and Y and that resulting map $H_{f,p} : T_p M \times T_p M \to \mathbb{R}$ is bilinear. $H_{f,p}$ is called the Hessian of f at p. Show that the assumption $df_p = 0$ is needed.

10. Show that for a smooth map the map $F : M \to N$ the (bundle) tangent map $TF : TM \to TN$ is smooth. It sometimes is supposed that the one can obtain a well defined map $F_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ by thinking of vector fields as derivations on functions and then letting $(F_* X) f = X(f \circ F)$ for $f \in C^\infty(N)$. Show why this is misguided. Recall that the proper definition of $F_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ would be $F_* X := TF \circ X \circ F^{-1}$ and is defined in case F is a diffeomorphism. What if F is merely surjective.

11. Show that if $\psi : M' \to M$ is a smooth covering map then so also is $T\psi : TM' \to TM$.

12. Define map $f : \mathbb{R}^{n\times n} \to sym(\mathbb{R}^{n\times n})$ by $f(A) := A^T A$ where $\mathbb{R}^{n\times n}$ and $sym(\mathbb{R}^{n\times n})$ are the manifolds of $n \times n$ matrices and $n \times n$ symmetric matrices respectively. Identifying $T_A \mathbb{R}^{n\times n}$ with $\mathbb{R}^{n\times n}$ and $T_{f(A)} sym(\mathbb{R}^{n\times n})$ with $sym(\mathbb{R}^{n\times n})$ in the natural way for each A. Calculate $T_I f : \mathbb{R}^{n\times n} \to sym(\mathbb{R}^{n\times n})$ using these identifications.

13. Let $f_1, ..., f_N$ be a set of smooth functions defined on an open subset of a smooth manifold. Show that if $df_1(p), ..., df_N(p)$ spans T_p^*M for some $p \in U$ then some ordered subset of $\{ f_1, ..., f_N\}$ provides a coordinate system on some open subset V of U containing p.

14. Let Δ_r be the vector space of derivations on $C^r(M)$ at $p \in M$ where $0 < r \leq \infty$ is a positive integer or ∞. Fill in the details in the following outline which studies Δ_r. It will be shown that Δ_r is not finite dimensional unless $r = \infty$.

    (a) We may assume that $M = \mathbb{R}^n$ and $p = 0$ is the origin. Let $\mathfrak{m}_r := \{f \in C^r(\mathbb{R}^n) : f(0) = 0\}$ and let $\mathfrak{m}_r^2$ be the subspace spanned by functions of the form fg for $f, g \in \mathfrak{m}_r$. We form the quotient space $\mathfrak{m}_r/\mathfrak{m}_r^2$ and consider its vector space dual $\left(\mathfrak{m}_r/\mathfrak{m}_r^2\right)^*$. Show that if $\delta \in \Delta_r$ then δ restricts to a linear functional on $\mathfrak{m}_r$ and is zero on all elements of $\mathfrak{m}_r^2$. Conclude that δ gives a linear functional on $\mathfrak{m}_r/\mathfrak{m}_r^2$. Thus we have a linear map $\Delta_r \to \left(\mathfrak{m}_r/\mathfrak{m}_r^2\right)^*$.

    (b) Show that the map $\Delta_r \to \left(\mathfrak{m}_r/\mathfrak{m}_r^2\right)^*$ given above has an inverse. Hint: For a $\lambda \in \left(\mathfrak{m}_r/\mathfrak{m}_r^2\right)^*$ take any linear map consider $\delta_\lambda(f) := \lambda([f - f(0)])$ where $f \in C^r(\mathbb{R}^n)$ and hence $[f - f(0)] \in \mathfrak{m}_r/\mathfrak{m}_r^2$. Conclude that by taking $r = \infty$ we have $T_0R^n = \Delta_\infty \cong \left(\mathfrak{m}_r/\mathfrak{m}_r^2\right)^*$. This case $r < \infty$ is different as we see next.

    (c) Let $r < \infty$. The goal from here on is to show that $\mathfrak{m}_r/\mathfrak{m}_r^2$ and hence $\left(\mathfrak{m}_r/\mathfrak{m}_r^2\right)^*$ are infinite dimensional. We start out with the case $\mathbb{R}^n = \mathbb{R}$. First show that if $f \in \mathfrak{m}_r$ then $f(x) = xg(x)$ for $g \in C^{r-1}(\mathbb{R})$. Also if $f \in \mathfrak{m}_r^2$ then $f(x) = x^2g(x)$ for $g \in C^{r-1}(\mathbb{R})$.

    (d) For each $r \in \{1, 2, 3, ...\}$ and each $\varepsilon \in (0, 1)$ define

    $$g_\varepsilon^r(x) := \begin{cases} x^{r+\varepsilon}, \text{ for } x > 0 \\ 0 \text{ for } x \leq 0 \end{cases}$$

    $g_\varepsilon^r(x) \in \mathfrak{m}_r$ but $g_\varepsilon^r(x) \notin C^{r+1}(\mathbb{R})$. Show that for any fixed $r \in \{1, 2, 3, ...\}$ the set of elements of the form $[g_\varepsilon^r] := g_{\ \varepsilon}^{\ r} + \mathfrak{m}_r^2$ for $\varepsilon \in (0, 1)$ is linearly independent in the quotient. Hint: Use induction on r. In the case of $r = 1$ it would suffice to show that if we are given $0 < \varepsilon_1 < \cdots < \varepsilon_l < 1$ and if $\sum_{i=1}^l a_j \ g_{\varepsilon_j}^1 \in \mathfrak{m}_r^2$ then $a_j = 0$ for all j.

    (Thanks to Lance Drager for donating this problem and its solution.)

15. Find the integral curves of the vector field on $\mathbb{R}^2$ given by $X(x, y) := x^2 \frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$

16. Find the integral curves (and hence the flow) for the vector field on $\mathbb{R}^2$ given by $X(x, y) := -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

17. Find an vector field on $S^2 - \{N\}$ which is not complete.

18. Using the usual spherical coordinates (ϕ, θ) on S^n calculate the bracket $[\phi\frac{\partial}{\partial\theta}, \theta\frac{\partial}{\partial\phi}]$.

19. Show that if X and Y are (time independent) vector fields that have flows φ_t^X and φ_t^Y then if $[X,Y]=0$, the flow of $X+Y$ is $\varphi_t^X \circ \varphi_t^Y$.

20. Recall that the tangent bundle of the open set $Gl(n,\mathbb{R})$ in $\mathbb{M}_{n\times n}(\mathbb{R})$ is identified with $Gl(n,\mathbb{R}) \times \mathbb{M}_{n\times n}(\mathbb{R})$. Consider the vector field on $Gl(n,\mathbb{R})$ given by $X : g \mapsto (g, g^2)$. Find the flow of X.

21. Let $Q_t = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $t \in (0, 2\pi]$. Let $\phi(t, P) := Q_t P$ where P is a plane in $\mathbb{R}^3$. Show that this defines a flow on $Gr(3,2)$. Find the local expression in some coordinate system of the vector field X^Q that gives this flow. Do the same thing for the flow induced by the matrices $R_t = \begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix}$ $t \in (0, 2\pi]$, finding the vector field X^R. Find the bracket $[X^R, X^Q]$.

Chapter 3

Immersion and Submersion.

Suppose we are given a smooth map $f : M \to N$. Near a point $p \in M$ the tangent map $T_pf : T_pM \to T_pN$ is a linear approximation of f. A very important invariant of a linear map is its rank which is the dimension of its image. The **rank** of a smooth map f at p is defined to be the rank of T_pf. It turns out that under certain conditions on the rank of f at p or near p we can draw conclusions about the behavior of f near p. The basic idea is that f behaves very much like T_pf. If $L : V \to W$ is a linear map on finite vector spaces then $\ker L$ and $L(V)$ are subspaces (and hence submanifolds). In this section we study the extent to which something similar happens for smooth maps between manifolds. In this chapter we make heavy use of some basic theorems of multivariable calculus such as the implicit and inverse mapping theorems as well as the constant rank theorem. These can be found in appendix C (see Theorems C.8, C.9 and C.12). Since the constant rank theorem is particularly useful and often not included in advanced calculus courses we include a proof in appendix C.

Proposition 3.1 *If $f : M \to N$ is a smooth map such that $T_pf : T_pM \to T_qN$ is an isomorphism for all $p \in M$ then $f : M \to N$ is a local diffeomorphism.*

The proof is a simple application of the inverse mapping theorem.

Definition 3.1 *Let $f : M \to N$ be C^r-map and $p \in M$. We say that p is a **regular point** for the map f if T_pf is a surjection and is called a **singular point** otherwise. A point q in N is called a **regular value** of f if every point in the inverse image $f^{-1}\{q\}$ is a regular point for f. This includes the case where $f^{-1}\{q\}$ is empty. A point of N that is not regular is called a **critical value**.*

It is a very useful fact that regular values are easy to come by because most values are regular. In order to make this precise we will introduce the notion of measure zero on a second countable smooth manifold. It is actually no problem to define a Lebesgue measure on such a manifold but for now the notion of measure zero is all we need.

Definition 3.2 *Let M be smooth n dimensional manifold M which is second countable. A subset $A \subset M$ is said to be of* ***measure zero*** *if for every admissible chart $(U, \mathtt{x})$ the set $\mathtt{x}(A \cap U)$ has Lebesgue measure zero in $\mathbb{R}^n$.*

In order for this to be a reasonable definition, the manifold must be second countable so that every atlas has a countable subatlas. This way we may be assured that every set that we have defined to be measure zero is the countable union of sets that are measure zero as viewed in a some chart. We also need to know that the local notion of zero measure is independent of the chart. This follows from

Lemma 3.1 *Let M be a second countable n-manifold. The image of a measure zero set under a differentiable map is of measure zero.*

Proof. Since M is Hausdorff and second countable, any set is contained in the countable union of coordinate charts. Thus we may assume that $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ and A is some Lebesgue measure zero subset of U. In fact, since A is certainly contained in the countable union of compact balls (all of which are translates of a ball at the origin) we may as well assume that $U = B(0, r)$ and that A is contained in a slightly smaller ball $B(0, r - \delta) \subset B(0, r)$. By the mean value theorem, there is a constant c depending only on f and its domain such that for $x, y \in B(0, r)$ we have $\|f(y) - f(x)\| \leq c \|x - y\|$. Let $\epsilon > 0$ be given. Since A has measure zero there is a sequence of balls $B(x_i, \epsilon_i)$ such that $A \subset \bigcup B(x_i, \epsilon_i)$ and

$$\sum vol(B(x_i, \epsilon_i)) < \frac{\epsilon}{2^n c^n}$$

Thus $f(B(x_i, \epsilon_i)) \subset B(f(x_i), 2c\epsilon_i)$ and while $f(A) \subset \bigcup B(f(x_i), 2c\epsilon_i)$ we also have

$$\begin{aligned}
\mathrm{vol}\left(\bigcup B(f(x_i), 2c\epsilon_i)\right) &\leq \\
\sum \mathrm{vol}(B(f(x_i), 2c\epsilon_i)) &\leq \sum \mathrm{vol}(B_1)(2c\epsilon_i)^n \\
&\leq 2^n c^n \sum \mathrm{vol}(B(x_i, \epsilon_i)) \\
&\leq \epsilon.
\end{aligned}$$

Thus the measure of A is less than or equal to ϵ. Since ϵ was arbitrary it follows that A has measure zero. ∎

Corollary 3.1 *Given a fixed atlas $\mathcal{A} = \{(U_\alpha, \mathtt{x}_\alpha)\}$ for M, if $\mathtt{x}_\alpha(A \cap U_\alpha)$ has measure zero for all α, then A has measure zero.*

We now state and prove the fantastically useful theorem of Sard.

Theorem 3.1 (Sard) *Let N be an n-manifold and M an m-manifold both assumed second countable. For a smooth map $f : N \to M$ the set of critical values has Lebesgue measure zero.*

Proof. Through the use of countable covers of the manifolds in question by charts, we may immediately reduce to the problem of showing that for a smooth map $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ the set of critical values $C \subset U$ has image $f(C)$ of measure zero. We will use induction on the dimension n. For $n = 0$, the set $f(C)$ is just a point (or empty) and so has measure zero. Now assume the theorem is true for all dimensions $j \leq n-1$. We seek to show that the truth of the theorem follows for $j = n$ also.

Let us use the following common notation: For any k-tuple of nonnegative integers $\alpha = (i_1, ..., i_k)$ we let

$$\frac{\partial^{|\alpha|} f}{\partial x^\alpha} := \frac{\partial^{i_1+...+i_k} f}{\partial x^{i_1} \cdots \partial x^{i_k}}$$

where $|\alpha| := i_1 + ... + i_k$. Now let

$$C_i := \{x \in U : \ \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) = 0 \text{ for all } |\alpha| \leq i\}.$$

Then

$$C = (C \backslash C_1) \cup (C_1 \backslash C_2) \cup \cdots \cup (C_{k-1} \backslash C_k) \cup C_k$$

so we will be done if we can show that

a) $f(C \backslash C_1)$ has measure zero,

b) $f(C_{j-1} \backslash C_j)$ has measure zero and

c) $f(C_k)$ has measure zero for some sufficiently large k.

Proof of a): We may assume that $m \geq 2$ since if $m = 1$ we have $C = C_1$. Now let $x \in C \backslash C_1$ so that some first partial derivative is not zero at $x = a$. By reordering we may assume that this partial is $\frac{\partial f}{\partial x^1}$ and so the map

$$(x^1, ..., x^n) \mapsto (f(x), x^2, ..., x^n)$$

restricts to a diffeomorphism ϕ on some open neighborhood containing x. Since we may always replace f by the equivalent map $f \circ \phi^{-1}$ we may go ahead and assume without loss of generality that f has the form

$$f : x \mapsto (x^1, f^2(x), ..., f^m(x)) := (x^1, h(x))$$

on some perhaps smaller neighborhood V containing a. The Jacobian matrix for f in V is of the form

$$\begin{bmatrix} 1 & 0 \\ * & Dh \end{bmatrix}$$

and so $x \in V$ is critical for f if and only if it is critical for h. Now $h(C \cap V) \subset \mathbb{R}^{m-1}$ and so by the induction hypothesis $h(C \cap V)$ has measure zero in $\mathbb{R}^{m-1}$. Now $f(C \cap V) \cap (\{x\} \times \mathbb{R}^{m-1}) \subset \{x\} \times h(C \cap V)$ which has measure zero in

$\{x\} \times \mathbb{R}^{m-1} \cong \mathbb{R}^{m-1}$ and so by Fubini's theorem $f(C \cap V)$ has measure zero. Since we may cover C by a countable number of sets of the form $C \cap V$ we conclude that $f(C)$ itself has measure zero.

Proof of (b): The proof of this part is quite similar to the proof of (a). Let $a \in C_{j-1} \backslash C_j$. It follows that some j-th partial derivative is not zero at a and after some permutation of the coordinate functions we may assume that

$$\frac{\partial}{\partial x^1} \frac{\partial^{|\beta|} f^1}{\partial x^\beta}(a) \neq 0$$

for some $j-1$- tuple $\beta = (i_1, ..., i_{j-1})$ where the function $g := \frac{\partial^{|\beta|} f^1}{\partial x^\beta}$ is zero at a since a is in C_{j-1}. Thus as before we have a map

$$x \mapsto (g(x), x^2, ..., x^n)$$

which restricts to a diffeomorphism ϕ on some open set V. We use ϕ, V as a chart about a. Notice that $\phi(C_{j-1} \cap V) \subset 0 \times \mathbb{R}^{n-1}$. We may use this chart ϕ to replace f by $g = f \circ \phi^{-1}$ which has the form

$$x \mapsto (x^1, h(x))$$

for some map $h : V \to \mathbb{R}^{m-1}$. Now by the induction hypothesis the restriction of g to

$$g_0 : \{0\} \times \mathbb{R}^{n-1} \cap V \to \mathbb{R}^m$$

has a set of critical values of measure zero. But each point from $\phi(C_{j-1} \cap V) \subset 0 \times \mathbb{R}^{n-1}$ is critical for g_0 since diffeomorphisms preserve criticality. Thus $g \circ \phi(C_{j-1} \cap V) = f(C_{j-1} \cap V)$ has measure zero.

Proof of (c): Let $I^n(r) \subset U$ be a cube of side r. We will show that if $k > (n/m) - 1$ then $f(I^n(r) \cap C_k)$ has measure zero. Since we may cover by a countable collection of such V the result follows. Now Taylor's theorem gives that if $a \in I^n(r) \cap C_k$ and $a + h \in I^n(r)$ then

$$|f(a+h) - f(a)| \leq c\,|h|^{k+1} \tag{3.1}$$

for some constant c that depends only on f and $I^n(r)$. We now decompose the cube $I^n(r)$ into R^n cubes of side length r/R. Suppose that we label these cubes which contain critical points of f as $D_1, ...D_N$. Let D_i contain a critical point a of f. Now if $y \in D$ then $|y - a| \leq \sqrt{n}r/R$ so using the Taylor's theorem remainder estimate above (3.1) with $y = a + h$ we see that $f(D_i)$ is contained in a cube $\widetilde{D}_i \subset \mathbb{R}^m$ of side

$$2c\left(\frac{\sqrt{n}r}{R}\right)^{k+1} = \frac{b}{R^{k+1}}$$

where the constant $b := 2c(\sqrt{n}r)^{k+1}$ is independent of the particular cube D from the decomposition and depends only on f and $I^n(r)$. The sum of the volumes of all such cubes $\widetilde{D}_i$ is

$$S \leq R^n \left(\frac{b}{R^{k+1}}\right)^m$$

which, under the condition that $m(k+1) > n$, may be made arbitrarily small by choosing R large (refining the decomposition of $I^n(r)$). The result now follows. ∎

Corollary 3.2 *If M and N are second countable manifolds then the set of regular values of a smooth map $f : M \to N$ is dense in N.*

3.1 Immersions

Definition 3.3 *A map $f : M \to N$ is called an **immersion at** $p \in M$ if $T_pf : T_pM \to T_{f(p)}N$ is an injection. $f : M \to N$ is called an **immersion** if f is an immersion at every $p \in M$.*

Figure **??** shows a simple illustration of an immersion of $\mathbb{R}^2$ into $\mathbb{R}^3$. This example is also an injective immersion (as far as the picture reveals) but an immersion can come back and cross itself. Being an immersion at p only requires that the restriction of the map to some small open neighborhood of p is injective.

Example 3.1 *We describe an immersion of the torus $T^2 := S^1 \times S^1$ into $\mathbb{R}^3$. We can represent points in T^2 as pairs $(e^{i\theta_1}, e^{i\theta_2})$. It is easy to see that for fixed $a, b > 0$ the following map is well defined:*

$$(e^{i\theta_1}, e^{i\theta_2}) \mapsto (x(e^{i\theta_1}, e^{i\theta_2}), y(e^{i\theta_1}, e^{i\theta_2}), z(e^{i\theta_1}, e^{i\theta_2}))$$

where

$$\begin{aligned} x(e^{i\theta_1}, e^{i\theta_2}) &= (a + b\cos\theta_1)\cos\theta_2 \\ y(e^{i\theta_1}, e^{i\theta_2}) &= (a + b\cos\theta_1)\sin\theta_2 \\ z(e^{i\theta_1}, e^{i\theta_2}) &= b\sin\theta_1 \end{aligned}$$

Exercise 3.1 *Show that the map of the above example is an immersion. Give conditions on a and b that guarantee that the map is a 1-1 immersion.*

Theorem 3.2 *Let $f : M \to N$ be a smooth map that is an immersion at p. Then there exist charts $(\mathtt{x}, U)$ with $p \in U$ and $(\mathtt{y}, V)$ with $f(p) \in V$ so that the corresponding coordinate expression for f is $(x^1, ..., x^k) \mapsto (x^1, ..., x^k, 0, ..., 0) \in \mathbb{R}^{k+m}$. Here, $k + m = n$ is the dimension of N and k is the rank of T_pf.*

Proof. Follows easily from theorem C.11. ∎

Theorem 3.3 *If $f : M \to N$ is an immersion (so an immersion at every point) and if f is a homeomorphism onto its image $f(M)$ using the relative topology, then $f(M)$ is a regular submanifold of N. In this case we call $f : M \to N$ an **embedding**.*

Proof. This follows from the last theorem plus a little point set topology (see the problems). ∎

Exercise 3.2 *Show that every injective immersion of a compact manifold is an embedding.*

Recall that a continuous map f is said to be proper if $f^{-1}(K)$ is compact whenever K is compact.

Exercise 3.3 *Show that a proper 1-1 immersion is an embedding. Hint: This is mainly a topological argument. You may assume (without loss of generality) that the spaces involved are Hausdorff and second countable. The slightly more general case of paracompact Hausdorff spaces follows.*

Definition 3.4 *Let S and M be smooth manifolds. An smooth map $f : S \to M$ will be called **smoothly universal** if for any smooth manifold N, a mapping $g : N \to S$ is smooth if and only if $f \circ g$ is smooth.*

Definition 3.5 *A **weak embedding** is a 1-1 immersion which is smoothly universal.*

Let $f : S \to M$ be a weak embedding and let $\mathcal{A}$ be the maximal atlas that gives the differentiable structure on S. Suppose we consider a different differentiable structure on S given by a maximal atlas $\mathcal{A}_2$. Now suppose that $f : S \to M$ is also a weak embedding with respect to $\mathcal{A}_2$. In seldom used pedantic notation we are supposing that both $f : (S, \mathcal{A}) \to M$ and $f : (S, \mathcal{A}_2) \to M$ are weak embeddings. From this it is easy to show that the identity map gives smooth maps $(S, \mathcal{A}) \to (S, \mathcal{A}_2)$ and $(S, \mathcal{A}_2) \to (S, \mathcal{A})$. This means that in fact $\mathcal{A} = \mathcal{A}_2$ so that the smooth structure of S is uniquely determined by the fact that f is a weak embedding.

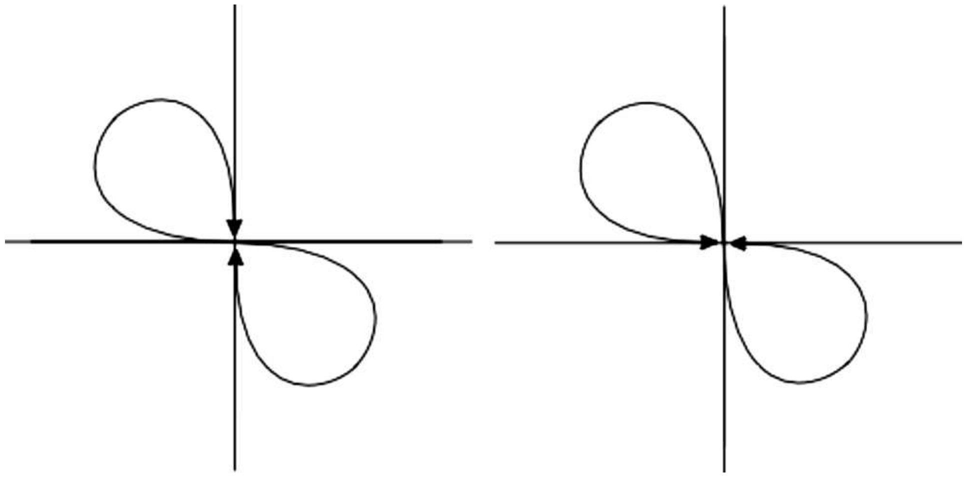
Exercise 3.4 *Show that every embedding is a weak embedding.*

In terms of 1-1 immersions we now have the following inclusions:

$$\begin{aligned}\{1-1 \text{ immersions}\} &\supset \{ \text{ weak embeddings}\} \\ &\supset \{ \text{ embeddings}\} \supset \{ \text{ proper embeddings}\}\end{aligned}$$

3.2 Immersed and Weakly Embedded Submanifolds

We have already seen the definition of a regular submanifold. The more general notion of a submanifold is supposed to be the “subobjects” in the category of smooth manifolds and smooth maps. Submanifolds are to manifolds what subsets are to sets in general. However, what exactly should be the definition of a submanifold? The fact is that there is some disagreement on this point. From the category theoretic point of view it seems natural that a submanifold of M



should be some kind of smooth map $I : S \to M$. This is not quite in line with our definition of regular submanifold which is, after all, a type of *subset* of M. There is considerable motivation to define submanifolds in general as certain subsets; perhaps the images of certain nice smooth maps. We shall follow this route.

Definition 3.6 *Let S be a subset of a smooth manifold M. If S is a smooth manifold such that the inclusion map $\iota : S \to M$ is an injective immersion then call S an* ***immersed submanifold****.*

If $f : N \to M$ is an injective immersion then $f(N)$ can be given a smooth structure so that it is an immersed submanifold. Indeed, we can simply transfer the structure from N via the bijection $f : N \to f(N)$. However, this may not be the only possible smooth structure on $f(N)$ which makes it an immersed submanifold. Thus it is imperative to specify what smooth structure is being used. Simply looking at the set is not enough.

Definition 3.7 *Let S be a subset of a smooth manifold M. If S is a smooth manifold such that the inclusion map $\iota : S \to M$ is a weak embedding then we say that S is a* ***weakly embedded submanifold****.*

From the properties of weak embeddings we know that for any given subset $S \subset M$ there is at most one smooth structure on S that makes it a weakly embedded submanifold.

Corresponding to each type of injective immersion considered so far we have in their images different notions of submanifold:

$$
\begin{aligned}
\{\text{immersed submanifolds}\} &\supset \{\text{ weakly embedded submanifolds}\} \\
&\supset \{\text{ regular manifolds}\} \supset \{\text{ proper submanifolds}\}
\end{aligned}
$$

We wish to further characterize the weakly embedded submanifolds.

Definition 3.8 *Let S be any subset of a smooth manifold M. For any $x \in S$, denote by $C_x(S)$ the set of all points of S that can be connected to x by a smooth curve with image entirely inside S.*

Definition 3.9 *We say that a subset S of an $n-$manifold M has **property W** if for each $s_0 \in S$ there exists a chart $(U, \mathtt{x})$ centered at s_0 such that $\mathtt{x}(C_{s_0}(U \cap S)) = \mathtt{x}(U) \cap (\mathbb{R}^k \times \{0\})$. Here $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$.*

Together, the next two propositions show that weakly embedded submanifolds are exactly those subsets which have property W. Our proof follows that of Michor [**?**] who refers to subsets with property W as initial submanifolds. With Michor's terminology the result will be that initial submanifolds are the same as the weakly embedded submanifolds.

Proposition 3.2 *If an injective immersion $I : S \to M$ is smoothly universal then the image $I(S)$ has property W. In particular, if $S \subset M$ is a weakly embedded submanifold of M then it has property W.*

Proof. Let $\dim(S) = k$ and $\dim M = n$. Choose $s_0 \in S$. Since I is an immersion we may pick a coordinate chart $(W, \mathtt{w})$ for S centered at s_0 and a chart $(V, \mathtt{v})$ for M centered at $I(s_0)$ such that

$$\mathtt{v} \circ I \circ \mathtt{w}^{-1}(y) = (y, 0).$$

Choose an $r > 0$ small enough that $B^k(0, 2r) \subset \mathtt{w}(W)$ and $B^n(0, 2r) \subset \mathtt{v}(V)$. Let $U = \mathtt{v}^{-1}(B^n(0, r))$ and $W_1 = \mathtt{w}^{-1}(B^k(0, r))$. Let $\mathtt{x} := \mathtt{v}|_U$. We show that the coordinate chart $(U, \mathtt{x})$ satisfies the property W of Definition 3.9.

$$\begin{aligned} \mathtt{x}^{-1}(\mathtt{x}(U) \cap (\mathbb{R}^k \times \{0\})) &= \mathtt{x}^{-1}\{(y, 0) : \|y\| < r\} \\ &= I \circ \mathtt{w}^{-1} \circ (\mathtt{x} \circ I \circ \mathtt{w}^{-1})^{-1}(\{(y, 0) : \|y\| < r\}) \\ &= I \circ \mathtt{w}^{-1}(\{y : \|y\| < r\}) = I(W_1) \end{aligned}$$

Now clearly $I(W_1) \subset I(S)$ but also we have

$$\begin{aligned} \mathtt{v} \circ I(W_1) &\subset \mathtt{v} \circ I \circ \mathtt{w}^{-1}(B^k(0, r)) \\ &= B^n(0, r) \cap \{\mathbb{R}^k \times \{0\}\} \subset B^n(0, r) \end{aligned}$$

so that $I(W_1) \subset \mathtt{v}^{-1}(B^n(0, r)) = U$. Thus $I(W_1) \subset U \cap I(S)$. Now since $I(W_1)$ is smoothly contractible to $I(s_0)$, every point of $I(W_1)$ is connected by a smooth curve to $I(s_0)$ completely contained in $I(W_1) \subset U \cap I(S)$. This implies that $I(W_1) \subset C_{I(s_0)}(U \cap I(S))$. Thus $\mathtt{x}^{-1}(\mathtt{x}(U) \cap (\mathbb{R}^k \times \{0\})) \subset C_{I(s_0)}(U \cap I(S))$ or

$$\mathtt{x}(U) \cap (\mathbb{R}^k \times \{0\}) \subset \mathtt{x}(C_{I(s_0)}(U \cap I(S))).$$

Conversely, let $z \in C_{I(s_0)}(U \cap I(S))$. By definition there must be a smooth curve $c : [0, 1] \to S$ starting at $I(s_0)$, ending at z and with $c([0, 1]) \subset U \cap I(S)$.

Since $I : S \to M$ is injective and smoothly universal there is a unique smooth curve $c_1 : [0,1] \to S$ with $I \circ c_1 = c$.
Claim: $c_1([0,1]) \subset W_1$.
Assume not. Then there is some number $t \in [0,1]$ with $c_1(t) \in \mathtt{w}^{-1}(\{r \le \|y\| < 2r\})$. Then

$$\begin{aligned}(\mathtt{v} \circ I)(c_1(t)) &\in (\mathtt{v} \circ I \circ \mathtt{w}^{-1})(\{r \le \|y\| < 2r\}) \\ &= \{(y,0) : r \le \|y\| < 2r\} \subset \{z \in \mathbb{R}^n : r \le \|z\| < 2r\}.\end{aligned}$$

Now this implies that $(\mathtt{v} \circ I \circ c_1)(t) = (\mathtt{v} \circ c)(t) \in \{z \in \mathbb{R}^n : r \le \|z\| < 2r\}$ which in turn implies the contradiction $c(t) \notin U$. The claim is proven.

Now the fact that $c_1([0,1]) \subset W_1$ implies $c_1(1) = I^{-1}(z) \in W_1$ and so $z \in I(W_1)$. As a result we have $C_{I(s_0)}(U \cap I(S)) = I(W_1)$ which together with the first half of the proof gives the result:

$$\begin{aligned} I(W_1) = \mathtt{x}^{-1}(\mathtt{x}(U) \cap (\mathbb{R}^k \times \{0\})) &\subset C_{I(s_0)}(U \cap I(S)) = I(W_1) \\ \Longrightarrow \quad \mathtt{x}^{-1}(\mathtt{x}(U) \cap (\mathbb{R}^k \times \{0\})) &= C_{I(s_0)}(U \cap I(S)) \\ \Longrightarrow \quad \mathtt{x}(U) \cap (\mathbb{R}^k \times \{0\}) &= \mathtt{x}(C_{I(s_0)}(U \cap I(S))). \end{aligned}$$

■

Conversely we have

Proposition 3.3 *If $S \subset M$ has property W then it has a unique smooth structure on S which makes it a weakly embedded submanifold of M.*

Proof. We are given that for every $s \in S$ there exists a chart $(U_s, \mathtt{x}_s)$ with $\mathtt{x}_s(s) = 0$ and with $\mathtt{x}(C_s(U \cap S)) = \mathtt{x}(U) \cap (\mathbb{R}^k \times \{0\})$. The charts on S will be the restrictions of the charts $(U_s, \mathtt{x}_s)$ to the sets $C_s(U \cap S)$. The overlap maps are smooth because they are restrictions of overlap maps on M to subsets of the form $V \cap (\mathbb{R}^k \times \{0\})$ for V open in $\mathbb{R}^n$. Notice however that the induced topology on S is finer than the subspace topology that S inherits from M. This is because the sets of the form $C_s(U \cap S)$ are not necessarily open in the subspace topology. Since it is a finer topology it is also Hausdorff.

It is clear that with this smooth structure on S, the inclusion $\iota : S \hookrightarrow M$ is an injective immersion.

We now show that the inclusion map $\iota : S \hookrightarrow M$ is smoothly universal and hence a weak embedding. By the comments following Definition 3.5 the smooth structure on S is unique. Let $g : N \to S$ be a map with $g(N) \subset S$ and suppose that $\iota \circ g$ is smooth. Given $x \in M$, choose a chart $(U_s, \mathtt{x}_s)$ where $s = g(x)$. The set $g^{-1}(U_s)$ is open since $\iota \circ g$ is continuous and $(\iota \circ g)^{-1}(U_s) = g^{-1} \circ \iota^{-1}(U_s) = g^{-1}(U_s)$. We may choose a chart $(V, \mathtt{y})$ centered at x with $V \subset g^{-1}(U_s)$ and we may arrange that $\mathtt{y}(V)$ is a ball centered at the origin. This means that $g(V)$ is smoothly contractible in $U_{g(x)} \cap S$ and hence $g(V) \subset C_{g(x)}(U_{g(x)} \cap S)$. But then

$$\mathtt{x}_s|_{C_s(U_s \cap S)} \circ g \circ \mathtt{y}^{-1} = \mathtt{x}_s \circ (\iota \circ g) \circ \mathtt{y}^{-1}$$

and so g is smooth because $\iota \circ g$ is smooth.

To be completely finished we need to show that with the topology induced by the charts (the submanifold topology) each connected component of S is second countable. We can give a quick proof but it depends on Riemannian metrics which we have yet to discuss. The idea is that on any paracompact smooth manifold, there are plenty of Riemannian metrics. A choice of Riemannian metric gives a notion of distant making every connected component a second countable metric space. Now if we put such a metric on M then it induces one on S (by restriction). Now this means that each component of S is also a separable metric space and hence a second countable Hausdorff topological space. ■

We say that two immersions $I_1 : N_1 \to M$ and $I_2 : N_2 \to M$ are **equivalent** if there exists a diffeomorphism $\Phi : N_1 \to N_2$ such that $I_2 \circ \Phi = I_1$; i.e. so that the following diagram commutes

$$\begin{tikzcd} N_1 \arrow[rr, "\Phi"] \arrow[dr] & & N_2 \arrow[dl] \\ & M & \end{tikzcd}.$$

If $I : N \to M$ is a weak embedding (resp. embedding) then there is a unique smooth structure on $S = I(N)$ such that S is a weakly embedded (reps. embedded) submanifold and $I : N \to M$ is equivalent to the inclusion $\iota : S \hookrightarrow M$ in the above sense.

We now come to a question that has perhaps been bothering the reader. Namely, which of these types of submanifold is meant when one just refers to a "submanifold". Unfortunately, there is no universally accepted convention. We shall follow the convention that "submanifold" when used alone without a qualifier is to mean a *regular* submanifold. What R. Sharpe [Shrp, Sharp] calls a "submanifold" refers to something more restrictive than the weakly embedded submanifolds but still less restrictive that the regular submanifolds. Sharpe's definition of submanifold seems designed to exclude examples like the following.

Example 3.2 *The image of the map* $I : (0, \infty) \to \mathbb{R}^2$ *given by* $I(t) := (\frac{1}{\ln(2+t^2)} \cos t, \frac{1}{\ln(2+t^2)} \sin t)$ *is a weakly embedded submanifold.*

The celebrated Whitney embedding theorem states that any smooth second-countable n-dimensional manifold can be embedded in a Euclidean of dimension 2n. We do not prove the full embedding theorem of Whitney but we will settle for the following easier result:

Theorem 3.4 *Suppose that* M *is an* $n-$ *manifold that has a finite atlas. Then there exists an injective immersion of* M *into* $\mathbb{R}^{2n+1}$. *Consequently, every* compact $n-$*dimensional smooth manifold can be embedded into* $\mathbb{R}^{2n+1}$.

Proof. Let M be a smooth manifold with a finite atlas. In particular, M is second countable. Initially, we will settle for an immersion into $\mathbb{R}^D$ for some possibly very large dimension D. Let $\{O_i, \varphi_i\}_{i \in N}$ be an atlas with cardinality $N < \infty$. The cover $\{O_i\}$ cover may be refined to two other covers $\{U_i\}_{i \in N}$ and $\{V_i\}_{i \in N}$ such that $\overline{U_i} \subset V_i \subset \overline{V_i} \subset O_i$. Also, we may find smooth functions $f_i : M \to [0, 1]$ such that

$$
\begin{aligned}
f_i(x) &= 1 \text{ for all } x \in U_i \\
\operatorname{supp}(f_i) &\subset O_i.
\end{aligned}
$$

Next we write $\varphi_i = (x_i^1, ...x_i^n)$ so that $x_i^j : O_i \to \mathbb{R}$ is the $j-$th coordinate function of the i-th chart and then let

$$
f_{ij} := f_i x_i^j \quad \text{(no sum)}
$$

which is defined and smooth on all of M after extension by zero.

Now we put the functions f_i together with the functions f_{ij} to get a map $i : M \to \mathbb{R}^{n+Nn}$:

$$
i = (f_1, ..., f_n, f_{11}, f_{12}, ..., f_{21}, ..., f_{nN}).
$$

Now we show that i is injective. Suppose that $i(x) = i(y)$. Now $f_k(x)$ must be 1 for some k since $x \in U_k$ for some k. But then also $f_k(y) = 1$ also and this means that $y \in V_k$ (why?). Now then, since $f_k(x) = f_k(y) = 1$ it follows that $f_{kj}(x) = f_{kj}(y)$ for all j. Remembering how things were defined we see that x and y have the same image under $\varphi_k : O_k \to \mathbb{R}^n$ and thus $x = y$.

To show that $T_x i$ is injective for all $x \in M$ we fix an arbitrary such x and then $x \in U_k$ for some k. But then near this x the functions $f_{k1,} f_{k2}, ..., f_{kn,}$ are equal to $x_k^1, ...x_k^n$ and so the rank of i must be *at least* n and in fact equal to n since $\dim T_x M = n$.

So far we have an injective immersion of M into $\mathbb{R}^{n+Nn}$.

We show that there is a projection $\pi : \mathbb{R}^D \to L \subset \mathbb{R}^D$ where $L \cong \mathbb{R}^{2n+1}$ is a $2n+1$ dimensional subspace of $\mathbb{R}^D$, such that $\pi \circ f$ is an injective immersion. The proof of this will be inductive. So suppose that there is an injective immersion f of M into $\mathbb{R}^d$ for some d with $D \geq d > 2n+1$. We show that there is a projection $\pi_d : \mathbb{R}^d \to L^{d-1} \cong \mathbb{R}^{d-1}$ such that $\pi_d \circ f$ is still an injective immersion. To this end, define a map $h : M \times M \times \mathbb{R} \to \mathbb{R}^d$ by $h(x, y, t) := t(f(x) - f(y))$. Now since $d > 2n + 1$, Sard's theorem 3.1 implies that there is a vector $y \in \mathbb{R}^d$ which is neither in the image of the map h nor in the image of the map $df : TM \to \mathbb{R}^d$. This y cannot be 0 since 0 is certainly in the image of both of these maps. Now if $pr_{\perp y}$ is projection onto the orthogonal complement of y, then $pr_{\perp y} \circ f$ is injective; for if $pr_{\perp y} \circ f(x) = pr_{\perp y} \circ f(y)$ then $f(x) - f(y) = ay$ for some $a \in \mathbb{R}$. But suppose $x \neq y$. Since f is injective we must have $a \neq 0$. This state of affairs is impossible since it results in the equation $h(x, y, 1/a) = y$ which contradicts our choice of y. Thus $pr_{\perp y} \circ f$ is injective.

Next we examine $T_x(pr_{\perp y} \circ f)$ for an arbitrary $x \in M$. Suppose that $T_x(pr_{\perp y} \circ f)v = 0$. Then $d(pr_{\perp y} \circ f)|_x v = 0$ and since $pr_{\perp y}$ is linear this amounts to $pr_{\perp y} \circ df|_x v = 0$ which gives $df|_x v = ay$ for some number $a \in \mathbb{R}$, and which cannot be 0 since f is assumed an immersion. But then $df|_x \frac{1}{a}v = y$ which also contradicts our choice of y.

We conclude that $pr_{\perp y} \circ f$ is an injective immersion. Repeating this process inductively we finally get a composition of projections $pr : \mathbb{R}^D \to \mathbb{R}^{2n+1}$ such that $pr \circ f : M \to \mathbb{R}^{2n+1}$ is an injective immersion. ■

3.2.1 Level sets as submanifolds

If $f : M \to N$ is a smooth map that has the same rank at each point, then we say it has **constant rank**. Similarly, if f has the same rank for each p in a open subset U then we say that f has constant rank on U.

Theorem 3.5 (Level Submanifold Theorem) *Let $f : M \to N$ be a smooth map and consider the level set $f^{-1}(q_0)$ for $q_0 \in N$. If f has constant rank k on an open neighborhood of each $p \in f^{-1}(q_0)$ then $f^{-1}(q_0)$ a closed regular submanifold of codimension k.*

Proof. Clearly $f^{-1}(q_0)$ is a closed subset of M. Let $p_0 \in f^{-1}(q_0)$ and consider a chart (U, φ) centered at p_0 and a chart (V, ψ) centered at q_0 with $f(U) \subset V$. We may choose U small enough that f has rank k on U. By Theorem C.12 we may compose with diffeomorphisms to replace (U, φ) by a new chart $(U', \mathtt{x})$ also centered at p_0 and also replace (V, ψ) by a chart $(V', \mathtt{y})$ centered at q_0 such that $\bar{f} := \mathtt{y} \circ f \circ \mathtt{x}^{-1}$ is given by $(a^1, ..., a^n) \mapsto (a^1, ..., a^k, 0, ..., 0)$. We show that

$$U' \cap f^{-1}(q_0) = \{p \in U' : x^1(p) = \cdots = x^k(p) = 0\}$$

If $p \in U' \cap f^{-1}(q_0)$ then $\mathtt{y} \circ f(p) = 0$ and $\mathtt{y} \circ f \circ \mathtt{x}^{-1}(x^1(p), ..., x^n(p)) = 0$ or

$$x^1(p) = \cdots = x^k(p) = 0$$

On the other hand, suppose that $p \in U'$ and $x^1(p) = \cdots = x^k(p) = 0$. Then we can reverse the logic to obtain that $\mathtt{y} \circ f(p) = 0$ and hence $f(p) = q_0$. Since p_0 was arbitrary we have verified the existence of a cover by charts adapted to $f^{-1}(q_0)$. ■

3.3 Submersions

Definition 3.10 *A map $f : M \to N$ is called a **submersion at** $p \in M$ if $T_pf : T_pM \to T_{f(p)}N$ is surjection. $f : M \to N$ is called a **submersion** if f is a submersion at every $p \in M$.*

Example 3.3 *The map of the punctured space $\mathbb{R}^3 - \{0\}$ onto the sphere S^2 given by $x \mapsto |x|$ is a submersion. To see this, use any spherical coordinates (ρ, ϕ, θ) on $\mathbb{R}^3 - \{0\}$ and the induced submanifold coordinates (ϕ, θ) on so S^2. Expressed with respect to these coordinates, the map becomes $(\rho, \phi, \theta) \mapsto (\phi, \theta)$ on the domain of the spherical coordinate chart. Here we ended up locally with a projection onto a second factor $\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ but this is clearly good enough to prove the point.*

As in the last example, to show that a map is a submersion at some p is is enough to find charts containing p and $f(p)$ so that the coordinate representative of the map is just a projection. Conversely, we have

Theorem 3.6 *Let M be an $m-$manifold and N an $n-$manifold and let $f : M \to N$ be a smooth map that is an submersion at p. Then there exist charts $(U, \mathtt{x})$ and $(V, \mathtt{y})$ containing p and $f(p)$ respectively and with $f(U) \subset V$ such that $\mathtt{y} \circ f \circ \mathtt{x}^{-1}$ is given by $(x^1, ..., x^k, ..., x^n) \mapsto (x^1, ..., x^k) \in \mathbb{R}^k$. Here k is both the dimension of N and the rank of $T_p f$.*

Proof. Follows directly from theorem C.10 of Appendix G. ■

In certain contexts, submersions, especially surjective submersions, are referred to as projections. We often denote such map by the letter π. Recall that if $\pi : M \to N$ is a smooth map then a smooth local **section** of π is a smooth map $\sigma : V \to M$ defined on an open set V such that $\pi \circ \sigma = id_V$. Also, we adopt the terminology that subsets of M of the form $\pi^{-1}(q)$ are called **fibers** of the submersion.

Proposition 3.4 *If $\pi : M \to N$ is a submersion then it is an open map and every point $p \in M$ is in the image of a smooth local section.*

Proof. Let $p \in M$ be arbitrary. We choose a chart $(U, \mathtt{x})$ centered at p and a chart $(V, \mathtt{y})$ centered at $\pi(p)$ such that $\mathtt{y} \circ \pi \circ \mathtt{x}^{-1}$ is of the form $(x^1, ..., x^l, x^{l+1}, ...x^m) \to (x^1, ..., x^l)$. By shrinking the domains if necessary we can arrange that $\mathtt{x}(U)$ has the form $A \times B \subset \mathbb{R}^l \times \mathbb{R}^k$ and $\mathtt{y}(V) = B \subset \mathbb{R}^l$. Then we may transfer the section $i_b : a \to (a, b)$ where $b = \mathtt{x}(p)$. More precisely we let $\sigma := \mathtt{x}^{-1} \circ i_b \circ \mathtt{y}$ on A.

Now let O be any open set in M. To show that $\pi(O)$ is open we pick any $q \in \pi(O)$ and choose $p \in \pi^{-1}(q)$. Now we choose a chart $(U, \mathtt{x})$ as above but with $U \subset O$. Then q is in the domain of a section which is open and contained in $\pi(O)$. ■

Proposition 3.5 *Let $\pi : M \to N$ be a surjective submersion. If $f : N \to P$ is any map then f is smooth if and only if $f \circ \pi$ is smooth.*

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "f\circ\pi"] & \\ N \arrow[r, "f"] & P \end{tikzcd}$$

Proof. One direction is trivial. For the other direction, assume that $f \circ \pi$ is smooth. We check for smoothness about an arbitrary point $q \in N$. Pick $p \in \pi^{-1}(q)$. By the previous proposition p is in the image of a smooth section $\sigma : V \to M$. This means that f and $(f \circ \pi) \circ \sigma$ agree on a neighborhood of q and since that later is smooth we are done. ■

Next suppose that we have a surjective submersion $\pi : M \longrightarrow N$ and consider a smooth map $g : M \to P$ which is constant on fibers. That is, we assume that if $p_1, p_2 \in \pi^{-1}(q)$ for some $q \in N$ then $f(p_1) = f(p_2)$. Clearly there is a unique induced map $f : N \to P$ so that $g = f \circ \pi$. By the above proposition f must be smooth. This we record as

Corollary 3.3 *If $g : M \to P$ is a smooth which is constant on the fibers of a surjective submersion $\pi : M \longrightarrow N$ then there is a unique smooth map $f : N \to P$ such that $g = f \circ \pi$.*

The following technical lemma is needed later and represents one more situation where second countability is needed:

Lemma 3.2 *Suppose that M is a second countable smooth manifold. If $f : M \longrightarrow N$ is a smooth map with constant rank that is also surjective then it is a submersion.*

Proof. Let $\dim M = m$, $\dim N = n$ and $\operatorname{rank}(f) = k$ and chose $p \in M$. Suppose that f is not a submersion so that $k < n$. We can cover M by a countable collection of charts $(U_\alpha, \mathtt{x}_\alpha)$ and cover N by charts $(V_i, \mathtt{y}_i)$ such that for every α there is an $i = i(\alpha)$ with $f(U_\alpha) \subset V_i$ and $\mathtt{y}_i \circ f \circ \mathtt{x}_\alpha^{-1}(x^1, ..., x^n) = (x^1, ..., x^k, 0, ..., 0)$. But this means that $f(U_\alpha)$ has Lebesgue measure zero. But $f(M) = \cup_\alpha f(U_\alpha)$ and so $f(M)$ is also of measure zero which contradicts the surjectivity of f. This contradiction means that f must be a submersion after all. ■

Proposition 3.6 *Let M and N be smooth manifolds of dimension m and n respectively with $n > m$. Consider any smooth map $f : M \to N$. Then if $q \in N$ is a regular value, the inverse image set $f^{-1}(q)$ is a regular submanifold.*

Proof. It is clear that since f must have maximal rank it also has constant rank (this also follows from the previous local result). We may now apply Theorem 3.5. ■

Example 3.4 (The unit sphere) *The set $S^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}$ is a codimension 1 submanifold of $\mathbb{R}^n$ since we can use the map $(x, y) \mapsto x^2 + y^2$ as our map and let $q = 1$.*

Given k functions $F^j(x, y)$ on $\mathbb{R}^l \times \mathbb{R}^k$ we define the locus

$$M := \{(x, y) \in \mathbb{R}^l \times \mathbb{R}^k : F^j(x, y) = c^j\}$$

where each c^j is a fixed number in the range of F^j. If the Jacobian determinant at $(x_0, y_0) \in M$;

$$\frac{\partial(F^1, ..., F^k)}{\partial(y^1, ..., y^k)}(x_0, y_0)$$

is not zero then near (x_0, y_0) then we can apply the theorem. We can see things more directly: Since the Jacobian determinant is nonzero, we can solve the equations $F^j(x, y) = c^j$ for $y^1, ..., y^k$ in terms of $x^1, ..., x^l$:

$$\begin{aligned} y^1 &= f^1(x^1, ..., x^l) \\ y^2 &= f^2(x^1, ..., x^l) \\ \\ y^k &= f^k(x^1, ..., x^l) \end{aligned}$$

and then $(x^1, ..., x^l) \longmapsto (x^1, ..., x^l, f^1(x), ..., f^k(x))$ parameterizes M near (x_0, y_0) in such a nice way that the inverse is a chart for M. This latter statement is due to the inverse mapping theorem in this case. If the Jacobian determinant never vanishes on M then we have a cover by charts and M is a submanifold of $\mathbb{R}^l \times \mathbb{R}^k$ of dimension l and codimension k.

Example 3.5 *The set of all square matrices $\mathbb{M}_{n\times n}$ is a manifold by virtue of the obvious isomorphism $\mathbb{M}_{n\times n} \cong \mathbb{R}^{n^2}$. The set $\mathfrak{sym}(n,\mathbb{R})$ of all symmetric matrices is an $n(n+1)/2$-dimensional manifold by virtue of the obvious 1-1 correspondence $\mathfrak{sym}(n,\mathbb{R}) \cong \mathbb{R}^{n(n+1)/2}$ given by using $n(n+1)/2$ independent entries in the upper triangle of the matrix as coordinates.*
Now the set $\mathrm{O}(n,\mathbb{R})$ of all $n \times n$ orthogonal matrices is a submanifold of $\mathbb{M}_{n\times n}$. We can show this using Theorem 3.6 as follows. Consider the map $f : \mathbb{M}_{n\times n} \to \mathfrak{sym}(n,\mathbb{R})$ given by $A \mapsto A^tA$. Notice that by definition of $\mathrm{O}(n,\mathbb{R})$ we have $f^{-1}(I) = \mathrm{O}(n,\mathbb{R})$. Let us compute the tangent map at any point $Q \in f^{-1}(I) = \mathrm{O}(n,\mathbb{R})$. The tangent space of $\mathfrak{sym}(n,\mathbb{R})$ at I is $\mathfrak{sym}(n,\mathbb{R})$ itself since $\mathfrak{sym}(n,\mathbb{R})$ is a vector space. Similarly, $\mathbb{M}_{n\times n}$ is its own tangent space. Under the identifications of section 2.5 we have

$$T_Qf \cdot v = \frac{d}{ds}(Q^t + sv^t)(AQ + sv) = v^tQ + Q^tv.$$

Now this map is clearly surjective onto $\mathfrak{sym}(n,\mathbb{R})$ when $Q = I$. On the other hand, for any $Q \in \mathrm{O}(n,\mathbb{R})$ consider the map $L_{Q^{-1}} : \mathbb{M}_{n\times n} \to \mathbb{M}_{n\times n}$ given by $L_{Q^{-1}}(B) = Q^{-1}B$. The map $T_QL_{Q^{-1}}$ is actually just $T_QL_{Q^{-1}} \cdot v = Q^{-1}v$ which is a linear isomorphism since Q is nonsingular. We have that $f \circ L_Q = f$ and so by the chain rule

$$\begin{aligned} T_Qf \cdot v &= T_If \circ T_Q(L_{Q^{-1}}) \cdot v \\ &= T_If \cdot Q^{-1}v \end{aligned}$$

which shows that T_Qf is also surjective.

The following proposition shows an example of the simultaneous use of Sard's theorem and theorem 3.6.

Proposition 3.7 *Let S be a connected submanifold of $\mathbb{R}^n$ and let L be a linear subspace of $\mathbb{R}^n$. Then there exist $x \in \mathbb{R}^n$ such that $(x + L) \cap S$ is a submanifold of S.*

Proof. Start with a line l through the origin that is normal to L. Let $pr : \mathbb{R}^n \to S$ be orthogonal projection onto l . The restriction $\pi := pr|_S \to l$ is easily seen to be smooth. If $\pi(S)$ were just a single point x then $\pi^{-1}(x)$ would be all of S. Now $\pi(S)$ is connected and a connected subset of $l \cong \mathbb{R}$ must contain an interval which means that $\pi(S)$ has positive measure. Thus by Sard's theorem there must be a point $x \in l$ that is a regular value of π. But then 3.6 implies that $\pi^{-1}(x)$ is a submanifold of S. But this is the conclusion since $\pi^{-1}(v) = (x + L) \cap S$. ■

We can generalize theorem 3.6 using the concept of transversality .

Definition 3.11 *Let $f : M \to N$ be a smooth map and $S \subset N$ a submanifold of N. We say that f is **transverse to** S if for every $p \in f^{-1}(S)$ we have*

$$T_{f(p)}N = T_{f(p)}S + T_p f(T_p M).$$

*If f is **transverse to** S we write $f \pitchfork S$.*

Theorem 3.7 *Let $f : M \to N$ be a smooth map and $S \subset N$ a submanifold of N of codimension k and suppose that $f \pitchfork S$. Then $f^{-1}(S)$ is a submanifold of M with codimension k. Furthermore we have $T_p(f^{-1}(S)) = T_{f(p)}f^{-1}(T_{f(p)}S)$ for all $p \in f^{-1}(S)$ and $\operatorname{codim}(f^{-1}(S)) = \operatorname{codim}(S)$.*

Proof. Let $p = f(s) \in S$ and choose a regular submanifold chart $(U, \mathtt{x})$ with $p \in U$ so that $\mathtt{x}(S \cap U) = \mathtt{x}(U) \cap (\mathbb{R}^{n-k} \times 0)$. If $\pi : \mathbb{R}^{n-k} \times \mathbb{R}^k \to \mathbb{R}^k$ is the projection then the transversality condition implies that 0 is a regular value of $\pi \circ \mathtt{x} \circ f$. Thus $(\pi \circ \mathtt{x} \circ f)^{-1}(0) = f^{-1}(S) \cap U$ is submanifold of U of codimension k. Since this is true for all $s \in f^{-1}(S)$ the result follows. ∎

It can also be shown that if S in the above theorem is only a weakly embedded submanifold then $f^{-1}(S)$ is also a weakly embedded submanifold of the same codimension.

We can also define when two maps are transverse to each other:

Definition 3.12 *If $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are smooth maps, we say that f_1 and f_2 are **transverse** at $q \in N$ if*

$$T_{f(p)}N = T_{p_1}f_1(T_{p_1}M) + T_{p_2}f_2(T_{p_2}M) \text{ whenever } f_1(p_1) = f_2(p_2) = q$$

(Note that f_1 is transverse to f_2 at any point not in the image of one of the maps f_1 and f_2). If f_1 and f_2 are transverse for all $q \in N$ then we sat that f_1 and f_2 are transverse and we write $f_1 \pitchfork f_2$.

One can check that if $f : M \to N$ is a smooth map and S is a submanifold of N then f and the inclusion $\iota : S \hookrightarrow N$ are transverse if and only if $f \pitchfork S$ according to definition 3.7.

If $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are smooth maps then we can consider the set

$$(f_1 \times f_2)^{-1}(\Delta) := \{(p_1, p_2) \in M_1 \times M_2 : f_1(p_1) = f_2(p_2)\}$$

which is the inverse image of the diagonal $\{(q_1, q_2) \in N \times N : q_1 = q_2\}$.

Corollary 3.4 (Transverse pullbacks) *If $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ are transverse smooth maps then $(f_1 \times f_2)^{-1}(\Delta)$ is a submanifold of $M_1 \times M_2$. If $g_1 : P \to M_1$ and $g_2 : P \to M_2$ are any two maps with the property $f_1 \circ g_1 = f_2 \circ g_2$ then the map $(g_1, g_2) : P \to (f_1 \times f_2)^{-1}(\Delta)$ given by $(g_1, g_2)(x) = (g_1(x), g_2(x))$ is smooth and is the unique smooth map such that $pr_1 \circ (g_1, g_2) = g_1$ and $pr_2 \circ (g_1, g_2) = g_2$.*

Proof. We leave the proof as an exercise. Hint: $f_1 \times f_2$ is transverse to Δ if and only if $f_1 \pitchfork f_2$. ∎

3.4 Morse Functions

If we consider a smooth function $f : M \to \mathbb{R}$ and assume that M is a compact manifold (without boundary) then f must achieve both a maximum at one or more points of M and a minimum at one or more points of M. Let p_e be one of these points. The usual argument shows that $df|_{p_e} = 0$ (Recall that under the usual identification of $\mathbb{R}$ with any of its tangent spaces we have $df|_{p_e} = T_{p_e} f$). Now let p be some point for which $df|_p = 0$, i.e. p is a critical point for f. Does f achieve either a maximum or a minimum at p? How does the function behave in a neighborhood of x_0? As the reader may well be aware, these questions are easier to answer in case the second derivative of f at p is nondegenerate. But what is the second derivative in this case?

Definition 3.13 *The* ***Hessian*** *matrix of f at one of its critical points p and with respect to coordinates $\mathtt{x} = (x^1, ..., x^n)$ is the matrix of second partials:*

$$[H_{f,p}]_{\mathtt{x}} = \begin{bmatrix} \frac{\partial^2 f\circ\mathtt{x}^{-1}}{\partial x^1 \partial x^1}(x_0) & \cdots & \frac{\partial^2 f\circ\mathtt{x}^{-1}}{\partial x^1 \partial x^n}(x_0) \\ \vdots & & \vdots \\ \frac{\partial^2 f\circ\mathtt{x}^{-1}}{\partial x^n \partial x^1}(x_0) & \cdots & \frac{\partial^2 f\circ\mathtt{x}^{-1}}{\partial x^n \partial x^n}(x_0) \end{bmatrix}$$

where $x_0 = \mathtt{x}(p)$. The critical point p is called ***nondegenerate*** *if H is nonsingular.*

Now any such matrix H is symmetric and by Sylvester's law of inertia this matrix is equivalent to a diagonal matrix whose diagonal entries are either 1 or -1. The number of -1's occurring in this diagonal matrix is called the **index** of the critical point. According to problem 9 we may define the Hessian $H_{f,p} : T_pM \times T_pM \to \mathbb{R}$ which is a symmetric bilinear form at each critical point p of f by letting $H_{f,p}(v.w) = X_p(Yf) = Y_p(Xf)$ for any vector fields X and Y which respectively take the values v and w at p. Now we may give a coordinate free definition of a nondegenerate point for f. Namely, p is nondegenerate point for f if and only if $H_{f,p}$ is a nondegenerate bilinear form. $H_{f,p}$ is a nondegenerate if for each fixed nonzero $v \in T_pM$ the map $H_{f,p}(v, .) : T_pM \to \mathbb{R}$ is a nonzero element of the dual space T_p^*M.

Exercise 3.5 *Show that the nondegeneracy is well defined by ether of the two definitions given above and the definitions agree. Hint: $H_{f,p}(\frac{\partial}{\partial x^i}\big|, \frac{\partial}{\partial x^j}\big|) = \frac{\partial^2 f\circ\mathtt{x}^{-1}}{\partial x^i \partial x^j}(\mathtt{x}(p))$.*

Exercise 3.6 *Show that nondegenerate critical points are isolated. Show by example that this need not be true for general critical points.*

The structure of a function near one of its nondegenerate critical points is given by the following famous theorem of M. Morse:

Theorem 3.8 (Morse Lemma) *Let $f : M \to \mathbb{R}$ be a smooth function and let x_0 be a nondegenerate critical point for f of index i. Then there is a local* *coordinate system* $(U, \mathtt{x})$ *containing* x_0 *such that the local representative* $f_U := f \circ \mathtt{x}^{-1}$ *for* f *has the form*

$$f_U(x^1, ..., x^n) = f(x_0) + h_{ij}x^i x^j$$

and where it may be arranged that the matrix $h = (h_{ij})$ *is a diagonal matrix of the form* $\mathrm{diag}(-1, ... - 1, 1, ..., 1)$ *for some number (perhaps zero) of ones and minus ones. The number of minus ones is exactly the index* i.

Proof. This is clearly a local problem and so it suffices to assume $f :: \mathbb{R}^n \to \mathbb{R}$ and also that $f(0) = 0$. Then our task is to show that there exists a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $f \circ \phi(x) = x^t h x$ for a matrix of the form described. The first step is to observe that if $g : U \subset \mathbb{R}^n \to \mathbb{R}$ is any function defined on a convex open set U and $g(0) = 0$ then

$$\begin{aligned} g(u_1, ..., u_n) &= \int_0^1 \frac{d}{dt} g(tu_1, ..., tu_n) dt \\ &= \int_0^1 \sum_{i=1}^n u_i \partial_i g(tu_1, ..., tu_n) dt \end{aligned}$$

Thus g is of the form $g = \sum_{i=1}^n u_i g_i$ for certain smooth functions $g_i, 1 \le i \le n$ with the property that $\partial_i g(0) = g_i(0)$. Now we apply this procedure first to f to get $f = \sum_{i=1}^n u_i f_i$ where $\partial_i f(0) = f_i(0) = 0$ and then apply the procedure to each f_i and substitute back. The result is that

$$f(u_1, ..., u_n) = \sum_{i,j=1}^n u_i u_j h^{ij}(u_1, ..., u_n) \tag{3.2}$$

for some functions h^{ij} with the property that $h^{ij}()$ is nonsingular at and therefore near 0. Next we symmetrize (h^{ij}) by replacing it with $\frac{1}{2}(h^{ij} + h^{ji})$ if necessary. This leaves the expression 3.2 untouched. Now the index of the matrix $(h^{ij}(0))$ is i and this remains true in a neighborhood of 0. The trick is to find a matrix $C(x)$ for each x in the neighborhood that effects the diagonalization guaranteed by Sylvester's theorem: $D = C(x)h(x)C(x)^{-1}$. The remaining details, including the fact that the matrix $C(x)$ may be chosen to depend smoothly on x, is left to the reader. ∎

3.5 Problems.

1. Show that a submersion always maps open set to open set (it is an open mapping). Further show that if M is compact and N connected then a submersion $f : M \to N$ must be surjective.

2. Prove Theorem 3.3.

3. Define a function $s : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}P^n$ by the rule that $s(x)$ is the line through x. Show that s is a submersion.

4. Given a smooth surjection $f : M \to N$, define a local section of f over an open $U \subset N$ to be a smooth map $\sigma : U \to M$ such that $f \circ \sigma = id_N$. Show that f is a submersion if and only if for each $p \in M$ there is a local section σ whose range contains p.

5. Show that if $p(x) = p(x_1, ..., x_n)$ is a homogeneous polynomial so that for some $m \in \mathbb{Z}_+$
$$p(tx_1, ..., tx_n) = t^m p(x_1, ..., x_n)$$
then as long as $c \neq 0$ the set $p^{-1}(c)$ is a $n-1$ dimensional submanifold of $\mathbb{R}^n$.

6. Suppose that $g : M \to N$ is transverse to a submanifold $W \subset N$. For another smooth map $f : Y \to M$ show that $f \pitchfork g^{-1}(N)$ if and only if $(g \circ f) \pitchfork W$.

7. Suppose that $c : [a, b] \to M$ is a smooth map. Show that given any compact subset $C \subset (a, b)$ and any $\epsilon > 0$ there is an immersion $\gamma : (a, b) \to M$ that agrees with c on the set C and such that
$$|\gamma(t) - c(t)| \leq \epsilon \text{ for all } t \in (a, b).$$

8. Show that there is a continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(B(0,1)) \subset f(B(0,1))$, $f(\mathbb{R}^2 \backslash B(0,1)) \subset f(\mathbb{R}^2 \backslash B(0,1))$ and $f_{\partial B(0,1)} = \mathrm{id}_{\partial B(0,1)}$ and with the properties that f is C^∞ on $\overline{B}(0,1)$ and on $\mathbb{R}^2 \backslash B(0,1)$ while f is not C^∞ on $\mathbb{R}^2$.

9. Construct an embedding of $\mathbb{R} \times S^n$ into $\mathbb{R}^{n+1}$

10. Construct an embedding of $G(n, k)$ into $G(n, k + l)$ for each $l \geq 1$.

11. Show that the map $f : \mathbb{P}^2 \to \mathbb{R}^3$ defined by $f([x, y, z]) = (yz, xz, xy)$ is an immersion at all but six points $p \in \mathbb{P}^2$.

12. Let $h : M \to \mathbb{R}^n$ be smooth and let $N \subset \mathbb{R}^n$ a regular submanifold. Prove that for each $\varepsilon > 0$ there exists a $v \in \mathbb{R}^n$, with $|v| < \varepsilon$, so that the map $p \mapsto h(p) + v$ is transverse to N. (Think about the map $M \times N \to \mathbb{R}^n$ given by $(p, y) \mapsto y - f(p)$.)

13. Define $\phi : S^1 \to \mathbb{R}$ by $e^{i\theta} \mapsto \theta$ for $0 \leq \theta < 2\pi$. Define $\lambda : \mathbb{R} \to S^1$ by $\theta \mapsto e^{i\theta}$. Show that λ is an immersion, that $\lambda \circ \phi$ is smooth but that ϕ is not differentiable (it isn't even continuous).

Chapter 4

Lie Groups I

4.1 Definitions and Examples

One approach to geometry is to view geometry as the study of invariance and symmetry. In our case, we are interested in studying symmetries of smooth manifolds, Riemannian manifolds, symplectic manifolds etc. Now the usual way to talk about symmetry in mathematics is by the use of the notion of a transformation group. The wonderful thing for us is that the groups that arise in the study of geometric symmetries are often themselves smooth manifolds. Such "group manifolds" are called Lie groups.

In physics, Lie groups play a big role in connection with physical symmetries and conservation laws (Noether's theorem). Within physics, perhaps the most celebrated role played by Lie groups is in particle physics and gauge theory. In mathematics, Lie groups play a prominent role in Harmonic analysis (generalized Fourier theory), group representations and in virtually every branch of geometry including Riemannian geometry, Cartan geometry, algebraic geometry, Kähler geometry, and symplectic geometry.

Definition 4.1 *A smooth manifold G is called a* ***Lie group*** *if it is a group (abstract group) such that the multiplication map $\mu : G \times G \to G$ and the inverse map $\nu : G \to G$ given respectively by $\mu(g, h) = gh$ and $\nu(g) = g^{-1}$ are C^∞ maps. If the group is abelian we sometimes opt to use the additive notation $g + h$ for the group operation.*

Example 4.1 $\mathbb{R}$ *is a one-dimensional (abelian) Lie group were the group multiplication is the usual addition $+$. Similarly, any real or complex vector space is a Lie group under vector addition.*

Example 4.2 *The circle $S^1 = \{z \in \mathbb{C} : |z|^2 = 1\}$ is a 1-dimensional (abelian) Lie group under complex multiplication. It is also traditional to denote this group by $U(1)$.*

Exercise 4.1 *Let $g \in G$ (a Lie group). Show that each of the following maps $G \to G$ is a diffeomorphism:*
1) $L_g : x \mapsto gx$ (left translation)
2) $R_g : x \mapsto xg$ (right translation)
3) $C_g : x \mapsto gxg^{-1}$ (conjugation).
4) $inv : x \mapsto x^{-1}$ (inversion)

If G and H are Lie groups then so is the product manifold $G \times H$ where multiplication is $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$. $G \times H$ is called the **product Lie group.**

Definition 4.2 *Let H be an abstract subgroup of a Lie group G. If H is a Lie group such that the inclusion map $H \hookrightarrow G$ is an immersion then we say that H is a **Lie subgroup** of G.*

Proposition 4.1 *If H is an abstract subgroup of a Lie group G that is also a regular submanifold then H is a closed Lie subgroup.*

Proof. The multiplication and inversion maps, $H \times H \to H$ and $H \to H$ are the restrictions of the multiplication and inversion maps on G and since H is a regular submanifold we obtain the needed smoothness of these maps. The harder part is to show that H is closed. So let $x_0 \in \overline{H}$ be arbitrary. Let $(U, \mathtt{x})$ be a chart adapted to H whose domain contains e. Let $\delta : G \times G \to G$ be the map $\delta(g_1, g_2) = g_1^{-1} g_2$ and choose an open set V such that $e \in V \subset \overline{V} \subset U$. By continuity of the map δ we can find an open neighborhood O of the identity element such that $O \times O \subset \delta^{-1}(V)$. Now if h_i is a sequence in H converging to $x_0 \in \overline{H}$ then $x_0^{-1} h_i \to e$ and $x_0^{-1} h_i \in O$ for all sufficiently large i. Since $h_j^{-1} h_i = \left(x_0^{-1} h_j\right)^{-1} x_0^{-1} h_i$ we have that $h_j^{-1} h_i \in V$ for sufficiently large i, j. Now for any sufficiently large fixed j we have

$$\lim_{i \to \infty} h_j^{-1} h_i = h_j^{-1} x_0 \in \overline{V} \subset U$$

Each $h_j^{-1} h_i$ is in H and since $U \cap H$ is closed in U we see that $h_j^{-1} x_0 \in U \cap H \subset H$ for all sufficiently large j. This shows that $x_0 \in H$ and since x_0 was arbitrary we are done. ■

By a **closed Lie subgroup** we shall always mean one that is a regular submanifold as in the previous theorem. It is a nontrivial fact that an abstract subgroup of a Lie group that is also a closed subset is automatically a closed Lie subgroup in this sense (see 4.5).

Example 4.3 *The product group $S^1 \times S^1$ is called the 2-torus. More generally, the torus groups are defined by $T^n = S^1 \times \underset{n\text{-times}}{\ldots} \times S^1$.*

Example 4.4 *S^1 embedded as $S^1 \times \{1\}$ in the torus $S^1 \times S^1$ is a closed subgroup.*

4.2 Linear Lie Groups

The group $Aut(\mathrm{V})$ of all linear automorphisms of a vector space V over a field $\mathbb{F}$ ($= \mathbb{R}$ or $\mathbb{C}$) is an open submanifold of the vector space $End(\mathrm{V})$ (the space of linear maps $\mathrm{V} \to \mathrm{V}$). This group is easily seen to be a Lie group and in that context it is usually denoted by $Gl(\mathrm{V})$ and referred to as the **general linear group of** V. In case $\mathrm{V} = \mathbb{R}^n$, the group is usually identified with the group of invertible $n \times n$ real matrices and is denoted $Gl(n, \mathbb{R})$ (or by $Gl(n)$). Similarly, if $\mathrm{V} = \mathbb{R}^n$, the group $Gl(\mathbb{C}^n)$ is identified with the invertible $n \times n$ complex matrices denoted $Gl(n, \mathbb{C})$. Lie groups which are subgroups of $Gl(\mathrm{V})$ for some vector space V are referred to as **linear Lie groups** are usually realized as matrix groups. These are the linear Lie groups and by choosing a basis we always have the associated group of matrices.

Recall that the determinant of a linear transformation from a vector space to itself is defined independent of any choice of basis.

Theorem 4.1 *Let* V *be an* $n-$*dimensional vector space over the field* $\mathbb{F}$ *which we take to be either* $\mathbb{R}$ *or* $\mathbb{C}$*. Let* β *be a nondegenerate* $\mathbb{R}-$*bilinear form on* V*. Each of the following groups is a closed (Lie) subgroup of* $Gl(\mathrm{V})$*:*
1. $Sl(\mathrm{V}) := \{A \in Gl(\mathrm{V}) : \det(A) = 1\}$

2. $Aut(\mathrm{V}, \beta) := \{A \in Gl(\mathrm{V}) : \beta(Av, Aw) = \beta(v, w)$ *for all* $v, w \in \mathrm{V}\}$

3. $SAut(\mathrm{V}, \beta) = Aut(\mathrm{V}, \beta) \cap Sl(\mathrm{V})$

Proof. They are clearly closed subgroups. The fact that they are Lie subgroups follows from theorem 4.5 from the sequel. However, as we shall see most of the specific cases arising from various choices of β can be proved to be Lie groups by direct argument. That they are Lie subgroups follows from proposition 4.1 once we show that they are regular submanifolds of the appropriate group $Gl(\mathrm{V}, \mathbb{F})$. We will return to this later once we have introduced another powerful theorem that will allow us to verify this without the use of the theorem 4.5. ■

Notice that even in the case that $\mathbb{F} = \mathbb{C}$, we have specified that β may only be $\mathbb{R}$-linear. There reason for this is that we wish to include the case that β is a Hermitian form (also called a sesquilinear form). Here, β satisfies $\beta(v, w) = \overline{\beta(w, v)}$ and is required to be linear in one slot and conjugate linear in the other slot: $\beta(av, w) = \bar{a}\beta(v, w)$ for $a \in \mathbb{C}$. Depending on whether $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ and on whether β is symmetric, Hermitian, or skewsymmetric, the notation for the linear groups takes on a special conventional forms. Also, when choosing a basis in order to represent the group in its matrix version, it is usually the case that one uses a basis under which the matrix which represents β takes on a canonical form. Let us look at the usual examples. Let $\dim \mathrm{V} = n$.

Example 4.5 *After choosing a basis the groups* $Gl(\mathrm{V}, \mathbb{F})$ *and* $Sl(\mathrm{V}, \mathbb{F})$ *become* *the matrix groups*

$$Gl(n, \mathbb{F}) := \{A \in M_{n\times n}(\mathbb{F}) : \det A \neq 0\}$$
$$Sl(n, \mathbb{F}) := \{A \in M_{n\times n}(\mathbb{F}) : \det A = 1\}$$

Example 4.6 (The (semi) Orthogonal Groups) *Here we consider the case where* V *is a real vector space and where the bilinear form* β *is symmetric and nondegenerate. Then* (V, β) *is a general scalar product space. In this case we write* $Aut(\mathrm{V}, \beta)$ *as* $O(\mathrm{V}, \beta)$ *and refer to it as the* ***semi-orthogonal*** *group associated to* β*. By Sylvester's law of inertia we may choose a basis so that* β *is represented by a diagonal matrix of the form*

$$\eta_{p,q} = \begin{bmatrix} 1 & 0 & \cdots & & \cdots & 0 \\ 0 & \ddots & 0 & & & \vdots \\ \vdots & 0 & 1 & \ddots & & \\ & & \ddots & -1 & 0 & \vdots \\ \vdots & & & 0 & \ddots & 0 \\ 0 & \cdots & & \cdots & 0 & -1 \end{bmatrix}$$

where there are p *ones and* q *minus ones down the diagonal. The group of matrices arising from* $O(\mathrm{V}, \beta)$ *with such a choice of basis is denoted* $O(p, q)$ *and consists exactly of the real matrices* A *satisfying* $A\eta_{p,q}A^t = \eta_{p,q}$*. These are called the semi-orthogonal matrix groups. With such an orthonormal choice of basis as above, the bilinear form (scalar product) is given as a canonical form on* $\mathbb{R}^n$ $(p + q = n)$ *:*

$$\langle x, y\rangle := \sum_{i=1}^{p} x^i y^i - \sum_{i=p+1}^{n} x^i y^i$$

and we have the alternative description

$$O(p, q) = \{Q \in Gl(n) : \langle Qx, Qy\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{R}^n\}$$

We write $O(n, 0)$ *as* $O(n)$ *and refer to it as the (real)* ***orthogonal*** *(matrix)* ***group****.*

Example 4.7 *There are also* ***complex orthogonal groups*** *(not to be confused with unitary groups). In matrix representation we have* $O(n, \mathbb{C}) := \{Q \in Gl(n, \mathbb{C}) : Q^tQ = I\}$*.*

Example 4.8 *In this example we consider the situation where* $\mathbb{F} = \mathbb{C}$ *and where* β *is complex linear in one argument (say the first one) and conjugate linear in the other. Thus* β *is a* ***sesquilinear*** *form and* (V, β) *is a complex scalar product space. In this case we write* $Aut(\mathrm{V}, \beta, \mathbb{F})$ *as* $U(\mathrm{V}, \beta)$ *and refer to it as the* ***semi-unitary group*** *associated to the sesquilinear form* β*. If* β *is positive definite,* *then we call it a* ***unitary group.*** *Again we may choose a basis for* V *such that* β *is represented by the canonical sesquilinear form on* $\mathbb{C}^n$

$$\langle x, y\rangle := \sum_{i=1}^{p} \bar{x}^i y^i - \sum_{i=p+1}^{p+q=n} \bar{x}^i y^i$$

We then obtain the semi-unitary matrix group

$$U(p,q) = \{A \in Gl(n,\mathbb{C}) : \langle Ax, Ay\rangle = \langle x, y\rangle \text{ for all } x, y \in \mathbb{R}^n\}$$

We write $U(n,0)$ *as* $U(n)$ *and refer to it as the* ***unitary*** *(matrix)* ***group****.*

In particular, $U(1) = S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

Example 4.9 (Symplectic groups) *We will describe both the real and the complex symplectic groups. Suppose now that* β *is a skewsymmetric* $\mathbb{C}-$*bilinear (resp.* $\mathbb{R}-$*bilinear) form on a* $2n$ *dimensional complex (resp. real) vector space* V*. The group* $Aut(\mathrm{V},\beta)$ *(resp.* $Aut(\mathrm{V},\beta)$*) is called the complex (resp. real) symplectic group and denoted by* $Sp(\mathrm{V},\mathbb{C})$ *(resp.* $Sp(\mathrm{V},\mathbb{R})$*). There exists a basis* $\{f_i\}$ *for* V *such that* β *is represented in canonical form by*

$$(v,w) = \sum_{i=1}^{n} v^i w^{n+i} - \sum_{j=1}^{n} v^{n+j} w^j.$$

and the symplectic matrix groups are given by

$$Sp(2n,\mathbb{C}) = \{A \in M_{2n\times 2n}(\mathbb{C}) : (Av, Aw) = (v,w)\}$$
$$Sp(2n,\mathbb{R}) = \{A \in M_{2n\times 2n}(\mathbb{R}) : (Av, Aw) = (v,w)\}$$

Exercise 4.2 *For* $\mathbb{F} = \mathbb{C}$ *or* $\mathbb{R}$*, show that* $A \in Sp(2n,\mathbb{F})$ *if and only if* $A^t J A = J$ *where*

$$J = \begin{pmatrix} 0 & id \\ -id & 0 \end{pmatrix}$$

The groups of the form $SAut(\mathrm{V}, b, \mathbb{F}) = Aut(\mathrm{V}, b, \mathbb{F}) \cap Sl(\mathrm{V}, \mathbb{F})$ are usually designated by use of the word "*special*". We have the **special orthogonal** and **special semi-orthogonal** groups $SO(n)$ and $SO(p,q)$, the **special unitary** and **special semi-unitary** groups $SU(n)$and $SU(p,q)$ etc.

Exercise 4.3 *Show that* $SU(2)$ *is simply connected.*

Much of the above can be generalized somewhat more. Recall that the algebra of quaternions $\mathbb{H}$ is a copy of $\mathbb{R}^4$ endowed with a multiplication described as follows: First let a generic elements of $\mathbb{R}^4$ be denoted by $x = (x^0, x^1, x^2, x^3)$, $y = (y^0, y^1, y^2, y^3)$ etc. Thus we are using $\{0, 1, 2, 3\}$ as our index set. Let the standard basis be denoted by $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$. We define a multiplication by taking these basis elements as generators and insisting on the following relations

$$\begin{aligned}
\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 &= -\mathbf{1} \\
\mathbf{ij} = -\mathbf{ji} &= \mathbf{k} \\
\mathbf{jk} = -\mathbf{kj} &= \mathbf{i} \\
\mathbf{ki} = -\mathbf{ik} &= \mathbf{j}
\end{aligned}$$

Of course, $\mathbb{H}$ is a vector space over $\mathbb{R}$ since it is just $\mathbb{R}^4$ with some extra structure. It is common to identify the $\mathbb{R}$-linear span of $\mathbf{1}$ with $\mathbb{R}$ and correspondingly write $\mathbf{1}$ as 1. As a ring, $\mathbb{H}$ is a division algebra which is very much like a field lacking only the property of commutativity. In particular, we shall see that every nonzero element of $\mathbb{H}$ has a multiplicative inverse. As we said, elements of the form $a\mathbf{1}$ for $a \in \mathbb{R}$ are identified with the corresponding real numbers and such quaternions are called **real** quaternions. By analogy with complex numbers the set of all quaternions of the form $x^1\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k}$ are called **imaginary** quaternions. For a given quaternion $x = x^0\mathbf{1} + x^1\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k}$ the quaternion $x^1\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k}$ is called the imaginary part of x and $x^0\mathbf{1} = x^0$ is called the real part of x. We also have a conjugation defined by

$$x \mapsto \bar{x} := x^0\mathbf{1} - x^1\mathbf{i} - x^2\mathbf{j} - x^3\mathbf{k}$$

Notice that $x\bar{x} = \bar{x}x$ is real and equal to $\left(x^0\right)^2 + \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2$. We denote the positive square root of this by $|x|$ so that $\bar{x}x = |x|^2$.

Exercise 4.4 *Verify the following for $x, y \in \mathbb{H}$ and $a, b \in \mathbb{R}$*

$$\begin{array}{cc}
\overline{ax + by} = a\bar{x} + b\bar{y} & \overline{(\bar{x})} = x \\
|xy| = |x|\,|y| & |\bar{x}| = |x| \\
 & \overline{xy} = \bar{y}\bar{x}
\end{array}$$

Now we can write down the inverse of a nonzero $x \in \mathbb{H}$:

$$x^{-1} = \frac{1}{|x|^2}\bar{x}$$

Notice the strong the analogy with complex number arithmetic.

Example 4.10 *The set of unit quaternions is $U(1, \mathbb{H}) := \{|x| = 1\}$. This set is closed under multiplication. As a manifold it is (diffeomorphic to) S^3. With quaternionic multiplication $S^3 = U(1, \mathbb{H})$ is a compact Lie group. Compare this to Example 4.2 where we saw that $U(1, \mathbb{C}) = S^1$. For the future, we unify things by letting $U(1, \mathbb{R}) := \mathbb{Z}_2 = S^0 \subset \mathbb{R}$. In other words, we take the 0-sphere to be the subset $\{-1, 1\}$ with its natural structure as a multiplicative group.*

$$\begin{aligned}
U(1, \mathbb{H}) &= S^3 \\
U(1, \mathbb{C}) &= S^1 \\
U(1, \mathbb{R}) := \mathbb{Z}_2 &= S^0
\end{aligned}$$

Exercise 4.5 *Prove the assertions in the last example.*

We now consider the $n-$fold product $\mathbb{H}^n$ which as a real vector space (and smooth manifold) is $\mathbb{R}^{4n}$. However, let us think of elements of $\mathbb{H}^n$ as column vectors with quaternion entries. We want to treat $\mathbb{H}^n$ as a vector space over $\mathbb{H}$ with addition defined just as for $\mathbb{R}^n$ and $\mathbb{C}^n$ but since $\mathbb{H}$ is not commutative we are not properly dealing with a vector space. In particular, we should decide whether scalars should multiply column vectors on the right or on the left. We choose to multiply on the right and this could take some getting used to but there is a good reason for our choice. This puts us into the category of right $\mathbb{H}$-modules were elements of $\mathbb{H}$ are the "scalars". The reader should have no trouble catching on and so we do not make formal definitions at this time (see Appendix D). For $v, w \in \mathbb{H}^n$ and $a, b \in \mathbb{H}$ we have

$$\begin{aligned} v(a+b) &= va + vb \\ (v+w)a &= va + wa \\ (va)\, b &= v\,(ab)\,. \end{aligned}$$

A map $A : \mathbb{H}^n \to \mathbb{H}^n$ is said to be $\mathbb{H}$-linear if $A(va) = A(v)a$ for all $v \in \mathbb{H}^n$ and $a \in \mathbb{H}$. There is no problem with doing matrix algebra with matrices with quaternion entries as long as one respects the noncommutativity of $\mathbb{H}$. For example, if $A = (a^i_j)$ and $B = (b^i_j)$ are matrices with quaternion entries then writing $C = AB$ we have

$$c^i_j = \sum a^i_k b^k_j$$

but we can *not* expect that $\sum a^i_k b^k_j = \sum b^k_j a^i_k$. For any $A = (a^i_j)$ the map $\mathbb{H}^n \to \mathbb{H}^n$ defined by $v \mapsto Av$ is $\mathbb{H}$-linear since $A\,(va) = (Av)\,a$.

Definition 4.3 *The set of all $m \times n$ matrices with quaternion entries is denoted $M_{m\times n}(\mathbb{H})$. The subset $Gl(n, \mathbb{H})$ is defined as the set of all $Q \in M_{m\times n}(\mathbb{H})$ such that the map $v \mapsto Qv$ is a bijection.*

We will now see that $Gl(n, \mathbb{H})$ is a Lie group isomorphic to a subgroup of $Gl(2n, \mathbb{C})$. First we defined a map $\iota : \mathbb{C}^2 \to \mathbb{H}$ as follows: For $(z_1, z_2) \in \mathbb{C}$ with $z_1 = x^0 + x^1\mathbf{i}$ and $z_2 = x^2 + x^3\mathbf{i}$, we let $\iota(z^1, z^2) = (x^0 + x^1\mathbf{i}) + \left(x^2 + x^3\mathbf{i}\right)\mathbf{j}$ where on the right hand side we interpret $\mathbf{i}$ as a quaternion. Note that $(x^0 + x^1\mathbf{i}) + \left(x^2 + x^3\mathbf{i}\right)\mathbf{j} = x^0 + x^1\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k}$. It is easily shown that this map is an $\mathbb{R}$-linear bijection and we use this map to identify $\mathbb{C}^2$ with $\mathbb{H}$. Another way of looking at this is that we identify $\mathbb{C}$ with the span of 1 and $\mathbf{i}$ in $\mathbb{H}$ and then every quaternion has a unique representation as $z^1 + z^2\mathbf{j}$ for $(z_1, z_2) \in \mathbb{C} \subset \mathbb{H}$. Now we extend this idea to square quaternionic matrices; we can write every $Q \in M_{m\times n}(\mathbb{H})$ in the form $A + B\mathbf{j}$ for $A, B \in \mathbb{C}^{m\times n}$ in a unique way. Now this representation makes it clear that $M_{m\times n}(\mathbb{H})$ has a natural complex vector space structure where the scalar multiplication is $z(A + B\mathbf{j}) = zA + zB\mathbf{j}$. Direct computation shows that

$$(A + B\mathbf{j})\,(C + D\mathbf{j}) = (AC - B\bar{D}) + (AD + B\bar{C})\mathbf{j}$$

for $A+B\mathbf{j} \in M_{m\times n}(\mathbb{H})$ and $C+D\mathbf{j} \in M_{n\times k}(\mathbb{H})$ and where we have used the fact that for $Q \in M_{m\times n}(\mathbb{H})$ we have $Q\mathbf{j} = \mathbf{j}\bar{Q}$. From this it is not hard to show that the map $\vartheta_{m\times n} : M_{m\times n}(\mathbb{H}) \to M_{2m\times 2n}(\mathbb{C})$ given by

$$\vartheta_{m\times n} : A + B\mathbf{j} \longmapsto \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$$

is an injective $\mathbb{R}-$linear map which respects matrix multiplication and thus is an $\mathbb{R}-$algebra isomorphism onto its image. We may identify $M_{m\times n}(\mathbb{H})$ with the subspace of $M_{2m\times 2n}(\mathbb{C})$ consisting of all matrices of the form $\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$ where $A, B \in \mathbb{C}^{m\times n}$. In particular, if $m = n$ the we obtain an injective $\mathbb{R}-$linear algebra homomorphism $\vartheta_{n\times n} : M_{n\times n}(\mathbb{H}) \to M_{2n\times 2n}(\mathbb{C})$ and thus the image of this map in $M_{2n\times 2n}(\mathbb{C})$ is another realization of the matrix algebra $M_{n\times n}(\mathbb{H})$. If we specialize to the case of $n = 1$ we get a realization of $\mathbb{H}$ as the set of all 2×2 complex matrices of the form $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$. This set of matrices is closed under multiplication and forms an algebra over the field $\mathbb{R}$. Let us denote this algebra of matrices by the symbol $\mathcal{R}^4$ since it is diffeomorphic with $\mathbb{H} \cong \mathbb{R}^4$. We now have an algebra isomorphism $\vartheta : \mathbb{H} \to \mathcal{R}^4$ under which the quaternions $\mathbf{1}$ $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ correspond to the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

respectively. Since $\mathbb{H}$ is a division algebra, each of its nonzero elements has a multiplicative inverse. Thus $\mathcal{R}^4$ must contain the matrix inverse of each of its nonzero elements. This can be seen directly:

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}^{-1} = \frac{1}{|z|^2 + |w|^2} \begin{pmatrix} \bar{z} & -w \\ \bar{w} & z \end{pmatrix}$$

Here we have used the easily verified fact that

$$\det \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = |z|^2 + |w|^2$$

Consider again the group of unit quaternions $U(1, \mathbb{H})$. We have already seen that as a smooth manifold $U(1, \mathbb{H})$ is S^3. However, under the isomorphism $\mathbb{H} \to \mathcal{R}^4 \subset M_{2\times 2}(\mathbb{C})$ just mentioned $U(1, \mathbb{H})$ manifests itself as $SU(2)$. We record this as

Proposition 4.2 $S^3 = U(1, \mathbb{H}) \cong SU(2)$. *The second isomorphism is given by*

$$x = z + w\mathbf{j} \mapsto \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$$

where $x = x^0 + x^1\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k}$ *and* $z = x^0 + x^1\mathbf{j}$ *and* $w = x^2 + x^3\mathbf{j}$.

Proof. The first equality has already been easily established. Notice that then

$$|x|^2 = |z|^2 + |w|^2 = \det \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$$

and so $x \in U(1, \mathbb{H})$ if and only if $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}$ has determinant one. On the other hand if $\det \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} = 1$ then

$$\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}^{-1} = \begin{pmatrix} \bar{z} & -w \\ \bar{w} & z \end{pmatrix} = \overline{\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}}^{t}$$

and so $\begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \in SU(2)$. ∎

Exercise 4.6 *Show that* $Q \in Gl(n, \mathbb{H})$ *if and only if* $\det(\vartheta_{n\times n}(Q)) \neq 0$.

The set of all elements of $Gl(2n, \mathbb{C})$ which are of the form $\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$ is a subgroup of $Gl(2n, \mathbb{C})$ and in fact a Lie group. Using this last exercise we see that we may identify $Gl(n, \mathbb{H})$ as a Lie group with this subgroup of $Gl(2n, \mathbb{C})$. We want to find a quaternionic analogue of $U(n, \mathbb{C})$ and so we define $b : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ by

$$b(v, w) = \bar{v}^t w.$$

Explicitly, if

$$v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \text{ and } w = \begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix}$$

then

$$\begin{aligned} b(v, w) &= \begin{bmatrix} v^1 & \cdots & v^n \end{bmatrix} \begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix} \\ &= \sum \bar{v}^i w^i \end{aligned}$$

Note that while b is obviously $\mathbb{R}$-bilinear, if $a \in \mathbb{H}$ then we have $b(va, w) = b(v, w)\bar{a}$ and $b(v, w) = b(v, w)a$. Notice that we consistently use *right* multiplication by quaternionic scalars. Thus b is the quaternionic analogue of an Hermitian scalar product.

Definition 4.4 *We define* $U(n, \mathbb{H})$*:*

$$U(n, \mathbb{H}) := \{Q \in Gl(n, \mathbb{H}) : b(Qv, Qw) = b(v, w) \text{ for all } v, w \in \mathbb{H}^n\}$$

$U(n, \mathbb{H})$ *is called the quaternionic unitary group.*

$U(n,\mathbb{H})$ is is sometimes called the symplectic group and denoted $Sp(n)$ but we will avoid this since we want no confusion with the symplectic groups we have already defined.

$U(n,\mathbb{H})$ is in fact a group since we may consider $\mathbb{H}^n$ a real vector space and b a nondegenerate $\mathbb{R}$-binear form with some special properties. Then $U(n,\mathbb{H})$ is just the group of linear maps which preserve this bilinear from. The image of $U(n,\mathbb{H})$ under the isomorphism $\vartheta_{n\times n}: M_{n\times n}(\mathbb{H}) \to \vartheta_{n\times n}(M_{n\times n}(\mathbb{H}))$ is denoted $USp(2n,\mathbb{C})$. Since it is easily established that $\vartheta_{n\times n}|_{U(n,\mathbb{H})}$ is a group homomorphism, the image $USp(2n,\mathbb{C})$ is a subgroup of $Gl(2n,\mathbb{C})$.

Exercise 4.7 *Show that* $\vartheta_{n\times n}(\bar{A}^t) = \left(\overline{\vartheta_{n\times n}(A)}\right)^t$.

Exercise 4.8 *Show that* $USp(2n,\mathbb{C})$ *is a Lie subgroup of* $Gl(2n,\mathbb{C})$.

Exercise 4.9 *Show that* $USp(2n,\mathbb{C}) = U(2n) \cap Sp(2n,\mathbb{C})$. *Hint: Show that*

$$\vartheta_{n\times n}(M_{n\times n}(\mathbb{H})) = \left\{A \in Gl(2n,\mathbb{C}) : JAJ^{-1} = \bar{A}\right\}$$

where $J = \begin{pmatrix} 0 & id \\ -id & 0 \end{pmatrix}$. *Next show that if* $A \in U(2n)$ *then* $JAJ^{-1} = \bar{A}$ *if and only if* $A^tJA = J$.

4.3 Lie Group Homomorphisms

Definition 4.5 *A smooth map* $f: G \to H$ *is called a* ***Lie group homomorphism*** *if*

$$f(g_1g_2) = f(g_1)f(g_2) \text{ for all } g_1, g_2 \in G \text{ and}$$
$$f(g^{-1}) = f(g)^{-1} \text{ for all } g \in G.$$

and a ***Lie group isomorphism*** *in case it has an inverse that is also a Lie group homomorphism. A Lie group isomorphism* $G \to G$ *is called a* ***Lie group automorphism.***

Example 4.11 *The inclusion* $\mathrm{SO}(n,\mathbb{R}) \hookrightarrow Gl(n,\mathbb{R})$ *is a Lie group homomorphism.*

Example 4.12 *The circle* $S^1 \subset \mathbb{C}$ *is a Lie group under complex multiplication and the map*

$$z = e^{i\theta} \to \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & I_{n-2} \end{bmatrix}$$

is a Lie group homomorphism of S^1 *into* $\mathrm{SO}(n)$.

Example 4.13 *The conjugation map $C_g : G \to G$ is a Lie group automorphism.*

Exercise 4.10 (*) *Show that the multiplication map $\mu : G \times G \to G$ has tangent map at $(e,e) \in G \times G$ given as $T_{(e,e)}\mu(v,w) = v+w$. Recall that we identify $T_{(e,e)}(G \times G)$ with $T_eG \times T_eG$.*

Exercise 4.11 *$Gl(n,\mathbb{R})$ is an open subset of the vector space of all $n \times n$ matrices $M_{n\times n}(\mathbb{R})$. Using the natural identification of $T_eGl(n,\mathbb{R})$ with $M_{n\times n}(\mathbb{R})$ show that as a map $M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ we have*

$$T_eC_g : x \mapsto gxg^{-1}$$

where $g \in Gl(n,\mathbb{R})$ and $x \in M_{n\times n}(\mathbb{R})$.

Example 4.14 *The map $t \mapsto e^{it}$ is a Lie group homomorphism from $\mathbb{R}$ to $S^1 \subset \mathbb{C}$.*

Remark 4.1 *It is an unfortunate fact that in this setting a map itself is sometimes referred to as a "subgroup". The term "one-parameter subgroup" from the next definition is such a case.*

Definition 4.6 *A homomorphism from the additive group $\mathbb{R}$ into a Lie group is called a* ***one-parameter subgroup****.*

Example 4.15 *We have seen that the torus $S^1 \times S^1$ is a Lie group under multiplication given by $(e^{i\tau_1}, e^{i\theta_1})(e^{i\tau_2}, e^{i\theta_2}) = (e^{i(\tau_1+\tau_2)}, e^{i(\theta_1+\theta_2)})$. Every homomorphism of $\mathbb{R}$ into $S^1 \times S^1$, that is, every one parameter subgroup of $S^1 \times S^1$ is of the form $t \mapsto (e^{tai}, e^{tbi})$ for some pair of real numbers $a, b \in \mathbb{R}$.*

Example 4.16 *The map $R : \mathbb{R} \to SO(3)$ given by*

$$\theta \mapsto \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a one parameter subgroup.

Recall that an $n \times n$ complex matrix A is called Hermitian (resp. skew-Hermitian) if $\bar{A}^t = A$ (resp. $\bar{A}^t = -A$).

Example 4.17 *Given an element g of the group $SU(2)$ we define the map $Ad_g : \mathfrak{su}(2) \to \mathfrak{su}(2)$ by $Ad_g : x \mapsto gxg^{-1}$. Now the skew-Hermitian matrices of zero trace can be identified with $\mathbb{R}^3$ by using the following matrices as a basis:*

$$\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

These are just $-i$ times the ***Pauli matrices*** $\sigma_1, \sigma_2, \sigma_3$ *and so the correspondence* $\mathfrak{su}(2) \to \mathbb{R}^3$ *is given by* $-xi\sigma_1 - yi\sigma_2 - iz\sigma_3 \mapsto (x, y, z)$. *Under this correspondence the inner product on* $\mathbb{R}^3$ *becomes the inner product* $(A, B) = trace(AB)$. *But then*

$$\begin{aligned}(\mathrm{Ad}_g A, \mathrm{Ad}_g B) &= \mathrm{trace}(gAgg^{-1}Bg^{-1}) \\ &= \mathrm{trace}(AB) = (A, B)\end{aligned}$$

so Ad_g *can be thought of as an element of* $O(3)$. *More is true;* Ad_g *acts as an element of* $SO(3)$ *and the map* $g \mapsto Ad_g$ *is then a homomorphism from* $SU(2)$ *to* $SO(\mathfrak{su}(2)) \cong SO(3)$. *This is a special case of the adjoint map studied later.*

Definition 4.7 *If a Lie group homomorphism* $\wp : \widetilde{G} \to G$ *is also a covering map then we say that* $\widetilde{G}$ *is a covering group and* $\wp$ *is a covering homomorphism. If* $\widetilde{G}$ *is simply connected then* $\widetilde{G}$ *(resp.* $\wp$*) is called the universal covering group (resp. universal covering homomorphism) of* G.

Exercise 4.12 *Show that if* $\wp : \widetilde{M} \to G$ *is a smooth covering map and* G *is a Lie group then* $\widetilde{M}$ *can be given a unique Lie group structure such that* $\wp$ *becomes a covering homomorphism.*

Example 4.18 *The group* Mob *of transformations of the complex plane given by* $T_A : z \mapsto \frac{az+b}{cz+d}$ *for* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl(2, \mathbb{C})$ *can be given the structure of a Lie group. The map* $\wp : Sl(2, \mathbb{C}) \to Mob$ *given by* $\wp : A \mapsto T_A$ *is onto but not injective. In fact, it is a (two fold) covering homomorphism. When do two elements of* $Sl(2, \mathbb{C})$ *map to the same element of* Mob*?*

4.4 The Lie algebra

Definition 4.8 *A vector field* $X \in \mathfrak{X}(G)$ *is called* ***left invariant*** *if and only if* $(L_g)_* X = X$ *for all* $g \in G$. *A vector field* $X \in \mathfrak{X}(G)$ *is called* ***right invariant*** *if and only if* $(R_g)_* X = X$ *for all* $g \in G$. *The set of left invariant (resp. right invariant) vectors Fields is denoted* $\mathfrak{L}(G)$ *or* $\mathfrak{X}^L(G)$ *(resp.* $\mathfrak{R}(G)$ *or* $\mathfrak{X}^R(G)$*).*

Recall that by definition $(L_g)_* X = TL_g \circ X \circ L_g^{-1}$ and so left invariance means that $TL_g \circ X \circ L_g^{-1} = X$ or that $TL_g \circ X = X \circ L_g$. Thus $X \in \mathfrak{X}(G)$ is left invariant if and only if the following diagram commutes for every $g \in G$.

$$\begin{array}{ccc} TG & \xrightarrow{TL_g} & TG \\ X \uparrow & & \uparrow X \\ G & \xrightarrow{L_g} & G \end{array}$$

There is a similar diagram for right invariance.

Remark 4.2 *As we mentioned previously,* L_{g*} *is sometimes used to mean* TL_g *and so left invariance of* X *would then amount to the requirement that for any fixed* $p \in G$ *we have* $L_{g*} X_p = X_{gp}$ *for all* $g \in G$.

Lemma 4.1 *$\mathfrak{X}^L(G)$ is closed under the Lie bracket operation.*

Proof. Suppose that $X, Y \in \mathfrak{X}^L(G)$. Then by Proposition 2.5 we have

$$\begin{aligned}(L_g)_*[X,Y] &= (L_g)_*(\mathcal{L}_X Y) = \mathcal{L}_{L_{g*}X} L_{g*}Y \\ &= [L_{g*}X, L_{g*}Y] = [X,Y]\end{aligned}$$

■

Corollary 4.1 *$\mathfrak{X}^L(G)$ is an n-dimensional Lie algebra under the bracket of vector fields (see definition 2.20).*

Given a vector $v \in T_eG$ we can define a smooth left (resp. right) invariant vector field L^v (resp. R^v) such that $L^v(e) = v$ (resp. $R^v(e) = v$) by the simple prescription

$$\begin{aligned} L^v(g) &= TL_g \cdot v \\ \text{(resp. } R^v(g) &= \ TR_g \cdot v) \end{aligned}$$

Furthermore, the map $v \mapsto L^v$ (resp. $v \mapsto R^v$) is a linear isomorphism from T_eG onto $\mathfrak{X}^L(G)$ (resp.$\mathfrak{X}^R(G)$) . The proof that this prescription gives smooth invariant vector fields is left to the reader (see Problem 3) . We now restrict attention to the the left invariant fields but keep in mind that essentially all of what we say for this case has analogies in the right invariant case. In any case we will discover a conduit (the adjoint map) between the two cases.

The linear isomorphism $T_eG \cong \mathfrak{X}^L(G)$ just discovered shows that $\mathfrak{X}^L(G)$ is, in fact, a finite dimensional vector space and using this isomorphism we can transfer the Lie algebra structure to T_eG. This is the content of the following

Definition 4.9 *For a Lie group G, define the bracket of any two elements $v, w \in T_eG$ by*

$$[v, w] := [L^v, L^w](e).$$

With this bracket, the vector space T_eG becomes a Lie algebra (see definition 2.21) and so we now have two Lie algebras, $\mathfrak{X}^L(G)$ and T_eG which are isomorphic by construction. The abstract Lie algebra isomorphic the either/both of them is often referred to as *the* Lie algebra of the Lie group G and denoted variously by $L(G)$, $\mathfrak{Lie}(G)$ or $\mathfrak{g}$. Of course, we are implying that $\mathfrak{Lie}(H)$ is denoted $\mathfrak{h}$ and $\mathfrak{Lie}(K)$ by $\mathfrak{k}$ etc. In some computations we will have to use a specific realization of $\mathfrak{g}$. Our default convention will be that $\mathfrak{g} = \mathfrak{Lie}(G) := T_eG$ with the bracket defined above and then occasionally identify this with the left invariant fields $\mathfrak{X}^L(G)$ under the vector field Lie bracket.

Definition 4.10 *Given two Lie algebras over a field $\mathbb{F}$, say $(\mathfrak{a}, [,]_{\mathfrak{a}})$ and $(\mathfrak{b}, [,]_{\mathfrak{b}})$, an $\mathbb{F}$-linear map σ is called a* ***Lie algebra homomorphism*** *if and only if*

$$\sigma([v, w]_{\mathfrak{a}}) = [\sigma v, \sigma w]_{\mathfrak{b}}$$

for all $v, w \in \mathfrak{a}$. A Lie algebra isomorphism is defined in the obvious way. A Lie algebra ***isomorphism*** *$\mathfrak{g} \to \mathfrak{g}$ is called an* ***automorphism*** *of $\mathfrak{g}$.*

It is not hard to show that the set of all automorphisms of $\mathfrak{g}$, denoted $\mathrm{Aut}(\mathfrak{g})$, forms a Lie group (actually a Lie subgroup of $Gl(\mathfrak{g})$).

Recall that if V is a finite dimensional vector space then each tangent space $T_x\mathrm{V}$ is naturally isomorphic to V. Now $Gl(n)$ is an open subset of the vector space of $n \times n$ real matrices $\mathbb{M}_{n\times n}$ and so we obtain natural vector space isomorphisms $T_{\,g}Gl(n) \cong \mathbb{M}_{n\times n}$ for all $g \in Gl(n)$. To move to the level of bundles we reconstitute these isomorphisms to get maps $T_gGl(n) \to \{g\} \times \mathbb{M}_{n\times n}$ which we can then bundle together to get a trivialization $T\,Gl(n) \to Gl(n) \times \mathbb{M}_{n\times n}$ (recall definition 2.15). One could use this trivialization to identify $T\,Gl(n)$ with $Gl(n) \times \mathbb{M}_{n\times n}$ and this trivialization is just a special case of the general situation: When U is an open subset of a vector space V, we have a trivialization $TU \cong U \times \mathrm{V}$. Further on, we introduce two more trivializations of $T\,Gl(n) \cong Gl(n) \times \mathbb{M}_{n\times n}$ defined using the (left or right) Maurer-Cartan form defined below. This will work for general Lie groups. Since these trivializations could also be used as identifying isomorphisms we had better introduce a bit of nonstandard terminology. Let us refer to the identification of $T\,Gl(n)$ with $Gl(n)\times\mathbb{M}_{n\times n}$, or more generally TU with $U\times\mathrm{V}$, as the canonical identification.

Let us explicitly recall one way to describe the isomorphism $T_gGl(n) \cong \mathbb{M}_{n\times n}$. If $v_g \in T_gG$ then there is some curve (of matrices) $c^g : t \mapsto c(t)$ such that $c^g(0) = g$ and $\dot{c}^g(0) = v_g \in T_gG$. By definition $\dot{c}^g(0) := T_0c^g \cdot \left.\frac{d}{dt}\right|_0$ which is based at g. If we just take the ordinary derivative we get the matrix that represents v_g: If $c(t)$ is given by

$$c(t) = \begin{bmatrix} g_1^1(t) & g_1^2(t) & \cdots \\ g_2^1(t) & \ddots & \\ \vdots & & g_n^n(t) \end{bmatrix}$$

then $\dot{c}(0) = v_g$ is identified with the matrix

$$a := \begin{bmatrix} \left.\frac{d}{dt}\right|_{t=0} g_1^1 & \left.\frac{d}{dt}\right|_{t=0} g_1^2 & \cdots \\ \left.\frac{d}{dt}\right|_{t=0} g_2^1 & \ddots & \\ \vdots & & \left.\frac{d}{dt}\right|_{t=0} g_n^n \end{bmatrix}.$$

As a particular case, we have a vector space isomorphism $\mathfrak{gl}(n) = T_{\,I}Gl(n) \cong \mathbb{M}_{n\times n}$ where I is the identity matrix in $Gl(n)$. This we want to use to identify $\mathfrak{gl}(n)$ with $\mathbb{M}_{n\times n}$. Now $\mathfrak{gl}(n) = T_IGl(n)$ has a Lie algebra structure and we would like to transfer the Lie bracket from $\mathfrak{gl}(n)$ to $\mathbb{M}_{n\times n}$ is such a way that this isomorphism becomes a Lie algebra isomorphism. Below we discover that the Lie bracket that we end up with for $\mathbb{M}_{n\times n}$ is the commutator bracket defined by $[A, B] := AB - BA$. This is so natural that we can safely identify the Lie algebra $\mathfrak{gl}(n)$ with $\mathbb{M}_{n\times n}$. Along these lines we will also be able to identify the Lie algebras of subgroups of $Gl(n)$ (or $Gl(n, \mathbb{C})$) with linear subspaces of $\mathbb{M}_{n\times n}$ (or $\mathbb{M}_{n\times n}(\mathbb{C})$).

Initially, the Lie algebra of $Gl(n)$ is given as the space of left invariant vectors fields on $GL(n)$. The bracket is the bracket of vector fields that we met earlier. This bracket is transferred to $T_I Gl(n)$ according to the isomorphism of $\mathfrak{X}(GL(n))$ with $T_I GL(n)$ given in this case by $X \mapsto X(I)$. The situation is that we have two Lie algebra isomorphisms

$$\mathfrak{X}(Gl(n)) \cong T_I Gl(n) \cong \mathbb{M}_{n\times n}$$

and the origin of all of the Lie algebra structure is $\mathfrak{X}(Gl(n))$. The plan is then to figure out what is the left invariant vector field that corresponds to a given matrix from $\mathbb{M}_{n\times n}$. This gives a direct route between $\mathfrak{X}(Gl(n))$ and $\mathbb{M}_{n\times n}$ allowing us to see what the correct bracket on $\mathbb{M}_{n\times n}$ should be. Note that a global coordinate system for $Gl(n)$ is given by the maps x_l^k which are defined by the requirement that $x_l^k(A) = a_l^k$ whenever $A = (a_j^i)$. Thus any vector fields $X, Y \in \mathfrak{X}(Gl(n))$ can be written

$$X = f_j^i \frac{\partial}{\partial x_j^i}$$
$$Y = g_j^i \frac{\partial}{\partial x_j^i}$$

for some functions f_j^i and g_j^i. Now let $(a_j^i) = A \in \mathbb{M}_{n\times n}$. The corresponding element of $T\,_I Gl(n)$ can be written $A_I = a_j^i \left.\frac{\partial}{\partial x_j^i}\right|_I$. Corresponding to A_I there is a left invariant vector field X^A which is given by $X^A(x) = T_I L_x \cdot v_A$ which in turn corresponds to the matrix $\left.\frac{d}{dt}\right|_0 xc(t) = xA$. Thus X^A is given by $X = f_j^i \frac{\partial}{\partial x_j^i}$ where $f_j^i(x) = xA = \left(x_k^i a_j^k\right)$. Similarly, let B be another matrix with corresponding left invariant vector field X^B. The bracket $[X^A, X^B]$ can now be computed as follows:

$$\begin{aligned}
[X^A, X^B] &= (f_j^i \frac{\partial g_l^k}{\partial x_j^i} - g_j^i \frac{\partial f_l^k}{\partial x_j^i}) \frac{\partial}{\partial x_l^k} \\
&= \left(x_r^i a_j^r \frac{\partial (x_s^k b_l^s)}{\partial x_j^i} - x_r^i b_j^r \frac{\partial (x_s^k a_l^s)}{\partial x_j^i} \right) \frac{\partial}{\partial x_l^k} \\
&= \left(x_r^k a_s^r b_l^s - x_r^k b_s^r a_l^s \right) \frac{\partial}{\partial x_l^k}
\end{aligned}$$

Evaluating at $I = (\delta_i^i)$ we have

$$\begin{aligned}
[X^A, X^B](I) &= \left(\delta_r^k a_s^r b_l^s - \delta_r^k b_s^r a_l^s \right) \left.\frac{\partial}{\partial x_l^k}\right|_I \\
&= \left(a_s^k b_l^s - b_s^k a_l^s \right) \left.\frac{\partial}{\partial x_l^k}\right|_I
\end{aligned}$$

which corresponds to the matrix $AB - BA$. Thus the proper Lie algebra structure on $\mathbb{M}_{n\times n}$ is given by the commutator $[A, B] = AB - BA$. In summary, we have

Proposition 4.3 *Under the canonical of identification of* $\mathfrak{gl}(n) = T_I Gl(n)$ *with* $\mathbb{M}_{n\times n}$ *the Lie bracket is the commutator bracket*

$$[A, B] = AB - BA$$

Similarly, under the identification of $T_{\mathrm{id}}Gl(\mathrm{V})$ *with* $\mathrm{End}(\mathrm{V})$ *the bracket is*

$$[A, B] = A \circ B - B \circ A.$$

If $G \subset Gl(n)$ is some matrix group then $T_I G$ may be identified with a linear subspace of $\mathbb{M}_{n\times n}$ and it turns out that this linear subspace will be closed under the commutator bracket and so we actually have an identification of Lie algebras: $\mathfrak{g}$ is identified with a subspace of $\mathbb{M}_{n\times n}$. It is often the case that G is defined by some matrix equation(s). By differentiating this equation we find the defining equation(s) for $\mathfrak{g}$ (the subspace of $\mathbb{M}_{n\times n}$). We first prove a general result for Lie algebras of closed subgroups and then we apply his to some matrix groups. Recall that if N is a submanifold of M and $\iota : N \hookrightarrow M$ is the inclusion map, then we identify $T_p N$ with $T_p\iota(T_p N)$ and $T_p\iota$ is an inclusion. Now if H is a closed Lie subgroup of a Lie group G and $v \in \mathfrak{h} = T_e H$ then v corresponds to a left invariant vector field on G which is obtained by using the left translation in G but also to a left invariant vector field on H obtain from left translation in H. The notation we have been using so far is not sensitive to the distinction so let us introduce an alternative notation.

Notation 4.1 *For a Lie group* G *we have the alternative notation* v^G *for the left invariant vector field whose value at* e *is* v*. If* H *is a closed Lie subgroup of* G *and* $v \in T_e H$ *then* $v^G \in \mathfrak{X}^L(G)$ *while* $v^H \in \mathfrak{X}^L(H)$.

Proposition 4.4 *Let* H *be a closed Lie subgroup of a Lie group* G*. Let*

$$\widetilde{\mathfrak{X}^L}(H) := \{X \in \mathfrak{X}^L(G) : X(e) \in T_e H\}$$

then then the restriction of elements of $\widetilde{\mathfrak{X}^L}(H)$ *to the submanifold* H *are elements of* $\mathfrak{X}^L(H)$*. This induces an isomorphism of Lie algebras of vector fields* $\widetilde{\mathfrak{X}^L}(H) \cong \mathfrak{X}^L(H)$*. For* $v, w \in \mathfrak{h}$ *we have*

$$[v, w]_{\mathfrak{h}} = [v^H, w^H]_e = [v^G, w^G]_e = [v, w]_{\mathfrak{g}}$$

and so $\mathfrak{h}$ *is a Lie subalgebra of* $\mathfrak{g}$*; the bracket on* $\mathfrak{h}$ *is the same as that inherited from* $\mathfrak{g}$*.*

Proof. The Lie bracket on $\mathfrak{h} = T_e H$ is given by $[v, w] := [v^H, w^H](e)$. Notice that if H is a closed Lie subgroup of a Lie group G then for $h \in H$ we have left translation by h as a map $G \to G$ and also as a map $H \to H$. The latter is the restriction of the former. To avoid notational clutter let us denote $L_h|_H$ by l_h. If $\iota : H \hookrightarrow G$ is the inclusion then we have $\iota \circ l_h = L_h \circ \iota$ and so

$T\iota \circ Tl_h = TL_h \circ T\iota$. Now if $v^H \in \mathfrak{X}^L(H)$ then we have

$$\begin{aligned} T_h\iota\left(v^H(h)\right) &= T_h\iota(T_e l_h(v)) = T\iota \circ Tl_h(v) \\ &= TL_h \circ T\iota(v) = T_e L_h\left(T_e\iota(v)\right) \\ &= T_e L_h\left(v\right) = v^G(h) = \left(v^G \circ \iota\right)(h) \end{aligned}$$

so that v^H and v^G are ι-related for any $v \in T_eH \subset T_eG$. Thus for $v, w \in T_eH$ we have

$$[v,w]_{\mathfrak{h}} = [v^H, w^H]_e = [v^G, w^G]_e = [v,w]_{\mathfrak{g}}$$

which is the formula we wanted. Next, notice that if we take $T_h\iota$ as an inclusion so that $T_h\iota\left(v^H(h)\right) = v^H(h)$ for all h then we have really shown that if $v \in T_eH$ then v^H is the restriction of v^G to H. Also it is easy to see that

$$\widetilde{\mathfrak{X}^L}(H) = \{v^G : v \in T_eH\}$$

and so the restrictions of elements of $\widetilde{\mathfrak{X}^L}(H)$ are none other than the elements of $\mathfrak{X}^L(H)$ and from what we have shown, the restriction map $\widetilde{\mathfrak{X}^L}(H) \to \mathfrak{X}^L(H)$ is given by $v^G \longmapsto v^H$ and is a surjective Lie algebra homomorphism. It also has kernel zero since if v^H is the zero vector field then $v = 0$ which implies that v^G is the zero vector field. ■

Because $[v,w]_{\mathfrak{h}} = [v,w]_{\mathfrak{g}}$, the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is a Lie algebra homomorphism. Examining the details of the previous proof we see that we have a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{X}^L}(H) & \xrightarrow{\cong} & \mathfrak{X}^L(H) \\ \nwarrow & & \nearrow \\ & \mathfrak{h} & \end{array}$$

of Lie algebra homomorphisms where the top horizontal map is restriction to H, the left diagonal map is $v \longmapsto v^G$ and the right diagonal map is $v \longmapsto v^H$. In practice, what this last proposition shows is that in order to find the Lie algebra of a closed subgroup $H \subset G$ we only need to find the subspace $\mathfrak{h} = T_eH$ since the bracket on $\mathfrak{h}$ is just the restriction of the bracket on $\mathfrak{g}$. The following is also easily seen to be a commutative diagram of Lie algebra homomorphisms:

$$\begin{array}{ccc} \widetilde{\mathfrak{X}^L}(H) & \hookrightarrow & \mathfrak{X}^L(G) \\ \uparrow & & \uparrow \\ \mathfrak{h} & \hookrightarrow & \mathfrak{g} \end{array}$$

where both vertical maps are $v \longmapsto v^G$.

Now since the Lie algebra of $Gl(n)$ is the set of all square matrices with the commutator bracket and since we have just shown that the bracket for the Lie algebra of a subgroup is just the restriction of the bracket on the containing group we see that the bracket on matrix subgroups can also be taken to be the commutator bracket.

Example 4.19 *Consider the orthogonal group* $\mathrm{O}(n) \subset Gl(n)$. *Given a curve of orthogonal matrices* $Q(t)$ *with* $Q(0) = I$ *and* $\frac{d}{dt}\big|_{t=0} Q(0) = A$ *we compute by differentiating the defining equation* $I = Q^t Q$.

$$
\begin{aligned}
0 &= \left.\frac{d}{dt}\right|_{t=0} Q^t Q \\
&= \left(\left.\frac{d}{dt}\right|_{t=0} Q \right)^t Q(0) + Q^t(0) \left(\left.\frac{d}{dt}\right|_{t=0} Q \right) \\
&= A^t + A
\end{aligned}
$$

So that the space of skewsymmetric matrices is contained in the tangent space $T_I\mathrm{O}(n)$. *But both* $T_I\mathrm{O}(n)$ *and the space of skewsymmetric matrices have dimension* $\frac{n(n-1)}{2}$ *so they are equal. This means that we can identify* $\mathfrak{o}(n)$ *with the space of skewsymmetric matrices with the commutator bracket (inherited from* $Gl(n)$*). One can easily check that the commutator bracket of two such matrices is skewsymmetric as expected.*

We have considered matrix groups as subgroups of $Gl(n)$ but it is often more convenient to consider subgroups of $Gl(n, \mathbb{C})$. Since $Gl(n, \mathbb{C})$ can be identified with a subgroup of $Gl(2n)$ this is only a slight change in viewpoint. The essential parts of our discussion go through for $Gl(n, \mathbb{C})$ without any essential change.

Example 4.20 *Consider the unitary group* $U(n) \subset Gl(n, \mathbb{C})$. *Given a curve of unitary matrices* $Q(t)$ *with* $Q(0) = I$ *and* $\frac{d}{dt}\big|_{t=0} Q(0) = A$ *we compute by differentiating the defining equation* $I = \bar{Q}^t Q$.

$$
\begin{aligned}
0 &= \left.\frac{d}{dt}\right|_{t=0} \bar{Q}^t Q \\
&= \left(\left.\frac{d}{dt}\right|_{t=0} \bar{Q} \right)^t Q(0) + \bar{Q}^t(0) \left(\left.\frac{d}{dt}\right|_{t=0} Q \right) \\
&= \bar{A}^t + A
\end{aligned}
$$

examining dimensions as before we see that we can identify $\mathfrak{u}(n)$ *with the space of skewhermitian matrices* ($\bar{A}^t = -A$) *under the commutator bracket.*

We would now like to relate Lie group homomorphisms to Lie algebra homomorphisms.

Proposition 4.5 *Let* $h : G_1 \to G_2$ *be a Lie group homomorphism. The map* $T_e h : \mathfrak{g}_1 \to \mathfrak{g}_2$ *is a Lie algebra homomorphism called the* ***Lie differential*** *which is denoted in this context by* $dh : \mathfrak{g}_1 \to \mathfrak{g}_2$ *or by* $Lh : \mathfrak{g}_1 \to \mathfrak{g}_2$.

Proof. For $v \in \mathfrak{g}_1$ and $x \in G$ we have

$$\begin{aligned} T_x h \cdot L^v(x) &= T_x h \cdot (T_e L_x \cdot v) \\ &= T_e(h \circ L_x) \cdot v \\ &= T_e(L_{h(x)} \circ h) \cdot v \\ &= T_e L_{h(x)}(T_e h \cdot v) \\ &= L^{dh(v)}(h(x)) \end{aligned}$$

so $L^v \backsim_h L^{dh(v)}$. Thus by 2.5 we have for any $v, w \in \mathfrak{g}_1$ that $L^{[v,w]} \backsim_h [L^{dh(v)}, L^{dh(w)}]$ or in other words, $[L^{dh(v)}, L^{dh(w)}] \circ h = Th \circ L^{[v,w]}$ which at e gives

$$[dh(v), dh(w)] = [v, w].$$

■

Theorem 4.2 *Invariant vector fields are complete. The integral curves through the identity element are the one-parameter subgroups.*

Proof. Let X be a left invariant vector field and $c : (a, b) \to G$ be the integral curve of X with $\dot{c}(0) = X(p)$. Let $a < t_1 < t_2 < b$ and choose an element $g \in G$ such that $gc(t_1) = c(t_2)$. Let $\Delta t = t_2 - t_1$ and define $\bar{c} : (a + \Delta t, b + \Delta t) \to G$ by $\bar{c}(t) = gc(t - \Delta t)$. Then we have

$$\begin{aligned} \bar{c}'(t) &= TL_g \cdot c'(t - \Delta t) = TL_g \cdot X(c(t - \Delta t)) \\ &= X(gc(t - \Delta t)) = X(\bar{c}(t)) \end{aligned}$$

and so $\bar{c}$ is also an integral curve of X. Now on the intersection $(a + \Delta t, b)$ of their domains, c and $\bar{c}$ are equal since they are both integral curve of the same field and since $\bar{c}(t_2) = gc(t_1) = c(t_2)$. Thus we can concatenate the curves to get a new integral curve defined on the larger domain $(a, b + \Delta t)$. Since this extension can be done again for the same fixed Δt, we see that c can be extended to (a, ∞). A similar argument shows that we can extend in the negative direction to get the needed extension of c to $(-\infty, \infty)$.

Next assume that c is the integral curve with $c(0) = e$. The proof that $c(s + t) = c(s)c(t)$ proceeds by considering $\gamma(t) = c(s)^{-1}c(s + t)$. Then $\gamma(0) = e$ and also

$$\begin{aligned} \gamma'(t) &= TL_{c(s)^{-1}} \cdot c'(s + t) = TL_{c(s)^{-1}} \cdot X(c(s + t)) \\ &= X(c(s)^{-1}c(s + t)) = X(\gamma(t)). \end{aligned}$$

By the uniqueness of integral curves we must have $c(s)^{-1}c(s + t) = c(t)$ which implies the result. Conversely, suppose $c : \mathbb{R} \to G$ is a one parameter subgroup and let $X_e = \dot{c}(0)$. There is a left invariant vector field X such that $X(e) = X_e$, namely, $X = L^{X_e}$. We must show that the integral curve through e of the field

X is exactly c. But for this we only need that $\dot{c}(t) = X(c(t))$ for all t. Now $c(t+s) = c(t)c(s)$ or $c(t+s) = L_{c(t)}c(s)$. Thus

$$\dot{c}(t) = \left.\frac{d}{ds}\right|_0 c(t+s) = (T_{c(t)}L) \cdot \dot{c}(0) = X(c(t))$$

and we are done. ∎

Lemma 4.2 *Let $v \in \mathfrak{g} = T_eG$ and the corresponding left invariant field L^v. Then with $\varphi^v(t) := \varphi_t^{L^v}(e)$ we have that*

$$\varphi^v(st) = \varphi^{sv}(t) \tag{4.1}$$

A similar statement holds with R^v replacing L^v.

Proof. Let $u = st$. We have that $\left.\frac{d}{dt}\right|_{t=0} \varphi^v(st) = \frac{du}{dt} \left.\frac{d}{du}\right|_{t=0} \varphi^v(u) \frac{du}{dt} = sv$ and so by uniqueness $\varphi^v(st) = \varphi^{sv}(t)$. ∎

Theorem 4.3 *Let G be a Lie group. For a smooth curve $c : \mathbb{R} \to G$ with $c(0) = e$ and $\dot{c}(0) = v$, the following are all equivalent:*

1. *$c(t) = \varphi_t^{L^v}(e)$ for all t.*
2. *$c(t) = \varphi_t^{R^v}(e)$ for all t.*
3. *c is a one parameter subgroup with $c(0) = v$.*
4. *$\varphi_t^{L^v} = R_{c(t)}$ for all t.*
5. *$\varphi_t^{R^v} = L_{c(t)}$ for all t.*

Proof. By definition $\varphi_t^{L^v}(e) = \exp(tv)$. We have already shown that 1 implies 3. The proof that 2 implies 3 would be analogous. We have also already shown that 3 implies 1.

Also, 4 implies 1 since then $\varphi_t^{L^v}(e) = R_{c(t)}(e) = c(t)$. Now assuming 1 we have

$$\begin{aligned}
c(t) &= \varphi_t^{L^v}(e) \\
\left.\frac{d}{dt}\right|_0 c(t) &= L^v(e) \\
\left.\frac{d}{dt}\right|_0 gc(t) &= \left.\frac{d}{dt}\right|_0 L_g(c(t)) \\
&= TL_g v = L^v(g) \text{ for any } g \\
\left.\frac{d}{dt}\right|_0 R_{c(t)}g &= L^v(g) \text{ for any } g \\
R_{c(t)} &= \varphi_t^{L^v}
\end{aligned}$$

The rest is left to the reader. ∎

Definition 4.11 (The Exponential Map) *For any $v \in \mathfrak{g} = T_eG$ we have the corresponding left invariant field L^v which has an integral curve through e that we denote by $\exp(tv)$. Thus the map $t \to \exp(tv)$ is a Lie group homomorphism from $\mathbb{R}$ into G that is a one-parameter subgroup. The map $v \mapsto \exp(1v) = \exp^G(v)$ is referred to as the **exponential map** $\exp^G : \mathfrak{g} \to G$.*

Lemma 4.3 *The map $\exp^G : \mathfrak{g} \to G$ is smooth.*

Proof. Consider the map $\mathbb{R} \times G \times \mathfrak{g} \to G \times \mathfrak{g}$ given by $(t, g, v) \mapsto (g \cdot \exp^G(tv), v)$. This map is easily seen to be the flow on $G \times \mathfrak{g}$ of the vector field $\widetilde{X} : (g, v) \mapsto (L^v(g), 0)$ and so is smooth. Now the restriction of this smooth flow to the submanifold $\{1\} \times \{e\} \times \mathfrak{g}$ is $(1, e, v) \mapsto (\exp^G(v), v)$ is also smooth, which clearly implies that $\exp^G(v)$ is smooth also. ■

Note that $\exp^G(0) = e$. In the following theorem we use the canonical identification of the tangent space of T_eG at the zero element (that is $T_0(T_eG)$) with T_eG itself.

Theorem 4.4 *The tangent map of the exponential map $\exp^G : \mathfrak{g} \to G$ is the identity at $0 \in T_eG = \mathfrak{g}$ and $\exp$ is a diffeomorphism of some neighborhood of the origin onto its image in G.*

$$T_e \exp = \mathrm{id} : T_eG \to T_eG$$

Proof. By lemma 4.3 we know that $\exp^G : \mathfrak{g} \to G$ is a smooth map. Also, $\frac{d}{dt}\big|_0 \exp^G(tv) = v$ which means the tangent map is $v \mapsto v$. If the reader thinks through the definitions carefully, he or she will discover that we have here used the identification of $\mathfrak{g}$ with $T_0\mathfrak{g}$. ■

Remark 4.3 *The "one-parameter subgroup" $\exp^G(tv)$ corresponding to a vector $v \in \mathfrak{g}$ is actually a homomorphism rather than a subgroup but the terminology is conventional. It is also an immersion whose image is an initial submanifold.*

From the definitions and Theorem 4.3 we have

$$\varphi_t^{L^v}(p) = p \exp tv$$
$$\varphi_t^{L^v}(p) = (\exp tv)\, p$$

for all $v \in \mathfrak{g}$, all $t \in \mathbb{R}$ and all $p \in G$.

Proposition 4.6 *For a (Lie group) homomorphism $h : G_1 \to G_2$ the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g}_1 & \overset{dh}{\to} & \mathfrak{g}_2 \\ \exp^{G_1} \downarrow & & \exp^{G_2} \downarrow \\ G_1 & \overset{h}{\to} & G_2 \end{array}$$

Proof. The curve $t \mapsto h(\exp^{G_1}(tv))$ is clearly a one parameter subgroup. Also,

$$\left.\frac{d}{dt}\right|_0 h(\exp^{G_1}(tv)) = dh(v)$$

so by uniqueness of integral curves $h(\exp^{G_1}(tv)) = \exp^{G_2}(tdh(v))$. ∎

Remark 4.4 *We will usually not index the maps and shall just write* exp *for the exponential map of any Lie group.*

If H is a Lie subgroup of G, then the inclusion $\iota : H \hookrightarrow G$ is an injective homomorphism and the above Proposition 4.6 tell us that the exponential map on $\mathfrak{h} \subset \mathfrak{g}$ is the restriction of the exponential map on $\mathfrak{g}$. Thus, to understand the exponential map for linear Lie groups, we must understand the exponential map for the general linear group. Let V be a finite dimensional vector space.It will be convenient to pick an inner product $\langle ., . \rangle$ on V and define the norm of $v \in \mathrm{V}$ by $\|v\| := \sqrt{\langle v, v \rangle}$. In case V is a complex vector space we use an hermitian inner product. We put a norm on set of linear transformations $L(\mathrm{V};\mathrm{V})$ by

$$\|A\| = \sup_{\|v\| \neq 0} \frac{\|Av\|}{\|v\|}$$

We have $\|A \circ B\| \leq \|A\| \|B\|$ which implies that $\left\|A^k\right\| \leq \|A\|^k$. If we use the identification of $\mathfrak{gl}(\mathrm{V})$ with the (or equivalently the identification of $\mathfrak{gl}(n, \mathbb{R})$ with the linear space of $n \times n$ matrices), then the exponential map is given by a power series

$$A \mapsto \exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

The sequence of partial sums $s_N := \sum_{k=0}^{N} \frac{1}{k!} A^k$ is a Cauchy sequence in the normed space $\mathfrak{gl}(\mathrm{V})$.

$$\begin{aligned} \left\| \sum_{k=0}^{N} \frac{1}{k!} A^k - \sum_{k=0}^{M} \frac{1}{k!} A^k \right\| &\leq \left\| \sum_{k=M}^{N} \frac{1}{k!} A^k \right\| \\ &\leq \sum_{k=M}^{N} \frac{1}{k!} \|A\|^k \end{aligned}$$

From this we see that

$$\lim_{M,N \to \infty} \left\| \sum_{k=0}^{N} \frac{1}{k!} A^k - \sum_{k=0}^{M} \frac{1}{k!} A^k \right\| = 0$$

and so $\{s_N\}$ is a Cauchy sequence. Since $\mathfrak{gl}(\mathrm{V})$ together with the given norm is known to be a Banach space we see that $\sum_{k=0}^{\infty} \frac{1}{k!} A^k$ converges. Now for a fixed

A the $\mathfrak{gl}(\mathrm{V})$ function $\alpha : t \mapsto \alpha(t) = \exp(tA)$ is the unique solution of the initial value problem

$$\alpha'(t) = A\alpha(t)$$
$$\alpha(0) = A$$

This can be seen by differentiating term by term

$$\begin{aligned}\frac{d}{dt}\exp(tA) &= \sum_{k=0}^{\infty}\frac{1}{k!}t^k A^k = \sum_{k=1}^{\infty}\frac{1}{(k-1)!}t^{k-1}A^k \\ &= A\sum_{k=1}^{\infty}\frac{1}{(k-1)!}t^{k-1}A^{k-1} = A\exp(tA)\end{aligned}$$

So under our identifications, this says that α is the integral curve corresponding to the left invariant vector field determined by A. Thus we have a concrete realization of the exponential map for $\mathfrak{gl}(\mathrm{V})$ and, by restriction, each Lie subgroup of $\mathfrak{gl}(\mathrm{V})$. Applying what we know about exponential maps in the abstract setting a general Lie group we have in this concrete case $\exp((s+t)A) = \exp(sA)\exp(tA)$ and $\exp(-tA) = (\exp(tA))^{-1}$. Let $A, B \in \mathfrak{gl}(\mathrm{V})$. Then

$$\begin{aligned}\exp(A)\exp(B) &= \left(\sum_{j=0}^{\infty}\frac{1}{j!}t^j A^j\right)\left(\sum_{k=0}^{\infty}\frac{1}{k!}t^k B^k\right) \\ &= \sum_{k=0}^{\infty}\frac{1}{j!k!}t^{j+k}A^j B^k\end{aligned}$$

On the other hand, suppose that $A \circ B = B \circ A$. Then we have

$$\begin{aligned}\exp(A+B) &= \sum_{m=0}^{\infty}\frac{1}{m!}t^m (A+B)^m = \sum_{m=0}^{\infty}\frac{1}{m!}\left(\sum_{j+k=m}^{\infty}\frac{m!}{j!k!}A^j B^k\right) \\ &= \sum_{k=0}^{\infty}\frac{1}{j!k!}t^{j+k}A^j B^k\end{aligned}$$

Thus in case A commutes with B, we have

$$\exp(A+B) = \exp(A)\exp(B)$$

The Lie algebra of a Lie group and the group itself are closely related in many ways. One observation is the following:

Proposition 4.7 *If G is a connected Lie group then for any open neighborhood $V \subset \mathfrak{g}$ of 0 the group generated by $\exp(V)$ is all of G.*

sketch of proof. Since $T_e \exp = \mathrm{id}$ we have that exp is an open map near 0. The subgroup H generated by $\exp(V)$ is a subgroup containing an open neighborhood of e. The complement is also open by exercise 4.13 below. Thus H is a connected component of G which contains the identity. ■

Exercise 4.13 *Let H be a subgroup of G and consider the cosets gH. Recall that we have a disjoint union $G = \cup gH$. Show that if H is open then so are all the cosets. Conclude that the complement H^c is also open and hence H is closed..*

Since most Lie groups of interest in practice are linear Lie groups, it will pay to understand the exponential map a bit better in this case. Let V be a finite dimensional vector space equiped with an inner product as before and also take the induced norm on $\mathfrak{gl}(\mathrm{V})$. By Problem 13 we can define a map $\log : U \to \mathfrak{gl}(\mathrm{V})$ where

$$U = \{B \in Gl(\mathrm{V}) : \|B\| < 1\}$$

by using power series:

$$\log B := \sum_{k=0}^{\infty} \frac{(-1)^k}{k}(B-I)^k$$

If we compute formally, then for $A \in \mathfrak{gl}(\mathrm{V})$

$$\begin{aligned}
\log(\exp A) &= \left(A + \frac{1}{2!}A^2\right) - \frac{1}{2}\left(A + \frac{1}{2!}A^2\right)^2 \\
&\quad + \frac{1}{3}\left(A + \frac{1}{2!}A^2\right)^3 + \cdots \\
&= A + \left(\frac{1}{2!}A^2 - \frac{1}{2}A^2\right) + \left(\frac{1}{3!}A^3 - \frac{1}{2}A^3 + \frac{1}{3}A^3\right) + \cdots
\end{aligned}$$

We will argue that the above makes sense if $\|A\| < \log 2$ and that there must be cancelations in the last line so that $\log(\exp A) = A$. In fact, $\|\exp A - I\| \leq e^{\|A\|} - 1$ and so the double series on the first line for $\log(\exp A)$ must converge absolutely if $e^{\|A\|} - 1 < 1$ or if $\|A\| < \log 2$. This means that we may free rearrange terms and expect the same cancelations as we find for the analogous calculation of $\log(\exp z)$ for complex z with $|z| < \log 2$. But since $\log(\exp z) = z$ for such z we have the desired conclusion. Similarly one may argue that

$$\exp(\log B) = B \text{ if } \|B - I\| < 1$$

This last argument appeals to what is sometimes called the substitution principle:

Proposition 4.8 (Substitution Principle) *Let* V *be a finite dimensional normed space and let $L(\mathrm{V},\mathrm{V})$ be endowed with the induced norm. Let $F(x)$ and $G(x)$ formal power series with real or complex coefficients. Then one may compute the formal power series $(F+G)(x), (FG)(x)$ and assuming $G(0) = 0$ we may also compute the composition formal power series $(F \circ G)(x)$. Consider the coresponding real or complex power series for F and G and suppose that the* *radii of convergence for $F(x)$ and $G(x)$ are r_1 and r_2 respectively. Then for $A \in L(\mathrm{V}, \mathrm{V})$ we have*

$$(F+G)(A) = F(A) + G(A) \text{ for } \|A\| < \min\{r_1, r_2\}$$
$$(FG)(A) = F(A)G(A) \text{ for } \|A\| < \min\{r_1, r_2\}$$
$$(F \circ G)(A) = F(G(A)) \text{ for } \|A\| < r_1 \text{ and } G(A) < r_2$$

Now we prove a remarkable theorem that shows how an algebraic assumption can have implications in the differentiable category. First we need some notation.

Notation 4.2 *If S is any subset of a Lie group G then we define*

$$S^{-1} = \{s^{-1} : s \in S\}$$

and for any $x \in G$ we define

$$xS = \{xs : s \in S\}.$$

Theorem 4.5 *An abstract subgroup H of a Lie group G is a (regular) submanifold if and only if H is a closed set in G. If follows that H is a (regular) Lie subgroup of G.*

Proof. First suppose that H is a (regular) submanifold. Then H is locally closed. That is, every point $x \in H$ has an open neighborhood U such that $U \cap H$ is a relatively closed set in H. Let U be such a neighborhood of the identity element e. We seek to show that H is closed in G. Let $y \in \overline{H}$ and $x \in yU^{-1} \cap H$. Thus $x \in H$ and $y \in xU$. Now this means that $y \in \overline{H} \cap xU$, and hence $x^{-1}y \in \overline{H} \cap U = H \cap U$. So $y \in H$ and we have shown that H is closed.

Now conversely, let us suppose that H is a closed abstract subgroup of G. Since we can always use the diffeomorphism to translate any point to the identity, it suffices to find a neighborhood U of e such that $U \cap H$ is a submanifold. The strategy is to find out what $\mathrm{Lie}(H) = \mathfrak{h}$ is likely to be and then exponentiate a neighborhood of $e \in \mathfrak{h}$.

First we will need to have an inner product on T_eG so choose any such. Then norms of vectors in T_eG makes sense. Choose a small neighborhood $\widetilde{U}$ of $0 \in T_eG = \mathfrak{g}$ on which exp is a diffeomorphism, say $\exp : \widetilde{U} \to U$, with inverse denoted by $\log_U$. Define the set $\widetilde{H}$ in $\widetilde{U}$ by $\widetilde{H} = \log_U(H \cap U)$.

Claim 4.1 *If h_n is a sequence in $\widetilde{H}$ converging to zero and such that $u_n = h_n / |h_n|$ converges to $v \in \mathfrak{g}$ then $\exp(tv) \in H$ for all $t \in \mathbb{R}$.*

Proof of claim: Note that $th_n/|h_n| \to tv$ while $|h_n|$ converges to zero. But since $|h_n| \to 0$ we must be able to find a sequence $k(n) \in \mathbb{Z}$ such that $k(n)|h_n| \to t$. From this we have $\exp(k(n)h_n) = \exp(k(n)|h_n| \frac{h_n}{|h_n|}) \to \exp(tv)$. But by the properties of $\exp$ proved previously, we have $\exp(k(n)h_n) = (\exp(h_n))^{k(n)}$. But $\exp(h_n) \in H \cap U \subset H$ and so $(\exp(h_n))^{k(n)} \in H$. But since H is closed we have $\exp(tv) = \lim_{n\to\infty}(\exp(h_n))^{k(n)} \in H$.

Claim 4.2 *The set W of all tv where v can be obtained as a limit $h_n/|h_n| \to v$ with $h_n \in \widetilde{H}$ is a vector space.*

Proof of claim: It is enough to show that if $h_n/|h_n| \to v$ and $h_n'/|h_n'| \to w$ with $h_n', h_n \in \widetilde{H}$ then there is a sequence of elements h_n'' from $\widetilde{H}$ with

$$h_n''/|h_n''| \to \frac{v+w}{|v+w|}.$$

This will follow from the observation that

$$h(t) = \log_U(\exp(tv)\exp(tw))$$

is in $\widetilde{H}$ and by exercise 4.10 we have that

$$\lim_{t\downarrow 0} h(t)/t = v + w$$

and so

$$\frac{h(t)/t}{|h(t)/t|} \to \frac{v+w}{|v+w|}.$$

The proof of the next claim will finish the proof of the theorem.

Claim 4.3 *Let W be the set from the last claim. Then $\exp(W)$ contains an open neighborhood of e in H. Let $W^\perp$ be the orthogonal compliment of W with respect to the inner product chosen above. Then we have $T_eG = W^\perp \oplus W$. It is not difficult to show that the map $\Sigma : W \oplus W^\perp \to G$ defined by*

$$v + w \mapsto \exp(v)\exp(w)$$

is a diffeomorphism in a neighborhood of the origin in T_eG. Now suppose that $\exp(W)$ does not contain an open neighborhood of e in H. Then we can choose a sequence $(v_n, w_n) \in W \oplus W^\perp$ with $(v_n, w_n) \to 0$ and $\exp(v_n)\exp(w_n) \in H$ and yet $w_n \neq 0$. The space $W^\perp$ is closed and the unit sphere in $W^\perp$ is compact so after passing to a subsequence we may assume that $w_n/|w_n| \to w \in W^\perp$ and of course $|w| = 1$. Since $\exp(v_n) \in H$ and H is at least an algebraic subgroup we see that since $\exp(v_n)\exp(w_n) \in H$, it must be that $\exp(w_n) \in H$ also. But then by the definition of W we have that $w \in W$ which contradicts the fact that $|w| = 1$ and $w \in W^\perp$.

■

4.5 The Adjoint Representation of a Lie group

Definition 4.12 *Fix an element $g \in G$. The map $C_g : G \to G$ defined by $C_g(x) = gxg^{-1}$ is a Lie group automorphism called the conjugation map and the tangent map $T_eC_g : \mathfrak{g} \to \mathfrak{g}$, denoted Ad_g, is called the **adjoint map**.*

Exercise 4.14 *$C_g : G \to G$ is a Lie group homomorphism. The proof is easy.*

Proposition 4.9 *The map $C : g \mapsto C_g$ is a Lie group homomorphism $G \to \operatorname{Aut}(G)$.*

The image of the map C inside $\operatorname{Aut}(G)$ is a subgroup called the group of inner automorphisms and is denoted by $\operatorname{Inn}(G)$.

Using proposition 4.5 we get the following

Corollary 4.2 $\operatorname{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ *is Lie algebra homomorphism.*

Proposition 4.10 *The map $\operatorname{Ad} : g \mapsto \operatorname{Ad}_g$ is a homomorphism $G \to Gl(\mathfrak{g})$ which is called the **adjoint representation** of G.*

Proof. We have

$$\begin{aligned}\operatorname{Ad}(g_1 g_2) &= T_e C_{g_1 g_2} = T_e(C_{g_1} \circ C_{g_2}) \\ &= T_e C_{g_1} \circ T_e C_{g_2} = \operatorname{Ad}_{g_1} \circ \operatorname{Ad}_{g_2}\end{aligned}$$

which shows that Ad is a group homomorphism. The smoothness follows from the following lemma applied to the map $C : (g, x) \mapsto C_g(x)$. ■

Lemma 4.4 *Let $f : M \times N \to N$ be a smooth map and define the partial map at $x \in M$ by $f_x(y) = f(x, y)$. Suppose that for every $x \in M$ the point y_0 is fixed by f_x:*

$$f_x(y_0) = y_0 \text{ for all } x.$$

The the map $A_{y_0} : x \mapsto T_{y_0} f_x$ is a smooth map from M to $Gl(T_{y_0} N)$.

Proof. It suffices to show that A_{y_0} composed with an arbitrary coordinate function from some atlas of charts on $Gl(T_{y_0} N)$ is smooth. But $Gl(T_{y_0} N)$ has an atlas consisting of a single chart. Namely, choose a basis $v_1, v_2, ..., v_n$ of $T_{y_0} N$ and let $v^1, v^2, ..., v^n$ the dual basis of $T^*_{y_0} N$, then $\chi^i_j : A \mapsto v^i(A v_j)$ is a typical coordinate function. Now we compose;

$$\begin{aligned}\chi^i_j \circ A_{y_0}(x) &= v^i(A_{y_0}(x) v_j) \\ &= v^i(T_{y_0} f_x \cdot v_j).\end{aligned}$$

Now it is enough to show that $T_{y_0} f_x \cdot v_j$ is smooth in x. But this is just the composition the smooth maps $M \to TM \times TN \cong T(M \times N) \to T(N)$ given by

$$\begin{aligned}&x \mapsto ((x, 0), (y_0, v_j)) \mapsto (\partial_1 f)(x, y_0) \cdot 0 + (\partial_2 f)(x, y_0) \cdot v_j \\ &= T_{y_0} f_x \cdot v_j.\end{aligned}$$

(The reader might wish to review the discussion leading up to Lemma 2.2).
■

Recall that for $v \in \mathfrak{g}$ we have the associated left invariant vector field L^v as well as the right invariant field R^v. Using this notation we have

Lemma 4.5 *Let $v \in \mathfrak{g}$. Then $L^v(x) = R^{\operatorname{Ad}_x v}$.*

Proof. $L^v(x) = T_e(L_x)\cdot v = T(R_x)T(R_{x^{-1}})T_e(L_x)\cdot v = T(R_x)T(R_{x^{-1}}\circ L_x)\cdot v = R^{\mathrm{Ad}(x)v}$. ■

We now go one step further and take the differential of Ad.

Definition 4.13 *For a Lie group G with Lie algebra $\mathfrak{g}$ define the* ***adjoint representation of*** $\mathfrak{g}$, *a map* $\mathrm{ad}:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ *by*

$$\mathrm{ad} = T_e\,\mathrm{Ad} = d\,(\mathrm{Ad})$$

Proposition 4.11 $\mathrm{ad}(v)w = [v,w]$ *for all* $v,w\in\mathfrak{g}$.

Proof. Let $v^1,...,v^n$ be a basis for $\mathfrak{g}$ so that $\mathrm{Ad}(x)w = \sum a_i(x)v^i$ for some functions a_i. Then we have

$$\begin{aligned}\mathrm{ad}(v)w &= T_e(\mathrm{Ad}(.)w)v\\ &= d(\sum a_i(.)v^i)v\\ &= \sum(\left.da_i\right|_e v)v^i\\ &= \sum(L^v a_i)(e)v^i\end{aligned}$$

On the other hand, by lemma 4.5

$$\begin{aligned}L^w(x) = R^{\mathrm{Ad}(x)w} &= R(\sum a_i(x)v^i)\\ &= \sum a_i(x)R^{v^i}(x)\end{aligned}$$

Then we have

$$[L^v,L^w] = [L^v,\sum a_i()R^{v^i}()] = 0+\sum L^v(a_i)R^{v^i}.$$

Finally, we have

$$\begin{aligned}[w,v] &= [L^w,L^v](e)\\ &= \sum L^v(a_i)(e)R^{v^i}(e) = \sum L^v(a_i)(e)v^i\\ &\mathrm{ad}(v)w.\end{aligned}$$

■

The map $\mathrm{ad}:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g}) = \mathrm{End}(T_eG)$ is given as the tangent map at the identity of Ad which is a Lie algebra homomorphism. Thus by Proposition 4.5 we have obtain

Proposition 4.12 $\mathrm{ad}:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ *is a Lie algebra homomorphism.*

Since ad is defined as the Lie differential of Ad, Proposition 4.6 tells us that the following diagram commutes for any Lie group G:

$$\begin{array}{ccc} \mathfrak{g} & \overset{\mathrm{ad}}{\to} & \mathfrak{gl}(\mathfrak{g})\\ \exp\downarrow & & \downarrow\exp\\ G & \overset{\mathrm{Ad}}{\to} & Gl(\mathfrak{g})\end{array}$$

On the other hand, for any $g \in G$, the map $C_g : x \mapsto gxg^{-1}$ is also a homomorphism and so Proposition 4.6 applies again giving the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{g} & \overset{\mathrm{Ad}_g}{\rightarrow} & \mathfrak{g} \\ \exp \downarrow & & \downarrow \exp \\ G & \overset{C_g}{\rightarrow} & G \end{array}$$

In other words,

$$\exp\left(tAd_g v\right) = g \exp(tv) g^{-1}$$

for any $g \in G, v \in \mathfrak{g}$ and $t \in \mathbb{R}$.

In the case of linear Lie groups $G \subset Gl(\mathrm{V})$ we have identified $\mathfrak{g}$ with a subspace of $\mathfrak{gl}(\mathrm{V})$ which is in turn identified with $L(\mathrm{V},\mathrm{V})$. In this case the exponential map is given by the power series as explained above. It is easy to show from the power series that $B \circ \exp(tA) \circ B^{-1} = \exp(tB \circ A \circ B^{-1})$ for any $A \in \mathfrak{gl}(\mathrm{V})$ and $B \in Gl(\mathrm{V})$. In this special set of circumstances we have

$$\mathrm{Ad}_B A = B \circ A \circ B^{-1}$$

This is seen as follows:

$$\begin{aligned} \mathrm{Ad}_B A &= \left.\frac{d}{dt}\right|_{t=0} B \circ \exp(tA) \circ B^{-1} \\ &= \left.\frac{d}{dt}\right|_{t=0} \exp(tB \circ A \circ B^{-1}) = B \circ A \circ B^{-1} \end{aligned}$$

Earlier we noted that for a general Lie group we always have $\mathrm{Ad} \circ \exp = \exp \circ \mathrm{ad}$. In the current context of linear Lie groups this can be written as

$$\exp(A) \circ B \circ \exp(-A) = \sum_{k=0}^{\infty} \frac{1}{k!} (\mathrm{ad}(A))^k B$$

for any $A \in \mathfrak{gl}(\mathrm{V})$ and any $B \in Gl(\mathrm{V})$.

We have already defined the group $\mathrm{Aut}(G)$ and the subgroup $\mathrm{Inn}(G)$. We have also defined $\mathrm{Aut}(\mathfrak{g})$ as the space of Lie algebra automorphisms of $\mathfrak{g}$. The image of $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is denoted $\mathrm{Inn}(\mathfrak{g})$ and elements of $\mathrm{Inn}(\mathfrak{g})$ are referred to as inner automorphisms of the Lie algebra $\mathfrak{g}$.

4.6 The Maurer-Cartan Form

Define the maps $\omega_G : TG \to \mathfrak{g}$ (resp. $\omega_G^{right} : TG \to \mathfrak{g}$) by

$$\omega_G(X_g) = TL_{g^{-1}} \cdot X_g$$
$$(\text{resp. } \omega_G^{right}(X_g) = TR_{g^{-1}} \cdot X_g).$$

ω_G is a $\mathfrak{g}$ valued 1-form called the (left-) **Maurer-Cartan form**. We will call ω_G^{right} the right Maurer-Cartan form but we will not be using it to the extent of ω_G.

As we have seen, $Gl(n)$ is an open set in a vector space and so its tangent bundle is trivial $TGl(n) \cong Gl(n) \times \mathbb{M}_{n\times n}$. A general abstract Lie group G is not an open subset of a vector space but we are still able to show that TG is trivial. There are two such trivializations obtained from the Maurer-Cartan forms. These are $\mathrm{triv}_L : TG \to G \times \mathfrak{g}$ and $\mathrm{triv}_R : TG \to G \times \mathfrak{g}$ and defined by

$$\begin{aligned}\mathrm{triv}_L(v_g) &= (g, \omega_G(v_g))\\ \mathrm{triv}_R(v_g) &= (g, \omega_G^{right}(v_g))\end{aligned}$$

for $v_g \in T_gG$. These maps are both trivializations. Thus we have the following:

Proposition 4.13 *The tangent bundle of a Lie group is trivial: $TG \cong G \times \mathfrak{g}$.*

Proof. It is easy to check that triv_L and triv_R are trivializations in the sense of definition 2.15. ∎

We will refer to triv_L and triv_R as the (left and right) **Maurer-Cartan trivializations**. How do these two trivializations compare? There is no special reason to prefer right multiplication. We could have used right invariant vector fields as our means of producing the Lie algebra and the whole theory would work 'on the other side' so to speak. What is the relation between left and right in this context? The bridge between left and right is the adjoint map.

Lemma 4.6 (Left-right lemma) *For any $v \in \mathfrak{g}$ the map $g \mapsto \mathrm{triv}_L^{-1}(g, v)$ is a left invariant vector field on G while $g \mapsto \mathrm{triv}_R^{-1}(g, v)$ is right invariant. Also,* $\mathrm{triv}_R \circ \mathrm{triv}_L^{-1}(g, v) = (g, \mathrm{Ad}_g(v))$.

Proof. The invariance is easy to check and is left as an exercise. Now the second statement is also easy:

$$\begin{aligned}&\mathrm{triv}_R \circ \mathrm{triv}_L^{-1}(g, v)\\ &= (g, TR_{g^{-1}}TL_g v) = (g, T(R_{g^{-1}}L_g)\cdot v)\\ &= (g, \mathrm{Ad}_g(v)).\end{aligned}$$

∎

It is often convenient to actually identify the tangent bundle TG of a Lie group G with $G \times \mathfrak{g}$. Of course we must specify which of the two trivializations described above is being invoked. Unless indicated otherwise we shall use the "left version" described above: $v_g \mapsto (g, \omega_G(v_g)) = (g, TL_g^{-1}(v_g))$.

Warning: It must be realized that we now have three natural ways to trivialize the tangent bundle of the general linear group. In fact, the usual one which we introduced earlier is actually the restriction to $TGl(n)$ of the Maurer-Cartan trivialization of the *abelian* Lie group $(\mathbb{M}_{n\times n}, +)$.

In order to use the (left) Maurer-Cartan trivialization as an identification effectively, we need to find out how a few basic operations look when this identification is imposed.

The picture obtained from using the trivialization produced by the left Maurer-Cartan form:

1. The tangent map of left translation $TL_g : TG \to TG$ takes the form "TL_g" : $(x, v) \mapsto (gx, v)$. Indeed, the following diagram commutes:

$$\begin{array}{ccc} TG & \xrightarrow{TL_g} & TG \\ \downarrow & & \downarrow \\ G \times \mathfrak{g} & \xrightarrow{\text{“}TL_g\text{”}} & G \times \mathfrak{g} \end{array}$$

where elementwise we have

$$\begin{tikzcd} v_x \arrow[r, "TL_g"] \arrow[d] & TL_g \cdot v_x \arrow[d] \\ \begin{array}{c}(x, TL_x^{-1} v_x) \\ = (x, v)\end{array} \arrow[r] & \begin{array}{c}(gx, TL_x^{-1} v_x) \\ = (gx, v)\end{array} \end{tikzcd}$$

2. The tangent map of multiplication: This time we will invoke two identifications. First, group multiplication is a map $\mu : G \times G \to G$ and so on the tangent level we have a map $T(G \times G) \to G$. Now recall that we have a natural isomorphism $T(G \times G) \cong TG \times TG$ given by $T\pi_1 \times T\pi_2 : (v_{(x,y)}) \mapsto (T\pi_1 \cdot v_{(x,y)}, T\pi_2 \cdot v_{(x,y)})$. If we also identify TG with $G \times \mathfrak{g}$ then $TG \times TG \cong (G \times \mathfrak{g}) \times (G \times \mathfrak{g})$ and we end up with the following "version" of $T\mu$:

$$\text{“}T\mu\text{”} : (G \times \mathfrak{g}) \times (G \times \mathfrak{g}) \to G \times \mathfrak{g}$$
$$\text{“}T\mu\text{”} : ((x, v), (y, w)) \mapsto (xy, TR_y v + TL_x w)$$

(See exercise 4.15).

3. The (left) Maurer-Cartan form is a map $\omega_G : TG \to T_eG = \mathfrak{g}$ and so there must be a "version", "ω_G", that uses the identification $TG \cong G \times \mathfrak{g}$. In fact, the map we seek is just projection:

$$\text{“}\omega_G\text{”} : (x, v) \mapsto v$$

4. The right Maurer-Cartan form is a little more complicated since we are using the isomorphism $TG \cong G \times \mathfrak{g}$ obtained from the *left* Maurer-Cartan form. The result follows from the "left-right lemma 4.6:

$$\text{“}\omega_G^{right}\text{”} : (x, v) \mapsto Ad_g(v)$$

The adjoint map is nearly the same thing as the right Maurer-Cartan form once we decide to use the (left) trivialization $TG \cong G \times \mathfrak{g}$ as an identification.

5. A vector field $X \in \mathfrak{X}(G)$ should correspond to a section of the product bundle $G \times \mathfrak{g} \to G$ which must have the form $\overleftrightarrow{X} : x \mapsto (x, F^X(x))$ for some smooth $\mathfrak{g}$-valued function $F^X \in C^\infty(G; \mathfrak{g})$. It is an easy consequence of the definitions that $F^X(x) = \omega_G(X(x)) = TL_x^{-1} \cdot X(x)$. We ought to be able to identify how $\overleftrightarrow{X}$ acts on $C^\infty(G)$. We have $(\overleftrightarrow{X} f)(x) = (x, F^X(x))f = (TL_x \cdot F^X(x))f$. Under this identification, a left invariant vector field becomes a constant section of $G \times \mathfrak{g}$. For example, if X is left invariant then the corresponding constant section is $x \mapsto (x, X(e))$.

Exercise 4.15 *Refer to 2. Show that the map "$T\mu$" defined so that the diagram below commutes is $((x, v), (y, w)) \mapsto (xy, TR_y v + TL_x w)$.*

$$\begin{array}{ccc} T(G \times G) & & \\ \cong TG \times TG & \overset{T\mu}{\to} & TG \\ \downarrow & & \downarrow \\ (G \times \mathfrak{g}) \times (G \times \mathfrak{g}) & \overset{\text{“}T\mu\text{”}}{\to} & G \times \mathfrak{g} \end{array}$$

4.7 Spinors and rotation

The matrix Lie group $SO(3)$ is the group of orientation preserving rotations of $\mathbb{R}^3$ acting by matrix multiplication on column vectors. The group $SU(2)$ is the group of complex 2×2 unitary matrices of determinant 1. We shall now expose an interesting relation between these groups. First recall the Pauli matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The real vector space spanned by $\sigma_1, \sigma_2, \sigma_3$ is the space of traceless Hermitian matrices. Let us temporarily denote the latter by $\widehat{\mathbb{R}^3}$. We have a linear isomorphism $\mathbb{R}^3 \to \widehat{\mathbb{R}^3}$ given by $(x^1, x^2, x^3) \mapsto x^1\sigma_1 + x^2\sigma_2 + x^3\sigma_3$ which we abbreviate to $\vec{x} \mapsto \widehat{x}$. Now it is easy to check that $\det(\widehat{x})$ is just $-|\vec{x}|^2$. In fact, if introduce an inner product on $\widehat{\mathbb{R}^3}$ by the formula $\langle \widehat{x}, \widehat{y} \rangle := -\frac{1}{2} tr(\widehat{x}\widehat{y})$ then the map $\vec{x} \mapsto \widehat{x}$ is an isometry. Next we notice that $SU(2)$ acts on $\widehat{\mathbb{R}^3}$ by $(g, \widehat{x}) \mapsto g\widehat{x}g^{-1} = g\widehat{x}g^*$ thus giving a representation ρ of $SU(2)$ in $\widehat{\mathbb{R}^3}$. It is easy to see that $\langle \rho(g)\widehat{x}, \rho(g)\widehat{y} \rangle = \langle \widehat{x}, \widehat{y} \rangle$ and so under the identification $\mathbb{R}^3 \leftrightarrow \widehat{\mathbb{R}^3}$ we see that $SU(2)$ act on $\mathbb{R}^3$ as an element of $O(3)$.

Exercise 4.16 *Show that in fact, the map $SU(2) \to O(3)$ has image $SO(3)$ and is actually a covering homomorphism onto $SO(3)$ with kernel $\{\pm I\} \cong \mathbb{Z}_2$.*

Exercise 4.17 *Show that the algebra generated by the matrices $\sigma_0, -i\sigma_1, -i\sigma_2, -i\sigma_3$ is isomorphic to the quaternion algebra and that the set of matrices $-i\sigma_1, -i\sigma_2, -i\sigma_3$ span a real vector space which is equal as a set to the traceless skew Hermitian matrices $\mathfrak{su}(2)$.*

Let $I = -i\sigma_1$, $J = -i\sigma_2$ and $-i\sigma_3 = K$. One can redo the above analysis using the isometry $\mathbb{R}^3 \to \mathfrak{su}(2)$ given by

$$\begin{aligned}(x^1, x^2, x^3) &\mapsto x^1 I + x^2 J + x^3 K \\ \vec{x} &\mapsto \widetilde{x}\end{aligned}$$

where this time $\langle \widetilde{x}, \widetilde{y} \rangle := \frac{1}{2} tr(\widetilde{x}\widetilde{y}^*) = -\frac{1}{2} tr(\widetilde{x}\widetilde{y})$. Notice that $\mathfrak{su}(2) = span\{I, J, K\}$ is the Lie algebra of $SU(2)$ the action $(g, \widehat{x}) \mapsto g\widehat{x}g^{-1} = g\widehat{x}g^*$ is just the adjoint action to be defined in a more general setting below. Anticipating this, let us write $Ad(g) : \widehat{x} \mapsto g\widehat{x}g^*$. This gives the map $g \mapsto Ad(g)$; a Lie group homomorphism $SU(2) \to SO(\mathfrak{su}(2), \langle,\rangle)$. Once again we get the same map $SU(2) \to \mathrm{O}(3)$ which is a Lie group homomorphism and has kernel $\{\pm I\} \cong \mathbb{Z}_2$. In fact, we have the following commutative diagram:

$$\begin{array}{ccc} SU(2) & = & SU(2) \\ \rho \downarrow & & Ad \downarrow \\ SO(3) & \cong & SO(\mathfrak{su}(2), \langle,\rangle) \end{array}$$

Exercise 4.18 *Check the details here!*

What is the differential of the map $\rho : SU(2) \to O(3)$ at the identity? Let $g(t)$ be a curve in $SU(2)$ with $\frac{d}{dt}\big|_{t=0} g = g'$. We have $\frac{d}{dt}(g(t)Ag^*(t)) = (\frac{d}{dt}g(t))Ag^*(t) + g(t)A(\frac{d}{dt}g(t))^*$ and so the map $ad : g' \mapsto g'A + Ag'^* = [g', A]$

$$\begin{aligned}\frac{d}{dt}\langle g\widehat{x}, g\widehat{y}\rangle &= \frac{d}{dt}\frac{1}{2}tr(g\widetilde{x}(g\widetilde{y})^*) \\ &\frac{1}{2}tr([g', \widetilde{x}], (\widetilde{y})^*) + \frac{1}{2}tr(\widetilde{x}, ([g', \widetilde{y}])^*) \\ &== \frac{1}{2}tr([g', \widetilde{x}], (\widetilde{y})^*) - \frac{1}{2}tr(\widetilde{x}, [g', \widetilde{y}]) \\ &= \langle [g', \widetilde{x}], \widetilde{y}\rangle - \langle \widetilde{x}, [g', \widetilde{y}]\rangle \\ &= .\langle ad(g')\widetilde{x}, \widetilde{y}\rangle - \langle \widetilde{x}, ad(g')\widetilde{y}\rangle\end{aligned}$$

From this is follows that the differential of the map $SU(2) \to O(3)$ takes $\mathfrak{su}(2)$ isomorphically onto the space $\mathfrak{so}(3)$. We have

$$\begin{array}{ccc} \mathfrak{su}(2) & = & \mathfrak{su}(2) \\ d\rho \downarrow & & ad \downarrow \\ \mathfrak{so}(3) & \cong & \mathfrak{so}(\mathfrak{su}(2), \langle,\rangle) \end{array}$$

where $\mathfrak{so}(\mathfrak{su}(2), \langle,\rangle)$ denotes the linear maps $\mathfrak{su}(2) \to \mathfrak{su}(2)$ skew-symmetric with respect to the inner product $\langle \widetilde{x}, \widetilde{y}\rangle := \frac{1}{2}tr(\widetilde{x}\widetilde{y}^*)$.

4.8 Lie Group Actions

The basic definitions for group actions were given earlier in definition 1.24 and 1.25. As before we give most of our definitions and results for left actions and ask the reader to notice that analogous statements can be made for right actions.

Definition 4.14 *Let $l : G \times M \to M$ be a left action where G is a Lie group and M a smooth manifold. If l is a smooth map then we say that l is a (smooth) Lie group action.*

As before, we also use either of the notations gx or $l_g(p)$ for $l(g, p)$. For right actions $r : M \times G \to M$ we write $pg = r_g(p) = r(p, g)$. A right action corresponds to a left action by the rule $gp := pg^{-1}$. Recall that for $p \in M$ the orbit of p is denoted Gp and we call the action transitive if $Gp = M$.

Definition 4.15 *Let l be a Lie group action as above. For a fixed $p \in M$ the **isotropy** group of p is defined to be*

$$G_p := \{g \in G : gp = p\}$$

*The isotropy group of p is also called the **stabilizer** of p.*

Exercise 4.19 *Show that G_p is a closed subset and abstract subgroup of G. This means that G_p is a closed Lie subgroup.*

Recalling the definition of a free action, it is easy to see that an action is free if and only if the isotropy subgroup of every point is the trivial subgroup consisting of the identity element alone.

Definition 4.16 *Suppose that we have Lie group action of G on M. If N is a subset of M and $gx \in x$ for all $x \in N$ then we say that N is an invariant subset. If N is also a submanifold then it is called an invariant submanifold.*

In this definition we include the possibility that N is an open submanifold. If N is an invariant subset of N then it is easy to set that $gN = N$ where $gN = l_g(N)$ for any g. Furthermore, if N is a submanifold then the action restricts to a Lie group action $G \times N \to N$.

If G is zero dimensional then by definition it is just a group with discrete topology and we recover the definition of discrete group action. We have already seen several examples of discrete group actions and now we list a few examples of more general Lie group actions.

Example 4.21 *In case $M = \mathbb{R}^n$ then the Lie group $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication. Similarly, $GL(n, \mathbb{C})$ acts on $\mathbb{C}^n$. More abstractly, $GL(V)$ acts on the vector space V. This action is smooth since Ax depends smoothly (polynomially) on the components of A and on the components of $x \in \mathbb{R}^n$.*

Example 4.22 *Any Lie subgroup of $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$ also by matrix multiplication. For example, $O(n, \mathbb{R})$ acts on $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ the orbit of x is the sphere of radius $\|x\|$. This is trivially true if $\|x\| = 0$. In general, if $\|x\| \neq 0$ then, $\|gx\| = \|x\|$ for any $g \in O(n, \mathbb{R})$. On the other hand, if $x, y \in \mathbb{R}^n$ and $\|x\| = \|y\| = r$ then let $\widehat{x} := x/r$ and $\widehat{y} := y/r$. Extend to orthonormal bases $(\widehat{x} = e_1, ..., e_n)$ and $(\widehat{y} = f_1, ..., f_n)$. Then there exists an orthogonal matrix S such that $Se_i = f_i$ for $i = 1, ..., n$. In particular, $S\widehat{x} = \widehat{y}$ and so $Sx = y$.*

Exercise 4.20 *From the last example we can restrict the action of $O(n,\mathbb{R})$ to a transitive action on S^{n-1}. Now $SO(n,\mathbb{R})$ also acts on $\mathbb{R}^n$ and by restriction on S^{n-1}. From the last example we know that $O(n,\mathbb{R})$ acts transitively on S^{n-1}. Show that the same is true for $SO(n,\mathbb{R})$ as long as $n > 1$.*

A Lie group acts on itself in an obvious way:

Definition 4.17 *For a Lie group G and a fixed element $g \in G$, the maps $L_g : G \to G$ and $R_g : G \to G$ are defined by*

$$L_g x = gx \text{ for } x \in G$$
$$R_g x = xg \text{ for } x \in G$$

*and are called **left translation** and **right translation** (by g) respectively.*

The maps $G \times G \to G$ given by $(g, x) \mapsto L_g x$ and $(g, x) \mapsto R_g x$ are Lie group actions.

Example 4.23 *If H is a Lie subgroup of a Lie group G then we can consider L_h for any $h \in H$ and thereby obtain an action of H on G.*

Recall that a subgroup H of a group G is called a normal subgroup if $gkg^{-1} \in K$ for any $k \in H$ and all $g \in G$. In other word, H is normal if $gHg^{-1} \subset H$ for all all $g \in G$ and it is easy to see that in this case we always have $gHg^{-1} = H$.

Example 4.24 *If H is a normal Lie subgroup of G, then G acts on H by conjugation:*

$$C_g h = ghg^{-1}$$

Suppose now that a Lie group G acts on smooth manifolds M and N. For simplicity we take both actions to be left action which we denote by l and λ respectively. A map $\Phi : M \to N$ such that $\Phi \circ l_g = \lambda_g \circ \Phi$ for all $g \in G$, is said to be an equivariant map (equivariant with respect to the given actions). This means that for all g the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & N \\ l_g \downarrow & & \downarrow \lambda_g \\ M & \xrightarrow{\Phi} & N \end{array}$$

If Φ is also a diffeomorphism then we have an **equivalence** of Lie group actions.

Example 4.25 *If $\phi : G \to H$ is a Lie group homomorphism then we can define an action of G on H by $\lambda(g, g_1) = \lambda_g(h) = L_{\phi(g)} h$. We leave it to the reader to verify that this is indeed a Lie group action. In this situation ϕ is equivariant with respect to the actions λ and L (left translation).*

Example 4.26 *Let $T^n = S^1 \times \cdots \times S^1$ be the $n-$torus where we identify S^1 with the complex numbers of unit modulus. Fix $k = (k_1, ..., k_n) \in \mathbb{R}^n$. Then $\mathbb{R}$ acts on $\mathbb{R}^n$ by $\tau^k(t, x) = t \cdot x := x + tk$. On the other hand, $\mathbb{R}$ acts on T^n by $t \cdot (z^1, ..., z^n) = (e^{itk_1} z^1, ..., e^{itk_n} z^n)$. The map $\mathbb{R}^n \to T^n$ given by $(x^1, ..., x^n) \mapsto (e^{ix^1}, ..., e^{ix^n})$ is equivariant with respect to these actions.*

Theorem 4.6 (Equivariant Rank Theorem) *Suppose that $f : M \to N$ is smooth and that a Lie group G acts on both M and N with the action on M being transitive. If f is equivariant then it has constant rank. In particular, each level set of f is a closed regular submanifold.*

Proof. Let the actions on M and N be denoted by l and λ respectively as before. Pick any two points $p_1, p_2 \in M$. Since G acts transitively on M there is a g with $l_g p_1 = p_2$. By hypothesis, we have the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ l_g \downarrow & & \downarrow \lambda_g \\ M & \xrightarrow{f} & N \end{array}$$

which, upon application of the tangent functor gives the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{T_{p_1} f} & N \\ T_{p_1} l_g \downarrow & & \downarrow T_{f(p_1)} \lambda_g \\ M & \xrightarrow{T_{p_2} f} & N \end{array}$$

Since the maps $T_{p_1} l_g$ and $T_{f(p_1)} \lambda_g$ are linear isomorphisms we see that $T_{p_1} f$ must have the same rank as $T_{p_1} f$. Since p_1 and p_2 were arbitrary we see that the rank of f is constant on M. ■

There are several corollaries of this neat theorem. For example, we know that $O(n, \mathbb{R})$ is the level set $f^{-1}(I)$ where $f : GL(n, \mathbb{R}) \to gl(n, \mathbb{R}) = M(n, \mathbb{R})$ is given by $f(A) = A^T A$. The group $O(n, \mathbb{R})$ acts on itself via left translation and we also let $O(n, \mathbb{R})$ act on $\mathfrak{gl}(n, \mathbb{R})$ by $Q \cdot A := Q^T A Q$ (adjoint action). One checks easily that f is equivariant with respect to these actions and since the first action (left translation) is certainly transitive we see that $O(n, \mathbb{R})$ is a closed regular submanifold of $GL(n, \mathbb{R})$. It follows from proposition 4.1 that $O(n, \mathbb{R})$ is a closed Lie subgroup of $GL(n, \mathbb{R})$. Similar arguments apply for $U(n, \mathbb{C}) \subset GL(n, \mathbb{C})$ and other linear Lie groups. In fact we have the following general corollary to Theorem 4.6 above.

Corollary 4.3 *If $\phi : G \to H$ is a Lie group homomorphism then the kernel $Ker(h)$ is a closed Lie subgroup of G.*

Proof. Let G act on itself and on H as in example 4.25. Then ϕ is equivariant and $\phi^{-1}(e) = Ker(h)$ is a closed Lie subgroup by Theorem 4.6 and Proposition 4.1. ■

We also have use for the

Corollary 4.4 *Let $l : G \times M \to M$ be a Lie group action and G_p the isotropy subgroup of some $p \in M$. Then G_p is a closed Lie subgroup of G.*

Proof. The orbit map $\theta_p : G \to M$ given by $\theta_p(g) = gp$ is an equivariant with respect to left translation on G and the given action on M. Thus by the equivariant rank theorem, G_p is a regular submanifold of G an then by Proposition 4.1 it is a closed Lie subgroup. ■

4.8.1 Proper Lie Group Actions

Definition 4.18 *Let $l : G \times M \to M$ be a smooth (or merely continuous) group action. If the the map $P : G \times M \to M \times M$ given by $(g, p) \mapsto (l_g p, p)$ is proper we say that the action is a **proper action**.*

It is important to notice that a proper action is *not* defined to be an action such that the defining map $l : G \times M \to M$ is proper.

We now give a useful characterization of a proper action. For any subset $K \subset M$, let $g \cdot K := \{gx : x \in K\}$.

Proposition 4.14 *Let $l : G \times M \to M$ be a smooth (or merely continuous) group action. Then l is a proper action if and only if the set*

$$G_K := \{g \in G : (g \cdot K) \cap K \neq \emptyset\}$$

is compact whenever K is compact.

Proof. We follow the proof from [Lee, John];

Suppose that l is proper so that the map P is a proper map. Let π_G be the first factor projection $G \times M \to G$. Then

$$\begin{aligned} G_K &= \{g : \text{ there exists a } x \in K \text{ such that } gx \in K\} \\ &= \{g : \text{ there exists a } x \in M \text{ such that } P(g, x) \in K \times K\} \\ &= \pi_G(P^{-1}(K \times K)) \end{aligned}$$

and so G_K is compact.

Next we assume that G_K is compact for all compact K. If $C \subset M \times M$ is compact then letting $K = \pi_1(C) \cap \pi_2(C)$ where π_1 and π_2 are first and second factor projections $M \times M \to M$ respectively we have

$$\begin{aligned} P^{-1}(C) &\subset P^{-1}(K \times K) \subset \{(g, x) : gp \in K\} \\ &\subset G_K \times K. \end{aligned}$$

Since $P^{-1}(C)$ is a closed subset of the compact set $G_K \times K$ it is compact. This means that P is proper since C was an arbitrary compact subset of $M \times M$. ■

Using this proposition, one can show that definition 1.29 for discrete actions is consistent with definition 4.18 above.

Proposition 4.15 *If G is compact then any smooth action $l : G \times M \to M$ is proper.*

Proof. Let $K \subset M \times M$ be compact. We find compact $C \subset M$ such that $K \subset C \times C$ as in the proof of proposition 4.14.

Claim: $P^{-1}(K)$ is compact. Indeed,

$$\begin{aligned} P^{-1}(K) &\subset P^{-1}(C \times C) = \cup_{c \in C} P^{-1}(C \times \{c\}) \\ &= \cup_{c \in C} \{(g, p) : (gp, p) \in C \times \{c\}\} \\ &= \cup_{c \in C} \{(g, c) : gp \in C\} \\ &\subset \cup_{c \in C} (G \times \{c\}) = G \times C \end{aligned}$$

Thus $P^{-1}(K)$ is a closed subset of the compact set $G \times C$ and hence is compact. ■

Exercise 4.21 *Prove the following*

i) If $l : G \times M \to M$ is a proper action and $H \subset G$ is a closed subgroup then the restricted action $H \times M \to M$ is proper.

ii) If N is an invariant submanifold for a proper action $l : G \times M \to M$ then the restricted action $G \times N \to N$ is also proper.

Let us now consider a Lie group action $l : G \times M \to M$ that is both *proper and free*. The map orbit map at p is the map $\theta_p : G \to M$ given by $\theta_p(g) = g \cdot p$. It is easily seen to be smooth and its image is obviously $G \cdot p$. In fact, if the action is free then each orbit map is injective. Also, θ_p is equivariant with respect to the left action of G on itself and the action l :

$$\begin{aligned} \theta_p(gx) &= (gx) \cdot p = g \cdot (x \cdot p) \\ &= g \cdot \theta_p(x) \end{aligned}$$

for all $x, g \in G$. It now follows from Theorem 4.6 (the equivariant rank theorem) that θ_p has constant rank and since it is injective it must be an immersion. Not only that, but it is a proper map. Indeed, for any compact $K \subset M$ the set $\theta_p^{-1}(K)$ is a closed subset of the set $G_{K \cup \{p\}}$ and since the later set is compact by Theorem 4.14, $\theta_p^{-1}(K)$ is compact. Now by exercise 3.3 obtain the result that θ_p is an embedding and each orbit is a regular submanifold of M.

It will be very convenient to have charts on M which fit the action of G in a nice way. See figure 4.1.

Definition 4.19 *Let M be an n-manifold and G a Lie group of dimension k. If $l : G \times M \to M$ is a Lie group action then an* ***action-adapted chart*** *on M is a chart $(U, \mathtt{x})$ such that*

i) $\mathtt{x}(U)$ is a product open set $V_1 \times V_1 \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$

ii) if an orbit has nonempty intersection with U then that intersection has the form

$$\{x^{k+1} = c^1, ..., x^n = c^{n-k}\}$$

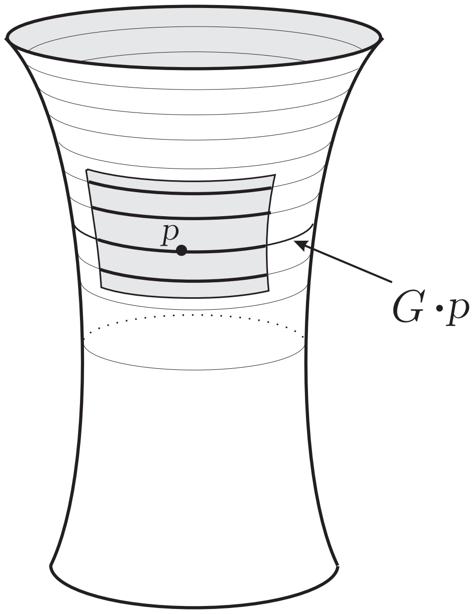


Figure 4.1: Action-adapted chart

for some constants $c^1, ..., c^{n-k}$.

Theorem 4.7 *If* $l : G \times M \to M$ *is a free and proper Lie group action then for every* $p \in M$ *there is an action-adapted chart centered at* p.

Proof. Let $p \in M$ be given. Since $G \cdot p$ is a regular submanifold we may choose a regular submanifold chart $(W, \mathtt{y})$ centered at p so that $(G \cdot p) \cap W$ is exactly given by $y^{k+1} = ... = y^n = 0$ in W. Let S be the complementary slice in W given by $y^1 = ... = y^k = 0$. Note that S is a regular submanifold. The tangent space T_pM decomposes as

$$T_pM = T_p\,(G \cdot p) \oplus T_pS$$

Let $\varphi : G \times S \to M$ be the restriction of the action l to the set $G \times S$. Also, let $i_p : G \to G \times S$ be the insertion map $g \mapsto (g, p)$ and let $j_e : S \to G \times S$ be the insertion map $s \mapsto (e, s)$. These insertion maps are embeddings and we have $\theta_p = \varphi \circ i_p$ and also $\varphi \circ j_e = \iota$ where ι is the inclusion $S \hookrightarrow M$. Now $T_e\theta_p(T_eG) = T_p(G \cdot p)$ since θ_p is an embedding. On the other hand, $T\theta_p = T\varphi \circ Ti_p$ and so the image of $T_{(e,p)}\varphi$ must contain $T_p(G \cdot p)$. Similarly, from the composition $\varphi \circ j_e = \iota$ we see that the image of $T_{(e,p)}\varphi$ must contain T_pS. It follows that $T_{(e,p)}\varphi : T_{(e,p)}(G \times S) \to T_pM$ is surjective and since $T_{(e,p)}(G \times S)$ and T_pM have the same dimension it is also injective.

By the inverse function theorem, there is neighborhood O of (e,p) such that $\varphi|\,O$ is a diffeomorphism. By shrinking O further if necessary we may assume that $\varphi(O) \subset W$. We may also arrange that O has the form of a product $O = A \times B$ for A open in G and B open in S. In fact, we can assume that there are diffeomorphisms $\alpha : I^k \to A$ and $\beta : I^{n-k} \to B$ where I^k and I^{n-k} are the open cubes in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ given respectively by $I^k = (-1,1)^k$ and $I^{n-k} = (-1,1)^{n-k}$ and where $\alpha(e) = 0 \in \mathbb{R}^k$ and $\beta(p) = 0 \in \mathbb{R}^{n-k}$. Let $U := \varphi\,(A \times B)$. The map $\varphi \circ (\alpha \times \beta) : I^k \times I^{n-k} \to U$ is a diffeomorphism and so its inverse is a chart. We must make one more adjustment. We must show that B can be chosen small enough that the intersection of each orbit with B is either empty or a single point. If this were not true then there would be a sequence of open sets B_i with compact closure (and with corresponding diffeomorphisms $\beta_i : I^k \to B_i$ as above) such that for every i there is a pair of distinct points $p_i, p_i' \in B_i$ with $g_i p_i = p_i'$ for some sequence $\{g_i\} \subset G$. Since manifolds are first countable and normal, we may assume that the sequence $\{B_i\}$ is a nested neighborhood basis which means that $B_{i+1} \subset B_i$ for all i and for each neighborhood V of p, we have $B_i \subset V$ for large enough i. This forces both p_i, and $p_i' = g_i p_i$ to converge to p. From this we see that the set $K = \{(g_i p_i, p_i), (p,p)\} \subset M \times M$ is compact. Recall that by definition the map $P : (g,x) \longmapsto (gx, x)$ is proper. Since $(g_i, p_i) = P^{-1}(g_i p_i, p_i)$ we see that $\{(g_i, p_i)\}$ is a subset of the compact set $P^{-1}(K)$. Thus after passing to a subsequence we have that (g_i, p_i) converges to (g,p) for some g and hence $g_i \to g$ and $g_i p_i \to gp$. But this means we have

$$gp = \lim_{i\to\infty} g_i p_i = \lim_{i\to\infty} p_i' = p$$

and since the action is free we conclude that $g = e$. But this is impossible since it would mean that for large enough i we would have $g_i \in A$ and in turn this would imply that

$$\varphi(g_i, p_i) = l_{g_i}(p_i) = p_i' = l_e(p_i') = \varphi(e, p_i')$$

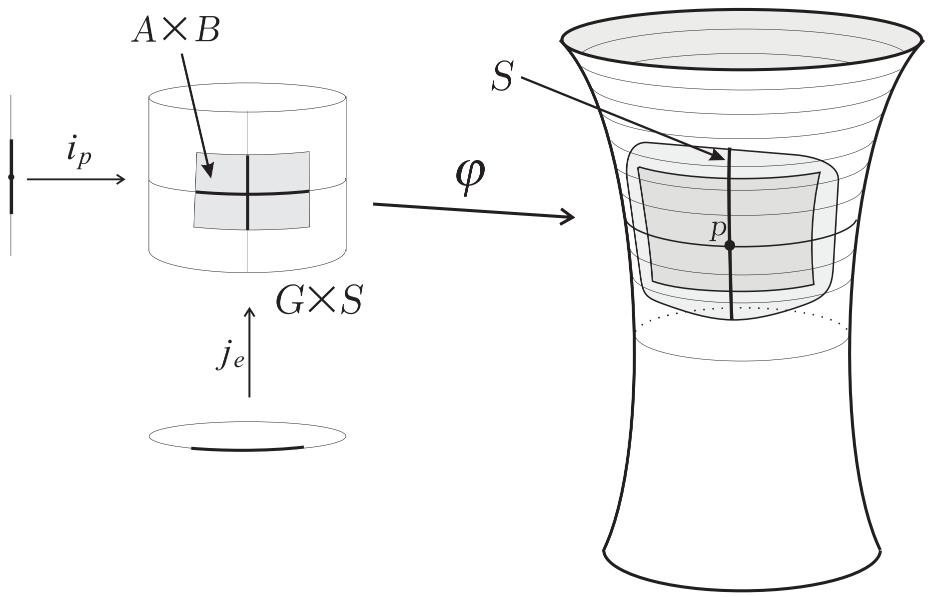
contradicting the injectivity of φ on $A \times B$. Thus after shrinking B we may assume that the intersection of each orbit with B is either empty or a single point. We leave it to the reader to check that with $\mathtt{x} := (\varphi \circ (\alpha \times \beta))^{-1} : U \to I^k \times I^{n-k} \subset \mathbb{R}^n$ we obtain a chart $(U, \mathtt{x})$ with the desired properties. ∎

For the next lemma we continue with the convention that I is the interval $(-1,1)$.

Lemma 4.7 *Let* $\mathtt{x} := (\varphi \circ (\alpha \times \beta))^{-1} : U \to I^k \times I^{n-k} = I^n \subset \mathbb{R}^n$ *be an action-adapted chart map obtained as in the proof of Theorem 4.7 above. Then given any* $p_1 \in U$*, there exists a diffeomorphism* $\psi : I^n \to I^n$ *such that* $\psi \circ \mathtt{x}$ *is an action-adapted chart centered at* p_1*.*

Proof. Clearly all we need to do is show that for any $a \in I^n$ there is a diffeomorphism $\psi : I^n \to I^n$ such that $\psi(a) = 0$. Let a^i be the $i-$th component of a. Let $\psi_i : I \to I$ be defined by

$$\psi_i := \phi \circ t_{-\phi(a_i)} \circ \phi$$



where $t_{-c}(x) := x - c$ and $\phi : (-1, 1) \to \mathbb{R}$ is the useful diffeomorphism $\phi : x \mapsto \tan(\frac{\pi}{2}x)$. The diffeomorphism we want is now $\psi(x) = (\psi_1(x^1), ..., \psi_1(x^n))$. ■

4.8.2 Quotients

If $l : G \times M \to M$ is a Lie group action, then there is a natural equivalence relation on M whereby the equivalence classes are exactly the orbits of the action. The quotient space space (or orbit space) is denoted $G\backslash M$ and we have the quotient map $\pi : M \to G\backslash M$. We put the quotient topology on $G\backslash M$ so that $A \subset G\backslash M$ is open if and only if $\pi^{-1}(A)$ is open in M. The quotient map is also open. Indeed, let $U \subset M$ be open. We want to show that $\pi(U)$ is open and for this it suffices to show that $\pi^{-1}(\pi(U))$ is open. But $\pi^{-1}(\pi(U))$ is the union $\cup_g l_g(U)$ and this is open since each $l_g(U)$ is open.

Lemma 4.8 *$G\backslash M$ is a Hausdorff space if the set $\Gamma := \{(gp, p) : g \in G, p \in M\}$ is a closed subset of $M \times M$.*

Proof. Let $\underline{p}, \underline{q} \in G\backslash M$ with $\pi(p) = \underline{p}$ and $\pi(q) = \underline{q}$. If $\underline{p} \neq \underline{q}$ then p and q are not in the same orbit. This means that $(p, q) \notin \Gamma$ and so there must be a product open set $U \times V$ such that $(p, q) \in U \times V$ and $U \times V$ disjoint from Γ. This means that $\pi(U)$ and $\pi(V)$ are disjoint neighborhoods of $\underline{p}$ and $\underline{q}$ respectively ■

Proposition 4.16 *If $l : G \times M \to M$ is a free and proper action then $G\backslash M$ is Hausdorff and paracompact.*

Proof. To show that $G\backslash M$ is Hausdorff we use the previous lemma. We must show that Γ is closed. But $\Gamma = P(G \times M)$ is closed since P is proper.

To show that $G\backslash M$ is paracompact it suffices to show that each connected component of $G\backslash M$ is second countable. This reduces the situation to the case where $G\backslash M$ is connected. In this case we can see that if $\{U_i\}$ is a countable basis for the topology on M then $\{\pi(U_i)\}$ is a countable basis for the topology on $G\backslash M$. ∎

We will shortly show that if the action is free and proper then $G\backslash M$ has a smooth structure which makes the quotient map $\pi : M \to G\backslash M$ a submersion. Before coming to this lets us note that if such a smooth structure exists then it is unique. Indeed, if $(G\backslash M)_{\mathcal{A}}$ is $G\backslash M$ with a smooth structure given by maximal atlas $\mathcal{A}$ and similarly for $(G\backslash M)_{\mathcal{B}}$ for another atlas $\mathcal{B}$ then we have the following commutative diagram:

$$\begin{array}{ccc} & M & \\ \pi \swarrow & & \searrow \pi \\ (G\backslash M)_{\mathcal{A}} & \xrightarrow{id} & (G\backslash M)_{\mathcal{B}} \end{array}$$

Since π is a surjective submersion, Proposition 3.5applies to show that $(G\backslash M)_{\mathcal{A}} \xrightarrow{id} (G\backslash M)_{\mathcal{B}}$ is smooth as is its inverse. This means that $\mathcal{A} = \mathcal{B}$.

Theorem 4.8 *If $l : G \times M \to M$ is a free and proper Lie group action then there is a unique smooth structure on the quotient $G\backslash M$ such that*

(i) the induced topology is the quotient topology and hence $G\backslash M$ is a smooth manifold,

(ii) the projection $\pi : M \to G\backslash M$ is a submersion,

(iii) $\dim(G\backslash M) = \dim(M) - \dim(G)$.

Proof. Let $\dim(M) = n$ and $\dim(G) = k$. We have already show that $G\backslash M$ is a paracompact Hausdorff space. All that is left is to exhibit an atlas such that the charts are homeomorphisms with respect to this quotient topology. Let $q \in G\backslash M$ and choose p with $\pi(p) = q$. Let $(U, \mathtt{x})$ be an action-adapted chart centered at p and constructed exactly as in Theorem 4.7. Let $\pi(U) = V \subset G\backslash M$ and let B be the slice $x^1 = \cdots = x^k = 0$. By construction $\pi|_B : B \to V$ is a bijection and in fact it is easy to check that $\pi|_B$ is a homeomorphism and $\sigma := (\pi|_B)^{-1}$ is the corresponding local section. Consider the map $\mathtt{y} = \pi_2 \circ \mathtt{x} \circ \sigma$ where π_2 is the second factor projection $\pi_2 : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$. This is a homeomorphism since $(\pi_2 \circ \mathtt{x})|_B$ is a homeomorphsim and $\pi_2 \circ \mathtt{x} \circ \sigma = (\pi_2 \circ \mathtt{x})|_B \circ \sigma$. We now have chart $(V, \mathtt{y})$.

Given two such charts $(V, \mathtt{y})$ and $(\bar{V}, \bar{\mathtt{y}})$ we must show that $\bar{\mathtt{y}}^{-1} \circ \mathtt{y}^{-1}$ is smooth. The $(V, \mathtt{y})$ and $(\bar{V}, \bar{\mathtt{y}})$ are constructed from associated action adapted charts $(U, \mathtt{x})$ and $(\bar{U}, \bar{\mathtt{x}})$ on M. Let $q \in V \cap \bar{V}$. As in the proof of Lemma 4.7 we may find diffeormorphsims ψ and $\bar{\psi}$ so that $(U, \psi \circ \mathtt{x})$ and $(\bar{U}, \bar{\psi} \circ \bar{\mathtt{x}})$ are action adapted charts centered at points $p_1 \in \pi^{-1}(q)$ and $p_2 \in \pi^{-1}(q)$ respectively. Corresponding to this modifications the charts $(V, \mathtt{y})$ and $(\bar{V}, \bar{\mathtt{y}})$ are modified to charts $(V, \mathtt{y}_\psi)$ and $(\bar{V}, \bar{\mathtt{y}}_{\bar{\psi}})$ centered at q where

$$\mathtt{y}_\psi := \pi_2 \circ \psi \circ \mathtt{x} \circ \sigma$$
$$\bar{\mathtt{y}}_{\bar{\psi}} := \pi_2 \circ \bar{\psi} \circ \bar{\mathtt{x}} \circ \sigma$$

One checks $\mathtt{y}_\psi \circ \mathtt{y}^{-1} = \pi_2 \circ \psi$ and similarly for $\bar{\mathtt{y}}_{\bar{\psi}} \circ \bar{\mathtt{y}}^{-1}$. From this it follows that the overlap map $\bar{\mathtt{y}}_{\bar{\psi}}^{-1} \circ \mathtt{y}_\psi$ will be smooth if and only if $\bar{\mathtt{y}}^{-1} \circ \mathtt{y}^{-1}$ is smooth. Thus we have reduced to the case where $(U, \mathtt{x})$ and $(\bar{U}, \bar{\mathtt{x}})$ are centered at $p_1 \in \pi^{-1}(q)$ and $p_2 \in \pi^{-1}(q)$ respectively. This entails that both $(V, \mathtt{y})$ and $(\bar{V}, \bar{\mathtt{y}})$ are centered at $q \in V \cap \bar{V}$. Now if we choose a $g \in G$ such that $l_g(p_1) = p_2$ then by composing with the diffeomorphism l_g we can reduce further to the case where $p_1 = p_2$. Here we use the fact that l_g takes the set of orbits to the set of orbits in a bijective manner and the special nature of our adapted charts with respect to these orbits. In this case the overlap map $\bar{\mathtt{x}} \circ \mathtt{x}^{-1}$ must have the form $(a, b) \mapsto (f(a, b), g(b))$ for some smooth functions f and g. It follows that $\bar{\mathtt{y}}^{-1} \circ \mathtt{y}^{-1}$ has the form $b \mapsto g(b)$. ■

Similar results hold for right actions. In fact, some of the most important examples of proper actions are usually presented as right actions (the right action associated to a principal bundle). In fact, we shall see situations where there is both a right and a left action in play.

Example 4.27 *Consider S^{2n-1} as the subset of $\mathbb{C}^n$ given by $S^{2n-1} = \{\xi \in \mathbb{C}^n : |\xi| = 1\}$. Here $\xi = (z^1, ..., z^n)$ and $|\xi| = \sum \bar{z}^i z^i$. Now we let S^1 act on S^{2n-1} by $(a, \xi) \longmapsto a\xi = (az^1, ..., az^n)$. This action is free and proper. The quotient is the complex projective space $\mathbb{C}P^{n-1}$.*

$$\begin{array}{c} S^{2n-1} \\ \downarrow \\ \mathbb{C}P^{n-1} \end{array}$$

These maps (one for each n) are called the Hopf maps. In this context S^1 is usually denoted by $U(1)$.
In the sequel we will be considering the similar right action $S^n \times U(1) \to S^n$. In this case we think of $\mathbb{C}^{n+1}$ as consisting of column vector and the action is given by $(\xi, a) \longmapsto \xi a$. Of course, since $U(1)$ is abelian this makes essentially no difference but in the next example we consider the quaternionic analogue where keeping track of order is important.

The quaternionic projective $\mathbb{H}P^{n-1}$ space is defined by analogy with $\mathbb{C}P^{n-1}$. The elements of $\mathbb{H}P^{n-1}$ as 1-dimensional subspaces of the the right $\mathbb{H}$-vector space $\mathbb{H}^n$. Lets use called these $\mathbb{H}$-lines for clarity. Each of these are of real dimension 4. Each element of $\mathbb{H}^n \backslash \{0\}$ determines an $\mathbb{H}$-line and the $\mathbb{H}$-line determined by $(\xi^1, ..., \xi^n)^t$ will be the same as that determined $(\widetilde{\xi}^1, ..., \widetilde{\xi}^n)^t$ if and only if there is a nonzero element $a \in \mathbb{H}$ so that $(\widetilde{\xi}^1, ..., \widetilde{\xi}^n)^t = (\xi^1, ..., \xi^n)^t a =$

$(\xi^1 a, ..., \xi^n a)^t$. This defines an equivalence relation $\sim$ on $\mathbb{H}^n \backslash \{0\}$ and thus we may also think of $\mathbb{H}P^{n-1}$ as $(\mathbb{H}^n \backslash \{0\}) / \sim$. The element of $\mathbb{H}P^{n-1}$ determined by $(\xi^1, ..., \xi^n)^t$ is denoted by $[\xi^1, ..., \xi^n]$. Notice that the subset $\{\xi \in \mathbb{H}^n : |\xi| = 1\}$ is S^{4n-1}. Just as for the complex projective spaces we observe that all such $\mathbb{H}$-lines contain points of S^{4n-1} and two points $\xi, \zeta \in S^{4n-1}$ determine the same $\mathbb{H}$-line if and only if $\xi = \zeta a$ for some a with $|a| = 1$. Thus we can think of $\mathbb{H}P^{n-1}$ as a quotient of S^{4n-1}. When viewed in this way, we also denote the equivalence class of $\xi = (\xi^1, ..., \xi^n)^t \in S^{4n-1}$ by $[\xi] = [\xi^1, ..., \xi^n]$. The equivalence classes are clearly the orbits of an action as described in the following example.

Example 4.28 *Consider S^{4n-1} considered as the subset of $\mathbb{H}^n$ given by $S^{4n-1} = \{\xi \in \mathbb{H}^n : |\xi| = 1\}$. Here $\xi = (\xi^1, ..., \xi^n)^t$ and $|\xi| = \sum \bar{\xi}^i \xi^i$. Now we define a* right *action of $U(1, \mathbb{H})$ on S^{4n-1} by $(\xi, a) \longmapsto \xi a = (\xi^1 a, ..., \xi^n a)^t$. This action is free and proper. The quotient is the quaternionic projective space $\mathbb{H}P^{n-1}$ and we have the quotient map denoted by $\wp$*

$$\begin{array}{c} S^{4n-1} \\ \wp \downarrow \\ \mathbb{H}P^{n-1} \end{array}$$

This map is also referred to as an Hopf map. Recall that $\mathbb{Z}_2 = \{1, -1\}$ acts on $S^{n-1} = \mathbb{R}^n$ on the right (or left) by multiplication and the action is a (discrete) proper and free action with quotient $\mathbb{R}P^{n-1}$ and so the above two examples generalize this.

For completeness we describe an atlas for $\mathbb{H}P^{n-1}$. View $\mathbb{H}P^{n-1}$ as the quotient $S^{4n-1}/\sim$ described above. Let

$$U_k := \{[\xi] \in S^{4n-1} \subset \mathbb{H}^n : \xi^k \neq 0\}$$

and define $\varphi_k : U_k \to \mathbb{H}^{n-1} \cong \mathbb{R}^{4n-1}$ by

$$\varphi_k([\xi]) = (\varphi_i([\xi_1, ..., \xi_n]) = (\xi_1 \xi_1^{-1}, ..., \widehat{1}, ..., \xi_n \xi_n^{-1})$$

where as for the real and complex cases the caret symbol $\hat{}$ indicates that we have omitted the 1 in the i-th slot so as to obtain an element of $\mathbb{H}^{n-1}$. Notice that we insist that the ξ_i^{-1} in this expression multiply from the right. The general pattern for the overlap maps become clear from the example $\varphi_3 \circ \varphi_2^{-1}$. Here have

$$\begin{aligned} \varphi_3 \circ \varphi_2^{-1}(y_1, y_3, ..., y_n) &= \varphi_3([y_1, 1, y_3, ..., y_n]) \\ &= \left(y_1 y_3^{-1}, y_3^{-1}, y_4 y_3^{-1}, ..., y_n y_3^{-1}\right) \end{aligned}$$

In the special case $n = 1$, we have an atlas of just two charts $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ and in close analogy with the complex case we have $U_1 \cap U_2 = \mathbb{H} \backslash \{0\}$ and $\varphi_1 \circ \varphi_2^{-1}(y) = \bar{y}^{-1} = \varphi_2 \circ \varphi_1^{-1}(y)$ for $y \in \mathbb{H} \backslash \{0\}$.

Exercise 4.22 *Show that by identifying $\mathbb{H}$ with $\mathbb{R}^4$ and modifying the stereographic charts on $S^3 \subset \mathbb{R}^4$ we can obtain an atlas for S^3 with overlap maps of the same form as for $\mathbb{H}P^1$ given above. Use this to show that $\mathbb{H}P^1 \cong S^3$.*

Combining the last exercise with previous results we have

$$\begin{aligned}\mathbb{R}P^1 &\cong S^0 = \mathbb{Z}_2\\ \mathbb{C}P^1 &\cong S^1\\ \mathbb{H}P^1 &\cong S^3\end{aligned}$$

4.9 Homogeneous Spaces

Let H be a closed Lie subgroup of a Lie group G. The we have a *right* action of H on G given by right multiplication $r : G \times H \to G$. The orbits of this right action are exactly the left cosets of the quotient G/H. The action is clearly free and we would like to show that it is also proper. Since we are now talking about a right action and G is the manifold on which we are acting, we need to show that the map $P_{right} : G \times H \to G \times G$ given by $(p, h) \mapsto (p, ph)$ is a proper map. The characterization of proper action becomes

$$H_K := \{h \in H : (K \cdot h) \cap K \neq \emptyset\}$$

is compact whenever K is compact. To this end let K be any compact subset of G. It will suffice to show that H_K is sequentially compact and to this end let $\{h_i\}_{i\in\mathbb{Z}_+}$ be a sequence in H_K. Then there must be a sequences $\{a_i\}$ and $\{b_i\}$ in K such that $a_i h_i = b_i$. Since K is compact and hence sequentially compact, we can pass to subsequences $\{a_{i(j)}\}_{j\in\mathbb{Z}_+}$ and $\{b_{i(j)}\}_{j\in\mathbb{Z}_+}$ so that $\lim_{j\to\infty} a_{i(j)} = a$ and $\lim_{j\to\infty} b_{i(j)} = b$. Here $i \mapsto i(j)$ is a monotonic map on positive integers; $\mathbb{Z}_+ \to \mathbb{Z}_+$. This means that $\lim_{j\to\infty} h_{i(j)} = \lim_{j\to\infty} a_{i(j)}^{-1} b_{i(j)} = a^{-1}b$. Thus the original sequence $\{h_i\}$ is shown to have a convergent subsequence and we conclude that the right action is proper. Using Theorem 4.8 (or its analogue for right actions) we obtain

Proposition 4.17 *Let H be a closed Lie subgroup of a Lie group G then*
i) the right action $G \times H \to G$ is free and proper
ii) the orbit space is the left coset space G/H and this has a unique smooth manifold structure such that the quotient map $\pi : G \to G/H$ is a surjection. Furthermore, $\dim(G/H) = \dim(G) - \dim(H)$.

If K is a normal Lie subgroup of G then the quotient is a group with multiplication given by $[g_1][g_2] = (g_1K)(g_2K) = g_1g_2K$. The normality of K is what makes this definition well defined. In this case, we may ask whether G/K is a Lie group. If K is closed then we know from the considerations above that G/K is a smooth manifold and that the quotient map is smooth. In fact, we have the following

Proposition 4.18 (Quotient Lie Groups) *If K is closed normal subgroup of a Lie group G then G/K is a Lie group and the quotient map $G \to G/K$ is a Lie group homomorphism. Furthermore, if $h : G \to H$ is a surjective Lie group homomorphism then* $\ker(h)$ *is a closed normal subgroup and the induced map $\widetilde{h} : G/\ker(h) \to H$ is a Lie group isomorphism.*

Proof. We have already observed that G/K is a smooth manifold and that the quotient map is smooth. After taking into account what we know from standard group theory the only this we need to prove for the first part is that the multiplication and inversion in the quotient are smooth. it is an easy exercise using corollary 3.3 to show that both of these maps are smooth.

Consider a Lie group homomorphism h as in the hypothesis of the proposition. It is standard that $\ker(h)$ is a normal subgroup and it is clearly closed. It is also easy to verify fact that the induced $\widetilde{h}$ map is an isomorphism. One can then use Corollary 3.3 to show that the induced map $\widetilde{h}$ is smooth. ∎

If a group G acts transitively on M (on the right or left) then M is called a **homogeneous space** with respect to that action. Of course it is possible that a single group G may act on M in more than one way and so M may be a homogeneous space in more than one way. We will give a few concrete examples shortly but we already have an abstract example on hand.

Theorem 4.9 *If H is a closed Lie subgroup of a Lie group G then the map $G \times G/H \to G$ given by $l : (g, g_1H) \to gg_1H$ is a transitive Lie group action. Thus G/H is a homogeneous space with respect to this action.*

Proof. The fact that l is well defined follows since if $g_1H = g_2H$ $g_2^{-1}g_1 \in H$ and so $gg_2H = gg_2g_2^{-1}g_1H = gg_1H$. We already know that G/H is a smooth manifold and $\pi : G \to G/H$ is a surjective submersion. We can form another submersion $id_G \times \pi : G \times G \to G \times G/H$ making the following diagram commute:

$$\begin{tikzcd} G \times G \arrow[r] \arrow[d, "id_G \times \pi"'] \arrow[dr] & G \arrow[d, "\pi"] \\ G \times G/H \arrow[r] & G/H \end{tikzcd}$$

Here the upper horizontal map is group multiplication and the lower horizontal map is the action l. Since the diagonal map is smooth, it follows from Proposition 3.5 that l is smooth. We see that l is transitive by observing that if $g_1H, g_2H \in G/H$ then

$$l_{g_2 g_1^{-1}}(g_1H) = g_2H$$

∎

It turns out that up to appropriate equivalence, the examples of the above type account for all homogeneous spaces. Before proving this let us look at some concrete examples.

Example 4.29 *Let $M = \mathbb{R}^n$ and $G = Euc(n, \mathbb{R})$ the group of Euclidean motions. We realize $Euc(n, \mathbb{R})$ as a matrix group*

$$Euc(n, \mathbb{R}) = \left\{ \begin{bmatrix} 1 & 0 \\ v & Q \end{bmatrix} : v \in \mathbb{R}^n \text{ and } Q \in O(n) \right\}$$

The action of $Euc(n, \mathbb{R})$ on $\mathbb{R}^n$ is given by the rule

$$\begin{bmatrix} 1 & 0 \\ v & Q \end{bmatrix} \cdot x = Qx + v$$

where x is written as a column vector. Notice that this action is not given by a matrix multiplication but one can use the trick of representing the points x of $\mathbb{R}^n$ by the $(n+1)\times 1$ column vectors $\begin{bmatrix} 1 \\ x \end{bmatrix}$ and then we have $\begin{bmatrix} 1 & 0 \\ v & Q \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ Qx+v \end{bmatrix}$. The action is easily seen to be transitive.

Example 4.30 *As in the previous example we take $M = \mathbb{R}^n$ but this time the group acting is the affine group $Aff(n,\mathbb{R})$ which we realize as a matrix group:*

$$Aff(n,\mathbb{R}) = \left\{ \begin{bmatrix} 1 & 0 \\ v & A \end{bmatrix} : v \in \mathbb{R}^n \text{ and } A \in GL(n,\mathbb{R}) \right\}$$

The action is

$$\begin{bmatrix} 1 & 0 \\ v & A \end{bmatrix} \cdot x = Ax + v$$

and this is again a transitive action.

Comparing these first two examples we see that we have made $\mathbb{R}^n$ into a homogeneous space in two different ways. It is sometime desirable to give different names and/or notations for $\mathbb{R}^n$ to distinguish how we are acting on the space. In the first example we might denote $\mathbb{E}^n$ (Euclidean space) and in the second case by $\mathbb{A}^n$ and refer to it as affine space. Note that, roughly speaking the action by $Euc(n,\mathbb{R})$ preserves all metric properties of figures such as curves defined in $\mathbb{E}^n$. On the other hand, $Aff(n,\mathbb{R})$ always sends lines to lines, planes to planes etc.

Example 4.31 *Let $M = \mathrm{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. This is the upper half complex plane. The group acting on H will be $Sl(2,\mathbb{R})$ and the action is given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

This action is transitive.

Example 4.32 *We have already seen in Example 4.20 that both $O(n)$ and $SO(n)$ act transitively on the sphere $S^{n-1} \subset \mathbb{R}^n$ so S^{n-1} is a homogeneous space in at least two (slightly) different ways. Also, both $SU(n)$ and $U(n)$ act transitively $S^{2n-1} \subset \mathbb{C}^n$.*

Example 4.33 *Let $V'_{n,k}$ denote the set of all k-frames for $\mathbb{R}^n$ where by a k-frame we mean an ordered set of k linearly independent vectors. Thus an $n-$frame is just an ordered basis for $\mathbb{R}^n$. This set can easily be given a smooth manifold structure. This manifold is called the (real)* ***Stiefel manifold of k-frames****. The Lie group $GL(n,\mathbb{R})$ acts (smoothly) on $V'_{n,k}$ by $g \cdot (e_1, ..., e_k) = (ge_1, ..., ge_k)$. To see that this action is transitive let $(e_1, ..., e_k)$ and $(f_1, ..., f_k)$ be two k-frames. Extend each to $n-$frames $(e_1, ..., e_k, ..., e_n)$ and $(f_1, ..., f_k, ..., f_n)$ since we consider elements of $\mathbb{R}^n$ as column vectors these two $n-$frames can be viewed as invertible $n \times n$ matrices E and F. The if we let $g := EF^{-1}$ then $gE = F$ which entails $g \cdot (e_1, ..., e_k) = (ge_1, ..., ge_k) = (f_1, ..., f_k)$.*

Example 4.34 *Let $V_{n,k}$ denote the set of all orthonormal k-frames for $\mathbb{R}^n$ where by an orthonormal k-frame we mean an ordered set of k orthonormal vectors. Thus an orthonormal $n-$frame is just an orthonormal basis for $\mathbb{R}^n$. This set can easily be given a smooth manifold structure and is called the* ***Stiefel manifold of orthonormal k-frames****. The group $O(n, \mathbb{R})$ act transitively on $V_{n,k}$ for reasons similar to those given in the last example.*

Theorem 4.10 *Let M be a homogeneous space via the transitive action $l : G \times M \to M$ and let G_p the isotropy subgroup of a point $p \in M$. Recall that G acts on G/G_p. If G/G_p is second countable (in particular if G is second countable) then there is an equivariant diffeomorphism $\phi : G/G_p \to M$ such that $\phi(gG_p) = g \cdot p$.*

Proof. We want to define ϕ by the rule $\phi(gG_p) = g \cdot p$ but we must show that this is well defined. This is a standard group theory argument; if $g_1 G_p = g_2 G_p$ then $g_1^{-1} g_2 \in G_p$ so that $\left(g_1^{-1} g_2\right) \cdot p = p$ or $g_1 \cdot p = g_2 \cdot p$. This map is a surjective by the transitivity of the action l. It is also injective since if $\phi(g_1 G_p) = \phi(g_2 G_p)$ then $g_1 \cdot p = g_2 \cdot p$ or $\left(g_1^{-1} g_2\right) \cdot p$ which by definition means that $g_1^{-1} g_2 \in G_p$ and then $g_1 G_p = g_2 G_p$. Notice that the following diagram commutes:

$$\begin{tikzcd} G \arrow[d] \arrow[dr, "\theta_p"] & \\ G/G_p \arrow[r, "\phi"] & M \end{tikzcd}$$

From Corollary 3.3 we see that ϕ is smooth.

To show that ϕ is a diffeomorphism it suffices to show that the rank of ϕ is equal to $\dim M$ or in other words that ϕ is a submersion. Since $\phi(gg_1 G_p) = (gg_1) \cdot p = g\phi(g_1 G_p)$ the map ϕ is equivariant and so has constant rank. By Lemma 3.2 ϕ is a submersion and hence in the present case a diffeomorphism. ∎

Without the technical assumption on second countability, the proof shows that we still have that $\phi : G/G_p \to M$ is a smooth equivariant bijection.

Exercise 4.23 *Show that if instead of the hypothesis of second countability in the last theorem we assume instead that θ_p has full rank at the identity then $\phi : G/G_p \to M$ is a diffeomorphism.*

We now look again at some of our examples of homogeneous spaces and apply this theorem.

Example 4.35 *Consider again Example 4.29. The isotropy group of the origin in $\mathbb{R}^n$ is the subgroup consisting of matrices of the form*

$$\begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$$

where $Q \in O(n)$. This group is clearly isomorphic to $O(n, \mathbb{R})$ and so by the above theorem we have an equivariant diffeomorphism

$$\mathbb{R}^n \cong \frac{Euc(n, \mathbb{R})}{O(n, \mathbb{R})}$$

Example 4.36 *Consider again Example 4.30. The isotropy group of the origin in $\mathbb{R}^n$ is the subgroup consisting of matrices of the form*

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

where $A \in Gl(n, \mathbb{R})$. This group is clearly isomorphic to $Gl(n, \mathbb{R})$ and so by the above theorem we have an equivariant diffeomorphism

$$\mathbb{R}^n \cong \frac{Aff(n, \mathbb{R})}{Gl(n, \mathbb{R})}$$

It is important to realize that there is an implied action on $\mathbb{R}^n$ which is different from that in the previous example.

Example 4.37 *Now consider the action of $Sl(2, \mathbb{R})$ on the upper half complex plane as in Example 4.31. Let us determine the isotropy subgroup for the point $i = \sqrt{-1}$. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in this subgroup then*

$$\frac{ai + b}{ci + d} = i$$

so that $bc - ad = 1$ and $bd + ac = 0$. Thus $A \in SO(2, \mathbb{R})$ (which is isomorphic as a Lie group to the circle $S^1 = U(1, \mathbb{C})$). Thus we have an equivariant diffeomorphism

$$H = \mathbb{C}_+ \cong \frac{Sl(2, \mathbb{R})}{SO(2, \mathbb{R})}$$

Example 4.38 *From example 4.32 we obtain*

$$S^{n-1} \cong \frac{O(n)}{O(n-1)}$$
$$S^{n-1} \cong \frac{SO(n)}{SO(n-1)}$$
$$S^{2n-1} \cong \frac{U(n)}{U(n-1)}$$
$$S^{2n-1} \cong \frac{SU(n)}{SU(n-1)}$$

Example 4.39 *Let $(\mathbf{e}_1, ..., \mathbf{e}_n)$ be the standard basis for $\mathbb{R}^n$. Under the action of $GL(n, \mathbb{R})$ on $V'_{n,k}$ given in Example 4.33, the isotropy group of the point which is the k-plane $\mathbf{e} = (\mathbf{e}_{k+1}, ..., \mathbf{e}_n)$ is the subgroup of $Gl(n, \mathbb{R})$ of the form*

$$\begin{pmatrix} A & 0 \\ 0 & id \end{pmatrix} \text{ for } A \in Gl(n-k, \mathbb{R})$$

We identify this group with $Gl(n-k, \mathbb{R})$ and then we obtain

$$V'_{n,k} \cong \frac{Gl(n, \mathbb{R})}{Gl(n-k, \mathbb{R})}$$

Example 4.40 *A similar analysis leads to an equivariant diffeomorphism*

$$V_{n,k} \cong \frac{O(n, \mathbb{R})}{O(n-k, \mathbb{R})}$$

where $V_{n,k}$ is the Stiefel manifold of orthonormal *k-planes of Example 4.34. Notice that taking $k = 1$ we recover Example 4.35.*

Exercise 4.24 *Show that if $k < n$ then we have $V_{n,k} \cong \frac{SO(n,\mathbb{R})}{SO(n-k,\mathbb{R})}$.*

Next we introduce a couple of standard results concerning connectivity.

Proposition 4.19 *Let G be a Lie group acting freely and properly on a smooth manifold M. Let the action be a left (reps. right) action. If both G and $M\backslash G$ (resp. M/G) are connected then M is connected.*

Proof. Assume for concreteness that the action is a left action and that G and $M\backslash G$. Suppose by way of contradiction that M is not connected. Then there are disjoint open set U and V whose union is M. Each orbit $G \cdot p$ is the image of the connected space G under the orbit map $g \mapsto g{\cdot}p$ and so is connected. this means that each orbit must be contained in one and only one of U and V. Now since the quotient map π is an open map, $\pi(U)$ and $\pi(V)$ are open from what we have just observe they must be disjoint and $\pi(U) \cup \pi(V) = M\backslash G$. This contradict the assumption that $M\backslash G$ is connected. ■

Corollary 4.5 *Let H be a closed Lie subgroup of G. Then if both H and G/H are connected then G is connected.*

Corollary 4.6 *For each $n \geq 1$ then groups $SO(n)$, $SU(n)$ and $U(n)$ are connected while the group $O(n)$ has exactly two components.*

Proof. $SO(1)$ and $SU(1)$ are both connected since they each contain only one element. $U(1)$ is the circle and so it too is connected. We use induction. Suppose that $SO(k)$, $SU(k)$ are connected for $1 \leq k \leq n-1$. We show that this implies that $SO(n)$, $SU(n)$ and $U(n)$ are connected. From example 4.38 we know that $S^{n-1} = SO(n)/SO(n-1)$. Since S^{n-1} and $SO(n-1)$ are connected (the second one by the induction hypothesis) we see that $SO(n)$ is connected. The same argument works for $SU(n)$ and $U(n)$.

Every element of $O(n)$ has either determinant 1 or -1. The subset $SO(n) \subset O(n)$ since it is exactly $\{g \in O(n) : \det g \neq 1\}$. If we fix an element a_0 with $\det a_0 = -1$ then $SO(n)$ and $a_0 SO(n)$ are disjoint, open and both connected since $g \mapsto a_0 g$ is a diffeomorphism which maps the first to the second. It is easy to show that $SO(n) \cup a_0 SO(n) = O(n)$. ∎

We close this chapter by relating the notion of a Lie group action with that of a Lie group representation. We given just a few basic definitions, some of which will be used in the next chapter.

Definition 4.20 *A representation of a Lie group G in a finite dimensional vector space* V *is a left Lie group action $\lambda : G \times \mathrm{V} \to \mathrm{V}$ such that for each $g \in G$ the the map $\lambda_g : v \mapsto \lambda(g, v)$ is linear. Thus a representation is a linear action.*

The map $G \to Gl(\mathrm{V})$ given by $g \mapsto \lambda(g) := \lambda_g$ is a Lie group homomorphism and will be denoted by the same letter λ as the action so that $\lambda(g)v := \lambda(g, v)$. In fact, given a Lie group homomorphism $\lambda : G \to Gl(\mathrm{V})$ we obtain a linear action by letting $\lambda(g, v) := \lambda(g)v$. Thus a representation is basically the same thing as a Lie group homomorphism into $Gl(\mathrm{V})$ and is often defined as such. The kernel of the action is the kernel of the associated homomorphism. A **faithful** representation is one that acts effectively and this means that the associated homomorphism has trivial kernel.

Exercise 4.25 *Show that if $\lambda : G \times \mathrm{V} \to \mathrm{V}$ is a map such that $\lambda_g : v \mapsto \lambda(g, v)$ is linear for all g, then λ is smooth if and only if $\lambda_g : G \to Gl(\mathrm{V})$ is smooth for every $g \in G$. (Assume* V *is finite dimensional as usual).*

We have already seen one important example of a Lie group representation. Namely, the adjoint representation. The adjoint representation came from first considering the action of G on itself given by conjugation which leaves the identity element fixed. The idea can be generalized:

Theorem 4.11 *Let $l : G \times M \to M$ be a (left) Lie group action. Suppose that $p_0 \in M$ is a fixed point of the action ($l_g(p_0) = p_0$ for all g). The map*

$$l_{(p_0)} : G \to Gl(T_{p_0} M)$$

given by

$$l_{(p_0)}(g) := T_{p_0} l_g$$

is a Lie group representation.

Proof. Since

$$\begin{aligned} l_{(p_0)}(g_1 g_2) &= T_{p_0}(l_{g_1 g_2}) = T_{p_0}(l_{g_1} \circ l_{g_2}) \\ &= T_{p_0} l_{g_1} \circ T_{p_0} l_{g_2} = l_{(p_0)}(g_1) l_{(p_0)}(g_2) \end{aligned}$$

we see that $l_{(p_0)}$ is a homomorphism. We must show that l' is smooth. By Exercise 4.25 this implies that the map $G \times T_{p_0}M \to T_{p_0}M$ given by $(g, v) \mapsto T_{p_0}l_g \cdot v$ is smooth. It will be enough to show that $g \mapsto \alpha(T_{p_0}l_g \cdot v)$ is smooth for any $v \in T_{p_0}M$ and any $\alpha \in T^*_{p_0}M$. This will follow if we can show that for fixed $v_0 \in T_{p_0}M$, the map $G \to TM$ given by $g \mapsto T_{p_0}l_g \cdot v_0$ is smooth. This map is a composition

$$G \to TG \times TM \cong T(G \times M) \stackrel{Tl}{\to} TM$$

where the first map is $g \mapsto (0_g, v_0)$ which is clearly smooth. ■

Definition 4.21 *For a Lie group action $l : G \times M \to M$ with fixed point p_0 the representation $l_{(p_0)}$ from the last theorem is called the **isotropy representation** for the fixed point.*

Suppose that V is an $\mathbb{F}$-vector space let $\mathcal{B} = \{v_1, ..., v_n\}$ be a basis for V. Then denote the matrix representative of λ_g with respect to $\mathcal{B}$ by $[\lambda_g]_{\mathcal{B}}$ we obtain a homomorphism $G \to Gl(n, \mathbb{F})$ given by $g \mapsto [\lambda_g]_{\mathcal{B}}$. In general, a Lie group homomorphism of a Lie group G into $Gl(n, \mathbb{F})$ is called a matrix representation of G. Notice that any Lie subgroup G of $Gl(\mathrm{V})$ acts on V in the obvious was simply by employing the definition of $Gl(\mathrm{V})$ as a set of linear transformation of V. We call this the **standard action** of linear Lie subgroup of $Gl(\mathrm{V})$ on V and the corresponding homomorphism is just the inclusion map $G \hookrightarrow Gl(n, \mathbb{F})$. Choosing a basis, the subgroup corresponds to a matrix group and the standard action becomes matrix multiplication on the left of $\mathbb{F}^n$ where the later is viewed as a space of column vectors. This action of a matrix group on column vectors is also referred to as a standard action.

Given a representation λ of G in a vector space V we have a representation λ^* of G in the dual space V^* by defining $\lambda^*(g) := \lambda(g^{-1})^t : \mathrm{V}^* \to \mathrm{V}^*$. Here we have by definition $\langle \lambda(g^{-1})^t v, w \rangle = \langle \lambda(g^{-1}) v, w \rangle$ for all $v, w \in \mathrm{V}$ and where $\langle .,. \rangle : \mathrm{V}^* \times \mathrm{V} \to \mathbb{F}$ is the natural bilinear pairing which defines the dual. This dual representation is also sometimes called the contragradient representation (especially when $\mathbb{F} = \mathbb{R}$).

Now let λ^{V} and λ^{W} be representations of a lie group G in $\mathbb{F}$-vector spaces V and W respectively. We can then form the direct product representation $\lambda^{\mathrm{V}} \oplus \lambda^{\mathrm{W}}$ by $\left(\lambda^{\mathrm{V}} \oplus \lambda^{\mathrm{W}}\right)_g := \lambda^{\mathrm{V}}_g \oplus \lambda^{\mathrm{W}}_g$ for $g \in G$ and where we have $\left(\lambda^{\mathrm{V}}_g \oplus \lambda^{\mathrm{W}}_g\right)(v, w) = (\lambda^{\mathrm{V}}_g v, \lambda^{\mathrm{W}} w)$.

One can also form the tensor product of representations. The definitions and basic facts about tensor products are given in the more general context of module theory in Appendix D. Here we given a quick recounting of the notion of a tensor product of vector spaces and then we defined tensor products of representations. Given to vector spaces V_1 and V_2 over some field $\mathbb{F}$. Consider the space $\mathcal{C}_{\mathrm{V}_1 \times \mathrm{V}_2}$ consisting of all bilinear maps $\mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{W}$ where W varies over all $\mathbb{F}$-vector spaces but V_1 and V_2 are fixed. A morphism from, say $\mu_1 : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{W}_1$ to $\mu_2 : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{W}_2$ is defined to be a map $\ell : \mathrm{W}_1 \to \mathrm{W}_2$ such that the diagram

$$
\begin{tikzcd}
 & \mathrm{W}_1 \arrow[dd, "\ell"] \\
\mathrm{V}_1 \times \mathrm{V}_2 \arrow[ur, "\mu_1"] \arrow[dr, "\mu_2"'] & \\
 & \mathrm{W}_2
\end{tikzcd}
$$

commutes.

That there exists vector space $\mathrm{T}_{\mathrm{V}_1,\mathrm{V}_2}$ together with a bilinear map $\otimes : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{T}_{\mathrm{V}_1,\mathrm{V}_2}$ that has the following universal property: For every bilinear map $\mu : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{W}$ there is a unique linear map $\widetilde{\mu} : \mathrm{T}_{\mathrm{V}_1,\mathrm{V}_2} \to \mathrm{W}$ such that the following diagram commutes:

$$
\begin{tikzcd}
V_1 \times \mathrm{V}_2 \arrow[r, "\mu"] \arrow[d, "\otimes"'] & \mathrm{W} \\
T_{\mathrm{V}_1,\mathrm{V}_2} \arrow[ur, "\tilde{\mu}"'] &
\end{tikzcd}
$$

If such a pair $(\mathrm{T}_{\mathrm{V}_1,\mathrm{V}_2}, \otimes)$ exists with this property then it is unique up to isomorphism in $\mathcal{C}_{\mathrm{V}_1 \times \mathrm{V}_2}$. In other words, if $\widehat{\otimes} : \mathrm{V}_1 \times \mathrm{V}_2 \to \widehat{\mathrm{T}}_{\mathrm{V}_1,\mathrm{V}_2}$ is another object with this universal property then there is a linear isomorphism $\mathrm{T}_{\mathrm{V}_1,\mathrm{V}_2} \cong \widehat{\mathrm{T}}_{\mathrm{V}_1,\mathrm{V}_2}$ such that the following diagram commutes:

$$
\begin{tikzcd}
 & T_{\mathrm{V}_1,\mathrm{V}_2} \arrow[dd, "\cong"] \\
\mathrm{V}_1 \times \mathrm{V}_2 \arrow[ur, "\otimes"] \arrow[dr, "\widehat{\otimes}"'] & \\
 & \widehat{T}_{\mathrm{V}_1,\mathrm{V}_2}
\end{tikzcd}
$$

We refer to any such universal object as a tensor product of V_1 and V_2. We will indicate the construction of a specific tensor product that we denote by $\mathrm{V}_1 \otimes \mathrm{V}_2$ with corresponding map $\otimes : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{V}_1 \otimes \mathrm{V}_2$. The idea is simple: We let $\mathrm{V}_1 \otimes \mathrm{V}_2$ be the set of all linear combinations of symbols of the form $v_1 \otimes v_2$ for $v_1 \in \mathrm{V}_1$ and $v_2 \in \mathrm{V}_2$, subject to the relations

$$
\begin{aligned}
(v_1 + v_2) \otimes v &= v_1 \otimes v + v_2 \otimes v \\
v \otimes (v_1 + v_2) &= v \otimes v_1 + v \otimes v_2 \\
r\,(v_1 \otimes v_2) &= rv_1 \otimes v_2 = v_1 \otimes rv_2
\end{aligned}
$$

The map $\otimes$ is then simply $\otimes : (v_1, v_2) \to v_1 \otimes v_2$. A somewhat more pedantically, let $F(\mathrm{V}_1 \times \mathrm{V}_2)$ denote the free vector space generated by the *set* $\mathrm{V}_1 \times \mathrm{V}_k$ (the elements of $V_1 \times V_k$ are treated as a basis for the space and so the free space has dimension equal to the cardinality of the set $V_1 \times V_2$). Next we define an equivalence relation "$\sim$" $F(V_1 \times V_2)$ generated by the relations

$$\begin{aligned} (av_1, v_2) &\sim a(v_1, v_2) \\ (v_1, av_2) &\sim a(v_1, v_2) \end{aligned}$$

$$\begin{aligned} (v + w, v_2) &\sim (v, v_2) + (w, v_2) \\ (v_1, v + w) &\sim (v_1, v) + (v_1, w) \end{aligned}$$

for $v_1, v \in V_1,\ v_2, w \in V_2$ and $a \in \mathbb{F}$. Then we let $V_1 \otimes V_2 := F(V_1 \times V_k)/\sim$ and denote the equivalence class of (v_1, v_2) by $v_1 \otimes v_2$.

Tensor products of several vector spaces at a time are constructed similarly to be a universal space in a category of multilinear maps. We may also form the tensor products two at a time and then use the easily proved fact $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$ which is then denoted by $V \otimes W \otimes U$. Again the reader is referred to Appendix D for more about tensor products.

Elements of the form $v_1 \otimes v_2$ generate $V_1 \otimes V_2$ and in fact, if $\{e_1, ..., e_r\}$ is a basis for V_1 and $\{f_1, ..., f_s\}$ is a basis for V_2 then set

$$\{e_i \otimes f_j : 1 \leq i \leq r, 1 \leq j \leq s\}$$

is a basis for $V_1 \otimes V_2$ which therefore has dimension $rs = \dim V_1 \dim V_2$.

One more observation: If $A : V_1 \to W_1$ is linear and $B : V_2 \to W_2$ is linear then can define a linear map $A \otimes B : V_1 \otimes V_2 \to W_1 \otimes W_2$. It is enough to define $A \otimes B$ on element of the form $v_1 \otimes v_2$ and then extend linearly:

$$A \otimes B(v_1 \otimes v_2) = Av_1 \otimes Bv_2$$

Notice that if A and B are invertible then $A \otimes B$ is invertible with $(A \otimes B)^{-1}(v_1 \otimes v_2) = A^{-1}v_1 \otimes B^{-1}v_2$. Pick bases for V_1 and V_2 as above and bases $\{e'_1, ..., e'_r\}$ and $\{f'_1, ..., f'_s\}$ for W_1 and W_2 respectively. The for $\tau \in V_1 \otimes V_2$ we can write $\tau = \tau^{ij} e_i \otimes f_j$ using the Einstein summation convention. We have

$$\begin{aligned} A \otimes B(\tau) &= A \otimes B(\tau^{ij} e_i \otimes f_j) \\ &= \tau^{ij} Ae_i \otimes Bf_j \\ &= \tau^{ij} A_i^k e'_k \otimes B_j^l f'_l \\ &= \tau^{ij} A_i^k B_j^l \, (e'_k \otimes f'_l) \end{aligned}$$

so that the matrix of $A \otimes B$ is given by $(A \otimes B)_{ij}^{kl} = A_i^k B_j^l$.

Now let λ^{V} and λ^{W} be representations of a lie group G in $\mathbb{F}$-vector spaces V and W respectively. We can form a representation of G in the tensor product space by letting $\left(\lambda^{V} \otimes \lambda^{W}\right)_g := \lambda_g^{V} \otimes \lambda_g^{W}$ for all $g \in G$. There is a variation on the tensor product that is useful when we have two groups involved. If λ^{V} is a representation of a lie group G_1 in $\mathbb{F}$-vector space V and λ^{W} is representation of a lie group G_2 in the $\mathbb{F}$-vector space W, then we can form a representation of a the Lie group $G_1 \times G_2$ also called the tensor product representation and denoted $\lambda^{\mathrm{V}} \otimes \lambda^{\mathrm{W}}$ as before. In this case the definition is $\left(\lambda^{\mathrm{V}} \otimes \lambda^{\mathrm{W}}\right)_{(g_1,g_2)} := \lambda^{\mathrm{V}}_{g_1} \otimes \lambda^{\mathrm{W}}_{g_2}$. Of course if it happens that $G_1 = G_2$ then have an ambiguity since $\lambda^{\mathrm{V}} \otimes \lambda^{\mathrm{W}}$ could be a representation of G or of $G \times G$. One usually determines which version is meant from the context. Alternatively one can use pairs to denote actions so for example that an action $\lambda : G \times \mathrm{V} \to \mathrm{V}$ is denoted (G, λ). Then the two tensor product representations would be $(G \times G, \lambda^{\mathrm{V}} \otimes \lambda^{\mathrm{W}})$ and $(G, \lambda^{\mathrm{V}} \otimes \lambda^{\mathrm{W}})$ respectively.

4.10 Problems

1. Show that $Sl(2, \mathbb{C})$ is simply connected and that $\wp : Sl(2, \mathbb{C}) \to Mob$ is a universal covering homomorphism. See example 4.18.

2. Show that if we consider $Sl(2, \mathbb{R})$ as a subset of $Sl(2, \mathbb{C})$ in the obvious way then $Sl(2, \mathbb{R})$ is a Lie subgroup of $Sl(2, \mathbb{C})$ and $\wp(Sl(2, \mathbb{R}))$ is a Lie subgroup of Mob. Show that if $T \in \wp(Sl(2, \mathbb{R}))$ then T maps the upper half plane of $\mathbb{C}$ onto itself (bijectively).

3. Show that for $v \in T_eG$, the field defined by $g \mapsto L^v(g) := TL_g \cdot v$ is automatically smooth.

4. Determine explicitly the map $T_I \,\mathrm{inv} : T_I Gl(n, \mathbb{R}) \to T_I Gl(n, \mathbb{R})$ where $\mathrm{inv} : A \mapsto A^{-1}$.

5. Let H be the set of real 3×3 matrices of the form
$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}.$$
Find a global chart for H and show that this and the usual matrix multiplication gives H the structure of a Lie group.

6. If G is a connected Lie group and $h : G \to H$ is a Lie group homomorphism with discrete kernel K then $K \subset Z(G)$ where $Z(G) = \{x \in G : xg = gx$ for all $g \in G\}$ is the center of G.

7. Show that for a Lie group G, the conjugation map $C_g : G \to G$ defined by $x \mapsto gxg^{-1}$ is a Lie group isomorphism. Show that the map $C : g \to Diff(G)$ is a group homomorphism. Note that we have not defined any Lie group structure on $Diff(G)$.

8. Consider that map $T_eC_g : T_eG \to T_eG$. Show that $g \mapsto T_eC_g$ is a Lie group homomorphism from G into $GL(T_eG)$.

9. Let $A \in \mathfrak{gl}(\mathrm{V}) = L(\mathrm{V}, \mathrm{V})$ for some finite dimensional vector space V. Show that if A has eigenvalues $\{\lambda_i\}_{i=1,\ldots,n}$ then $\mathrm{ad}(A)$ has eigenvalues $\{\lambda_j - \lambda_k\}_{j,k=1,\ldots,n}$. Hint: Choose a basis for V such that A is represented by an unper triangular matrix. Show that this induces a basis for $\mathfrak{gl}(\mathrm{V})$ such that with the aapropriate ordering, $\mathrm{ad}(A)$ is upper triangular.

10. Fix a nonzero vector $w \in \mathbb{R}^3$ with length $\theta = \|w\|$. Let $L_w : \mathbb{R}^3 \to \mathbb{R}^3$ denote the linear transformation $v \mapsto w \times v$ where $\times$ is the cross product. Show that for any right handed orthonormal basis $\{e_1, e_2, e_3\}$ we have

$$\begin{aligned}
\exp(L_w)e_1 &= \cos\theta\, e_1 + \sin\theta\, e_2 \\
\exp(L_w)e_2 &= -\sin\theta\, e_1 + \cos\theta\, e_2 \\
\exp(L_w)e_3 &= e_3
\end{aligned}$$

11. Let L_w be as in the previous problem. Show that

$$\exp L_w = I + \frac{\sin\theta}{\theta} L_w + \frac{1-\cos\theta}{\theta^2} L_w^2$$

where $\frac{\sin\theta}{\theta}$ and $\frac{1-\cos\theta}{\theta^2}$ are defined in the obvious way using power series.

12. Let $A, B \in \mathfrak{gl}(\mathrm{V})$ where V is a finite dimensional vector space over the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and show that the following statments are equivalent:

a) $[A, B] = 0$.

b) $\exp sA$ and $\exp tB$ commute for all $s, t \in \mathbb{F}$.

c) $\exp(sA + tB) = \exp(sA)\exp(tB)$ for all $s, t \in \mathbb{F}$.

13. Let V be a finite dimensional normed space over the field $\mathbb{R}$ (resp. $\mathbb{C}$). Show that if $\sum_{n=0}^{\infty} a_n x^n$ is an absolutely convergent real (resp. complex) power series with radius of convergence R, then

$$\sum_{n=0}^{\infty} a_n A^n$$

converges (absolutely) in the normed space $\mathfrak{gl}(\mathrm{V})$ for $\|A\| < R$.

Chapter 5

Fiber Bundles

5.1 General Fiber Bundles

Definition 5.1 *Let F, M and E be C^r manifolds and let $\pi : E \to M$ be a C^r map. The quadruple (E, π, M, F) is called a (locally trivial) C^r* ***fiber bundle*** *if for each point $p \in M$ there is an open set U containing p and a C^r diffeomorphism $\phi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes:*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

In differential geometry attention is usually focused on C^∞ fiber bundles (smooth fiber bundles) but the continuous case is also of interest. We will restrict ourselves to the smooth case but the reader should realize that most of the definitions and theorems have analogous C^0 versions where the spaces are assumed merely to be Hausdorff topological spaces and the maps are only assumed to be continuous. The reader is asked to keep this in mind when reading through this chapter. From this point on, all maps and spaces will be smooth unless otherwise indicated.

Definition 5.2 *If (E, π, M, F) is a smooth fiber bundle then E is called the* ***total space****, π is called the* ***bundle projection****, M is called the* ***base space*** *and F is called the* ***typical fiber****. For each $p \in M$, the set $E_p := \pi^{-1}(p)$ is called the* ***fiber*** *over p.*

Because the quadruple notation is cumbersome, it is common to denote a fiber bundle by a single symbol. For example, we could write $\xi = (E, \pi, M, F)$. In the literature, it is common to see E refer both to the total space and to the fiber bundle itself (an abuse of notation). The map π is also a common way to reference the fiber bundle.

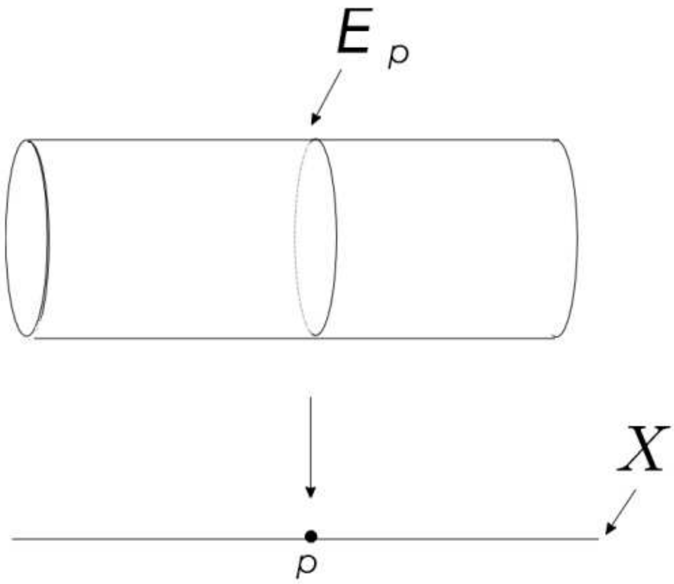


Figure 5.1: Circle bundle. Schematic for fiber bundle.

Example 5.1 *For smooth manifolds M and F we have the projections $pr_1 : M \times F \to M$ and $pr_2 : M \times F \to F$. Then $(M \times F, pr_1, M, F)$ is a fiber bundle called the trivial bundle.*

Exercise 5.1 *Show that if $\xi = (E, \pi, M, F)$ is a (smooth) fiber bundle then $\pi : E \to M$ is a submersion and each fiber $\pi^{-1}\{p\}$ is a regular submanifold which is diffeomorphic to F.*

Exercise 5.2 *Show that if $\xi = (E, \pi, M, F)$ is a fiber bundle and both F and M are connected then E is connected.*

Definition 5.3 *A (global)* ***smooth section*** *of a fiber bundle $\xi = (E, \pi, M, F)$ is a smooth map $\sigma : M \to E$ such that $\pi \circ \sigma = id_M$. A* ***local smooth section*** *over an open set U is a smooth map $\sigma : U \to E$ such that $\pi \circ \sigma = id_U$. The set of smooth sections of ξ is denoted $\Gamma(\xi)$ or sometimes by $\Gamma(E)$ or $\Gamma(\pi)$.*

A very important point is that a fiber bundle may not have any global smooth sections.

There are various categories of bundles with corresponding notions of morphism. We give two very general definitions and modify them as needed.

Definition 5.4 (Bundle Morphism (type I)) *Let $\xi_1 = (E_1, \pi_1, M, F_1)$ and $\xi_2 = (E_2, \pi_2, M, F_2$ be smooth fiber bundles with the same base space M. A*

bundle morphism over *M from ξ_1 to ξ_2 is a smooth map $h : E_1 \to E_2$ such that the following diagram commutes:*

$$\begin{tikzcd} E_1 \arrow[rr, "h"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd}$$

This type of morphism is also called an $M-$morphism or a morphism over M. If h is also a diffeomorphism, then h is called a bundle isomorphism over M and in this case the bundles are said to be isomorphic (over M) or ***equivalent****. A bundle isomorphism from a bundle to itself is called a* ***bundle automorphism****.*

Definition 5.5 (Bundle Morphism (type II)) *Let $\xi_1 = (E_1, \pi_1, M_1, F_1)$ and $\xi_2 = (E_2, \pi_2, M_2, F_2)$ be smooth fiber bundles. A bundle morphism from ξ_1 to*

ξ_2 is a pair of smooth maps $\widehat{f} : E_1 \to E_2$ and $f : M_1 \to M_2$ such that the following diagram commutes:

$$\begin{array}{ccc} E_1 & \xrightarrow{\widehat{f}} & E_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M_1 & \xrightarrow{f} & M_1 \end{array}$$

We write $(\widehat{f}, f) : \xi_1 \to \xi_2$. If both $\widehat{f}$ and f are diffeomorphisms then we call $(\widehat{f}, f)$ a bundle isomorphism. In this case we say the bundles are isomorphic over f and write $\xi_1 \cong \xi_2$.

Note that $\widehat{f}$ determines f and so it is also proper to refer to $\widehat{f}$ as the bundle morphism and we sometimes say that $\widehat{f}$ is a bundle morphism along (or over) f.

A bundle chart essentially gives a *local* bundle isomorphism which is of type I but it is sometimes more natural to consider charts which are local type II isomorphisms. We will call these type II bundle charts. These are of the form $(\phi,\mathtt{x})$ where $\phi : \pi^{-1}U \to V \times F$ and $\mathtt{x} : U \to V$ are smooth diffeomorphisms such that the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}U & \xrightarrow{\phi} & V \times F \\ \downarrow & & \downarrow \\ U & \xrightarrow{\mathtt{x}} & V \end{array}$$

Usually, the pair $(U,\mathtt{x})$ is a chart on the base manifold. The two types of bundle charts are equivalent since one may always compose a type II chart with $(\mathtt{x}^{-1}, id_F)$ to obtain a type I bundle chart.

More restricted notions of bundle morphism can be obtained by making requirements such as that the induced maps on fibers $\left.\widehat{f}\right|_{\pi_1^{-1}(p)} : \pi_1^{-1}(p) \to$

$\pi_2^{-1}(p)$ are smooth-diffeomorphisms. In this case we are restricting to a class of bundles whose typical fibers are all the same (or all diffeomorphic).

If two bundles are equivalent, via a bundle isomorphism h (of type I), then there is a natural bijection between the spaces of sections given by $\sigma \mapsto h \circ \sigma$ and this means that one quick way to conclude that two bundles are not equivalent is by showing that one bundle has global sections while the other does not.

The maps $\phi : \pi^{-1}(U) \to U \times F$ occurring in the definition are said to be **local trivializations** of the bundle. It is easy to see that such a local trivialization must be a map of the form $\phi = (\pi, \Phi)$ where $\Phi : \pi^{-1}(U) \to F$ is a smooth map with the property that $\Phi|_{E_p} : E_p \to F$ is a diffeomorphism. This second factor map Φ, is called the principal part of the local trivialization. A pair (U, ϕ) where ϕ is a local trivialization over $U \subset M$ is called a **bundle charts**. (Clearly, a local trivialization and a bundle chart are essentially the same thing). A family $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ of bundle charts such that the family $\{U_\alpha\}$ covers M is said to be a **bundle atlas** and the existence of such an atlas is enough to give the bundle structure. Given two such bundle charts (U_α, ϕ_α) and (U_β, ϕ_β) we have $\phi_\alpha = (\pi, \Phi_\alpha) : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ and $\phi_\beta = (\pi, \Phi_\beta) : \pi^{-1}(U_\beta) \to U_\beta \times F$. If $U_\alpha \cap U_\beta$ is not empty then $\pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta) = \pi^{-1}(U_\alpha \cap U_\beta)$ is not empty and we have overlap maps

$$\phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$$

With the notation $E_p := \pi^{-1}(p)$ one can check that $\Phi_\alpha|_{E_p} : E_p \to F$ is a diffeomorphism for each $p \in U_\alpha$ and hence $\Phi_\alpha|_{E_p} \circ \Phi_\beta|_{E_p}^{-1} : F \to F$ is a diffeomorphism for all $p \in U_\alpha \cap U_\beta$. We then obtain a map $\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$ defined by

$$\Phi_{\alpha\beta}(p) = \Phi_\alpha|_{E_p} \circ \Phi_\beta|_{E_p}^{-1}$$

If follows that

$$\phi_\alpha \circ \phi_\beta^{-1}(p, s) = (s, \Phi_{\alpha\beta}(p)(s))$$

The functions $\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$ are called **transition maps** or **transition functions**. Given a bundle atlas, the corresponding transition functions clearly satisfy the following “cocycle conditions”:

$$\begin{aligned}
\Phi_{\alpha\alpha}(p) &= e \text{ for } p \in U_\alpha \\
\Phi_{\alpha\beta}(p) &= \Phi_{\beta\alpha}^{-1}(p) \text{ for } p \in U_\alpha \cap U_\beta \\
\Phi_{\alpha\beta}(p) \circ \Phi_{\beta\gamma}(p) \circ \Phi_{\gamma\alpha}(p) &= e \text{ for } p \in U_\alpha \cap U_\beta \cap U_\gamma
\end{aligned}$$

Notation 5.1 *We will often denote $\Phi_{\alpha\beta}(p)\,(s)$ by $\Phi_{\alpha\beta}|_p\,(s)$ which is, in many contexts, more transparent.*

$\mathrm{Diff}(F)$ is a group and we have a group action $(\phi, s) \mapsto \phi(s)$. However, $\mathrm{Diff}(F)$ too big for our purposes and we have certainly not attempted to give $\mathrm{Diff}(F)$ a Lie group structure. Even if we were to somehow extend the notion of Lie group sufficiently to include $\mathrm{Diff}(F)$, it would be infinite dimensional and thereby take us out of the circle of ideas we have been developing. Because of this, the transition functions $\Phi_{\alpha\beta}$ above which could be called "raw transition functions", might not be appropriate for our needs. We remedy this below by bringing Lie groups into the picture. First we give a simple example of a nontrivial bundle.

Example 5.2 *The circle S^1 can be considered as a quotient $\mathbb{R}/\sim$ where x is equivalent to y if and only if $x-y$ is an integer multiple of 2π. For this example we put an equivalence relation on $\mathbb{R}\times(-1,1)$ according to the prescription $(x,t)\sim(x+2\pi n,(-1)^n t)$ for any integer n. The quotient $(\mathbb{R}\times(-1,1))/\sim$ can easily be seen to be a smooth manifold and is none other than the familiar* ***Mobius band*** *which we denote by MB. Define a map $\pi : MB \to \mathbb{R}/\sim = S^1$ by $\pi([x,t]) = [x]$. We show that this is a fiber bundle by exhibiting an atlas consisting of three bundle charts. We use three bundle charts instead of two in order that the overlaps be connected sets. Let $U_1 = \{[x]\in\mathbb{R}/\sim : -2\pi/3 < x < 2\pi/3\}$ and $U_2 = \{[x]\in\mathbb{R}/\sim : 0 < x < 4\pi/3\}$ and $U_3 = \{[x]\in\mathbb{R}/\sim : 2\pi/3 < x < 2\pi\}$. Then $U_1\cup U_2 = \mathbb{R}/\sim = S^1$. Now for $i=2,3$ define $\phi_i : \pi^{-1}(U_i)\to U_i\times S^1$ by*

$$\phi_i([x,t]) = ([x],t)$$

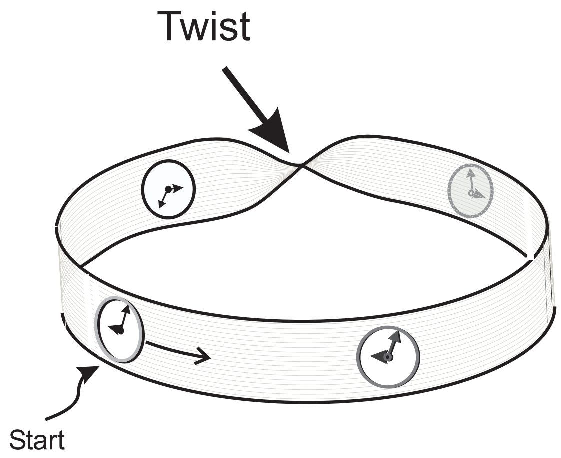
where (x,t) is the unique representative of $[x,t]$ in the set $(0,2\pi)\times(-1,1)$. For $\phi_1 : \pi^{-1}(U_1)\to U_1\times S^1$ we define $\phi_1([x,t]) = ([x],t)$ where (x,t) is the unique representative of $[x,t]$ in the set $(-2\pi/3,2\pi/3)\times(-1,1)$. One can check that $\phi_2\circ\phi_3^{-1} = \phi_3\circ\phi_2^{-1} = id$ on the overlap $\pi^{-1}(U_2)\cap\pi^{-1}(U_3)$. Now consider the overlap $\pi^{-1}(U_1)\cap\pi^{-1}(U_2)$. If $[x,t]\in\pi^{-1}(U_1)\cap\pi^{-1}(U_2)$, then $[x,t]$ is uniquely represented by some $(x,t)\in(0,2\pi/3)\times(-1,1)$ and in view of the definitions we see that $\phi_2\circ\phi_3^{-1} = id$ also. Finally we consider $\phi_1\circ\phi_3^{-1}$. If $[x,t]\in\pi^{-1}(U_1)\cap\pi^{-1}(U_3)$ then it has a unique representative (x,t) in $(4\pi/3,2\pi)\times(-1,1)$ and then $\phi_3^{-1}([x],t) = [x,t]$. Now for ϕ_1 we need to represent $[x,t]$ properly. We use the fact that $[x,t] = [x+2\pi,-t]$ and $(x-2\pi,-t)\in(-2\pi/3,0)\times(-1,1)$ so that $\phi_1([x+2\pi,-t]) = ([x+2\pi],-t) = ([x],-t)$. In short we have $\phi_1\circ\phi_3^{-1}([x,t]) = ([x],-t)$. From these considerations and the fact that in general $\phi_1\circ\phi_3^{-1}(p,s) = (p,\Phi_{\alpha\beta}(p)(s))$ we see that

$$\begin{aligned}
\Phi_{12}(p) &= id_{(-1,1)} \in \operatorname{Diff}(-1,1) \text{ for } p\in U_1\cap U_2\\
\Phi_{23}(p) &= id_{(-1,1)} \in \operatorname{Diff}(-1,1) \text{ for } p\in U_2\cap U_3\\
\Phi_{13}(p) &= -id_{(-1,1)} \in \operatorname{Diff}(-1,1) \text{ for } p\in U_1\cap U_3
\end{aligned}$$

A "twist" occurs on the overlap $\pi^{-1}(U_1)\cap\pi^{-1}(U_3)$. There is no way to construct an atlas for this bundle without having such a twist on at least one of the overlaps.

Notice that if we define an action λ of $\mathbb{Z}_2 = \{1,-1\}$ on the interval $(-1,1)$ by $\lambda(g,x)\mapsto gx$ then we can describe the transition functions in the last example by

$$\Phi_{\alpha\beta}(p)(s) = \lambda(g_{\alpha\beta}(p),x)$$



where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}_2$ is given by

$$\begin{aligned} g_{12} &= 1 \text{ on } U_1 \cap U_2 \\ g_{23} &= 1 \text{ on } U_2 \cap U_3 \\ g_{13} &= -1 \text{ on } U_1 \cap U_3 \end{aligned}$$

and in this case the $g_{\alpha\beta}$ satisfy a cocycle condition like the $\Phi_{\alpha\beta}$. Now this is convenient since we understand $\mathbb{Z}_2$ very well. It is a zero dimensional Lie group. Inspired by this we now seek to get Lie groups into the formalism which will alleviate our concerns about the group $\mathrm{Diff}(F)$ mentioned above.

Definition 5.6 *Let $\{U_\alpha\}$ be an open cover of a smooth manifold M and let G be a Lie group. A G-cocyle on $\{U_\alpha\}$ is the assignment of a smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ to every nonempty intersection $U_\alpha \cap U_\beta$ such that the **cocycle conditions** hold:*

$$\begin{aligned} g_{\alpha\alpha}(p) &= e \quad \text{for } p \in U_\alpha \\ g_{\alpha\beta}(p) &= (g_{\beta\alpha}(p))^{-1} \quad \text{for } p \in U_\alpha \cap U_\beta \\ g_{\alpha\beta}(p) g_{\beta\gamma}(p) g_{\gamma\alpha}(p) &= e \quad \text{for } p \in U_\alpha \cap U_\beta \cap U_\gamma \end{aligned}$$

The idea that we wish to pursue is that of representing the action of the raw transition maps by using Lie group actions. There is a subtle point here that the reader should not miss. Consider the following fact: If $\lambda : G \times F \to F$ is a group action, then by letting $K = \{g : \lambda_g(p) = p \text{ for all } p \in F\}$ (the kernel of the action) we obtain an effective action of G/K on F. Things are not so simple on the global level of bundles and this is related to the notion of spin structure. The best way to explain what is at stake is by the use of the notion of a principal bundle which we introduce later. Even before we get to that point, we will mention some things that will provide some idea as to why we need to be careful about ineffective actions.

We start out assuming that the action is effective:

Definition 5.7 *Let $\xi = (E, \pi, M, F)$ be a fiber bundle and G a Lie group. Suppose that we have an* effective *left action $\lambda : G \times F \to F$. Let $\{(\phi_\alpha, U_\alpha)\}$ be a bundle atlas for ξ. Suppose that for every nonempty intersection $U_\alpha \cap U_\beta$ there exists a smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that $\lambda(g_{\alpha\beta}(p), s) = \Phi_{\alpha\beta}|_p (s)$ for all $p \in U_\alpha \cap U_\beta$ and $s \in F$. Then the atlas $\{(\phi_\alpha, U_\alpha)\}$ is called a (G, λ)-**bundle atlas**. If the action λ is understood or standard in some way, one also speaks of a $G-$**bundle atlas**.*

Because the action λ in the above definition is assumed effective, it follows that the family $\{g_{\alpha\beta}\}$ satisfies the cocycle conditions of definition 5.6. Notice that if we had not assumed the action to be effective, then the maps $g_{\alpha\beta}$ would not be unique and may not satisfy a cocycle condition (although they would do so modulo the kernel of the action). Thus if we had not assumed effectiveness then we would have had to make the requirement that $\{g_{\alpha\beta}\}$ be a cocycle part of the definition (see [Mich]). For example, we might choose to call the larger family $\{(\phi_\alpha, U_\alpha), g_{\alpha\beta}, \lambda\}$ a (G, λ)-bundle atlas. Indeed, for an ineffective action it is conceivable that there could be a different cocycle $\{g'_{\alpha\beta}\}$ such that $\lambda(g'_{\alpha\beta}(p), s) = \Phi_{\alpha\beta}|_p (s)$ for all $s \in F$. Then $\{(\phi_\alpha, U_\alpha), g_{\alpha\beta}, \lambda\}$ and $\{(\phi_\alpha, U_\alpha), g'_{\alpha\beta}, \lambda\}$ would be different (but possibly equivalent) (G, λ)-bundle atlases.

Definition 5.8 *Let $\xi = (E, \pi, M, F)$ be a fiber bundle and G a Lie group. Suppose that we have an* effective *left action $\lambda : G \times F \to F$. Two (G, λ)-bundle atlas for ξ, say $\{(\phi_\alpha, U_\alpha)\}$ and $\{(\phi'_\alpha, U'_\alpha)\}$, are **strictly equivalent** if the union of the atlases is also a (G, λ)-bundle atlas. A* strict *equivalence class of atlases is referred to as an effective (G, λ)-**structure** on ξ and we say that ξ together with this (G, λ)-structure is an effective (G, λ)-**bundle**. Again, if the action is standard or understood then it is common to speak of a G-structure and refer to ξ is a G-bundle*

If there is no chance of confusion we will drop the adjective "effective". Readers familiar with the notion of a spin structure may be worried that we have lost something important by excluding ineffective actions but we assure the reader that we will be able to recover an appropriate notion of ineffective (G, λ)-structure once we introduce the notions of principal and associated bundle. The reader is warned that some standard expositions on fiber bundles allow ineffective actions right from the start but in some cases assertions are made which would only be true in the effective case! It is interesting to note that in his famous book on the subject, [St], Norman Steenrod restricts himself to effective left actions although he does this in one sentence early in the book which could easily be overlooked. For an exposition that does not assume effective actions but seems to be careful about the issues we have raised, see [Mich].

Notice that an alternative way to say that $\lambda(g_{\alpha\beta}(p), s) = \Phi_{\alpha\beta}|_p (s)$ is $\phi_\alpha \circ \phi_\beta^{-1}(p, s) = (p, \lambda(g_{\alpha\beta}(p), s))$. The maps $g_{\alpha\beta}$ are also called **transition functions** or **transition cocycles** for the (G, λ)-bundle atlas. Let us describe ways in which a $(G, \lambda)-$bundle atlas can come about. First, it could be that we may find a bundle atlas $\{(U_\alpha, \phi_\alpha)\}$ such that all of the transition maps are elements of a subgroup $G \subset \mathrm{Diff}(F)$. Also, G may *already* be a Lie group and it may be that the map $\mathrm{Diff}(F) \times F \to F$ given by $(\Phi, s) \mapsto \Phi(s)$ when restricted to $G \times F \to F$, is already known to be a (necessarily effective) Lie group action. Denoting this restricted action by λ we have $\lambda(\Phi_{\alpha\beta}(p), s) = \Phi_{\alpha\beta}|_p(s)$ so in this case we just have $g_{\alpha\beta} = \Phi_{\alpha\beta}$.

Since the action is effective, the corresponding homomorphism $G \to \mathrm{Diff}(F)$ given by $g \mapsto \lambda_g$ is injective and so the reader my feel that the group G can be replaced by its image in $\mathrm{Diff}(F)$. Indeed, by transferring the Lie group structure by brute force onto this image one may indeed replace G by its image. However, it is not always desirable nor is it always convenient to do so.

Theorem 5.1 ***(Fiber Bundle Construction Theorem)****Let M and F be smooth manifolds and G a Lie group. Let $\{U_\alpha\}_{\alpha \in A}$ be a cover of M and $\{g_{\alpha\beta}\}$ a $G-$cocycle $\{g_{\alpha\beta}\}$ for the cover. For every action $\lambda : G \times F \to F$, there exists a fiber bundle with bundle-atlas $\{(U_\alpha, \phi_\alpha)\}$ satisfying $\phi_\alpha \circ \phi_\beta^{-1}(p, s) = (p, \lambda(g_{\alpha\beta}(p), s))$ on nonempty overlaps $U_\alpha \cap U_\beta$.*

Proof. On the union $\Sigma := \bigcup_\alpha \{\alpha\} \times U_\alpha \times F$ define an equivalence relation such that

$$(\alpha, p, s) \in \{\alpha\} \times U_\alpha \times F$$

is equivalent to $(\beta, p', s') \in \{\beta\} \times U_\beta \times F$ if and only if $p = p'$ and $s' = g_{\alpha\beta}(p) \cdot s$. Notice that $p = p'$ is possible only in case $U_\alpha \cap U_\beta \neq \emptyset$. The the first member of the triple is only needed to make the union above disjoint. The cocycle conditions ensure that the equivalence relation is well defined.

The total space of our bundle is then $E := \Sigma/\sim$. The set Σ is essentially the disjoint union of the product spaces $U_\alpha \times F$ and so has an obvious topology. We then give $E := \Sigma/\sim$ the quotient topology. The bundle projection π is induced by $(\alpha, p, v) \mapsto p$. To get our trivializations we define

$$\phi_\alpha(e) := (p, y) \text{ for } e \in \pi^{-1}(U_\alpha)$$

where (p, y) is the unique member of $U_\alpha \times F$ such that $(\alpha, p, y) \in e$. The point here is that $(\alpha, p_1, y_1) \sim (\alpha, p_2, y_2)$ only if $(p_1, y_1) = (p_2, y_2)$. Now suppose $U_\alpha \cap U_\beta \neq \emptyset$. Then for $p \in U_\alpha \cap U_\beta$ the element $\phi_\beta^{-1}(p, y)$ is in $\pi^{-1}(U_\alpha \cap U_\beta) = \pi^{-1}(U_\alpha) \cap \pi^{-1}(U_\beta)$ and so $\phi_\beta^{-1}(p, y) = [(\beta, p, y_1)]$. But since $\phi_\beta^{-1}(p, y)$ is in $\pi^{-1}(U_\alpha)$ it must be equal to $[(\alpha, q, y_2)]$ for some y_2. This means that $p = q$ and $y_2 = g_{\alpha\beta}(x) \cdot y_1 = \lambda(g_{\alpha\beta}(x), y_1)$. From this it is not hard to see that

$$\phi_\alpha \circ \phi_\beta^{-1}(p, y) = (p, g_{\alpha\beta}(p) \cdot y))$$

We leave the routine verification of the regularity of these maps and the existence of the smooth structure to the reader. ∎

Notice that the previous theorem is true even without the assumption that the action is effective.

Definition 5.9 *Let $\xi = (E, \pi, M, F)$ be a smooth fiber bundle and $f : N \to M$ be a smooth map. The* ***pull-back bundle*** *$f^*\xi = (f^*E, f^*\pi, M, F)$ (or* ***induced*** *bundle) is defined as follows: The total space f^*E is the set $\{(q, e) \in N \times E$ such that $f(q) = \pi(e)\}$. Then we define $f^*\pi$ as the restriction to f^*E of the projection $pr_1 : N \times E \to N$.*

Notice that the second factor projection map $N \times E \to E$ restricts to a map $\widetilde{f} : f^*E \to E$ which is a bundle morphism over the map f :

$$\begin{array}{ccc} f^*E & \stackrel{\widetilde{f}}{\to} & E \\ \downarrow & & \downarrow \\ N & \stackrel{f}{\to} & M \end{array}$$

Definition 5.10 *Let $\xi = (E, \pi, M, F)$ be a smooth fiber bundle and $f : N \to M$ be a smooth map. A* ***section of*** *ξ* ***along*** *f is a map $\sigma : N \to E$ such that $\pi \circ \sigma = f$.*

If $\sigma : N \to E$ is a section of ξ along f then the map $\sigma' : N \to f^*E$ given by $p \mapsto (p, \sigma(p))$ is a section of the pull-back bundle $f^*\xi$. Conversely it is not hard to show that *all* sections of $f^*\xi$ have this form.

If $\phi = (\pi, \Phi)$ is a trivialization of the bundle ξ over the open set U then $(f^*\pi, \Phi \circ pr_2)$ is a trivialization of $f^*\xi$ over the open set $f^{-1}(U)$. Thus a bundle atlas on ξ induces a bundle atlas on $f^*\xi$.

5.2 Vector Bundles

The tangent and cotangent bundles are examples of a general type of fiber bundle called a vector bundle. Roughly speaking, a vector bundle is a parameterized family of vector spaces. We shall need both complex vector bundles and real vector bundles and so to facilitate definitions we let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let V be a finite dimensional $\mathbb{F}$-vector space. The simplest examples of vector bundles over a manifold M are the **product bundles** which consist of a Cartesian product $M \times \mathrm{V}$ together with the projection onto the first factor $pr_1 : M \times \mathrm{V} \to M$. Each set of the form $\{x\} \times \mathrm{V} \subset M \times \mathrm{V}$ inherits an $\mathbb{F}$-vector space structure from that of V in the obvious way. We think of $M \times \mathrm{V}$ as copies of V parameterized by M.

Definition 5.11 *Let V be a finite dimensional $\mathbb{F}$-vector space. A smooth $\mathbb{F}-$**vector bundle** with typical fiber V is a fiber bundle (E, π, M, V) such that for each $x \in M$ the set $E_x := \pi^{-1}\{x\}$ has the structure of a vector space over the field $\mathbb{F}$ isomorphic to the fixed vector space V and such that every $p \in M$ is in the domain of a bundle chart (U, ϕ) such that for each $x \in U$ the map $\Phi|_{E_x} : E_x \to \mathrm{V}$ is a vector space isomorphism.*

Definition 5.12 *(Terminology) We refer to a vector bundle as a complex vector bundle (resp. real vector bundle) if $\mathbb{F} = \mathbb{C}$ (resp.$\mathbb{F} = \mathbb{R}$)*

A bundle chart of the sort described in the definition is called a **vector bundle chart** (VB-chart). A family $\{(U_\alpha, \phi_\alpha)\}$ of vector bundle charts such that $\{U_\alpha\}$ is an open cover of M is called a **vector bundle atlas** for $\pi : E \to M$. The definition of a vector bundle guarantees that such an atlas exist. The dimension of the typical fiber V is called the **rank** of the vector bundle. Note that if we have a surjection $\pi : E \to M$ and a VB-atlas then we certainly have a vector bundle so in practice, if one is trying to show that a surjection is a vector bundle then one just exhibits a VB-atlas.

Remark 5.1 *By choosing a basis for* V *one gets an isomorphism with* $\mathbb{C}^k$ *or* $\mathbb{R}^k$ *as the case may be. Composing with this isomorphism we can convert the* V*-valued VB-charts into* $\mathbb{C}^k$ *or* $\mathbb{R}^k$ *valued VB-charts. Thus we could have assumed from the start that we were dealing with one of these standard vector spaces but it is not alway natural to do so since our vector space may arise in a specific way (it could be a Lie algebra or perhaps a space of algebraic tensors) and may not have a prefered choice of basis.*

The simplest example of a vector bundle is a trivial product bundle of the form $pr_1 : M \times \mathrm{V} \to M$ where the linear structure on the fibers is the obvious one: $a(p, v) + b(p, w) := (p, av + bw)$.

Exercise 5.3 *Show that the tangent and cotangent bundles over an* $n-$*manifold are vector bundles with typical fiber* $\mathbb{R}^k$*. (The cotangent bundle may be viewed as having typical fiber* $\left(\mathbb{R}^k\right)^*$*)*

Definition 5.13 *Let* ξ_1 *and* ξ_2 *be* $\mathbb{F}$*-vector bundles. A bundle morphism* $(\widehat{f}, f) : \xi_1 \to \xi_2$ *is called a* ***vector bundle morphism*** *if the restrictions to fibers;* $\left.\widehat{f}\right|_{\pi_1^{-1}(p)} \pi_1^{-1}(p) \to \pi_2^{-1}(f(p))$ *are* $\mathbb{F}-$*linear. If* ξ_1 *and* ξ_2 *have the same base space* M*, then we obtain the definition of a vector bundle morphism over* M *by specializing to the case* $f = id_M$*. We then also have the corresponding notions of vector bundle isomorphism and automorphism (for both type I and II bundle morphisms).*

A vector bundle is said to be **trivial** if it is vector bundle isomorphic to a product bundle $pr_1 : M \times \mathrm{V} \to M$.

Definition 5.14 *Let* $\pi : E \to M$ *be a rank* k *vector bundle modeled on* V *and fix an* l*-dimensional subspace* V' *of* V*. If* $E' \subset E$ *is a submanifold with the property that for every* $p \in M$ *there is a VB-chart* (U, ϕ) *such that*

$$\phi(\pi^{-1}(U) \cap E') = U \times \mathrm{V}' \subset U \times \mathrm{V}$$

then $(E', \pi|_{E'}, M, \mathrm{V}')$ *is called called a rank* l *vector subbundle of* (E, π, M, V)*. Charts with this property are said to be adapted to the subbundle.*

The triple $(E', \pi|_{E'}, M, \mathrm{V}')$ is in fact a vector bundle and every adapted VB- chart (U, ϕ) on E gives rise to a chart on $(E', \pi|_{E'}, M, \mathrm{V}')$. Namely, (U, ϕ')

where ϕ' is the resptriction of ϕ to $\pi^{-1}(U)\cap E' = \pi|_{E'}^{-1}(U)$. By picking a a basis for V' and extending to a basis for V one may take V to be $\mathbb{R}^k$ and V' to be $\mathbb{R}^l$ embedded in $\mathbb{R}^k$ as $\mathbb{R}^l \times \{0\} \subset \mathbb{R}^k$. If $h : E_1 \to E_2$ is a bundle homomorphism over M then

$$\ker h := \cup_{p\in M} \ker h|_{E_{1p}}$$

is a subset of E_1. This subset is not necessarily (the total space of) a subbundle but if the rank l of $h|_{E_{1p}}$ is independent of p then we say that the bundle map has rank l and in this case $\ker h$ is a vector subbundle. Similarly if h has constant rank in this sense, then the image $\operatorname{Im} h$ is a vector subbundle of E_2. Both of these facts follow from

Proposition 5.1 *Suppose that $h : E_1 \to E_2$ is a bundle homomorphism over M of constant rank r that E_1 and E_2 have typical fibers V_1 and V_2 repectively. Fix a rank r linear map $A : \mathrm{V}_1 \to \mathrm{V}_2$. Then for every $p \in M$ the is a chart (U, ϕ) for E_1 with $p \in U$ and a chart (U, ψ) for E_2 such that $\psi\circ h\circ\phi^{-1} : U\times \mathrm{V}_1 \to U\times \mathrm{V}_2$ has the form*

$$(p, v) \mapsto (p, Av)$$

It follows that $\ker h$ is a vector subbundle with typical fiber $\ker A$ and $\operatorname{Im} h$ is a vector subbundle of E_2 with typical fiber $\operatorname{Im} A$.

Before we prove this let us first make an observation. Notice that only the rank of the linear map $A : \mathrm{V}_1 \to \mathrm{V}_2$ in this last proposition is important and we may replace A by any linear map of the same rank. The reason for this that if $B : \mathrm{V}_1 \to \mathrm{V}_2$ is any other linear map with the same rank as A then there exist linear isomorphisms α and β such that $B = \beta A\alpha^{-1}$. In particular if one has choosen bases and identified V_1 with $\mathbb{R}^{k_1}$ and V_2 with $\mathbb{R}^{k_2}$ then we may take A to be a map of the form

$$(x^1, ..., x^{k_1}) \mapsto (x^1, ..., x^r, 0, ..., 0)$$

so that $\ker A$ is a copy of $\mathbb{R}^{k_1-r}$ and $\operatorname{Im} h$ is a copy of $\mathbb{R}^r$.

Proof. Using the fact that what we need to prove is entirely local we can assume that our task is to show that for any smooth map $h : U\times\mathbb{F}^{k_1} \to U\times\mathbb{F}^{k_2}$ of the form $(p, v) \to (p, h_p v)$ with A_p a linear map of rank r and where $p \mapsto A_p$ is smooth, we may find maps ψ and ϕ such that $\psi \circ h \circ \phi^{-1} : U \times \mathbb{F}^{k_1} \to U \times \mathbb{F}^{k_2}$ is given by $(p, x^1, ..., x^{k_1}) \mapsto (p, x^1, ..., x^r, 0, ..., 0)$. There exists linear isomorphisms $\alpha : \mathbb{F}^{k_1} \to \mathbb{F}^{k_1}$ and $\beta : \mathbb{F}^{k_2} \to \mathbb{F}^{k_2}$ such that $\beta \circ h_p \circ \alpha^{-1}$ is given by a $k_2 \times k_1$ matrix of the form

$$\begin{bmatrix} A_{11}(p) & A_{12}(p) \\ A_{21}(p) & A_{22}(p) \end{bmatrix}$$

were by $A_{11}(p)$ is an $r \times r$ matrix depending on p which is rank r and hence invertible at some fixed p_0. By shrinking U if needed we may assume that $A_{11}(p)$ is invertible for all $p \in U$. Thus we may as well assume from the start that h_p

is represented by a matrix of this form. Now consider the map $\phi_p : \mathbb{F}^{k_1} \to \mathbb{F}^{k_1}$ whose matrix is given by

$$\begin{bmatrix} A_{11}(p) & A_{12}(p) \\ 0 & id_{(k_1-r)\times(k_1-r)} \end{bmatrix}_{k_1\times k_1}$$

Then $h_p \circ \phi_p^{-1}$ has a matrix of the form

$$\begin{bmatrix} id_{r\times r} & 0 \\ A_{21}(p)A_{11}^{-1}(p) & C \end{bmatrix}_{k_2\times k_1}$$

and since this matrix must have rank r we see that $C = 0$. Let $M_p := A_{21}(p)A_{11}^{-1}(p)$ and let ψ_p be the linear map $\mathbb{R}^{k_2} \to \mathbb{R}^{k_2}$ with matrix

$$\begin{bmatrix} id_{r\times r} & 0 \\ -M_p & id_{(k_2-r)\times(k_2-r)} \end{bmatrix}$$

Then $\psi_p \circ h_p \circ \phi_p^{-1}$ has matrix of the form $\begin{bmatrix} id_{r\times r} & 0 \\ 0 & 0 \end{bmatrix}$. Now define $\phi(p, v) = (p, \phi_p v)$, $h(p, x) := (p, h_p x)$ and $\psi(p, v) := (p, \psi_p v)$ for $p \in U$ and $x \in \mathbb{F}^{k_1}$ and $v \in \mathbb{F}^{k_2}$. Notice that ψ_p, h_p and ϕ_p^{-1} each depend smoothly on p. The map $\psi \circ h \circ \phi^{-1}$ has the required form. ■

Proposition 5.2 *Let $\rho_0 : Gl(\mathrm{V}) \times \mathrm{V} \to \mathrm{V}$ be the standard action of $Gl(\mathrm{V})$ on the $\mathbb{F}$-vector space V. A fiber bundle with typical fiber V is an $\mathbb{F}$-vector bundle if and only if it admits a ρ_0-bundle atlas (a $Gl(\mathrm{V})$-bundle atlas). Furthermore, if $\rho : Gl(\mathrm{V}) \times \mathrm{V} \to \mathrm{V}$ is any effective action which acts linearly then any fiber bundle (E, π, M, V) that has a ρ-atlas is a vector bundle.*

Proof. That a vector bundle has a $Gl(\mathrm{V})$-structure follows directly from the definition. All that remains to show is the second part of the theorem since this will imply the remainder of the first part. Let $\rho : G \times \mathrm{V} \to \mathrm{V}$ be any effective Lie group action which acts linearly and suppose that (E, π, M, V) that has a ρ-structure. Let (U_α, ϕ_α) and (U_β, ϕ_β) be ρ-compatible bundle charts and $\phi_1 = (\pi, \Phi_\alpha)$ and $\phi_2 = (\pi, \Phi_\beta)$. Fix $p \in U_\alpha \cap U_\beta$. For $v, w \in \mathrm{V}$ and $a, b \in \mathbb{F}$ we have

$$\begin{aligned} \Phi_{\alpha\beta}(p)(av + bw) &= \rho(g_{\alpha\beta}(p), av + bw) \\ &= a\rho(g_{\alpha\beta}(p), v) + b\rho(g_{\alpha\beta}(p), w) \\ & a\Phi_{\alpha\beta}(p)(v) + b\Phi_{\alpha\beta}(p)(w) \end{aligned}$$

which shows that $\Phi_{\alpha\beta}(p) \in Gl(\mathrm{V})$ for all $p \in U_\alpha \cap U_\beta$. We transfer the vector space structure from V to E_p via $\Phi_\alpha|_{E_p}^{-1}$ and note that this is well defined by Proposition 2.1. With this linear structure on the fibers it is now easy to verify that (E, π, M, V) is a vector bundle. ■

Theorem 5.2 ***(Vector Bundle Construction Theorem)*** *Let $\{U_\alpha\}_{\alpha \in A}$ be a cover of M and let $\{g_{\alpha\beta}\}$ be a $G-$cocycle $\{g_{\alpha\beta}\}$ for a Lie group G. If G acts linearly on the vector space* V *(by say ρ) then there exists a vector bundle with a VB-atlas $\{(U_\alpha, \phi_\alpha)\}$ satisfying $\phi_\alpha \circ \phi_\beta^{-1}(p, v) = (p, g_{\alpha\beta}(p) \cdot v))$ on nonempty overlaps $U_\alpha \cap U_\beta$. In other words, there exists a vector bundle with ρ-atlas.*

Proof. This is essentially a special case of Theorem 5.1. We only need to check linearity of the ϕ_α on fibers. ∎

Perhaps some clarification is in order. In the case of a vector bundle, the raw transition maps $\Phi_{\alpha\beta}$ take values in the general linear group $Gl(\mathrm{V})$ which is a Lie group. They correspond to a ρ_0-structure where ρ_0 is the standard linear action of $Gl(\mathrm{V})$ on V (the standard representation). They automatically satisfy the cocycle condition and are $Gl(\mathrm{V})$ valued. The more general transition maps that define a $(G, \lambda)-$structure ($G-$structure) are G-valued. It is important to note that G may be small compared to $Gl(\mathrm{V})$ and certainly need not be thought of as a subset of $Gl(\mathrm{V})$. For example, the tensor bundles have $Gl(\mathrm{V})$-structures coming from tensor representations but the tensor bundles themselves generally have rank greater than k. Because in the vector bundle case, the $\Phi_{\alpha\beta}$ arise directly from a VB-atlas and act by the standard action we will call these **standard** transition maps and the corresponding $Gl(\mathrm{V})$-structure will be called the **standard** $Gl(\mathrm{V})$**-structure**. *The standard $Gl(\mathrm{V})$-structure is the structure that a vector bundle has simply by virtue of being an $\mathbb{F}$-vector bundle with typical fiber* V.

We already know what it means for two vector bundles over M to be equivalent. Of course any two vector bundles that are equivalent in a natural way can be thought of as the same. Since we can and often do construct our bundles according to the above recipe it will pay to know something about when two vector bundles over M are isomorphic based on their respective transition functions. Notice that standard transition functions are easily recovered from every $(G, \lambda)-$atlas by the formula $\lambda(g_{\alpha\beta}(p), s) = \Phi_{\alpha\beta}|_p (s)$.

Proposition 5.3 *Two bundles $\pi : E \to M$ and $\pi' : E' \to M$ with standard transition maps $\{\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl(\mathrm{V})\}$ and $\{\Phi'_{\alpha\beta} : U_{\alpha\beta} \to Gl(\mathrm{V})\}$ over the same cover $\{U_\alpha\}$ are isomorphic (over M) if and only if there are $Gl(\mathrm{V})$-valued functions f_α defined on each U_a such that*

$$\Phi'_{\alpha\beta}(x) = f_\alpha(x)\Phi_{\alpha\beta}(x)f_\alpha^{-1}(x) \text{ for } x \in U_\alpha \cap U_\beta \tag{5.1}$$

Proof. Given a vector bundle isomorphism $f : E \to E'$ over M let $f_\alpha := \phi'_\alpha f \phi_\alpha^{-1}$. Check that this works.

Conversely, given functions f_α satisfying equations 5.1, define $\widetilde{f}_\alpha : U_\alpha \times \mathrm{V} \to E|_{U_a}$ by $(x, v) \mapsto f_\alpha(x)v$. Now we define $f : E \to E'$ by

$$f(e) := \left((\phi'_\alpha)^{-1} \circ \widetilde{f}_\alpha \circ \phi_\alpha\right)(e) \text{ for } e \in E|_{U_a}$$

The conditions 5.1 insure that f is well defined on the overlaps $E|_{U_a} \cap E|_{U_\beta} = E|_{U_a \cap U_\beta}$. One easily checks that this is a vector bundle isomorphism. ∎

We can use this construction to arrive at several often used vector bundles.

Example 5.3 *Given an atlas* $\{(U_\alpha, \mathtt{x}_\alpha)\}$ *for a smooth manifold* M *we can let* $g_{\alpha\beta}(p) = T_p\mathtt{x}_\alpha \circ T_p\mathtt{x}_\beta^{-1}$ *for all* $p \in U_\alpha \cap U_\beta$. *The bundle constructed according to the recipe of Theorem 5.2 is a vector bundle which is (naturally isomorphic to) the tangent bundle* TM. *If we let* $g^*_{\alpha\beta}(p) = (T_p\mathtt{x}_\beta \circ T_p\mathtt{x}_\alpha^{-1})^t$ *then we arrive at the cotangent bundle* T^*M.

Example 5.4 *Let* $[n]$ *denote the set* $\{1, ..., n\}$. *Consider the space* $T^r_s(\mathbb{R}^n)$ *consisting of functions* $[n]\times...\times[n]\times..\times[n] \to \mathbb{R}$ *of the form* $t : (i_1, ..., i_r, j_1, ..., j_s) \mapsto t^{\ i_1...i_r}\big|_{j_1...j_s}$. *Clearly* $T^r_s(\mathbb{R}^n)$ *is a finite dimensional vector space isomorphic to* $\mathbb{R}^N$ *for* $N = n^r n^s$. *For* $g_{\alpha\beta}(p)$ *as in the last example we let* $(T^r_s g_{\alpha\beta})(p)$ *be the element of* $Gl(T^r_s(\mathbb{R}^n))$ *defined by*

$$T^r_s g_{\alpha\beta}(p) : (t')^{\ i_1...i_r}\big|_{j_1...j_s} \longmapsto g^{i_1}_{\ k_1} \cdots g^{i_r}_{\ k_r} t^{\ k_1...k_r}\big|_{l_1...l_s} \left(g^{-1}\right)^{l_1}_{\ j_1} \cdots \left(g^{-1}\right)^{l_s}_{\ j_s} \tag{5.2}$$

where $(g^i_j) := g_{\alpha\beta}(p) \in Gl(\mathbb{R}^n)$. *Using the maps* $\bar{g}_{\alpha\beta} := T^r_s g_{\alpha\beta}$ *as the cocycles we arrive, by the above construction, at a vector bundle* $T^r_s(TM)$. *This is called a tensor bundle and we give a slightly different construction of this bundle below when we study tensor fields.*

Proposition 5.4 *Let* $\pi : E \to M$ *be an* $\mathbb{F}-$*vector bundle with typical fiber* V *and with VB-atlas* $\{(U_\alpha, \phi_\alpha)\}$. *Let* $s_\alpha : U_\alpha \to \mathrm{V}$ *be a collection of maps such that whenever* $U_\alpha \cap U_\beta \neq \emptyset$, *then* $s_\alpha(p) = \phi_{\alpha\beta}(p)s_\beta(p)$ *for all* $p \in U_\alpha \cap U_\beta$. *Then there is a global section* s *such that* $s|_{U_\alpha} = s_\alpha$ *for all* α.

Proof. Let $\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathrm{V}$ be the trivializations that give rise to the cocycle $\{\Phi_{\alpha\beta}\}$. Let $\gamma_\alpha(p) := (p, s_\alpha(p))$ for $p \in U_\alpha$ and then let $s|_{U_\alpha} := \phi_\alpha^{-1} \circ \gamma_\alpha$. This gives a well defined section s because for $x \in U_\alpha \cap U_\beta$ we have

$$\begin{aligned}
\phi_\alpha^{-1} \circ \gamma_\alpha(p) &= \phi_\alpha^{-1}(x, s_\alpha(p)) \\
&= \phi_\alpha^{-1}(p, \Phi_{\alpha\beta}(p)s_\beta(p)) \\
&= \phi_\alpha^{-1} \circ \phi_\alpha \circ \phi_\beta^{-1}(p, s_\beta(p)) \\
&= \phi_\beta^{-1}(p, s_\beta(p)) = \phi_\beta^{-1} \circ \gamma_\beta(p)
\end{aligned}$$

■

Suppose we have two vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$. We give two constructions of the **Whitney sum** bundle $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \to M$. This is a globalization of the direct sum construction of vector spaces. In fact, the first construction simply takes $E_1 \oplus E_2 = \bigsqcup_{p\in M} E_{1p} \oplus E_{2p}$. Now we have a vector bundle atlas $\{(\phi_\alpha, U_\alpha)\}$ for π_1 and a vector bundle atlas $\{(\psi_\alpha, U_\alpha)\}$ for π_2. We have assumed that both atlases have the same family of open sets (we can arrange this by taking a common refinement). Now let $\phi_\alpha \oplus \psi_\alpha : (v_p, w_p) \mapsto (p, pr_2 \circ \phi_\alpha(v_p), pr_2 \circ \psi_\alpha(w_p))$ for all $(v_p, w_p) \in (E_1 \oplus E_2)|_{U_\alpha}$. Then $\{(\phi_\alpha \oplus \psi_\alpha, U_\alpha)\}$ is an atlas for $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \to M$.

Another method of constructing this bundle is to take the cocycle $\{g_{\alpha\beta}\}$ for π_1 and the cocycle $\{h_{\alpha\beta}\}$ for π_2 and then let $g_{\alpha\beta} \oplus h_{\alpha\beta} : U_\alpha \cap U_\beta \to$

$Gl(\mathbb{F}^{k_1} \times \mathbb{F}^{k_2})$ be defined by $(g_{\alpha\beta} \oplus h_{\alpha\beta})(x) = g_{\alpha\beta}(x) \oplus h_{\alpha\beta}(x) : (v, w) \longmapsto (g_{\alpha\beta}(x)v, h_{\alpha\beta}(x)w)$. The maps $g_{\alpha\beta} \oplus h_{\alpha\beta}$ form a cocycle which determines a bundle by the construction of proposition 5.2 which is (isomorphic to) $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \to M$.

The pullback of a vector bundle $\pi : E \to M$ by a smooth map $f : N \to M$ is naturally a vector bundle where the linear structure on each fiber $(f^*E)_q = \{q\} \times E_p$ is the obvious one induced from E_p. Put another way, we put the unique linear structure on each fiber that makes the bundle map $\widetilde{f} : f^*E \to E$ linear on fibers. Thus we call f^*E the **pullback vector bundle**.

Example 5.5 *Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be vector bundles and let $\triangle : M \to M \times M$ be the diagonal map given by $x \mapsto (x, x)$. From π_1 and π_2 one can construct a bundle $\pi_{E_1 \times E_2} : E_1 \times E_2 \to M \times M$ by $\pi_{E_1 \times E_2}(e_1, e_2) := (pr_1(e_1), pr_2(e_2))$. The Whitney sum bundle defined previously may also be defined as the pull-back $\triangle^* \pi_{E_1 \times E_2} : \triangle^*(E_1 \times E_2) \to M$. Of course one would write $E_1 \oplus E_2$ for $\triangle^*(E_1 \times E_2)$.*

Every vector bundle has global sections. An obvious example is the **zero section** which maps each $x \in M$ to the zero element 0_x of the fiber E_x.

Exercise 5.4 *Show that the range of the zero section of a vector bundle $E \to M$ is a submanifold of E that is diffeomorphic to M.*

Definition 5.15 *If $\xi = (E, \pi, M, \mathrm{V})$ is a vector bundle and $p \in M$ then a vector space basis for the fiber E_p is called a **frame** at p.*

Definition 5.16 *Let $\pi : E \to M$ be a rank k vector bundle. A k-tuple $\sigma = (\sigma_1, ..., \sigma_k)$ of sections of E over an open set U is called a (local) **frame field** over U if for all $p \in U$, $\{\sigma_1(p), ..., \sigma_k(p)\}$ is a frame at p.*

If we choose a fixed basis $\{\mathbf{e}_i\}_{i=1,\ldots,k}$ for the typical fiber V, then a choice of a local frame field over an open set $U \subset M$ is equivalent to a local trivialization (a vector bundle chart). Namely, if ϕ is such a trivialization over U then defining $\sigma_i(p) = \phi^{-1}(p, \mathbf{e}_i)$, we have that $\sigma_\phi = (\sigma_1, ..., \sigma_k)$ is a moving frame over U. Conversely, if $\sigma_\phi = (\sigma_1, ..., \sigma_k)$ is a moving frame over U then every $v \in \pi^{-1}(U)$ has the form $v = \sum v^i \sigma_i(p)$ for a unique p and unique numbers $v^i(p)$. Then the map $f : U \times \mathrm{V} \to \pi^{-1}(U)$ defined by $(p, v) \mapsto \sum v^i \sigma_i(p)$ is a diffeomorphism and its inverse $\phi = f^{-1}$ is a trivialization. Thus if there is a global frame field, then the vector bundle is trivial.

Definition 5.17 *A manifold M is said to be **parallelizable** if TM is trivial; i.e., if TM has a global frame field.*

For example, one can show that S^2 is not parallelizable while $S^2 \times \mathbb{R}$ is parallelizable. Also, $T^2 = S^1 \times S^1$ is parallelizable.

If G is a Lie subgroup of $Gl(\mathrm{V})$ and G acts in the standard way on V, that is, if the action is the restriction of the standard action $\rho_0|_G$, then a $\rho_0|_G$ −structure is called a reduction to the group G. Put another way, one has achieved such a reduction if one can find a cocycle of standard transition maps $\{\Phi_{\alpha\beta}\}$ which take values in G (acting in the standard way on V). By a slight extension, an effective (G,ρ) −structure on E can be considered as a reduction of the standard $Gl(\mathrm{V})$-structure.

Definition 5.18 *A* ***Riemannian metric*** *on a real vector bundle $\pi : E \to M$ is map $p \mapsto g_p(.,.)$ which assigns to each $p \in M$ a positive definite scalar product $g_p(.,.)$ on the fiber E_p that is smooth in the sense that $p \mapsto g_p(s_1(p), s_2(p))$ for all smooth sections s_1 and s_2. A real vector bundle together with a Riemannian metric is referred to as a* ***Riemannian vector bundle****.*

For example, a Riemannian metric on the tangent bundle of a smooth manifold is what one means by a Riemannian metric on the manifold. A smooth manifold with a Riemannian metric is called a Riemannian manifold and such will be studied later in this book. Now if a rank k real vector bundle $\pi : E \to M$ has a Riemannian metric, then this induces a reduction of the standard structure group to the subgroup $O(k)$ as follows: We start with an arbitrary VB-atlas $\{(U_\alpha, \phi_\alpha)\}$. Each chart (U_α, ϕ_α) defines a frame field $(\sigma_1^\alpha, ...\sigma_k^\alpha)$ on U_α. One can perform a Gramm-Schmidt process on the $\sigma_1^\alpha(p), ...\sigma_k^\alpha(p)$ simultaneously for all $p \in U_\alpha$ so that we have a new frame at each p (an orthonormal frame) $(e_1^\alpha(p), ...e_k^\alpha(p))$ where $e_j^\alpha(p) = A_j^i(p)\sigma_i^\alpha(p)$ for all $p \in U_\alpha$ and such that the matrix entries $A_j^i(p)$ depend smoothly on p. Thus $(e_1^\alpha, ...e_k^\alpha)$ is a smooth frame field called an **orthonormal frame field**. One then replaces the original chart (U_α, ϕ_α) by a new chart (U_α, ϕ'_α)which is the inverse of the map $(p, v) \mapsto \sum v^i e_i(p)$. Make this replacement for each (U_α, ϕ_α) to obtain a new atlas. Any two of these orthonormal frame fields, say $(e_1^\alpha, ...e_k^\alpha)$ and $\left(e_1^\beta, ...e_k^\beta\right)$ are related by

$$e_j^\alpha(p) = Q_j^i(p)e_i'^\beta(p)$$

for some smooth matrix function Q_j^i. One now checks that the transition maps for this new atlas (which is still a subatlas for the maximal VB-atlas) take values in $O(k)$. Indeed,

$$\begin{aligned}
\phi'_\beta \circ \phi_\alpha'^{-1}(p, v) &= \phi'_\beta(\sum v^j e_j^\alpha(p)) \\
&= \phi'_\beta(\sum v^j Q_j^i(p)e_i'^\beta(p)) \\
&= (p, \Phi'_\beta\big|_p \sum v^j Q_j^i(p)e_i'^\beta(p)) \\
&= (p, \sum v^j Q_j^i(p)\ \Phi'_\beta\big|_p e_i'^\beta(p)) \\
&= (p, \sum v^j Q_j^i(p)\mathbf{e}_i)
\end{aligned}$$

from which we see that $\Phi'_{\alpha\beta}(p)(v) = \Phi'_{\alpha\beta}(p)(v^j\mathbf{e}_j) = \sum v^j Q_j^i(p)\mathbf{e}_i$. Since $Q_j^i(p)$ is orthogonal for all p we have $\Phi'_{\alpha\beta}(p) \in O(k)$. The converse is also true. Namely, a reduction to structure group $O(k)$ (acting in the standard way) is tantamount to the introduction of a Riemannian metric.

Exercise 5.5 *Prove this last statement.*

Exercise 5.6 *Let E be a complex vector bundle of complex rank k. Define by analogy the notion of a Hermitian metric on E and show that every Hermitian metric on E corresponds to the reduction of the standard $Gl(k,\mathbb{C})$ structure to a $U(n)$-structure.*

Proposition 5.5 *On every real vector bundle E there can be defined a Riemannian metric. Similarly, on any complex vector bundle there exists a Hermitian metric.*

Proof. We prove the Riemannian case since the Hermitian case is entirely analogous. The proof uses the fact that a strict convex combination of positive definite scalar products is a positive definite scalar product. This allows us to use a partition of unity argument. Let $\{(U_\alpha, \phi_\alpha)\}$ be a VB-atlas. Let (U_α, ϕ_α) be given VB-chart. Endow V with an inner product. On the trivial bundle $U_\alpha \times \mathrm{V} \to U_\alpha$ there certainly exists a Riemannian metric given on each fiber by $\langle (p,v),(p,w)\rangle_0 = \langle v, w\rangle$. We may transfer this to the bundle $\pi^{-1}(U_\alpha) \to U_\alpha$ by using the map ϕ_α^{-1} thus obtaining a metric g_α on this restricted bundle over U_α. We do this for every VB-chart in the atlas. The trick is to piece these together in a smooth way. For that we take a smooth partition of unity (U_α, ρ_α) subordinate to a cover the cover $\{U_\alpha\}$. Let

$$\mathrm{g}(p) = \sum \rho_\alpha(p) g_\alpha(p).$$

The sum is finite at each $p \in M$ since the partition of unity is locally finite and the functions $\rho_\alpha \mathrm{g}_\alpha$ are extended to be zero outside of the corresponding U_α. The fact that $\rho_\alpha \geq 0$ and $\rho_\alpha > 0$ at p for at least one α easily gives the result that g is positive definite at each p and so it is a Riemannian metric on E. ∎

Example 5.6 (Canonical line bundle) *Recall that $\mathbb{R}P^n$ is the set of all lines through the origin in $\mathbb{R}^{n+1}$. Define the subset $\mathbb{L}(\mathbb{R}P^n)$ of $\mathbb{R}P^n \times \mathbb{R}^{n+1}$ consisting of all pairs (l, v) such that $v \in l$ (think about this). This set together with the map $\pi_{\mathbb{R}P^n} : (l,v) \mapsto l$ is a rank one vector bundle.*

Example 5.7 (Tautological Bundle) *Let $G(n,k)$ denote the Grassmann manifold of k-planes in $\mathbb{R}^n$. Let $\gamma_{n,k}$ be the subset of $G(n,k)\times\mathbb{R}^n$ consisting of pairs (P, v) where P is a k-plane (k-dimensional subspace) and v is a vector in the plane P. The projection $\pi : \gamma_{n,k} \to G(n,k)$ is simply $(P,v) \mapsto P$. The result is a vector bundle $(\gamma_{n,k}, \pi_{n,k}, G(n,k), \mathbb{R}^k)$. We leave it to the reader to discover an appropriate VB-atlas.*

Note well that these tautological vector bundles are not just trivial bundles and in fact their topology or twistedness (for large n) is of the utmost importance for classifying vector bundles. One may take the inclusions $...\mathbb{R}^n \subset \mathbb{R}^{n+1} \subset ... \subset \mathbb{R}^\infty$ to construct inclusions $G(n,k) \subset G(n+1,k)$...and $\gamma_{n,k} \subset \gamma_{n+1,k}$ from which a "universal bundle" $\gamma_n \to G(n)$ is constructed with the property that every rank k vector bundle E over X is the pull-back by some map $f : X \to G(n)$:

$$\begin{array}{ccccc} E \cong & f^*\gamma_n & \to & \gamma_n \\ & \downarrow & & \downarrow \\ & X & \xrightarrow{f} & G(n) \end{array}$$

Exercise 5.7 *To each point on a sphere attach the space of all vectors normal to the sphere at that point. Show that this* normal bundle *is in fact a (smooth) vector bundle.*

Exercise 5.8 *Fix a nonnegative integer j. Let $Y = \mathbb{R}\times(-1,1)$ and let $(x_1, y_1) \sim (x_2, y_2)$ if and only if $x_1 = x_2 + jk$ and $y_1 = (-1)^{jk} y_2$ for some integer k. Show that $E := Y/\sim$ is a vector bundle of rank 1 that is trivial if and only if j is even. Prove or at least convince yourself that this is the Mobius band when j is odd.*

5.3 Tensor Products of Vector Bundles

Given two vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ with respective typical fibers V_1 and V_2 we let

$$E_1 \otimes E_2 := \bigcup_{p\in M} E_{1p} \otimes E_{2p} \text{ (a disjoint union)}$$

Then we have a projection map $\pi : E_1 \otimes E_2 \to M$ given by mapping any element in a fiber $E_{1p} \otimes E_{2p}$ to the base point p. Now we show how to construct a VB-atlas for $E_1 \otimes E_2$ from and atlas on each of E_1 and E_2. The smooth structure and topology can be derived from the atlas as usual in such a way as to may all the relevant maps smooth. We leave the verification of this to the reader. As usual we can assume that the atlases are based on the same open cover. Thus suppose that $\{(U_\alpha, \phi_a)\}$ is VB-atlas for E_1 while $\{(V_\alpha, \psi_a)\}$ is a VB-atlas for E_2. Let $\phi_a \otimes \psi_a : \pi^{-1}(U_\alpha) \to U_\alpha \times (\mathrm{V}_1 \otimes \mathrm{V}_2)$ be defined by $(\phi_a \otimes \psi_a)(\tau_p) := (p, \Phi_a(p) \otimes \Psi_a(p)(\tau_p))$ if $\tau_p \in E_{1p} \otimes E_{2p} \subset E_1 \otimes E_2$. To clarify, the map $\Phi_a(p) \otimes \Psi_a(p) : E_{1p} \otimes E_{2p} \to \mathrm{V}_1 \otimes \mathrm{V}_2$ in this formula is the tensor product map of two linear maps as described above.. To see what the transition maps look like we compute

$$\begin{aligned} &(\phi_a \otimes \psi_\beta) \circ (\phi_a \otimes \psi_\beta)^{-1}(\tau_p) \\ &= (p, \left(\Phi_a(p) \circ \Phi_\beta^{-1}(p) \otimes \Psi_a(p) \circ \Psi_\beta(p)\right)(\tau_p)) \\ &= (p, (\Phi_{a\beta}(p) \otimes \Psi_{a\beta}(p))(\tau_p)) \end{aligned}$$

Thus the transition maps are given by $p \to \Phi_{a\beta}(p) \otimes \Psi_{a\beta}(p)$ which is a map from U_α to $Gl(\mathrm{V}_1 \otimes \mathrm{V}_2)$. The group $Gl(\mathrm{V}_1 \otimes \mathrm{V}_2)$ acts on $\mathrm{V}_1 \otimes \mathrm{V}_2$ in a standard way and this is the standard structure group of the bundle as we have just seen. However, it is also true that the bundle $E_1 \otimes E_2$ has $Gl(\mathrm{V}_1) \times Gl(\mathrm{V}_2)$ as a structure group via a tensor product representation. Indeed, if ι_1 denotes the standard representation of $Gl(\mathrm{V}_1)$ in V_1 and ι_2 denotes the standard representation of $Gl(\mathrm{V}_2)$ in V_2 then we have a tensor product representation $\iota_1 \otimes \iota_2$ of $Gl(\mathrm{V}_1) \times Gl(\mathrm{V}_2)$ in $\mathrm{V}_1 \otimes \mathrm{V}_2$. Using the $Gl(\mathrm{V}_1) \times Gl(\mathrm{V}_2)$-valued cocycle $p \mapsto h_{\alpha\beta}(p) := (\Phi_{a\beta}(p), \Psi_{a\beta}(p))$ together with $\iota_1 \otimes \iota_2$ we see that by definition

$$(\iota_1 \otimes \iota_2)(h_{\alpha\beta}(p), \tau) = (\Phi_{a\beta}(p) \otimes \Psi_{a\beta}(p))\,(\tau)$$

Furthermore, if $\mathrm{V}_1 = \mathrm{V}_2 = \mathrm{V}$ then the tensor product representation $\iota_1 \otimes \iota_2$ is usually defined as a representation of $Gl(\mathrm{V})$ rather than $Gl(\mathrm{V}) \times Gl(\mathrm{V})$ and so $E_1 \otimes E_2$ would have a $(Gl(\mathrm{V}), \iota_1 \otimes \iota_2)$-structure. We can reconstruct the self same vector bundle using any of these representation-cocycle pairs using Lemma **??**. In fact, it is quite common that we have different representations by one group: Suppose that we have two faithful representations λ_1 and λ_2 of a Lie groups G acting on V_1 and V_2 respectively. If $\{g_{\alpha\beta}\}$ is a cocycle of transition maps then we can use the pair $\{g_{\alpha\beta}, \lambda_1\}$ in Lemma **??** to form a vector bundle E_1 which has a (G, λ_1)-structure by construction. Similarly we can construct a vector bundle E_2 with (G, λ_2)-structure. Now if we use $\lambda_1 \otimes\, \lambda_2$ and the same cocycle $\{g_{\alpha\beta}\}$ we obtain a bundle which, as a vector bundle, is $E_1 \otimes E_2$ but by construction has a $(G, \lambda_1 \otimes\, \lambda_2)$ structure. This is the case in the following exercise:

Exercise 5.9 *Suppose that E is a vector bundle with a (G, λ)-structure given by a (G, λ)-atlas with a corresponding cocycle of transition functions. Show how one may use Theorem 5.2 to construct bundles isomorphic to E^*, $E \otimes E$ and $E \otimes E^*$ which will have a $(G, \lambda^*)-$structure, a $(G, \lambda \otimes \lambda)-$structure and a $(G, \lambda \otimes \lambda^*)-$structure respectively.*

We have now seen that various new vector bundles can be constructed starting with one or more vector bundles. Most of the operations of linear algebra extend to the vector bundle category. We can unify our thinking on these matters by introducing the notion of a C^∞ functor. With $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ the set of all $\mathbb{F}$-vector spaces together with linear maps is a category that we denote by $lin(\mathbb{F})$. The set of morphisms from V to W is the space of $\mathbb{F}$-linear maps $L(\mathrm{V}, \mathrm{W})$ (also denoted $Hom(\mathrm{V}, \mathrm{W})$).

Definition 5.19 *A covariant C^∞ functor $\mathcal{F}$ of one variable on $lin(\mathbb{F})$ consists of a map denoted again by $\mathcal{F}$ which assigns to every $\mathbb{F}$-vector space an $\mathbb{F}$-vector space $\mathcal{F}\mathrm{V}$ and a map also denoted by $\mathcal{F}$ which assigns to every linear map $A \in L(\mathrm{V}, \mathrm{W})$ a linear map $\mathcal{F}A \in L(\mathcal{F}\mathrm{V}, \mathcal{F}\mathrm{W})$ such that*

(i) $\mathcal{F} : L(\mathrm{V}, \mathrm{W}) \to L(\mathcal{F}\mathrm{V}, \mathcal{F}\mathrm{W})$ *is smooth,*

(ii) $\mathcal{F}(id_{\mathrm{V}}) = id_{\mathcal{F}\mathrm{V}}$ *for all* $\mathbb{F}$*-vector spaces* V

(iii) $\mathcal{F}(A \circ B) = \mathcal{F}A \circ \mathcal{F}B$ *for all* $A \in L(\mathrm{U}, \mathrm{V})$ *and* $B \in L(\mathrm{V}, \mathrm{W})$ *and vector spaces* U, V *and* W.

As an example we have the C^∞ functor which assigns to each V the $k-$fold direct sum $\oplus^k \mathrm{V} = \mathrm{V} \oplus \cdots \oplus \mathrm{V}$ and to each linear map $A \in L(\mathrm{V},\mathrm{W})$ the map

$$\oplus^k A : \oplus^k \mathrm{V} \to \oplus^k \mathrm{V}$$

given by $\oplus^k A(v_1, ..., v_k) := (Av_1, ..., Av_k)$. Similarly there is the functor which assigns to each V the $k-$fold tensor product $\otimes^k \mathrm{V} = \mathrm{V} \otimes \cdots \otimes \mathrm{V}$ and to each $A \in L(\mathrm{V},\mathrm{W})$ the map $\otimes^k A : \otimes^k \mathrm{V} \to \oplus^k \mathrm{V}$ given on homogeneous elements by $\left(\otimes^k A\right)(v_1 \otimes \cdots \otimes v_k) := Av_1 \otimes \cdots \otimes Av_k$.

One can also consider C^∞ covariant functors of several variables. For example, we may map each pair of vector spaces (V,W) to the tensor product $\mathrm{V} \otimes \mathrm{W}$ and to each pair $(A, B) \in L(\mathrm{V},\mathrm{V}') \times L(\mathrm{W},\mathrm{W}')$ the map $A \otimes B : \mathrm{V} \otimes \mathrm{W} \to \mathrm{V}' \otimes \mathrm{W}'$.

There is also a similar notion of *contravariant* C^∞ functor:

Definition 5.20 *A contravariant C^∞ functor $\mathcal{F}$ of one variable on $lin(\mathbb{F})$ consists of a map denoted again by $\mathcal{F}$ which assigns to every $\mathbb{F}$-vector space an $\mathbb{F}$-vector space $\mathcal{F}\mathrm{V}$ and a map also denoted by $\mathcal{F}$ which assigns to every linear map $A \in L(\mathrm{V},\mathrm{W})$ a linear map $\mathcal{F}A \in L(\mathcal{F}\mathrm{W}, \mathcal{F}\mathrm{V})$ (notice the reversal) such that*

(i) $\mathcal{F} : L(\mathrm{V},\mathrm{W}) \to L(\mathcal{F}\mathrm{W}, \mathcal{F}\mathrm{V})$ *is smooth,*

(ii) $\mathcal{F}(id_{\mathrm{V}}) = id_{\mathcal{F}\mathrm{V}}$ *for all* $\mathbb{F}$*-vector spaces* V

(iii) $\mathcal{F}(A \circ B) = \mathcal{F}B \circ \mathcal{F}A$ *for all* $A \in L(\mathrm{U},\mathrm{V})$ *and* $B \in L(\mathrm{V},\mathrm{W})$ *and vector spaces* U,V *and* W.

The map that assigns to each vector space its dual and to each map it dual (transpose) is a contravariant C^∞ functor $\mathcal{F}$. Now clearly one may define the notion of a C^∞ functor of several variables which may be covariant in some variables and contravariant in others. For example, consider the functor of two variables that assigns to each pair (V,W) the space $\mathrm{V} \otimes \mathrm{W}^*$ and to each pair $(A, B) \in L(\mathrm{V}_1, \mathrm{V}_2) \times L(\mathrm{W}_1, \mathrm{W}_2)$ the map $A \otimes B : \mathrm{V}_1 \otimes \mathrm{W}_2^* \to \mathrm{V}_1 \otimes \mathrm{W}_1^*$.

Theorem 5.3 *Let $\mathcal{F}$ be a C^∞ functor of m variables on $lin(\mathbb{F})$ and let $E_1, ..., E_k$ be $\mathbb{F}-$vector bundles with respective typical fibers $\mathrm{V}_1, .., \mathrm{V}_m$. Then the set*

$$E := \mathcal{F}(E_1, ..., E_m) := \cup_p \mathcal{F}(\left.E_1\right|_p, ..., \left.E_m\right|_p)$$

together with the map $\pi : E \to M$ which takes every element in $\mathcal{F}(\left.E_1\right|_p, ..., \left.E_m\right|_p)$ to p is natural a vector bundle. If the ranks of $E_1, ..., E_m$ are $k_1, ..., k_m$ respectively then the typical fiber of E will be $\mathcal{F}(\mathrm{V}_1, .., \mathrm{V}_m)$.

Proof. We will only prove the case of $k = 2$ with covariant first variable and contravariant second variable. This should make it clear how the general case would go while keeping the notational complexity under control.

Given vector bundles $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$, the total space of the constructed bundle is $\cup_p \mathcal{F}(\left.E_1\right|_p, \left.E_2\right|_p)$ with the obvious projection which we call π. Let (ϕ_α, U_α) be a VB-atlas for E_1 and (ψ_α, U_α) a VB-atlas for E_2 (we have arranged that both atlases use the same cover by going to a common refinement as usual). For each p let E_p denote the fiber $\mathcal{F}(\left.E_1\right|_p, \left.E_2\right|_p)$. Fix α and for each $p \in U_\alpha$ define $\left.\Theta_\alpha\right|_p : L(E_p, \mathcal{F}(\mathrm{V}_1, \mathrm{V}_2))$ by

$$\left.\Theta_\alpha\right|_p := \mathcal{F}(\left.\Phi_\alpha\right|_p, \left.\Psi_\alpha\right|_p^{-1})$$

where $\phi_\alpha = (\pi_1, \Phi_\alpha)$ and $\psi_\alpha = (\pi_2, \Psi_\alpha)$. Then define $\Theta_\alpha : \pi^{-1}U_\alpha \to \mathcal{F}(\mathrm{V}_1, \mathrm{V}_2)$ by $\Theta_\alpha(e) = \left.\Theta_\alpha\right|_p(e)$ whenever $e \in \mathcal{F}(\left.E_1\right|_p, \left.E_2\right|_p)$. Next define

$$\theta_\alpha = (\pi, \Theta_\alpha) : \pi^{-1}U_\alpha \to U_\alpha \times \mathcal{F}(\mathrm{V}_1, \mathrm{V}_2).$$

The family $\{(\theta_\alpha, U_\alpha)\}$ is to be a VB atlas for E. We check the transition maps:

$$\begin{aligned}
\Theta_{\alpha\beta}(p) &= \left.\Theta_\alpha\right|_p \circ \left.\Theta_\beta^{-1}\right|_p \\
&= \mathcal{F}(\left.\Phi_\alpha\right|_p, \left.\Psi_\alpha\right|_p^{-1}) \circ \mathcal{F}(\left.\Phi_\beta\right|_p, \left.\Psi_\beta\right|_p^{-1})^{-1} \\
&= \mathcal{F}(\left.\Phi_\alpha\right|_p, \left.\Psi_\alpha\right|_p^{-1}) \circ \mathcal{F}(\left.\Phi_\beta\right|_p^{-1}, \left.\Psi_\beta\right|_p) \\
&= \mathcal{F}(\left.\Phi_\alpha\right|_p \circ \left.\Phi_\beta\right|_p^{-1}, \left.\Psi_\beta\right|_p \circ \left.\Psi_\alpha\right|_p^{-1}) \\
&= \mathcal{F}(\Phi_{\alpha\beta}(p), \Psi_{\beta\alpha}(p))
\end{aligned}$$

(Remember that the functor is contravariant in the second variable). Now we can see that from the properties of $\Phi_{\alpha\beta}, \Psi_{\beta\alpha}$ and the definition of C^∞ functor that $\mathcal{F}(\Phi_{\alpha\beta}(p), \Psi_{\beta\alpha}(p)) \in Gl(\mathcal{F}(\mathrm{V}_1, \mathrm{V}_2))$ and the maps $\Theta_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl(\mathcal{F}(\mathrm{V}_1, \mathrm{V}_2))$ are smooth. ■

5.4 Principal and Associated Bundles

Let $\pi : E \to M$ be a vector bundle with typical fiber V and for every $p \in M$ let $Gl(\mathrm{V}, E_p)$ denote the set of linear isomorphisms from V to E_p. If we choose a fixed basis$\{\mathbf{e}_i\}_{i=1,...,k}$ for V then we can identify each frame $u = (u_1, ..., u_k)$ at p with the element of $Gl(\mathrm{V}, E_p)$ given by $u(v) := \sum v^i u_i$ where $v = \sum v^i \mathbf{e}_i$. With this identification, notice that if $\sigma_\alpha := \sigma_{\phi_\alpha}$ is the local section coming from a VB-chart (U_α, ϕ_α) as described above then we have

$$\sigma_\alpha(p) = \left.\Phi_\alpha\right|_{E_p} \text{ for } p \in U_\alpha$$

Now let

$$L(E) := \cup_{p \in M} Gl(\mathrm{V}, E_p) \text{ (disjoint union)}$$

it will shortly be clear that $L(E)$ is a smooth manifold and the total space of a fiber bundle. Let $\wp : L(E) \to M$ be the projection map defined by $\wp(u) = p$ when $u \in Gl(\mathrm{V}, E_p)$. Now observe that $Gl(\mathrm{V})$ acts on the right of the set $L(E)$. Indeed, the action $L(E) \times Gl(\mathrm{V}) \to L(E)$ is given by $r : (u, g) \longmapsto ug = u \circ g$.

If we pick a fixed basis for V as above then we may view g as a matrix and an element $u \in Gl(\mathrm{V}, E_p)$ as a frame $u = (u_1, ..., u_k)$. In this case we have

$$ug = (u_i g_1^i, ..., u_i g_k^i)$$

It is easy to see that the orbit of a frame at p is exactly the set $\wp^{-1}(p) = Gl(\mathrm{V}, E_p)$ and that the action is free. For each VB-chart (U, ϕ) for E let σ_ϕ be the associated frame field. Define $f_\phi : U \times Gl(\mathrm{V}) \to \wp^{-1}(U)$ by $f_\phi(p, g) = \sigma_\phi(p)g$. It is easy to check that this is a bijection. Let $\widetilde{\phi} : \wp^{-1}(U) \to U \times Gl(\mathrm{V})$ be the inverse of this map. We have $\widetilde{\phi} = (\wp, \widetilde{\Phi})$ where is $\widetilde{\Phi}$ a uniquely determined map. Starting with a VB atlas $\{(U_\alpha, \phi_\alpha)\}$ for E, we construct a family $\{\widetilde{\phi}_\alpha : \wp^{-1}(U_\alpha) \to U_\alpha \times Gl(\mathrm{V})\}$ of trivializations which gives a fiber bundle atlas $\{(U_\alpha, \widetilde{\phi}_\alpha)\}$ for $L(E)$ and simultaneously induces the smooth structure.

Definition 5.21 *Let $\pi : E \to M$ be a vector bundle with typical fiber* V. *The fiber bundle $(L(E), \wp, M, Gl(\mathrm{V}))$ constructed above is called the* ***linear frame bundle*** *of E and is usually denoted simply by $L(E)$. The frame bundle for the tangent bundle of a manifold M is often denoted by $L(M)$ rather than by $L(TM)$.*

Notice that any VB-atlas for E induces an atlas on $L(E)$ according to our considerations above. We have

$$\begin{aligned}
\widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1}(p, g) &= \widetilde{\phi}_\alpha(\sigma_\beta(p)g) \\
&= \widetilde{\phi}_\alpha(\Phi_\beta|_{E_p} g) \\
&(p, \Phi_{\alpha\beta}(p)g).
\end{aligned}$$

Thus the transition functions of $L(E)$ are given by the standard transition functions of E acting by left multiplication on $Gl(\mathrm{V})$. In other words, the bundle $L(E)$ has a $Gl(\mathrm{V})$-structure where the action of $Gl(\mathrm{V})$ on the typical fiber V is the standard one and the cocycle corresponding to the bundle atlas for $L(E)$ that we constructed from a VB-atlas $\{(U_\alpha, \phi_\alpha)\}$ for original vector bundle E is the very cocycle $\Phi_{\alpha\beta}$ coming from this atlas on E.

We need to make one more observation concerning the right action of $Gl(\mathrm{V})$ on $L(E)$. Take a VB- chart for E, say (U, ϕ), and let us look again at the associated chart $(\wp^{-1}(U), \widetilde{\phi})$ for $L(E)$. First, consider the trivial bundle $pr_1 : U \times Gl(\mathrm{V}) \to U$ and define the obvious *right* action on the total space $\widetilde{\phi}\left(\wp^{-1}(U)\right) = U \times Gl(\mathrm{V})$ by $((p, g_1), g) \to (p, g_1 g) := (p, g_1) \cdot g$. Then this action is transitive on the fibers of this trivial bundle which are exactly the orbits of the action. Of course since $Gl(\mathrm{V})$ acts on $L(E)$ and preserves fibers it also acts by restriction on $\wp^{-1}(U)$.

Proposition 5.6 *The bundle map $\widetilde{\phi} : \wp^{-1}(U) \to U \times Gl(\mathrm{V})$ is equivariant with respect to the right actions described above.*

Proof. We first look at the inverse.

$$\widetilde{\phi}^{-1}(p, g_1)g = \sigma_\phi(p)g_1 g = \widetilde{\phi}^{-1}(p, g_1 g)$$

Now to see this from the point of view of $\widetilde{\phi}$ rather than its inverse just let $u \in \wp^{-1}(U) \subset L(E)$ and let (p, g_1) be the unique pair such that $u = \widetilde{\phi}^{-1}(p, g_1)$. Then

$$\begin{aligned}\widetilde{\phi}(ug) &= \widetilde{\phi}\left(\widetilde{\phi}^{-1}(p, g_1)g\right) = \widetilde{\phi}\widetilde{\phi}^{-1}(p, g_1 g) \\ &= (p, g_1 g) = (p, g_1) \cdot g = \widetilde{\phi}(u) \cdot g\end{aligned}$$

■

A section of $L(E)$ over an open set U in M is just a frame field over U. A global frame field is a global section of $L(E)$ and clearly a global section exists if and only if E is trivial.

The linear frame bundles associated to vector bundles are example of principal bundles. For a frame bundle $F(E)$ the following things standout: The typical fiber is diffeomorphic to the structure group $Gl(\mathrm{V})$ and we constructed an atlas which showed that $L(E)$ had a $Gl(\mathrm{V})$-structure where the action was left multiplication. Furthermore there is a right $Gl(\mathrm{V})$ action on the total space $L(E)$ which has the fibers as orbits. The charts have the form $(\wp^{-1}(U), \widetilde{\phi})$ and derive from charts on E and $\widetilde{\phi}$ is equivariant in a sense that $\widetilde{\phi}(ug) = \widetilde{\phi}(u)g$ where if $\widetilde{\phi}(u) = (p, g_1)$ then $(p, g_1)g := (p, g_1 g)$ by definition. These facts motivate the concept of a principal bundle.

Definition 5.22 *Let $\wp : P \to M$ be a smooth fiber bundle with typical fiber a Lie group G. The bundle $(P, \wp, M, G)$ is called a* ***principal** $G-$**bundle*** *if there is a smooth free right action of G on P such*

(i) The action preserves fibers; $\wp(ug) = \wp(u)$ for all $u \in P$ and $g \in G$.

(ii) For each $p \in M$ there exists a bundle chart (U, ϕ) with $p \in U$ and such that if $\phi = (\wp, \Phi)$ then
$$\Phi(ug) = \Phi(u)g$$
for all $u \in P$ and $g \in G$.

If the group G is understood then we may refer $(P, \wp, M, G)$ simple as a ***principal bundle****.*

Charts of the form described in (ii) in the definition are called **principal bundle charts** and an atlas consisting of principal bundle charts is called a **principal bundle atlas**. If $\wp(u_1) = \wp(u_2)$ then $\Phi(u_1) = \Phi(u_2)g$ where $g := \Phi(u_1)\Phi(u_2)^{-1}$ and so

$$\begin{aligned}\phi(u_1) &= (\wp(u_1), \Phi(u_1)) = \\ (\wp(u_2), \Phi(u_2)g) &= (\wp(u_2 g), \Phi(u_2 g)) \\ &= \phi(u_2 g)\end{aligned}$$

Since ϕ is bijective we see that $u_1 = u_2 g$ and so we conclude that the fibers are actually the orbits of the right action.

Notice that if (ϕ_α, U_α) and (ϕ_β, U_β) are overlapping principal bundle charts with $\phi_\alpha = (\wp, \Phi_\alpha)$ and $\phi_\beta = (\wp, \Phi_\beta)$ then

$$\Phi_\alpha(ug)\Phi_\beta(ug)^{-1} = \Phi_\alpha(u)gg^{-1}\Phi_\beta(u)^{-1} = \Phi_\alpha(u)\Phi_\beta(u)^{-1}$$

so that the map $u \mapsto \Phi_\alpha(u)\Phi_\beta(u)^{-1}$ is constant on fibers. This means there is a smooth function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that $g_{\alpha\beta}(p) = \Phi_\alpha(u)\Phi_\beta(u)^{-1}$ where u is any element in the fiber at p.

Lemma 5.1 *Let (ϕ_α, U_α) and (ϕ_β, U_β) be overlapping principal bundle charts. For each $p \in U_\alpha \cap U_\beta$*

$$\Phi_\alpha|_{\wp^{-1}(p)} \circ \left(\Phi_\beta|_{\wp^{-1}(p)} \right)^{-1} (g) = g_{\alpha\beta}(p)g$$

where the $g_{\alpha\beta}$ are given as above.

Proof. Let $\left(\Phi_\beta|_{\wp^{-1}(p)} \right)^{-1} (g) = u$. Then $g = \Phi_\beta(u)$ and so $\Phi_\alpha|_{\wp^{-1}(p)} \circ \Phi_\beta|_{\wp^{-1}(p)} (g) = \Phi_\alpha(u)$. On the other hand, $u \in \wp^{-1}(p)$ and so

$$\begin{aligned} g_{\alpha\beta}(p)g &= \Phi_\alpha(u)\Phi_\beta(u)^{-1}g = \Phi_\alpha(u)\Phi_\beta(u)^{-1}\Phi_\beta(u) \\ &= \Phi_\alpha(u) = \Phi_\alpha|_{\wp^{-1}(p)} \circ \Phi_\beta|_{\wp^{-1}(p)} (g) \end{aligned}$$

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From this lemma we see that the structure group of a principal bundle is G acting on itself by left translation. Conversely if $(P, \wp, M, G)$ is a fiber bundle with a $G-$atlas with G acting by left translation then $(P, \wp, M, G)$ is a principal bundle. To see this we only need to exhibit the free right action. Let $u \in P$ and choose a chart from (ϕ_α, U_α) the $G-$atlas. Then let $ug := \phi_\alpha^{-1}(p, \Phi_\alpha(u)g)$ where $p = \wp(u)$. We need to show that this is well defined so let (ϕ_β, U_β) be another such bundle chart with $p = \wp(u) \in U_\beta \cap U_\alpha$. Then if $u_1 := \phi_\beta^{-1}(p, \Phi_\beta(u)g)$ we have

$$\begin{aligned} \phi_\alpha(u_1) = \phi_\alpha\phi_\beta^{-1}(p, \Phi_\beta(u)g) = \\ (p, g_{\alpha\beta}(p)\Phi_\beta(u)g) = (p, \Phi_\alpha(u)g) \end{aligned}$$

so that $u_1 = \phi_\alpha^{-1}(p, \Phi_\alpha(u)g) = ug$. It is easy to see that this action is free. Furthermore, since

$$\begin{aligned} \phi_\alpha^{-1}(p, \Phi_\alpha(ug)) &= \phi_\alpha^{-1} \circ \phi_\alpha (ug) \\ &= ug := \phi_\alpha^{-1}(p, \Phi_\alpha(u)g) \end{aligned}$$

we see that $\Phi_\alpha(ug) = \Phi_\alpha(u)g$ as required by the definition of principal bundle.

Obviously the frame bundles of vector bundles are examples of principal bundles. We also have the Hopf Bundles described in the next example and the following exercise.

Example 5.8 (Hopf Bundles) *Recall the Hopf map $\wp : S^{2n-1} \to \mathbb{C}P^{n-1}$. The quadruple $(S^{2n-1}, \wp, \mathbb{C}P^{n-1}, U(1))$ is a principal fiber bundle. We have already defined the left action of $U(1)$ on S^{2n-1} in Example 4.27. Since $U(1)$ is abelian we may take this action to also be a right action. Recall that in this context, we have $S^{2n-1} = \{\xi \in \mathbb{C}^n : |\xi| = 1\}$ where for $\xi = (z^1, ..., z^n)$ we have $|\xi| = \sum \bar{z}^i z^i$. The right action of $U(1) = S^1$ on S^{2n-1} is $(\xi, g) \longmapsto \xi g = (z^1 g, ..., z^n g)$. It is clear that $\wp(\xi a) = \wp(\xi)$. To finish the verification that $(S^{2n-1}, \wp, \mathbb{C}P^{n-1}, U(1))$ is a principal bundle we exhibit appropriate principal bundle charts. For each $k = 1, 2, ..., n$ we let $U_k := \{[z^1, ..., z^n] \in \mathbb{C}P^{n-1} : z^k \neq 0\}$ and we let $\psi_k : \wp^{-1}(U_k) \to U_k \times U(1)$ be defined by $\psi_k := (\wp, \Psi_k)$ where*

$$\Psi_k(\xi) = \Psi_k(z^1, ..., z^n) := \left|z^k\right|^{-1} z^k$$

we leave it to the reader to show that $\psi_k := (\wp, \Psi_k)$ is a diffeomorphism. Now we have for $g \in U(1)$

$$\begin{aligned}\Psi_k(\xi g) &= \left|z^k g\right|^{-1} \left(z^k g\right) = \left|z^k\right|^{-1} \left(z^k g\right) \\ &= \left(\left|z^k\right|^{-1} z^k\right) g = \Psi_k(\xi) g\end{aligned}$$

as desired. Let us compute the transition cocycle $\{g_{ij}\}$. For $p = [\xi] \in U_i \cap U_i$ have

$$g_{ij}(p) = \Psi_i(\xi)\Psi_j(\xi)^{-1} = \left|z^i\right|^{-1} z^i \left(z^j\right)^{-1} \left|z^j\right| \in U(1)$$

Exercise 5.10 *By analogy with the above example show that we have principal bundles $(S^{n-1}, \wp, \mathbb{R}P^{n-1}, \mathbb{Z}_2)$ and $(S^{4n-1}, \wp, \mathbb{H}P^{n-1}, Sp(1))$. Show that in the quaternionic case $g_{ij}(p) = \left|q^i\right|^{-1} q^i \left(q^j\right)^{-1} \left|q^j\right|$ for $p = [q^1, ..., q^n]$ and that the order matters in this case.*

If (U, ϕ) is principal bundle chart for a principal bundle $(P, \wp, M, G)$, then for each fixed $g \in G$, the map $\sigma_{\phi,g} : p \mapsto \phi^{-1}(p, g)$ is a smooth local section. Conversely if $\sigma : U \to P$ is a smooth local section then we let $f_\sigma : U \times G \to \wp^{-1}(U)$ be defined by $f_\sigma(p, g) = \sigma(p)g$. This is a diffeomorphism and if we denote the inverse by $\phi : \wp^{-1}(U) \to U \times G$ we have $\phi = (\wp, \Phi)$ for a uniquely determined smooth map $\Phi : U \to G$. If $p = \wp(u)$ we have

$$\begin{aligned}\phi(ug) &= (p, \Phi(ug)) \\ ug &= \phi^{-1}(p, \Phi(ug))\end{aligned}$$

while

$$\begin{aligned}\phi^{-1}(p, \Phi(u)g) &= f_\sigma\left(p, \Phi(u)g\right) \\ &= \sigma(p)\left(\Phi(u)g\right) = \left(\sigma(p)\Phi(u)\right)g \\ &= f_\sigma\left(p, \Phi(u)\right)g = \phi^{-1}\left(p, \Phi(u)\right)g \\ &= ug = \phi^{-1}(p, \Phi(ug))\end{aligned}$$

since ϕ^{-1} is a bijection we have $\Phi(ug) = \Phi(u)g$. Thus the section σ has given rise to a principal bundle chart (U, ϕ).

Proposition 5.7 *If $\wp : P \to M$ is a surjective submersion and a Lie group G acts freely on P such that for each $p \in M$ the orbit of p is exactly $\wp^{-1}(p)$ then $(P, \wp, M, G)$ is a principal bundle.*

Proof. Let us assume (without loss) that the action is a right action since it can always be converted into such by group inversion if needed. We use Proposition 3.4: For each point $p \in M$ there is a local section $\sigma : U \to P$ on some neighborhood U containing p. Consider the map $f_\sigma : U \times G \to \wp^{-1}(U)$ given by $f_\sigma(p, g) = \sigma_\alpha(p)g$. One can check that this map is injective and has an invertible tangent map at each point of U. Thus by the inverse mapping theorem one may choose a possibly smaller open neighborhood of p on which f_σ is a fiber preserving diffeomorphism. Choose a family of local sections $\{\sigma_\alpha : U_\alpha \to P\}$ such that $\cup_\alpha U_\alpha = M$ and such that for each α the map $\phi_\alpha := f_{\sigma_\alpha}^{-1} : \wp^{-1}(U_\alpha) \to U \times G$ is a fiber preserving diffeomorphism and hence bundle chart. The same argument given before the statement of the proposition works and shows that the bundle charts (U_α, ϕ_α) are actually principal bundle charts. ∎

Combining this with our results on proper free actions we obtain the following

Corollary 5.1 *If a Lie group G acts properly and freely on M (on the right) then $(M, \pi, M/G, G)$ is a principal bundle. In particular, if H is a closed subgroup of a Lie group G, then $(G, \pi, G/H, H)$ is a principal bundle (with structure group H).*

Definition 5.23 *Let $(P_1, \wp_1, M_1, G)$ and $(P_2, \wp_2, M_2, G)$ be two principal G-bundles. A bundle morphism $\widetilde{f} : P_1 \to P_2$ over f is called a **principal G-bundle morphism** if*

$$\widetilde{f}(u \cdot g) = \widetilde{f}(u) \cdot g$$

for all $g \in G$ and $u \in P$.

Exercise 5.11 *Show that if $(P_1, \wp_1, M_1, G)$ and $(P_2, \wp_2, M_2, G)$ are principal G-bundles and $\widetilde{f} : P_1 \to P_2$ is a principal G-bundle morphism over a* diffeomorphism *f then $\widetilde{f}$ is a diffeomorphism.*

If $M_1 = M_2 = M$ and the induced map f is the identity $id_M : M \to M$ then from the last exercise $\widetilde{f}$ is a diffeomorphism and hence a bundle equivalence (or bundle isomorphism over M) with the property $\widetilde{f}(u \cdot g) = \widetilde{f}(u) \cdot g$ for all $g \in G$ and $u \in P$. In this case we call $\widetilde{f}$ a **principal G-bundle equivalence** and the two bundles are **equivalent** principal bundles. A principal G-bundle equivalence from a principal bundle to itself is called a **principal bundle automorphism** or also a **(global) gauge transformation**.

We have seen that a principal $G-$bundle atlas $\{(U_\alpha, \phi_\alpha)\}$ is associated to a cocycle $\{g_{\alpha\beta}\}$. From this cocycle and the left action of G on itself we may construct a bundle which has $\{g_{\alpha\beta}\}$ as transition cocycle. In fact, recall that in the construction we formed the total space by putting an equivalence relation on the set $\Sigma := \bigcup_\alpha \{\alpha\} \times U_\alpha \times G$ where $(\alpha, p, g) \in \{\alpha\} \times U_\alpha \times G$ is equivalent to $(\beta, p', g') \in \{\beta\} \times U_\beta \times G$ if and only if $p = p'$ and $g' = g_{\alpha\beta}(p) \cdot g$. Now if we define a right action on the total space of the constructed bundle by $[\alpha, p, g_1] \cdot g = [\alpha, p, g_1 g]$ then this is well defined, smooth and makes the constructed bundle a principal $G-$bundle equivalent to the original principal $G-$bundle.

Exercise 5.12 *Prove the last assertion above.*

Thus we see that G-cocycles on a smooth manifold M give rise to principal G-bundles and conversely. If we start with two G-cocycles on M then we make ask whether the principal G-bundles constructed from these cocycles are equivalent or not. First notice that the constructed bundles will have principal bundle atlases with the respective original transition cocycles. Thus we are led to the following related question: What conditions on the transition cocycles arising from principal bundle atlases on two principal G-bundles will ensure that the bundles are equivalent principal G-bundles? By restricting the trivializing maps to open sets of a common refinement we obtain new atlases and so we may as well assume from the start that the respective principal bundle atlases are defined on the same cover of M.

Proposition 5.8 *Let $(P_1, \wp_1, M, G)$ and $(P_2, \wp_2, M, G)$ be principal G-bundles with principal bundle atlases $\{(\phi_\alpha, U_\alpha)\}$ and $\{(\phi'_\alpha, U_\alpha)\}$ respectively. Then $(P_1, \wp_1, M, G)$ is equivalent to $(P_2, \wp_2, M, G)$ if and only if there exist a family of (smooth) maps $\tau_\alpha : U_\alpha \to G$ such that $g'_{\alpha\beta}(p) = (\tau_\alpha(p))^{-1} g_{\alpha\beta}(p)\tau_\alpha(p)$ for all $p \in U_\alpha \cap U_\beta$ and for all nonempty intersections $U_\alpha \cap U_\beta$. (Here $\{g_{\alpha\beta}\}$ is the cocycle associated to $\{(\phi_\alpha, U_\alpha)\}$ and $\{g'_{\alpha\beta}\}$ is the cocycle associated to $\{(\phi'_\alpha, U_\alpha)\}$).*

Sketch of Proof. First suppose that P_1 and P_2 are equivalent principal G-bundles and let $\widetilde{f} : P_1 \to P_2$ be a bundle equivalence. Let $p \in U_\alpha$ and choose some $u \in \wp_1^{-1}(p)$ so that $\widetilde{f}(u) \in \wp_2^{-1}(p)$. Write $\phi_\alpha = (\wp_1, \Phi_\alpha)$ and $\phi'_\alpha = (\wp_2, \Phi'_\alpha)$. One can easily show that $\Phi_\alpha(u)(\Phi'_\alpha(\widetilde{f}(u)))^{-1}$ is an element of G that is independent of the choice of $u \in \wp_1^{-1}(p)$. Define $\tau_\alpha : U_\alpha \to G$ by

$$\tau_\alpha(p) := \Phi_\alpha(u)(\Phi'_\alpha(\widetilde{f}(u)))^{-1}$$

where $u \in \wp_1^{-1}(p)$. Do this for all α. Suppose that $p \in U_\alpha \cap U_\beta$. Then we have $(\tau_\beta(p))^{-1} = \Phi'_\alpha(\widetilde{f}(u))\,(\Phi_\alpha(u))^{-1}$. Using the definitions of $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ we have immediately

$$g'_{\alpha\beta}(p) = (\tau_\alpha(p))^{-1} g_{\alpha\beta}(p)\tau_\alpha(p)$$

Conversely, given the maps $\tau_\alpha : U_\alpha \to G$ satisfying $g'_{\alpha\beta}(p) = (\tau_\alpha(p))^{-1} g_{\alpha\beta}(p)\tau_\alpha(p)$ we define, for each α, a map $f_\alpha : \wp_1^{-1}(U_\alpha) \to \wp_2^{-1}(U_\alpha)$ by

$$f_\alpha(u) := (\Phi'_\alpha)^{-1}\left(p, (\tau_\alpha(p))^{-1}\Phi_\alpha(u)\right)$$

Next check that $f_\alpha(p) = f_\beta(p)$ so that there is a well defined map $\widetilde{f} : P_1 \to P_2$ such that $f_\alpha(p) = \widetilde{f}(p)$ whenever $p \in U_\alpha$. Finally, check that $\widetilde{f}(u \cdot g) = \widetilde{f}(u) \cdot g$.
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Let $\wp : P \to M$ be a principal $G-$bundle and suppose that we are given a smooth left action $\lambda : G \times F \to F$ on some smooth manifold F. Define a right action of G on $P \times F$ according to

$$(u, y) \cdot g := (ug, g^{-1}y) = (ug, \lambda(g^{-1}, y))$$

Denote the orbit space of this action by $P \times_\lambda F$ (or $P \times_G F$). Let $\widetilde{\wp} : P \times F \to P \times_\lambda F$ denote the quotient map. One may check that there is a unique map $\pi : P \times_\lambda F \to M$ such that $\pi \circ \widetilde{\wp} = \wp$ and so we have a commutative diagram:

$$\begin{array}{ccc} P \times F & \xrightarrow{pr_1} & P \\ \widetilde{\wp} \downarrow & & \downarrow \wp \\ P \times_\lambda F & \xrightarrow{\pi} & M \end{array}$$

Next we show that $(P \times_\lambda F, \pi, M, F)$ is a fiber bundle (actually a $(G, \lambda)-$bundle). It is said to be **associated** to the principal bundle P. Bundles constructed in this way are called **associated bundles**.

Theorem 5.4 *Referring to the above diagram and notations, $P \times_\lambda F$ is a smooth manifold and*
i) $(P \times_\lambda F, \pi, M, F)$ is a fiber bundle and for every principal bundle atlas $\{(U_\alpha, \phi_\alpha)\}$ there is a corresponding bundle atlas $\{(U_\alpha, \widetilde{\phi}_\alpha)\}$ for $P \times_\lambda F$ such that

$$\widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1}(p, y) = (p, \lambda(g_{\alpha\beta}(p), y)) \text{ if } p \in U_\alpha \cap U_\beta \text{ and } y \in F$$

ii) $(P \times F, \widetilde{\wp}, P \times_\lambda F, G)$ is principal bundle with the right action given by $(u, y) \cdot g := (ug, g^{-1}y)$.
iii) $P \times F \xrightarrow{pr_1} P$ is a principal bundle morphism along π.

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be a principal bundle atlas for $\wp : P \to M$. Note that $\widetilde{\wp}\left(\wp^{-1}(U_\alpha) \times F\right) = \pi^{-1}(U_\alpha)$. For each α, define $\widetilde{\Phi}_\alpha : \pi^{-1}(U_\alpha) \to F$ by requiring that $\widetilde{\Phi}_\alpha \circ \widetilde{\wp}(u, y) = \Phi_\alpha(u) \cdot y$ for all $(u, y) \in \wp^{-1}(U_\alpha) \times F$ and then let $\widetilde{\phi}_\alpha := (\pi, \widetilde{\Phi}_\alpha)$ on $\pi^{-1}(U_\alpha)$. The map $\widetilde{\phi}_\alpha$ is clearly surjective onto $U_\alpha \times F$ since given $(p, y) \in U_\alpha \times F$ we may choose $u_0 \in \wp^{-1}(p)$ with $\Phi_\alpha(u_0) = e$ and then $\widetilde{\phi}_\alpha(\widetilde{\wp}(u_0, y)) = (p, \Phi_\alpha(u_0) \cdot y) = (p, y)$. We want to show next that $\widetilde{\phi}_\alpha$ is injective. We define an inverse for $\widetilde{\phi}_\alpha$. For every $p \in U_\alpha$ let $\sigma_\alpha(p) := \phi_\alpha^{-1}(p, e)$. Then we have

$$\sigma_\alpha(p) \cdot \Phi_\alpha(u) = \phi_\alpha^{-1}(p, e) \cdot \Phi_\alpha(u) = \phi_\alpha^{-1}(p, \Phi_\alpha(u)) = u$$

Define $\eta_\alpha : U_\alpha \times F \to \pi^{-1}(U_\alpha)$ by $\eta_\alpha(p, y) := \widetilde{\wp}(\sigma_\alpha(p), y)$. Now

$$\begin{aligned} \eta_\alpha \circ \widetilde{\phi}_\alpha(\widetilde{\wp}(u, y)) &= \eta_\alpha(p, \Phi_\alpha(u) \cdot y) = \widetilde{\wp}(\sigma_\alpha(p), \Phi_\alpha(u) \cdot y) \\ &= \widetilde{\wp}(\sigma_\alpha(p) \cdot \Phi_\alpha(u), y) = \widetilde{\wp}(u, y) \end{aligned}$$

Thus η_α is a right inverse for $\widetilde{\phi}_\alpha$ and so $\widetilde{\phi}_\alpha$ is injective. It is easily checked that η_α is also a left inverse for $\widetilde{\phi}_\alpha$. Indeed, $\widetilde{\phi}_\alpha \circ \eta_\alpha(p,y) = (p, \widetilde{\Phi}_\alpha(\sigma_\alpha(p)) \cdot y) = (p,y)$.

Next we check the overlaps.

$$\begin{aligned}
\widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1}(p,y) &= \widetilde{\phi}_\alpha \circ \eta_\beta(p,y) = \widetilde{\phi}_\alpha\left(\widetilde{\wp}(\sigma_\beta(p),y)\right) \\
&= (p, \Phi_\alpha(\sigma_\beta(p)) \cdot y) = (p, \Phi_\alpha(\phi_\beta^{-1}(p,e)) \cdot y) \\
&= (p, \Phi_\alpha|_p \circ \Phi_\beta|_p^{-1}(e)) \cdot y) \\
&= (p, g_{\alpha\beta}(p) \cdot e \cdot y) = (p, g_{\alpha\beta}(p) y)
\end{aligned}$$

This shows that the transitions mappings have the stated form and that the overlap maps $\widetilde{\phi}_\alpha \circ \widetilde{\phi}_\beta^{-1}$ are smooth. The family $\{(U_\alpha, \phi_\alpha)\}$ provides both the induced smooth structure and is also a bundle atlas.

Since $\phi_\alpha \circ \widetilde{\wp}(u,p) = (\pi, \widetilde{\Phi}_\alpha) \circ \widetilde{\wp}(u,p) = (\wp(u), \Phi_\alpha(u)y)$ in the domain of every bundle chart (U_α, ϕ_α), it follows that $\widetilde{\wp}$ is smooth.

We leave it to the reader to verify that $(P \times F, \widetilde{\wp}, P \times_\lambda F, G)$ is a principal G-bundle. Notice that while the map $pr_1 : P \times F \to P$ is clearly a bundle map along π we also have $pr_1((u,y) \cdot g) = pr_1\left((u \cdot g, g^{-1}y)\right) = u \cdot g = pr_1(u,y) \cdot g$ and so that pr_1 is in fact a principal bundle morphism. ∎

We have now seen that given a principal $G-$bundle one may construct various fiber bundles with G-structures. Let us look at the converse situation. Suppose that (E, π, M, F) is a fiber bundle. Suppose that this bundle has a $(G,\lambda)-$atlas $\{(U_\alpha, \phi_\alpha)\}$ with associated $G-$valued cocycle of transition functions $\{\, g_{\alpha\beta}\}$. Using Theorem 5.1 one may construct a bundle with typical fiber G by using left translation as the action. The resulting bundle is then a principal bundle $(P, \wp, M, G)$ and it turns out that $P \times_\lambda F$ is equivalent to the original bundle E. If (E, π, M, V) a vector bundle and we use the standard $Gl(\mathrm{V})$-cocycle $\{\Phi_{\alpha\beta}\}$ associated to a VB-atlas then the principal bundle obtained by the above construction is (equivalent to) the linear frame bundle $F(E)$. Letting $Gl(\mathrm{V})$ act on V according to the standard action we have $F(E) \times_{Gl(\mathrm{V})} \mathrm{V}$ which is equivalent to the original bundle (E, π, M, V). More generally, if $\rho : G \to Gl(\mathrm{V})$ is a Lie group homomorphism treating ρ as a linear action we can form $P \times_\rho F$. Clearly what we have is another way of looking at bundle construction. The principal bundle takes the place of the cocycle of transition maps. For example, if we let $Gl(\mathrm{V})$ act on V^* according to $\rho(g,v) = g \cdot v := \left(g^{-1}\right)^t v$ then $F(E) \times_\rho \mathrm{V}^*$ is (equivalent to) the dual bundle E^*.

5.5 Degrees of locality

There is an interplay in geometry and topology between local and global data. We now look at one aspect of this. To set up our discussion, suppose that s is a section of a smooth vector bundle $\pi : E \to M$ with typical fiber V. For simplicity we choose a basis and identify the typical fiber with $\mathbb{R}^k$. Let us focus our attention near a point p which is contained in an open set U over which the bundle is trivialized. The trivialization provides a local frame field $\{\sigma_1, ..., \sigma_k\}$

on U. A section σ has a local expression $s = \sum s^i \sigma_i$ for some smooth functions s^i. Now the component functions s^i together give a smooth map $(s_i) : U \to \mathbb{R}^k$. We may assume without loss of generality that U is the domain of a chart $\mathtt{x}$ for the manifold M. The map $(s_i) \circ \mathtt{x}^{-1} : \mathtt{x}(U) \to \mathbb{R}^k$ has a Taylor expansion centered at $\mathtt{x}(p)$. It will be harmless to refer to this as the Taylor expansion around p. Now, we say that two sections σ_1 and σ_2 have the "same k-jet at p" if in any chart these two sections have Taylor expansions which agree up to and including terms of order k. This puts an equivalence relation on sections defined near p.

Consider two vector bundles $E_1 \to M$ and $E_2 \to M$. Suppose we have a map $\mu : \Gamma(M, E_1) \to \Gamma(M, E_2)$ that is not necessarily linear over $C^\infty(M)$ or even $\mathbb{R}$. We say that μ is **local** if the support of $F(s)$ is contained in the support of s for all $s \in \Gamma(M, E_1)$. We can ask for more refined kinds of locality. For any $s \in \Gamma(M, E_1)$ we have a section $\mu(s)$ and its value $\mu(s)(p) \in E_2$ at some point $p \in M$. What determines the value $\mu(s)(p)$? Let us consider in turn the following situations:

1. It just might be the case that whenever two sections s_1 and s_2 agree on some neighborhood of p then $\mu(s_1)(p) = \mu(s_2)(p)$. So all that matters for determining $\mu(s)(p)$ is the behavior of s in any arbitrarily small open set containing p. To describe this we say that $\mu(s)(p)$ only depends on the "germ" of s at p.

2. Certainly if s_1 and s_2 agree on some neighborhood of p then they both have the same Taylor expansion at p (as seen in any local VB-charts). The reverse is not true however. Suppose that whenever two section s_1 and s_2 have Taylor series that agree up to and including terms of order k then $\mu(s_1)(p) = \mu(s_2)(p)$. Then we say that $(\mu(s))(p)$ depends only on the $k-$jet of s at p.

3. Finally, it might be the case that $\mu(s_1)(p) = \mu(s_2)(p)$ exactly when $s_1(p) = s_2(p)$.

Of course it is also possible that none of the above hold at any point. Notice that as we go down the list we are saying that the information needed to determine $\mu(s)(p)$ is becoming more and more localized in some sense. A vector field can be viewed as an $\mathbb{R}$-linear map $C^\infty(M) \to C^\infty(M)$ and since $Xf(p) = Xg(p)$ exactly when $df(p) = dg(p)$ we see that $Xf(p)$ depends only on the $1-$jet of f at p. But this cannot be the whole story since two functions might have the same differential without sharing the same 1-jet at p since they might not agree at the 0-th jet level (if may be that $f(p) \neq g(p)$).

We now restrict our attention to (local) $\mathbb{R}$-linear maps $L : \Gamma(M, E_1) \to \Gamma(M, E_2)$. We start at the bottom, so to speak, with $0-$th order operators. One way that $0-$th order operators arise is from bundle maps. If $\tau : E_1 \to E_2$ is a bundle map (over the identity $M \to M$) then we get an induced map $\Gamma\tau : \Gamma(M, E_1) \to \Gamma(M, E_2)$ on the level of sections: If $s \in \Gamma(M, E_1)$ then

$$\Gamma\tau(s) := s \circ \tau$$

Notice the important property that $\Gamma\tau(fs) = f\Gamma\tau(s)$ and so $\Gamma\tau$ is $C^\infty(M)$ linear (a module homomorphism). Conversely, if $L : \Gamma(M, E_1) \to \Gamma(M, E_2)$ is $C^\infty(M)$ linear then $(Ls)(p)$ depends only on the value of $s(p)$ and as we see presently this means that L determines a bundle τ map such that $\Gamma\tau = L$. We shall prove a bit more general result which extends to multi-linear maps.

Proposition 5.9 *Let $p \in M$ and $\tau : \Gamma(M, E_1) \times \cdots \times \Gamma(M, E_N) \to \Gamma(M, E)$ be a $C^\infty(M)-$multilinear map. Let $s_i, \bar{s}_i \in \Gamma(M, E_i)$ smooth sections such that $s_i(p) = \bar{s}_i(p)$ for $1 \le i \le N$; Then we have that*

$$\tau(s_1, ..., s_N)(p) = \tau(\bar{s}_1, ..., \bar{s}_N)(p)$$

Proof. The proof will follow easily if we can show that $\tau(s_1, ..., s_N)(p) = 0$ whenever one of $s_i(p)$ is zero. We shall assume for simplicity of notation that $N = 3$. Now suppose that $s_1(p) = 0$. If $e_1, ..., e_N$ is a frame field over $U \subset M$ (with $p \in U$) then $s_1|_U = \sum s^i e_i$ for some smooth functions $s^i \in C^\infty(U)$. Let β be a bump function with support in U. Then $\beta\ s_1|_U$ and $\beta^2\ s_1|_U$ extend by zero to elements of $\Gamma(M, E_1)$which we shall denote by βs_1 and $\beta^2 s_1$. Similarly, βe_i and $\beta^2 e_i$ are globally defined sections, βs^i is a global function and $\beta s_1 = \sum \beta s^i \beta e_i$. Thus

$$\begin{aligned}
\beta^2 \tau(s_1, s_2, s_3) &= \tau(\beta^2 s_1, s_2, s_3) \\
&= \tau(\sum \beta s^i \beta e_i, s_2, s_3) \\
&= \sum \beta \tau(s^i \beta e_i, s_2, s_3).
\end{aligned}$$

Now since $s_1(p) = 0$ we must have $s^i(p) = 0$. Also recall that $\beta(p) = 1$. Plugging p into the formula above we obtain $\tau(s_1, s_2, s_3)(p) = 0$. A similar argument holds when $s_2(p) = 0$ or $s_3(p) = 0$.

Assume that $s_i(p) = \bar{s}_i(p)$ for $1 \le i \le 3$. Then we have

$$\begin{aligned}
&\tau(\bar{s}_1, \bar{s}_2, \bar{s}_3) - \tau(s_1, s_2, s_3) \\
&= \tau(\bar{s}_1 - s_1, \bar{s}_2, \bar{s}_3) + \tau(s_1, s_2, s_3) + \tau(s_1, \bar{s}_2 - s_2, s_3) \\
&+ \tau(\bar{s}_1, s_2, \bar{s}_3) + \tau(\bar{s}_1, s_2, \bar{s}_3 - s_3)
\end{aligned}$$

Since $\bar{s}_1 - s_1, \bar{s}_2 - s_2$ and $\bar{s}_3 - s_3$ are all zero at p we obtain the result. ∎

By the above, linearity over $C^\infty(M)$ on the level of sections corresponds to bundle maps on the vector bundle level. Thus whenever we have a $C^\infty(M)-$multilinear map $\tau : \Gamma(M, E_1) \times \cdots \times \Gamma(M, E_N) \to \Gamma(M, E)$ we also have an $\mathbb{R}-$multilinear map $E_{1p} \times \cdots \times E_{Np} \to E_p$ (which we shall often denote by the symbol τ_p):

$$\tau_p(v_1, ..., v_N) := \tau(s_1, ..., s_N)(p) \text{ for any sections } s_i \text{ with } s_i(p) = v_i$$

The individual maps $E_{1p} \times \cdots \times E_{Np} \to E_p$ combine to give a vector bundle morphism

$$E_1 \oplus \cdots \oplus E_N \to E$$

If an $\mathbb{R}$-multilinear map $\tau : \Gamma(M, E_1) \times \cdots \times \Gamma(M, E_N) \to \Gamma(M, E)$ is actually $C^\infty(M)-$linear in one or more variable then we say that τ is **tensorial** in those variables. If τ is tensorial in all variables say that τ is **tensorial**.

It is worth our while to look more closely at the case $N = 1$ above. Suppose that $\tau : \Gamma(M, E) \to \Gamma(M, F)$ is tensorial ($C^\infty-$linear). Then for each p we get a linear map $\tau_p : E_p \to F_p$ and so a bundle morphism $E \to F$ but also we may consider the assignment $p \mapsto \tau_p$ as a section of the bundle whose fiber at p is the space of linear transformations $L(E_p, F_p)$ (sometimes denoted $\mathrm{Hom}(E_p, F_p)$). This bundle is denoted $L(E, F)$ or $\mathrm{Hom}(E, F)$ and is isomorphic to the bundle $F \otimes E^*$.

5.6 Sheaves

Now we have seen that the section $\Gamma(M, E)$ of a vector bundle form a module over the smooth functions $C^\infty(M)$. It is important to realize that having a vector bundle at hand not only provides a module but a family of modules parameterized by the open subsets of M. How are these modules related to each other?

Consider a local section $\sigma : M \to E$ of a vector bundle E. Given any open set $U \subset M$, we may always produce the restricted section $\sigma|_U : U \to E$. This gives us a family of sections; one for each open set U. To reverse the situation, suppose that we have a family of sections $\sigma_U : U \to E$ where U varies over the open sets (or just a cover of M). When is it the case that such a family is just the family of restrictions of some (global) section $\sigma : M \to E$? To help with these kinds of questions and to provide a language that will occasionally be convenient we will introduce another formalism. This is the formalism of sheaves and presheaves. The formalism of sheaf theory is convenient for conveying certain concepts concerning the interplay between the local and the global. This will be our main use. More serious use of sheaf theory is found in cohomology theory and is especially useful in complex geometry. Sheaf theory also provides a very good framework within which to develop the foundations of supergeometry which is an extension of differential geometry that incorporates the important notion of "fermionic variables". A deep understanding of sheaf theory is not necessary for most of what we do here and it would be enough to acquire a basic familiarity with the definitions.

Definition 5.24 *A* ***presheaf of abelian groups (resp. rings etc.)*** *on a manifold (or more generally a topological space) M is an assignment $U \to \mathcal{M}(U)$ to each open set $U \subset M$ together with a family of abelian group homomorphisms (resp. ring homomorphisms etc.) $r^U_V : \mathcal{M}(U) \to \mathcal{M}(V)$ for each nested pair $V \subset U$ of open sets and such that*

Presheaf 1 $r^V_W \circ r^U_V = r^U_W$ *whenever $W \subset V \subset U$.*

Presheaf 2 $r^V_V = \mathrm{id}_V$ *for all open $V \subset M$.*

Definition 5.25 *Let $\mathcal{M}$ be a presheaf and $\mathcal{R}$ a presheaf of rings. If for each open $U \subset M$ we have that $\mathcal{M}(U)$ is a module over the ring $\mathcal{R}(U)$ and if the multiplication map $\mathcal{R}(U) \times \mathcal{M}(U) \to \mathcal{M}(U)$ commutes with the restriction maps r_W^U then we say that $\mathcal{M}$ is a presheaf of modules over $\mathcal{R}$.*

Definition 5.26 *A presheaf homomorphism $h : \mathcal{M}_1 \to \mathcal{M}_2$ is an is an assignment to each open set $U \subset M$ an abelian group (resp. ring, module, etc.) morphism $h_U : \mathcal{M}_1(U) \to \mathcal{M}_2(U)$ such that whenever $V \subset U$ then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}_1(U) & \stackrel{h_U}{\to} & \mathcal{M}_2(U) \\ r_V^U \downarrow & & r_V^U \downarrow \\ \mathcal{M}_1(V) & \stackrel{h_V}{\to} & \mathcal{M}_2(V) \end{array} .$$

Note we have used the same notation for the restriction maps of both presheaves.

Definition 5.27 *We will call a presheaf a* ***sheaf*** *if the following properties hold whenever $U = \bigcup_{U_\alpha \in \mathcal{U}} U_\alpha$ for some collection of open sets $\mathcal{U}$.*

Sheaf 1 *If $s_1, s_2 \in \mathcal{M}(U)$ and $r_{U_\alpha}^U s_1 = r_{U_\alpha}^U s_2$ for all $U_\alpha \in \mathcal{U}$ then $s_1 = s_2$.*

Sheaf 2 *If $s_\alpha \in \mathcal{M}(U_\alpha)$ and whenever $U_\alpha \cap U_\beta \neq \emptyset$ we have*

$$r_{U_\alpha \cap U_\beta}^U s_\alpha = r_{U_\alpha \cap U_\beta}^U s_\beta$$

then there exists $s \in \mathcal{M}(U)$ such that $r_{U_\alpha}^U s = s_\alpha$.

If we need to indicate the space M involved we will write $\mathcal{M}_M$ instead of $\mathcal{M}$.

Definition 5.28 *A morphism of sheaves is a morphism of the underlying presheaf.*

The assignment $C^\infty(.,M) : U \mapsto C^\infty(U)$ is a presheaf of rings. This sheaf will also be denoted by $\mathcal{C}_M^\infty$. The best and most important example of a sheaf of modules over $C^\infty(.,M)$ is the assignment $\Gamma(E,.) : U \mapsto \Gamma(E,U)$ for some vector bundle $E \to M$ and where by definition $r_V^U(s) = s|_V$ for $s \in \Gamma(E,U)$. In other words r_V^U is just the restriction map. Let us denote this (pre)sheaf by $\Gamma_E : U \mapsto \Gamma_E(U) := \Gamma(E,U)$.

Notation 5.2 *Many if not most of the constructions operations we introduce for sections of vector bundles are really also operations appropriate to the (pre)sheaf category. Naturality with respect to restrictions is one of the features that is often not even mentioned (sometime this is precisely because it seems obvious). This is the inspiration for a slight twist on our notation.*

	Global	local	Sheaf notation
functions on M	$C^\infty(M)$	$C^\infty(U)$	C_M^∞
Vector fields on M	$\mathfrak{X}(M)$	$\mathfrak{X}(U)$	$\mathfrak{X}_M$
Sections of E	$\Gamma(E)$	$\Gamma(U,E)$	Γ_E

where $C_M^\infty : U \mapsto C_M^\infty(U) := C^\infty(U), \mathfrak{X}_M : U \mapsto \mathfrak{X}_M(U) := \mathfrak{X}(U)$ and so on.

Notation 5.3 *For example, when we say that $D : C^\infty_M \to C^\infty_M$ is a derivation we mean that D is actually a family of algebra derivations $D_U : C^\infty_M(U) \to C^\infty_M(U)$ indexed by open sets U such that we have naturality with respect to restrictions. I.e. the diagrams of the form below for $V \subset U$ commute:*

$$\begin{array}{ccc} C^\infty_M(U) & \overset{D_U}{\to} & C^\infty_M(U) \\ r^U_V \downarrow & & r^U_V \downarrow \\ C^\infty_M(V) & \overset{D_V}{\to} & C^\infty_M(V) \end{array}.$$

It is easy to see that all of the following examples are sheaves. In each case the maps $r^U_{U_\alpha}$ are just the restriction maps.

Example 5.9 (Sheaf of holomorphic functions) *Sheaf theory really shows its strength in complex analysis. This example is one of the most studied. However, we have not studied notion of a complex manifold and so this example is for those readers with some exposure to complex manifolds. Let M be a complex manifold and let $\mathcal{O}_M(U)$ be the algebra of holomorphic functions defined on U. Here too, $\mathcal{O}_M$ is a sheaf of modules over itself. Where as the sheaf $\mathcal{C}^\infty_M$ always has global sections, the same is not true for $\mathcal{O}_M$. The sheaf theoretic approach to the study of obstructions to the existence of global holomorphic functions has been very successful.*

Recall that $s_1 \in \Gamma_E(U)$ and $s_2 \in \Gamma_E(V)$ determine the same germ of sections at p if there is an open set $W \subset U \cap V$ such that $r^U_W s_1 = r^V_W s_2$. Now on the union

$$\bigcup_{p \in U} \Gamma_E(U)$$

we impose the equivalence relation $s_1 \sim s_2$ if and only if s_1 and s_2 determine the same germ of sections at p. The set of equivalence classes (called germs of section at p) is an abelian group in the obvious way and is denoted Γ^E_p. If we are dealing with a sheaf of rings then Γ^E_p is a ring. The set $\Gamma_E((U)) = \bigcup_{p \in U} \Gamma^E_p$ is called the sheaf of germs and can be given a topology so that the projection $pr : \Gamma_E((U)) \to M$ defined by the requirement that $pr([s]) = p$ if and only if $[s] \in \mathcal{S}^E_p$ is a local homeomorphism.

More generally, let $\mathcal{M}$ be a presheaf of abelian groups on M. For each $p \in M$ we define the direct limit group

$$\mathcal{M}_p = \varinjlim_{p \in U} \mathcal{M}(U)$$

with respect to the restriction maps r^U_V.

Definition 5.29 *$\mathcal{M}_p$ is a set of equivalence classes called germs at p. Here $s_1 \in \mathcal{M}(U)$ and $s_2 \in \mathcal{M}(V)$ determine the same germ of sections at p if there is an open set $W \subset U \cap V$ containing p such that $r^U_W s_1 = r^V_W s_2$. The germ of $s \in \mathcal{M}(U)$ at p is denoted s_p.*

Now we take the union $\widetilde{\mathcal{M}} = \bigcup_{p\in M} \mathcal{M}_p$ and define a surjection $\pi : \widetilde{\mathcal{M}} \to M$ by the requirement that $\pi(s_p) = p$ for $s_p \in \mathcal{M}_p$. The space $\widetilde{\mathcal{M}}$ is called the **sheaf of germs** generated by $\mathcal{M}$. We want to topologize $\widetilde{\mathcal{M}}$ so that π is continuous and a local homeomorphism but first a definition.

Definition 5.30 (étalé space) *A topological space Y together with a continuous surjection $\pi : Y \to M$ that is a local homeomorphism is called an **étalé space**. A local section of an étalé space over an open subset $U \subset M$ is a map $s_U : U \to Y$ such that $\pi \circ s_U = \mathrm{id}_U$. The set of all such sections over U is denoted $\Gamma(U, Y)$.*

Definition 5.31 *For each $s \in \mathcal{M}(U)$ we can define a map (of sets) $\widetilde{s} : U \to \widetilde{\mathcal{M}}$ by*

$$\widetilde{s}(x) = s_x$$

and we give $\widetilde{\mathcal{M}}$ the smallest topology such that the images $\widetilde{s}(U)$ for all possible U and s are open subsets of $\widetilde{\mathcal{M}}$.

With the above topology $\widetilde{\mathcal{M}}$ becomes an étalé space and all the sections $\widetilde{s}$ are continuous open maps. Now if we let $\widetilde{\mathcal{M}}(U)$ denote the sections over U for this étalé space, then the assignment $U \to \widetilde{\mathcal{M}}(U)$ is a presheaf that is always a sheaf.

Proposition 5.10 *If $\mathcal{M}$ was a sheaf then $\widetilde{\mathcal{M}}$ is isomorphic as a sheaf to $\mathcal{M}$.*

Exercise 5.13 *Reformulate theorem C.11 in terms of germs of maps.*

5.7 Problem set

1. Show that $S^n \times \mathbb{R}$ is parallelizable.

2. Let $\pi : E \to M$ be a smooth vector bundle and let 0_p denote the zero element of the fiber E_p. The map $\mathbf{0} : p \to 0_p$ is a section of $\pi : E \to M$. Show that $\mathbf{0}$ is an embedding of M into E. Thus we sometimes identify the image $\mathbf{0}(M) \subset E$ with M itself. Both the map $\mathbf{0}$ and its image are called the **zero section** of $\pi : E \to M$.

3. Let $X := [0, 1] \times \mathbb{R}^n$. Fix a linear isomorphism $L : \mathbb{R}^n \to \mathbb{R}^n$ and consider the quotient space $E = X/\sim$ where the equivalence relation is simplest the such that $(0, v) \sim (1, Lv)$. Show that E is total space of a smooth vector bundle over the circle S^1.

4. Exhibit the vector bundle charts for the pull-back bundle construction of Defintion 5.9.

5. Show that the space of sections of a vector bundle is a finitely generated module. Show that if the bundle is trivial then the space of sections is finitely generated free module.

6. Let $E_1 \to N$ and $E \to M$ be smooth vector bundles. Show that if $F : E_1 \to E$ is a vector bundle homomorphism along a map $f : N \to M$ then $E_1 \to N$ is strongly isomorphic to the pull-back bundle $f^*E \to N$.

7. Let $E_1 \to N$ and $E_2 \to M$ and suppose that $\widetilde{f}$ is a bundle morphism over M. Show that if $rank(\widetilde{f}\Big|_{E_{1p}})$ is independent of $p \in M$ (i.e. $\widetilde{f}$ is constant rank) then there is a naturally defined bundle ker $\widetilde{f}$ which has fibers ker $\widetilde{f}\Big|_{E_{1p}}$. Similarly, under the same assumptions, there is a bundle coker $\widetilde{f}\Big|_{E_{1p}}$.

8. Show that $S^n \times S^1$ is parallelizable.

Chapter 6

Tensors

Tensor fields (usually referred to simply as tensors) can be introduced in rough and ready way by describing their local expressions in charts and then going on to explain how such expressions are transformed under a change of coordinates. With this approach one can gain proficiency with tensor calculations in short order and this is usually the way physicists and engineers are introduced to tensors. The cost is that this approach hides much of the underlying algebraic and geometric structure. We will not pursue this approach but rather we present tensor fields as fields of multilinear maps. It will be convenient to define the notion of an algebraic tensor on a vector space or module. The reader who has looked over the material in Appendix D will find this chapter easier to understand.

Definition 6.1 *If* V *and* W *be modules over a commutative ring with unity* R *then an algebraic* W*-valued* ***tensor*** *on* V *of type* (r,s) *is a multilinear mapping of the form*

$$\Lambda : \underbrace{\mathrm{V}^* \times \mathrm{V}^* \cdots \times \mathrm{V}^*}_{r-\text{times}} \times \underbrace{\mathrm{V} \times \mathrm{V} \times \cdots \times \mathrm{V}}_{s-\text{times}} \to \mathrm{W}.$$

The set of all W valued tensors on V will be denoted $T^r{}_s(\mathrm{V};\mathrm{W})$. Elements of $T^r{}_s(\mathrm{V};\mathrm{W})$ are said to be contravariant of order r and covariant of order s. In case V is a vector space over the field $\mathbb{R}$, let us agree to denote $T^r{}_s(\mathrm{V};\mathbb{R})$ by $T^r{}_s(\mathrm{V})$. As special cases we have $T^0{}_1(\mathrm{V};\mathsf{R}) = \mathrm{V}^*$ and $T^1{}_0(\mathrm{V};\mathsf{R}) = \mathrm{V}^{**}$. As for vector spaces there is a natural map from V to V^{**} given by $v \mapsto \widetilde{v}$ where $\widetilde{v} : \alpha \mapsto \alpha(v)$. If, as in the vector space case, this map is an isomorphism we say that V is reflexive and we identify V with V^{**}.

Example 6.1 *One always has the special tensor* $\delta \in T^1{}_1(\mathrm{V};\mathsf{R})$ *defined by*

$$\delta(a,v) = a(v)$$

for $a \in \mathrm{V}^*$ *and* $v \in \mathrm{V}$*. In some contexts, this tensor is refered to as the Kroneck delta tensor.*

Exercise 6.1 *Show that the* $C^\infty(M)$ *module of sections of a vector bundle* $E \to M$ *is a reflexive module. (It is important here that we are only considering vector bundles with finite dimensional fibers).*

We now consider the relationship between $T^r{}_s(\mathrm{V};\mathrm{W})$ and abstract tensor product spaces described in Appendix D. We first specialize to the case of $\mathsf{R}-$valued tensors, i.e. to the case where $\mathrm{W} = \mathsf{R}$. Recall that the k-th tensor power of an $\mathsf{R}-$module V is defined to be $\mathrm{V}^{\otimes k} := \mathrm{V} \otimes \cdots \otimes \mathrm{V}$. We always have a module homomorphism

$$\mathrm{V}^{\otimes r} \otimes (\mathrm{V}^*)^{\otimes s} \to T^{r}{}_{s}(\mathrm{V};\mathsf{R}) \tag{6.1}$$

whereby an element $u_1 \otimes \cdots \otimes u_r \otimes \beta^1 \otimes \cdots \otimes \beta^s \in \mathrm{V}^{\otimes r} \otimes (\mathrm{V}^*)^{\otimes s}$ corresponds to the algebraic tensor on V given by

$$\left(\alpha^1, ..., \alpha^r, v_1, ..., v_s\right) \mapsto \alpha^1(u_1) \cdots \alpha^r(u_r)\, \beta^1(v_1) \cdots \beta^s(v_s)$$

We will identify $u_1 \otimes \cdots \otimes u_r \otimes \beta^1 \otimes \cdots \otimes \beta^s$ with this multilinear map. If V is a (finite dimensional) vector space then the map 6.1 is an isomorphism. In fact, it is also true that if V is the space of sections of some vector bundle over M (with finite dimensional fibers) then V is a $C^\infty(M)-$module and the map 6.1 is still an isomorphism and so, for example, if $E \to M$ is a vector bundle and $E^* \to M$ the dual bundle then we have both

$$E_p \otimes E_p^* \cong T^1{}_1(E_p)$$

and

$$\Gamma E_{C^\infty(M)} \otimes \Gamma E^* \cong T^1{}_1(\Gamma E)$$

On the other hand, since the map 6.1 is not always an isomorphism for general modules and since no analogous isomorphism exists in the case of tangent spaces to infinite dimensional manifolds such as those discussed in [L1], it becomes important to ask to what extent the isomorphism 6.1 is needed in differential geometry. Serge Lang has written a very fine differential geometry book for manifolds modelled on Banach spaces [L1] without the help of such an isomorphism. The message seems to be that it is simply multilinear maps that are most needed and not the abstract tensor product spaces. In any case, we still can and will consider $u_1 \otimes \cdots \otimes u_r \otimes \beta^1 \otimes \cdots \otimes \beta^s$ to be an element of $T^r{}_s(\mathrm{V};\mathsf{R})$ as described above.

Another thing to notice is that if 6.1 is an isomorphism for all r and s then in particular $\mathrm{V} \cong T^0{}_1(\mathrm{V};\mathsf{R}) = L(\mathrm{V}^*;\mathsf{R}) = \mathrm{V}^{**}$ where this isomorphism is given by $v \mapsto \widetilde{v}$ where $\widetilde{v} : \alpha \mapsto \alpha(v)$ which just means that V must be reflexive. Corollary D.3 of Appendix D states that for a finitely generated free module, being reflexive is enough to insure that 6.1 is an isomorphism for all r and s.

Remark 6.1 *The reader may have wondered about the possibility of multilinear maps where the* V *factors and the* V^* *are interlaced, such as* $\Upsilon : \mathrm{V} \times \mathrm{V}^* \times \mathrm{V} \times$

V* $\times$ V* $\to$ W. *Of course, such things exist and this example would be an element of what we might denote by* $T_1{}^1{}_1{}^2(\mathrm{V};\mathrm{W})$. *We would still say that an element of* $T_1{}^1{}_1{}^2(\mathrm{V};\mathrm{W})$ *was contravariant of order* $1+2=3$ *and covariant of order* 2. *We can and eventually will agree to associate to each such object a unique element of* $T^3{}_2(\mathrm{V};\mathrm{W})$ *by simply keeping the relative order among the* V *variables and among the* V* *variables separately, but shifting all* V *variables to the left of the* V* *variables.*

Definition 6.2 *For* $S \in T^{r_1}{}_{s_1}(\mathrm{V})$ *and* $T \in T^{r_2}{}_{s_2}(\mathrm{V})$ *we define the* ***tensor product*** $S \otimes T \in T^{r_1+r_2}{}_{s_1+s_2}(\mathrm{V})$ *by*

$$S \otimes T(\theta^1, ..., \theta^{r_1+r_2}, \mathrm{v}_1, .., \mathrm{v}_{s_1+s_2})$$
$$:= S(\theta^1, ..., \theta^{r_1}, \mathrm{v}_1, .., \mathrm{v}_{s_1}) T(\theta^1, ..., \theta^{r_2}, \mathrm{v}_1, .., \mathrm{v}_{s_2})$$

This is consistent with the map 6.1. We can also extend this to products of several tensors at a time.While it is easy to see that the tensor product defineed above is associative, it is *not* commutative.

Let $T^*{}_*(\mathrm{V})$ denote the direct sum of all spaces of the form $T^{r_1+r_2}{}_{s_1+s_2}(\mathrm{V})$ where we take $T^0{}_0(\mathrm{V}) = \mathsf{R}$. The tensor product gives $T^*{}_*(\mathrm{V})$ the structure of an algebra over R as long as we make the definition that $r \otimes T := rT$ for $r \in \mathsf{R}$.

Proposition 6.1 *Let* V *be a free* R*-module with basis* $\{e_1, ..., e_n\}$ *and corresponding dual basis* $\{\varepsilon^1, ..., \varepsilon^n\}$ *for* V*. *Then the indexed set*

$$\{e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s} : i_1, ..., i_r, j_1, ..., j_s = 1, ..., n\}$$

is a basis for $T^r{}_s(\mathrm{V};\mathsf{R})$. *If* $\tau \in T^r{}_s(\mathrm{V};\mathsf{R})$ *then*

$$\tau = \sum_{\substack{i_1,...,i_r=1 \\ j_1,...,j_s=1}}^{n} \tau^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \, e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s}$$

where $\tau^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} = \tau(\varepsilon^{i_1}, ..., \varepsilon^{i_r}, e_{j_1}, ..., e_{j_s})$.

Proof. Suppose $\sum \tau^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \, e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s} = 0$ for some n^{r+s} elements $\tau^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}$ of R. This is an equality of multilinear maps and if we apply both sides to $(\varepsilon^{k_1}, ..., \varepsilon^{k_r}, e_{l_1}, ..., e_{l_s})$ then we obtain $\tau^{k_1 \ldots k_r}{}_{l_1 \ldots l_s} = 0$ and since our choices were arbitrary we see that all n^{r+s} elements $\tau^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}$ are equal to 0. On the other hand, if $\tau \in T^r{}_s(\mathrm{V};\mathsf{R})$ then it is easy to check that $\tau = \sum \tau(\varepsilon^{i_1}, ..., \varepsilon^{i_r}, e_{j_1}, ..., e_{j_s}) e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s}$ and so in particular the elements span $T^r{}_s(\mathrm{V};\mathsf{R})$. ■

As a special case of this proposition we have that if $\tau \in T^1{}_1(\mathrm{V};\mathsf{R})$, then $\tau = \sum \tau^i{}_j \; e_i \otimes \varepsilon^j$ where $\tau^i{}_j = \tau(\varepsilon^i, e_j)$.

Example 6.2 *It is easy to show that for any basis (with corresponding dual basis) as above the components of the Kronecker delta tensor* δ *has components* $\delta^i_j = 1$ *if* $i \neq j$ *and* $\delta^i_i = 0$.

It is easy to show that if $S \in T^1{}_2(\mathrm{V})$ and $T \in T^2{}_2(\mathrm{V})$ then $S \otimes T$ has components given by

$$(S \otimes T)^{abc}{}_{defg} = S^a{}_{de}\ T^{bc}{}_{fg}$$

More generally, if $S \in T^{r_1}{}_{s_1}(\mathrm{V})$ and $T \in T^{r_2}{}_{s_2}(\mathrm{V})$ then

$$(S \otimes T)^{a_1 \dots a_{r_1} \alpha_1 \dots \alpha_{r_2}}{}_{b_1 \dots b_{s_1} \beta_1 \dots \beta_{s_2}} = S^{a_1 \dots a_{r_1}}{}_{b_1 \dots b_{s_1}}\ T^{\alpha_1 \dots \alpha_{r_1}}{}_{\beta_1 \dots \beta_{s_2}}$$

One can make various reinterpretations of tensors.

Example 6.3 *Elements of $T^r{}_s(\mathrm{V})$ can be interpreted as members of $T_s(\mathrm{V}; T^r(\mathrm{V}))$ according to the natural prescription*

$$\tau(v_1, ..., v_s)(\alpha^1, ..., \alpha^r) := \tau(\alpha^1, ..., \alpha^r, v_1, ..., v_s)$$

similarly elements of $T^r{}_s(\mathrm{V})$ can be interpreted as members of $T^r(\mathrm{V}; T_s(\mathrm{V}))$.

Example 6.4 *Elements of $T_{r_1+r_2}(\mathrm{V})$ can be interpreted as members of $T_{r_1+r_2}(\mathrm{V}; T_{r_2}(\mathrm{V}))$ by*

$$\tau(v_1, ..., v_{r_1})(u_1, ..., u_{r_2}) := \tau(v_1, ..., v_{r_1}, u_1, ..., u_{r_2})$$

One can easily see from the above examples that many reinterpretation are possible. One of the most common is reinterpretations is where one interprets elements of $T_2(\mathrm{V},\mathsf{R})$ as elements of $T_2(\mathrm{V}, \mathrm{V}^*)$ according to

$$\tau(v)(u) = \tau(v, u) \text{ for } u, v \in \mathrm{V}$$

Now suppose that V is free with basis $e_1, ..., e_n$ and dual basis $\varepsilon^1, ..., \varepsilon^n$. Every basis element $e_i \otimes \varepsilon^j \in T^1_1(\mathrm{V}, \mathsf{R})$ can be though of as an $\mathsf{R}-$linear map $\mathrm{V} \to \mathrm{V}$ by the prescription $(e_i \otimes \varepsilon^j)^{map}(v) := \varepsilon^j(v)e_i$. Now we can extend this idea to an arbitrary $\tau \in T^1_1(\mathrm{V}, \mathsf{R})$ so that if $\tau = \tau^i_j e_i \otimes \varepsilon^j$ then we interpret τ as the map

$$\tau^{map} : v \mapsto \tau^i_j \varepsilon^j(v) e_i$$

The resulting map τ^{map} is independent of the basis chosen and is such that if $\tau = w \otimes \alpha$ then this map becomes $\tau^{map}(v) = (w \otimes \alpha)^{map}(v) = \alpha(v)w$. Thus we get a map $T^1_1(\mathrm{V}, \mathsf{R}) \to L(\mathrm{V}, \mathrm{V})$ given by $\tau \to \tau^{map}$ which can be shown to be natural in a certain sense and is an isomorphism. Note that we are not claiming such an isomorphism in the case that V is not a finite dimensional free module. When this isomorphism exists, it is popular to use this isomorphism to identify $T^1_1(\mathrm{V}, \mathsf{R})$ with $L(\mathrm{V}, \mathrm{V})$.

Exercise 6.2 *Show that under the identification of $T^1_1(\mathrm{V}, \mathsf{R})$ with $L(\mathrm{V}, \mathrm{V})$ we can interpret the Kronecker delta tensor as the identifty map.*

Definition 6.3 *A covariant tensor $\tau \in T_r(\mathrm{V}, \mathrm{W})$ is said to be symmetric if*

$$\tau(v_1, ..., v_r) = \tau(v_{\sigma 1}, ..., v_{\sigma r})$$

for all $v_1, ..., v_r$ and all permutations σ of the letters $\{1, 2, ..., r\}$. Similarly we can define a symmetric contravariant tensor.

Definition 6.4 *A covariant tensor $\tau \in T_r(\mathrm{V},\mathrm{W})$ is said to be alternating if*

$$\tau(v_1, ..., v_r) = \mathrm{sgn}(\sigma)\tau(v_{\sigma 1}, ..., v_{\sigma r})$$

for all $v_1, ..., v_r$ and all permutations σ of the letters $\{1, 2, ..., r\}$ and where $\mathrm{sgn}(\sigma) = 1$ *is σ is an even permutation and -1 if it is an odd permutation . Similarly we can define alternating contravariant tensor.*

Notice that if V is free R-module with basis $\{e_1, ..., e_n\}$ and corresponding dual basis $\{\varepsilon^1, ..., \varepsilon^n\}$ for V^* then if

$$\begin{gathered} v = v^i e_i \\ w = w^i e_i \\ \text{and} \\ \alpha = \alpha_i \varepsilon^i \end{gathered}$$

then for $\tau \in T^1{}_2(\mathrm{V};\mathsf{R})$ we have

$$\tau(\alpha, v, w) = \tau^i{}_{jk}\, \alpha_i v^j w^k \tag{6.2}$$

More generally, if we express elements $v_1, ..., v_s \in \mathrm{V}$ and $\alpha_1, ..., \alpha_r \in \mathrm{V}^*$ in terms of our basis and its dual basis then for $\tau \in T^r{}_s(\mathrm{V};\mathsf{R})$, we have a analogous general expression for $\tau(\alpha_1, ..., \alpha_r, v_1, ..., v_s)$ in terms of the components of the tensor and its arguments.

Exercise 6.3 *Show that if $\{e_1, ..., e_n\}$ is a basis for* V *and $\{\bar{e}_1, ..., \bar{e}_n\}$ is another such basis with*

$$\bar{e}_i = A_i^k e_k$$

then the basis dual $\{\bar{\varepsilon}^1, ..., \bar{\varepsilon}^n\}$ dual to $\{\bar{e}_1, ..., \bar{e}_n\}$ is related to the basis dual to $\{e_1, ..., e_n\}$ by $\bar{\varepsilon}^i = \left(A^{-1}\right)^i_k \varepsilon^k$. Show that if $\tau^i{}_{jk}$ is the comonents of τ with respect to the first basis (and its dual) and if $\bar{\tau}^i{}_{jk}$ are the components with respect to the second basis then

$$\bar{\tau}^i{}_{jk} = \tau^a{}_{bc}\, A_j^b A_k^c \left(A^{-1}\right)^i_a$$

What is the analogous statement for a tensor $\tau \in T^r{}_s(\mathrm{V};\mathsf{R})$.

Covariant tensors on modules have a nice property with respect to linear maps (module homomorphisms). If $\ell : \mathrm{U} \to \mathrm{V}$ is a linear map then we define the pull-back $\ell^* : T^0{}_s(\mathrm{V};\mathrm{W}) \to T^0{}_s(\mathrm{U};\mathrm{W})$ according to

$$(\ell^*\tau)(u_1, ..., u_s) := \tau(\ell u_1, ..., \ell u_s)$$

It is easy to show that ℓ^* is linear. If and we also have $\ell : \mathrm{U}_1 \to \mathrm{U}_2$ and $\lambda : \mathrm{U}_2 \to \mathrm{V}$ then

$$(\lambda \circ \ell)^* : T^0{}_s(\mathrm{V};\mathrm{W}) \to T^0{}_s(\mathrm{U}_1;\mathrm{W})$$

and

$$(\lambda \circ \ell)^* = \ell^* \circ \lambda^*$$

Thus the pull-back $\ell \to \ell^*$ defines a *contravariant* functor in the category of W−valued *covariant* tensors. (Because of this one might wish that covariant tensors were called contravariant and vice versa but the traditional terminology is entrenched). Suppose that $(e_1, ..., e_n)$ is a basis for V and that $(f_1, ..., f_m)$ is a basis for W. If $\ell e_i = \ell_i^k f_k$ then we have

$$\begin{aligned}(\ell^*\tau)_{i_1 \dots i_s} &= (\ell^*\tau)(e_{i_1}, ..., e_{i_s}) \\ &= \tau(\ell e_{i_1}, ..., \ell e_{i_s}) \\ &= \tau(\ell_{i_1}^{k_1} f_{k_1}, ..., \ell_{i_1}^{k_s} f_{k_s}) \\ &= \ell_{i_1}^{k_1} \cdots \ell_{i_s}^{k_s} \tau(f_{k_1}, ..., f_{k_s}) \\ &\tau_{k_1 \dots k_s} \ell_{i_1}^{k_1} \cdots \ell_{i_s}^{k_s}\end{aligned}$$

which give the component form of the pull-back operation in terms of the matrix $\left(\ell_i^k\right)$.

6.1 Contraction

Definition 6.5 *Let* $\{e_1, ..., e_n\} \subset \mathrm{V}$ *be a basis for* V *and* $\{\varepsilon^1, ..., \varepsilon^n\} \subset \mathrm{V}^*$ *the dual basis. If* $\tau \in T^r_s(\mathrm{V})$ *we define* $C^k_l \tau \in T^{r-1}_{s-1}(\mathrm{V})$

$$\begin{aligned}&C^i_j \tau(\theta^1, ..., \theta^{r-1}, \mathrm{w}_1, .., \mathrm{w}_{s-1}) \\ &= \sum_{a=1}^n \tau(\theta^1, ..., \underset{k-th\ position}{\varepsilon^a}, \dots, \theta^{r-1}, \mathrm{w}_1, \dots, \underset{l-th\ position}{e_a}, \dots, \mathrm{w}_{s-1}).\end{aligned}$$

Write the components of $\tau \in T^r_s(\mathrm{V})$ with respect to our basis as $\tau^{i_1 \dots i_r}_{j_1 \dots j_s}$. Suppose we pick out an upper index, say i_k, and also a lower index, say j_l. We obtain the components of the contracted tensor by:

$$\left(C^k_l \tau\right)^{i_1 .. \widehat{i_k} \dots i_r}_{j_1 \dots \widehat{j_l} \dots j_s} := \tau^{i_1 .. a \dots i_r}_{j_1 \dots a .. j_s} \text{ (sum over } a)$$

Here the caret means omission. In practice, one often just writes $T^{i_1 .. \widehat{i_k} \dots i_r}_{j_1 \dots \widehat{j_l} \dots j_s}$ instead of $\left(C^k_l T\right)^{i_1 .. \widehat{i_k} \dots i_r}_{j_1 \dots \widehat{j_l} \dots j_s}$ as long is it has been made clear how the contraction was carried out. Notice that we always contact an upper index with a lower index.

Consider a tensor of the form $\mathrm{v}_1 \otimes \mathrm{v}_2 \otimes \eta^1 \otimes \eta^2 \in T^2{}_2(\mathrm{V})$. We can define the 1,1 contraction of $\mathrm{v}_1 \otimes \mathrm{v}_2 \otimes \eta^1 \otimes \eta^2$ as the tensor obtained as

$$C^1_1(\mathrm{v}_1 \otimes \mathrm{v}_2 \otimes \eta^1 \otimes \eta^2) = \eta^1(\mathrm{v}_1)\mathrm{v}_2 \otimes \eta^2.$$

Similarly we can define

$$C^1_2(\mathrm{v}_1 \otimes \mathrm{v}_2 \otimes \eta^1 \otimes \eta^2) = \eta^2(\mathrm{v}_1)\mathrm{v}_2 \otimes \eta^1.$$

In general, we define C^i_j on "monomials" $\mathrm{v}_1 \otimes \mathrm{v}_2 ... \otimes \mathrm{v}_k \otimes \eta^1 \otimes \eta^2 \otimes ... \otimes \eta^l$ by an obvious extension of the above. This action of contraction on monomials can be use to give a basis free presentation of contraction. Contraction generalizes the notion of the trace of a linear transformation.

A common use of contration involve first taking the tensor product and the performing a contraction of a contravariant slot of one with a covariant slot of the the other. One often performs several contactions. For example, we may form a tensor that is given in components as

$$\tau^{ac}{}_{efg} = S^a{}_{ke}\ T^{kc}{}_{fg} \text{ (sum over } k)$$

6.2 Bottom up approach to tensors fields.

There are two approaches to tensor fields on smooth manifolds that turn out to be equivalent (at least in the case of finite dimensional manifolds). We start with the "bottom up" approach where we apply multilinear algebra first to individual tangents spaces. The second approach directly defines tensors as elements of $T^r{}_s(\mathfrak{X}(M))$.

Given a rank k vector bundle $\xi = (E, \pi, M)$ let $T^r{}_s(E, M) = \bigsqcup_{p \in M} T^r{}_s(E_p)$. We wish to construct a bundle $T^r{}_s(\xi)$ which has $T^r{}_s(E, M)$ as total space and $T^r{}_s(E_p)$ as fiber over p. If (U, ϕ) is a VB chart for ξ then we can construct a VB chart for $T^r{}_s(\xi)$. Recall that ϕ has the form $\phi = (\pi, \Phi)$ where $\Phi : \pi^{-1}U \to \mathbb{F}^k$ and where $\Phi_p := \Phi|_{E_p} : E_p \to \mathbb{F}^k$ is a linear isomorphism for each p. We obtain a map $\Phi^{r,s}_p : T^r{}_s(E_p) \to T^r{}_s(\mathbb{F}^k)$ by

$$(\Phi^{r,s}_p \tau_p)(\alpha_1, ..., \alpha_r, v_1, ..., v_s) := \tau_p(\left(\Phi^{-1}_p\right)^* \alpha_1, ..., \left(\Phi^{-1}_p\right)^* \alpha_r, \Phi_p v_1, ..., \Phi_p v_s)$$

These maps at each p combine to give a map $\Phi^{r,s} : \pi^{-1}U \to T^r{}_s(\mathbb{F}^k)$ which is smooth (exercise). Now our chart for $T^r{}_s(\xi)$ is

$$\phi^{r,s} := (\pi, \Phi^{r,s}) : \pi^{-1}U \to U \times T^r{}_s(\mathbb{F}^k)$$

If desired, one can choose once and for all an isomorphism $T^r{}_s(\mathbb{F}^k) \cong \mathbb{F}^{k^{r+s}}$. A VB atlas $\{(U_\alpha, \phi_\alpha)\}$for $\xi = (E, \pi, M)$ gives a VB atlas $\{(U_\alpha, \phi^{r,s}_\alpha)\}$ for $T^r{}_s(\xi)$.

Exercise 6.4 *Show that there is a natural vector bundle isomorphism $T^r{}_s(\xi) \cong (\otimes^r E) \otimes (\otimes^s E^*)$.*

We leave it to the interested reader to prove the following useful theorem.

Proposition 6.2 *A section Υ of $T^r{}_s(\xi)$ is smooth A map $\Upsilon : M \to T^r{}_s(\xi)$ if and only if $p \mapsto \Upsilon(p)(\alpha_1(p), ..., \alpha_r(p), X_1(p), ..., X_s(p))$ is smooth for all smooth sections $p \mapsto \alpha_i(p)$ and $p \mapsto X_i(p)$ of $E^* \to M$ and $E \to M$ respectively.*

The set of smooth sections of $T^r{}_s(\xi))$ is denoted $\Gamma(T^r{}_s(\xi))$.

Exercise 6.5 *If $\Upsilon \in \Gamma(T^r{}_s(\xi))$ then define a multilinear map $\Upsilon : (\Gamma E^*)^k \times (\Gamma E)^l \to C^\infty(M)$ (denoted by the same symbol!) by $\Upsilon(\alpha_1, ..., \alpha_r, X_1, ..., X_s)$ is defined by*

$$\Upsilon(\alpha_1, ..., \alpha_r, X_1, ..., X_s) : p \mapsto \Upsilon_p(\alpha_1(p), ..., \alpha_r(p), X_1(p), ..., X_s(p)).$$

Show that this map is multilinear over $C^\infty(M)$. This is an extension of idea thinking of a $1-$form α as a map $\mathfrak{X}(M) \to C^\infty(M)$ so that $\alpha(X)$ is a smooth function.

Like most linear algebraic structures existing at the level of the a single fiber E_p the notion of tensor product is easily extended to the level of sections: For $\tau \in \Gamma(T^{r_1}{}_{s_1}(E))$ and $\eta \in \Gamma(T^{r_2}{}_{s_2}(E))$ we *define* the (consolidated) **tensor product** $\tau \otimes \eta \in \Gamma(T^{r_1+r_2}{}_{s_1+s_2}(E))$ by

$$\begin{aligned}&(\tau \otimes \eta)(p)(\alpha^1, ..., \alpha^{r_1+r_2}, v_1, ..., v_{s_1+s_2}) \\ &= \tau(\alpha^1, ..., \alpha^{r_1}, v_1, ..., v_{s_1})\eta(\alpha^{r_1+1}, ...\alpha^{r_1+r_2}, v_{s_1+1}, ..., v_{s_1+s_2})\end{aligned}$$

for all $\alpha^i \in E_p^*$ and $v_i \in E_p$. In other words, we define $(\tau \otimes \eta)(p)$ by using $\tau_p \otimes \eta_p$ which is already defined since E_p is a vector space. Of course, for sections $\omega^i \in \Gamma(E^*)$ and $X_i \in \Gamma(E)$ we have

$$(\tau \otimes \eta)(\omega^1, ..., \omega^{r_1+r_2}, X_1, ..., X_{s_1+s_2}) \in C^\infty(M)$$

Now let us assume that $E \to M$ is a vector bundle of rank m. Let $s_1, ..., s_m$ be a local frame field for E over an open set U and let $\sigma^1, ..., \sigma^r$ be the frame field of the dual bundle $E^* \to M$ so that $\sigma^i(s_j) = \delta^i_j$. Consider the set

$$\{\sigma^{i_1} \otimes \cdots \otimes \sigma^{i_r} \otimes s_{j_1} \otimes \cdots \otimes s_{j_s} : i_1, ..., i_r, j_1, ..., j_s = 1, ..., m\}$$

If $\Upsilon \in \Gamma(T^r{}_s(E))$ then, in an obvious way, we have functions $\Upsilon^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \in C^\infty(U)$ defined by $\Upsilon^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} = \Upsilon(\sigma^{i_1}, ..., \sigma^{i_r}, s_{j_1}, ..., s_{j_s})$ and it is easy to deduce that Υ (restricted to U) has the expansion

$$\Upsilon = \Upsilon^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}\, \sigma^{i_1} \otimes \cdots \otimes \sigma^{i_r} \otimes s_{j_1} \otimes \cdots \otimes s_{j_s}$$

Exercise 6.6 *Show that the component functions for $\tau \otimes \eta$ are given by*

$$\begin{aligned}&(\tau \otimes \eta)^{i_1 \ldots i_{r_1+r_2}}{}_{j_1, \ldots, j_{s_1+s_2}} \\ &= \tau^{i_1 \ldots i_{r_1}}{}_{j_1, \ldots, j_{s_1}}\ \eta^{i_{r_1+1} \ldots i_{r_2}}{}_{j_{s_1+1} \ldots j_{s_2}}\end{aligned}$$

In the case of the tangent bundle TM we have special terminology and notation.

Definition 6.6 *The bundle $T^r{}_s(TM)$ is called the $(r,s)-$**tensor bundle**.*

Definition 6.7 *A* $(r,s)-$ ***tensor*** τ_p *at* p *is a real valued* $r+s$*-linear map*

$$\tau_p : \left(T_p^*M\right)^r \times \left(T_pM\right)^s \to \mathbb{R}.$$

or in other words an element of $T^r{}_s(T_pM)$*. The space of sections* $\Gamma(T^r{}_s(TM))$ *is denoted by* $\mathfrak{T}^r{}_s(M)$ *and its elements are referred to as* $r-$*contravariant* $s-$*covariant* ***tensor fields*** *or also,* ***type*** (r,s)***-tensor fields.***

In summary, a tensor field is a smooth assignment of a multilinear map on each tangent space of the manifold. For convenience we will sometimes refer to elements $T^r{}_s(T_pM)$ as **point tensors**. An important point is that a tangent vector may be considered as a contravariant tensor, or a tensor of type $(1,0)$ according to the prescription $X_p(\alpha_p) := \alpha_p(X_p)$ for each $\alpha_p \in T_p^*M$. This type of definition by duality is common in differential geometry and eventually this type of thing is done without comment. That said, we realize that if p is in a coordinate chart $(U,\mathtt{x})$ as before then the vectors $\frac{\partial}{\partial x^1}\big|_p, ..., \frac{\partial}{\partial x^n}\big|_p$ form a basis for T_pM and using the identification $T_pM = (T_pM^*)^*$ just mentioned we may form a basis for $T^r{}_s(T_pM)$ consisting of all tensors of the form

$$\left.\frac{\partial}{\partial x^{i_1}}\right|_p \otimes \cdots \otimes \left.\frac{\partial}{\partial x^{i_r}}\right|_p \otimes dx^{j_1}\big|_p \otimes \cdots \otimes dx^{j_s}\big|_p .$$

For example, a $(1,1)-$tensor A_p at p could be expressed in coordinate form as

$$A_p = A^i{}_j \left.\frac{\partial}{\partial x^i}\right|_p \otimes dx^j\big|_p .$$

Now the notation for coordinate expressions is already quite cluttered to the point of being intimidating and we shall take steps to alleviate this shortly but first behold the coordinate expression for a $(r,s)-$tensor (at p) :

$$\tau_p = \sum \tau^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \left.\frac{\partial}{\partial x^{i_1}}\right|_p \otimes \cdots \otimes \left.\frac{\partial}{\partial x^{i_r}}\right|_p \otimes dx^{j_1}\big|_p \otimes \cdots \otimes dx^{j_s}\big|_p$$

Of course the manifold in question could be an open submanifold U of M so for any such open set we have the tensor fields over that set such as $\mathfrak{T}^r_s(U)$. The open subsets are partially ordered by inclusion $V \subset U$ and the tensor fields on these are related in a nice and somewhat obvious way at least if we think of our tensor fields as literally fields of point tensors. This we now do. Let $r^U_V : \mathfrak{T}^r_s(U) \to \mathfrak{T}^r_s(V)$ denote the obvious restriction map. The assignment $U \to \mathfrak{T}^r_s(U)$ is an example of a (pre) sheaf. This means that we have the following easy to verify facts.

1. $r^V_W \circ r^U_V = r^U_W$ whenever $W \subset V \subset U$.
2. $r^V_V = \mathrm{id}_V$ for all open $V \subset M$.
3. Let $U = \bigcup_{U_\alpha \in \mathcal{U}} U_\alpha$ for some collection of open sets $\mathcal{U}$. If $s_1, s_2 \in \mathfrak{T}^r_s(U)$ and $r^U_{U_\alpha} s_1 = r^U_{U_\alpha} s_2$ for all $U_\alpha \in \mathcal{U}$ then $s_1 = s_2$.

4. Let $U = \bigcup_{U_\alpha \in \mathcal{U}} U_\alpha$ for some collection of open sets $\mathcal{U}$. If $s_\alpha \in \mathfrak{T}^r_s(U_\alpha)$ and whenever $U_\alpha \cap U_\beta \neq \emptyset$ we have
$$r^U_{U_\alpha \cap U_\beta} s_\alpha = r^U_{U_\alpha \cap U_\beta} s_\beta$$
then there exists a $s \in \mathfrak{T}^r_s(U)$ such that $r^U_{U_\alpha} s = s_\alpha$.

The reader should realize that we get a presheaf of local sections whenever we have a vector bundle.

We shall define several operations on spaces of tensor fields. We would like each of these to be natural with respect to restriction. We already have one such operation; the tensor product. If $\Upsilon_1 \in \mathfrak{T}^{r_1}_{s_1}(U)$ and $\Upsilon_2 \in \mathfrak{T}^{r_2}_{s_2}(U)$ and $V \subset U$ then $r^U_V(\Upsilon_1 \otimes \Upsilon_2) = r^U_V \Upsilon_1 \otimes r^U_V \Upsilon_2$. A $(k,l)-$tensor field τ may generally be expressed in a chart $(U, \mathtt{x})$ as

$$\tau = \sum \tau^{i_1 \ldots i_r}{}_{j_1 \ldots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

where $\tau^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}$ are now functions, $\frac{\partial}{\partial x^i} \in \mathfrak{X}(U)$ and $dx^j \in \mathfrak{X}^*(U)$. Actually, it is the restriction of τ to U that can be written in this way but because of the naturality with respect to restriction of all the operations we introduce, it is generally safe to use the same letter to denote a tensor field and its restriction to an open set such as a coordinate neighborhood. In accordance with our definitions above, a map $\tau : U \to T^r{}_s(M)$ is a C^∞ tensor field (over U) if whenever $\alpha_1, ..., \alpha_r$ are C^∞ 1-forms over U and $X_1, ..., X_s$ are C^∞ vector fields over U then the function

$$p \mapsto \tau(p)(\alpha_1(p), ..., \alpha_r(p), X_1(p), ..., X_s(p))$$

is smooth. It is easy to show that such a map is smooth if and only if the components $\tau^{i_1 \ldots i_r}{}_{j_1 \ldots j_s}$ are $C^\infty(U)$ for every choice of coordinates $(U, \mathtt{x})$.

Exercise 6.7 *Suppose that we have two charts* $(U, \mathtt{x})$ *and* $(V, \bar{\mathtt{x}})$. *If* $\tau \in \mathfrak{T}^1{}_2(M)$ *has components* τ^i_{jk} *in the first chart and* $\bar{\tau}^i_{jk}$ *in the second chart then on the overlap* $U \cap V$ *we have*
$$\bar{\tau}^i_{jk} = \tau^a_{bc} \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial x^b}{\partial \bar{x}^j} \frac{\partial x^c}{\partial \bar{x}^k}$$
where
$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^a} dx^a \text{ and } \frac{\partial}{\partial \bar{x}^i} = \frac{\partial \bar{x}^b}{\partial x^i} \frac{\partial}{\partial x^b}$$

This last exercise reveals the transformation law for tensor fields in $\mathfrak{T}^1{}_2(M)$ and there is obviously an analogous law for tensor fields from $\mathfrak{T}^r{}_s(M)$ for any values of r and s. In some presentations, tensor fields are defined in terms of such transformations laws. It should be emphasized again that there are two slightly different ways of reading local expressions like the above. We may think of all of these functions as living on the manifold in the domain $U \cap V$. In this interpretation we read the above as

$$\bar{\tau}^i_{jk}(p) = \tau^l_{ab}(p)\frac{\partial x^a}{\partial \bar{x}^j}(p)\frac{\partial x^b}{\partial \bar{x}^k}(p)\frac{\partial \bar{x}^i}{\partial x^l}(p) \text{ for each } p \in U \cap V$$

This is the default modern viewpoint. Alternatively, we could take $\frac{\partial x^j}{\partial \bar{x}^m}$ to be functions on $\bar{\mathtt{x}}(U \cap V)$ and write $\frac{\partial x^j}{\partial \bar{x}^m}(\bar{x}^1, ..., \bar{x}^n)$ while $\frac{\partial \bar{x}^i}{\partial x^l}$ would then refer to $\frac{\partial \bar{x}^i}{\partial x^l} \circ \mathtt{x} \circ \bar{\mathtt{x}}^{-1}(\bar{x}^1, ..., \bar{x}^n)$ so that both sides of the equation are functions of variables which we abusively write as $(\bar{x}^1, ..., \bar{x}^n)$. The first version seems theoretically pleasing but for specific calculations using familiar coordinates such as polar coordinates, the second version is often convenient. For example, suppose that a tensor τ has components with respect to rectangular coordinates on $\mathbb{R}^2$ given by τ^i_{jk} where for indexing purposes we take $(x, y) = (u^1, u^2)$ and $(r, \theta) = (v^1, v^2)$. Then

$$\bar{\tau}^i_{jk} = \tau^l_{ab}\frac{\partial u^a}{\partial v^j}\frac{\partial u^b}{\partial v^k}\frac{\partial v^i}{\partial u^l}$$

could be read so that $\frac{\partial u^j}{\partial v^j} = \frac{\partial u^j}{\partial v^j}(v^1, v^2)$ while $\frac{\partial v^i}{\partial u^i} = \frac{\partial v^i}{\partial u^i}(u^1(v^1, v^2), u^2(v^1, v^2))$. Of course, the charts are there to "identify" patches in Euclidean space with open sets on the manifold so these viewpoints are really somehow the same after all.

Exercise 6.8 *If in rectangular coordinates on $\mathbb{R}^2$ a tensor field with components (arranged as a matrix) given by*

$$\left(\tau^i{}_j\right) = \begin{bmatrix} \tau^1{}_1 = 1 & \tau^2{}_1 = xy \\ \tau^1{}_2 = xy & \tau^2{}_2 = 1 \end{bmatrix}$$

then what are the components of the same tensor in polar coordinates?

Example 6.5 *In definition 5.18 we introduced the notion of a Riemannian metric on a real vector bundle. We saw that such metrics always exist. The most important case is where the bundle is the tangent bundle TM of a manifold M. In this case we say that we have a Riemannian metric* on M. *Thus a Riemannian metric on M is an element of $\mathfrak{T}^0{}_2(M)$.*

Example 6.6 *A thin flat slab of material has associated with it a rank 2 tensor called the **stress tensor**. This tensor is symmetric. If rectangular coordinates (x, y) are imposed and if the material is homogeneous then the tensor has constant component functions (T_{ij}), $T_{ij} = -T_{ji}$. In polar coordinates (r, θ) the same tensor has components*

$$(\bar{T}_{ij}(r, \theta)) = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}\begin{pmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

Exercise 6.9 *If a thin and flat material object (a "thin film") is both homogeneous and isotropic then in rectangular coordinates we have $(T_{ij}) = (c\delta_{ij})$ for*

some constant c. Show that in this case the matrix expression for the stress tensor in polar coordinates is $\begin{pmatrix} \cos^2\theta - \left(\sin^2\theta\right) r & \cos\theta\sin\theta + (\sin\theta)\, r\cos\theta \\ -(\sin\theta)\, r\cos\theta - r^2\cos\theta\sin\theta & -\left(\sin^2\theta\right) r + r^2\cos^2\theta \end{pmatrix}$
.

We now introduce the pull-back of a covariant tensor field.

Definition 6.8 *If $f : M \to N$ is a smooth map and τ is a $s-$covariant tensor on N then we define the* ***pull-back*** $f^*\tau \in \mathfrak{T}^0{}_s(M)$ *by*

$$f^*\tau(v_1, ..., v_s)(p) = \tau(Tf \cdot v_1, ..., Tf \cdot v_s)$$

for all $v_1, ..., v_s \in T_pM$ and any $p \in M$.

It is not hard to see that $f^* : \mathfrak{T}^0{}_s(N) \to \mathfrak{T}^0{}_s(M)$ is linear over $\mathbb{R}$ and for any $h \in C^\infty(N)$ and $\tau \in \mathfrak{T}^0{}_s(N)$ we have $f^*(h\tau) = (h \circ f)\, f^*\tau$.

Let us discover the local expression for pull-back. Choose a chart $(U, \mathtt{x})$ on M and a chart $(V, \mathtt{y})$ on N and assume that $f(U) \subset V$. By a common abuse of notation $\frac{\partial y^i}{\partial x^j} := \frac{\partial\left(y^i \circ f\right)}{\partial x^j}$. We have $Tf\,\frac{\partial}{\partial x^i} = \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k}$

$$(f^*\tau)_{i_1\ldots i_s} = f^*\tau(\frac{\partial}{\partial x^{i_1}}, ..., \frac{\partial}{\partial x^{i_s}})$$

$$\begin{aligned}
(f^*\tau)_{i_1\ldots i_s} &= f^*\tau(\frac{\partial}{\partial x^{i_1}}, ..., \frac{\partial}{\partial x^{i_s}}) \\
&= \tau(Tf\,\frac{\partial}{\partial x^{i_1}}, ..., Tf\,\frac{\partial}{\partial x^{i_s}}) \\
&= \tau(\frac{\partial y^{k_1}}{\partial x^{i_1}}\frac{\partial}{\partial y^{k_1}}, ..., \frac{\partial y^{k_s}}{\partial x^{i_s}}\frac{\partial}{\partial y^{k_s}}) \\
&= \tau(\frac{\partial y^{k_1}}{\partial x^{i_1}}\frac{\partial}{\partial y^{k_1}}, ..., \frac{\partial y^{k_s}}{\partial x^{i_s}}\frac{\partial}{\partial y^{k_s}}) \\
&= \tau_{k_1\ldots k_s}\frac{\partial y^{k_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{k_s}}{\partial x^{i_s}}
\end{aligned}$$

This looks similar to a transformation law for a tensor but here f is not a change of coordinates and need not be a diffeomorphism.

Exercise 6.10 *Let f be as above. Show that for $\tau_1 \in \mathfrak{T}^0{}_{s_1}(M)$ and $\tau_2 \in \mathfrak{T}^0{}_{s_2}(M)$ we have $f^*(\tau_1 \otimes \tau_2) = f^*\tau_1 \otimes f^*\tau_2$.*

The notion of pull-back can be extended to contravariant tensors and tensors of mixed covariance if $f : M \to N$ is a diffeomorphism. For such a diffeomorphism, let $\left(Tf^{-1}\right)^* : T_p^*M \to T_p^*N$ denote the dual of the map $Tf^{-1} : T_pN \to T_pM$.

Definition 6.9 *If $f : M \to N$ is a diffeomorphism and τ is an (r,s) tensor field on N then we define the* ***pull-back*** $f^*\tau \in \mathfrak{T}^r{}_s(M)$ *by*

$$
\begin{aligned}
&f^*\tau(a_1,....,a_r,v_1,...,v_s)(p) \\
&:= \tau(\left(Tf^{-1}\right)^* a_1,...., \left(Tf^{-1}\right)^* a_r, Tf\cdot v_1,...,Tf\cdot v_s)
\end{aligned}
$$

for all $v_1,...,v_s \in T_pM$ and $a_1,....,a_r \in T_p^*M$ and any $p \in M$. The ***push-forward*****??** *is then defined for $\tau \in \mathfrak{T}^r{}_s(M)$ as $f_*\tau := (f^{-1})^*\tau$.*

6.3 Top down approach to tensor fields

Specializing what we learned from Exercise 6.5 to the case of the tangent bundle, we see that a tensor field gives us a $C^\infty(M)$-multilinear map based on the module $\mathfrak{X}(M)$. This observation leads to an alternate definition of a tensor field over M. In this "top down" view we simply *define* an (r,s)-tensor field to be a $C^\infty(M)$-multilinear map

$$\mathfrak{X}(M)^r \times \mathfrak{X}^*(M)^s \to C^\infty(M).$$

In this view, a tensor field is an element of $T^r{}_s(\mathfrak{X}(M))$. For example, a global covariant 2- tensor field on a manifold M is a map $\tau : \mathfrak{X}(M)\times\mathfrak{X}(M) \to C^\infty(M)$ such that

$$
\begin{aligned}
\tau(f_1X_1 + f_2X_2, Y) &= f_1\tau(X_1,Y) + f_2\tau(X_2,Y) \\
\tau(Y, f_1X_1 + f_2X_2) &= f_1\tau(Y,X_1) + f_2\tau(Y,X_2)
\end{aligned}
$$

for all $f_1, f_2 \in C^\infty(M)$ and all $X_1, X_2, Y \in \mathfrak{X}(M)$. As we shall see, it turns out that such $C^\infty(M)$-multilinear maps determine tensor fields in the sense of the previous section.

Exercise 6.11 *Let $\xi = (E,\pi,M)$ be a smooth vector bundle. Produce an analogous top down approach corresponding to $T^r{}_s(\xi)$.*

It is not obvious what is the relation between $T^r{}_s(\mathfrak{X}(M))$ and $T^r{}_s(\mathfrak{X}(U))$ for some proper open subset $U \subset M$. It we take a top down appraoch to tensor fields then we must work to recover the presheaf aspects. We do this in the next section indirectly by showing how the top down approach gives back tensors as sections. Another comment is that both $\mathfrak{X}(U)$ and $T^r{}_s(\mathfrak{X}(U))$ are finite dimensional free modules if U is a chart domain or the domain of a frame field. The reason is that a local frame field and its dual frame field provide a module basis for $\mathfrak{X}(U)$ and $\mathfrak{X}^*(U)$ and the latter really is the dual of the first in the module sense. On the other hand, $\mathfrak{X}(M)$ and $T^r{}_s(\mathfrak{X}(M))$ are not generally free unless M is parallelizable.

6.4 Matching the two approaches to tensor fields.

If we define a tensors field as we first did, that is, as a field of point tensors, then we immediately obtain a tensor as defined in the top down approach. On the other hand if τ is initially defined as a $C^\infty(M)-$multilinear map, then how should we recover the field of tensors on the tangent spaces[1]? Answering this is our next goal.

Proposition 6.3 *Let $p \in M$ and $\tau \in T^r{}_s(\mathfrak{X}(M))$. Let $\theta_1, ..., \theta_r$ and $\bar{\theta}_1, ..., \bar{\theta}_r$ be smooth 1-forms such that $\theta_i(p) = \bar{\theta}_i(p)$ for $1 \leq i \leq r$; also let $X_1, ..., X_s$ and $\bar{X}_1, ..., \bar{X}_s$ be smooth vector fields such that $X_i(p) = \bar{X}_i(p)$ for $1 \leq i \leq s$. Then we have that*

$$\tau(\theta_1, ..., \theta_r, X_1, ..., X_s)(p) = \tau(\bar{\theta}_1, ..., \bar{\theta}_r, \bar{X}_1, ..., \bar{X}_s)(p)$$

Proof. The proof will follow easily if we can show that $\tau(\theta_1, ..., \theta_r, X_1, ..., X_s)(p) = 0$ whenever one of $\theta_1(p), ..., \theta_r(p), X_1(p), ..., X_s(p)$ is zero. We shall assume for simplicity of notation that $r = 1$ and $s = 2$. Now suppose that $X_1(p) = 0$. If $(U, \mathtt{x})$ with $\mathtt{x} = (x^1, ..., x^n)$, then $X_1|_U = \sum \xi^i \frac{\partial}{\partial x^i}$ for some smooth functions $\xi^i \in C^\infty(U)$. Let β be a cut-off function with support in U. Then $\beta\ X_1|_U$ and $\beta^2\ X_1|_U$ extend by zero to elements of $\mathfrak{X}(M)$ which we shall denote by βX_1 and $\beta^2 X_1$. Similarly, $\beta \frac{\partial}{\partial x^i}$ and $\beta^2 \frac{\partial}{\partial x^i}$ are globally defined vector fields, $\beta \xi^i$ is a global function and $\beta X_1 = \sum \beta \xi^i \beta \frac{\partial}{\partial x^i}$. Thus

$$\begin{aligned}
\beta^2 \tau(\theta_1, X_1, X_2) &= \tau(\theta_1, \beta^2 X_1, X_2) \\
&= \tau(\theta_1, \sum \beta \xi^i \beta \frac{\partial}{\partial x^i}, X_2) \\
&= \sum \beta \tau(\theta_1, \xi^i \beta \frac{\partial}{\partial x^i}, X_2).
\end{aligned}$$

Now since $X_1(p) = 0$ we must have $\xi^i(p) = 0$. Also recall that $\beta(p) = 1$. Plugging p into the formula above we obtain $\tau(\theta_1, X_1, X_2)(p) = 0$. A similar argument holds when $X_1(p) = 0$ or $\theta_1(p) = 0$.

Assume that $\bar{\theta}_1(p) = \theta_1(p)$, $\bar{X}_1(p) = \bar{X}_1(p)$ and $X_2(p) = \bar{X}_2(p)$. Then we have

$$\begin{aligned}
&\tau(\bar{\theta}_1, \bar{X}_1, \bar{X}_2) - \tau(\theta_1, X_1, X_2) \\
&= \tau(\bar{\theta}_1 - \theta_1, \bar{X}_1, \bar{X}_2) + \tau(\theta_1, \bar{X}_1, \bar{X}_2) + \tau(\theta_1, \bar{X}_1 - X_1, \bar{X}_2) \\
&+ \tau(\bar{\theta}_1, X_1, \bar{X}_2) + \tau(\bar{\theta}_1, X_1, \bar{X}_2 - X_2)
\end{aligned}$$

Since $\bar{\theta}_1 - \theta_1$, $\bar{X}_1 - X_1$ and $\bar{X}_2 - X_1$ are all zero at p we obtain the result that $\tau(\bar{\theta}_1, \bar{X}_1, \bar{X}_2)(p) = \tau(\theta_1, X_1, X_2)(p)$. ∎

Thus we have a natural correspondence between $T^r{}_s(\mathfrak{X}(M))$ and $\mathfrak{T}^r_s(M)$ (the latter being smooth sections of the bundle $T^r{}_s(TM) \to M$). It is just as

[1] This is exactly where things might not go so well if the manifold is not finite dimensional. What we need is the existence of smooth cut-off functions. Some Banach manifolds support cut-off functions but not all do.

easy to give a similar correspondence between $T^r{}_s(\Gamma(\xi))$ and $\Gamma(T^r{}_s(\xi))$ for some vector bundle $\xi = (E, \pi, M)$ where we view $\Gamma(\xi)$ as a $C^\infty(M)$ module.

We end this section with some warnings. It may seem that there is a simple way to obtain a pull-back by a smooth map $f : M \to N$ entirely from the top down or module theoretic view. In fact, one often sees expressions like the

$$f^*\tau(X_1, ..., X_s) = \tau(f_*X_1, ..., f_*X_s) \quad \text{(problematic expression!)}$$

This looks cute but invites misunderstanding. The left hand side takes fields $X_1, ..., X_s$ as arguments and on the right hand side if we consider τ as a multilinear map $\mathfrak{X}(N) \times \cdots \times \mathfrak{X}(N) \to C^\infty(N)$ then f_*X_i must be fields. But the push forward map f_* is generally not defined and even if it were the above expression would seem to be an equality of a function on M with a function on N. Note that $\mathfrak{X}(M)$ is a $C^\infty(M)$-module, while $\mathfrak{X}(N)$ is a $C^\infty(N)$-module. The above expression may be taken to mean something like $f^*\tau(X_1, ..., X_s)(p) = \tau(Tf \cdot X_1(p), ..., Tf \cdot X_s(p))$ but now the right hand side has tangent vectors as arguments and we are back to the bottom up approach! A correct statement is the following:

Proposition 6.4 *Let $f : M \to N$ be smooth map. Let τ be a $(s, 0)$-tensor field.*

If τ and $f^*\tau$ as interpreted as elements of $T_s(\mathfrak{X}(N))$ and $T_s(\mathfrak{X}(M))$ respectively then

$$f^*\tau(X_1, ..., X_s) = \tau(Y_1, ..., Y_s) \circ f$$

whenever Y_i is $f-$related to X_i for $i = 1, ..., s$.

Despite these troubles, we can use definition 6.9 to make sense of both push-forward and pull-back in the case that f is a diffeomorphism.

6.5 Tensor Derivations

We would like to be able to define derivations of tensor fields . In particular we would like to extend the Lie derivative to tensor fields. For this purpose we introduce the following definition which will be useful not only for extending the Lie derivative but can also be used in several other contexts. Recall the presheaf of tensor fields $U \mapsto \mathfrak{T}^r_s(U)$ on a manifold M.

Definition 6.10 *A tensor derivation is a collection of maps $\mathcal{D}^r_s|_U : \mathfrak{T}^r_s(U) \to \mathfrak{T}^r_s(U)$, all denoted by $\mathcal{D}$ for convenience, such that*

1. *$\mathcal{D}$ is a presheaf map for $\mathfrak{T}^r_s$ considered as a presheaf of vector spaces over $\mathbb{R}$. In particular, for all open U and V with $V \subset U$ we have*

$$(\mathcal{D}\Upsilon)_V = \mathcal{D}\left(\Upsilon|_V\right)$$

for all $\Upsilon \in \mathfrak{T}^r_s(U)$. I.e., the restriction of $\mathcal{D}\Upsilon$ to V is just $\mathcal{D}\left(\Upsilon|_V\right)$.

2. *$\mathcal{D}$ commutes with contractions.*

3. *$\mathcal{D}$ satisfies a derivation law. Specifically, for $\Upsilon_1 \in \mathfrak{T}^r_s(U)$ and $\Upsilon_2 \in \mathfrak{T}^j_k(U)$ we have*
$$\mathcal{D}(\Upsilon_1 \otimes \Upsilon_2) = \mathcal{D}\Upsilon_1 \otimes \Upsilon_2 + \Upsilon_1 \otimes \mathcal{D}\Upsilon_2.$$

For smooth $n-$manifolds, the conditions 2 and 3 imply that for $\Upsilon \in \mathfrak{T}^r_s(U)$, $\alpha_1, ..., \alpha_r \in \mathfrak{X}^*(U)$ and $X_1, ..., X_s \in \mathfrak{X}(U)$ we have

$$\begin{aligned}\mathcal{D}(\Upsilon(\alpha_1, ..., \alpha_r, X_1, ..., X_s)) &= \mathcal{D}\Upsilon(\alpha_1, ..., \alpha_r, X_1, ..., X_s) \\ &\quad+ \sum_i \Upsilon(\alpha_1, ..., \mathcal{D}\alpha_i, ..., \alpha_r, X_1, ..., X_s) \\ &\quad+ \sum_i \Upsilon(\alpha_1, ..., \alpha_r, X_1, ..., ..., \mathcal{D}X_i, ..., X_s).\end{aligned} \tag{6.3}$$

This follows by noticing that

$$\Upsilon(\alpha_1, ..., \alpha_r, X_1, ..., X_s) = C(\Upsilon \otimes (\alpha_1 \otimes \cdots \otimes \alpha_r \otimes X_1 \otimes \cdots \otimes X_s))$$

(where C is the repreated contraction) and then applying 2 and 3.

Proposition 6.5 *Let M be a smooth manifold and suppose we have a map on globally defined tensor fields $\mathcal{D} : \mathfrak{T}^r_s(M) \to \mathfrak{T}^r_s(M)$ for all r, s nonnegative integers such that 2 and 3 above hold for $U = M$. Then there is a unique induced tensor derivation that agrees with $\mathcal{D}$ on global sections.*

Proof. We need to define $\mathcal{D} : \mathfrak{T}^r_s(U) \to \mathfrak{T}^r_s(U)$ for arbitrary open U as a derivation. Let δ be a function that vanishes on a neighborhood of V of $p \in U$. We claim that $(\mathcal{D}\delta)(p) = 0$. To see this let β be a cut-off function equal to 1 on a neighborhood of p and zero outside of V. Then $\delta = (1 - \beta)\delta$ and so

$$\begin{aligned}\mathcal{D}\delta(p) &= \mathcal{D}((1-\beta)\delta)(p) \\ &= \delta(p)\mathcal{D}(1-\beta)(p) + (1-\beta(p))\mathcal{D}\delta(p) = 0\end{aligned}$$

Now given $\tau \in \mathfrak{T}^r_s(U)$, let β be a cut-off function with support in U and equal to 1 on neighborhood of $p \in U$. Then $\beta\tau \in \mathfrak{T}^r_s(M)$ after extending by zero. Now define
$$(\mathcal{D}\tau)(p) = \mathcal{D}(\beta\tau)(p).$$
Now to show that this is well defined let β_2 be any other cut-off function with support in U and equal to 1 on neighborhood of $p_0 \in U$. Then we have

$$\begin{aligned}&\mathcal{D}(\beta\tau)(p_0) - \mathcal{D}(\beta_2\tau)(p_0) \\ &= \mathcal{D}(\beta\tau) - \mathcal{D}(\beta_2\tau))(p_0) = \mathcal{D}((\beta - \beta_2)\tau)(p_0) = 0\end{aligned}$$

where that last equality follows from our claim above with $\delta = \beta - \beta_2$. Thus $\mathcal{D}$ is well defined on $\mathfrak{T}^r_s(U)$. We now show that $\mathcal{D}\tau$ so defined is an element of $\mathfrak{T}^r_s(U)$. Let $(U', \mathtt{x})$ be a chart with $p \in U' \subset U$. Then we can write $\tau|_{U'} \in \mathfrak{T}^r_s(U')$ as

$\tau_{U'} = \sum \tau^{j_1,\dots,j_r}_{i_1,\dots,i_s} dx^{i_1} \otimes \cdots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}}$. We can use this to show that $\mathcal{D}\tau$ as defined is equal to a global section in a neighborhood of p and so must be a smooth section itself since the choice of $p \in U$ was arbitrary. To save on notation let us take the case $r = 1$, $s = 1$. Then $\tau_{U_\alpha} = \tau^i_j dx^j \otimes \frac{\partial}{\partial x^i}$. Let β be a cut-off function equal to one in a neighborhood of p and zero outside of U'. Now extend each of the sections $\beta\tau^i_j$, βdx^j and $\beta\frac{\partial}{\partial x^i}$ to global sections and apply $\mathcal{D}$ to $\beta^3\tau = \left(\beta\tau^i_j\right)\left(\beta dx^j\right) \otimes \left(\beta\frac{\partial}{\partial x^i}\right)$ to get

$$\begin{aligned}
&= \mathcal{D}(\beta^3\tau) = \mathcal{D}(\beta\tau^i_j \beta dx^j \otimes \beta\frac{\partial}{\partial x^i}) \\
&= \mathcal{D}(\beta\tau^i_j)\beta dx^j \otimes \beta\frac{\partial}{\partial x^i} + \beta\tau^i_j \mathcal{D}(\beta dx^j) \otimes \beta\frac{\partial}{\partial x^i} \\
&+ \beta\tau^i_j \beta dx^j \otimes \mathcal{D}(\beta\frac{\partial}{\partial x^i})
\end{aligned}$$

Now by assumption $\mathcal{D}$ takes smooth global sections to smooth global sections so both sides of the above equation are smooth. On the other hand, independent of the choice of β we have $\mathcal{D}(\beta^3\tau)(p) = \mathcal{D}(\tau)(p)$ by definition and valid for all p in a neighborhood of p_0. Thus $\mathcal{D}(\tau)$ is smooth and is the restriction of a smooth global section. This gives a unique derivation $\mathcal{D} : \mathfrak{T}^r_s(U) \to \mathfrak{T}^r_s(U)$ for all U satisfying the naturality conditions 1,2 and 3. We leave it to the reader to check this last statement. ■

Exercise 6.12 *Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two tensor derivations (so satisfying 1,2, and 3 of definition 6.10) that agree on functions and vector fields. Then $\mathcal{D}_1 = \mathcal{D}_2$. Hint: For $\alpha \in \mathfrak{X}^*(U) = \mathfrak{T}^0_1(U)$ we must have $(\mathcal{D}_i\alpha)(X) = \mathcal{D}_i(\alpha(X)) - \alpha(\mathcal{D}_i)$ for $i = 1, 2$. Then also both $\mathcal{D}_1$ and $\mathcal{D}_2$ must obey the formula 6.3 .*

Theorem 6.1 *Furthermore, if $\mathcal{D}_U$ can be defined on $C^\infty(U)$ and $\mathfrak{X}(U)$ for each open $U \subset M$ so that*

1. *$\mathcal{D}_U(fg) = \mathcal{D}_U f \otimes g + f \otimes \mathcal{D}_U g$ for all $f, g \in C^\infty(U)$,*
2. *for each $f \in \mathfrak{F}(M)$ we have $(\mathcal{D}_M f)|_U = \mathcal{D}_U\ f|_U$,*
3. *$\mathcal{D}_U(f \otimes X) = \mathcal{D}_U f \otimes X + f \otimes \mathcal{D}_U X$ for all $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$,*
4. *for each $X \in \mathfrak{X}(M)$ we have $(\mathcal{D}_M X)|_U = \mathcal{D}_U\ X|_U$,*

then there is a unique tensor derivation D on the presheaf of all tensor fields that is equal to $\mathcal{D}_U$ on $C^\infty(U)$ and $\mathfrak{X}(U)$ for all U.

Sketch of Proof. Define $\mathcal{D}$ on $\mathfrak{X}^*(U)$ by requiring

$$\mathcal{D}_U(\alpha \otimes X) = \mathcal{D}_U\alpha \otimes X + \alpha \otimes \mathcal{D}_U X$$

so that after contraction we see that we must have $(\mathcal{D}_U\alpha)(X) = \mathcal{D}_U(\alpha(X)) - \alpha(\mathcal{D}_U X)$. Now define $\mathcal{D}_U$ by formula 6.3 and verify we really have a map

$\mathfrak{T}^r_s(U) \to \mathfrak{T}^r_s(U)$. Check that $\mathcal{D}_U$ commutes with contraction $C : \mathfrak{T}^1_1(U) \to C^\infty(U)$ for decomposable tensors $\alpha \otimes X \in \mathfrak{T}^1_1(U)$. Use the fact that locally, every element of $\mathfrak{T}^1_1$ can be written as a sum of decomposable tensors. Next extend to $\mathfrak{T}^r_s$ along the lines of the case $\mathfrak{T}^1_2(U)$ and the contraction C^1_2:

$$\begin{aligned}\left(\mathcal{D}_U C^1_2 \tau\right)(X) &= \mathcal{D}_U\left(\left(C^1_2\tau\right)(X)\right) - \left(C^1_2\tau\right)\mathcal{D}_U X \\ &= \mathcal{D}_U\left(C(\tau(\cdot, X, \cdot))\right) - C(\tau(\cdot, \mathcal{D}_U X, \cdot)) \\ &= C\left(\mathcal{D}_U(\tau(\cdot, X, \cdot))\right) - \tau(\cdot, \mathcal{D}_U X, \cdot)) \\ &= C\left((\mathcal{D}_U\tau)(\cdot, X, \cdot)\right) = \left(C^1_2 \mathcal{D}_U \tau\right)(X)\end{aligned}$$

Clearly, the general case would involve a profusion of parentheses! ∎

Corollary 6.1 *The Lie derivative $\mathcal{L}_X$ can be extended to a tensor derivation for any $X \in \mathfrak{X}(M)$.*

This last corollary extends the Lie derivative to tensor fields but now present a different way of extending the Lie derivative to tensor fields that is equivalent to what we have just done. First let $\Upsilon \in \mathfrak{T}^r_s(M)$ and recall that if $\phi : M \to M$ is a diffeomorphism then we can define $\phi^*\Upsilon \in \mathfrak{T}^r_s(M)$ by

$$\begin{aligned}&(\phi^*\Upsilon)(p)(a, ..., a^r, v_1, ..., v_s) \\ &= \Upsilon(\phi(p))(T^*\phi^{-1} \cdot a^1, ..., T^*\phi^{-1} \cdot a^r, T\phi \cdot v_1, ..., T\phi \cdot v_s).\end{aligned}$$

Now if X is a complete vector field on M we can define

$$\mathcal{L}_X\Upsilon = \left.\frac{d}{dt}\right|_0 (\varphi^{X*}_t\Upsilon)$$

just as we did for vector fields. If X is not complete the it would be better to use the following careful pointwise formula analogous to 2.6:

$$\begin{aligned}&\left(\mathcal{L}_X\Upsilon\right)(p)(\alpha^1, ..., \alpha^r, v_1, ..., v_s) \\ &= \left.\frac{d}{dt}\right|_0 \Upsilon(\phi_t(p))(T^*\phi_t^{-1} \cdot a^1, ..., T^*\phi_t^{-1} \cdot a^r, T\phi_t \cdot v_1, ..., T\phi_t \cdot v_s)\end{aligned}$$

where $\phi_t = \phi^X_t$. We leave it as a problem that this definition coincides with our first definition of the Lie derivative of a tensor field.

The Lie derivative on tensor fields is natural with respect to diffeomorphisms in the sense that for any diffeomorphism $\phi : M \to N$ and any vector field X we have

$$\mathcal{L}_{\phi_* X}\phi_*\tau = \phi_*\mathcal{L}_X\Upsilon.$$

This property is not shared by some other important derivations such as the covariant derviative that we define later in this book.

Exercise 6.13 *Show that the Lie derivative is natural with respect to diffeomorphisms in the above sense by using the fact that it is natural on functions and vector fields.*

6.6 Problem set

1. Show that if $\tau \in V \otimes V^*$ has the same components τ^i_j with respect to every basis then $\tau^i_j = a\delta^i_j$ for some $a \in \mathbb{R}$.

2. Define $\tau : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ by $\tau(X,Y) = XYf$. Show that τ does not define a tensor field.

3. Show that while a single algebraic tensor τ_p at a point on a manifold can always be extended to a smooth tensor field, it is not the case that one may always extend a (smooth) tensor field defined on an open subset to a smooth tensor field on the whole manifold.

4. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $(x,y) \mapsto (x+2y, y)$. Let $\tau := x\frac{\partial}{\partial x} \otimes dy + \frac{\partial}{\partial y} \otimes dy$. Compute $\phi_*\tau$ and $\phi^*\tau$.

5. Let $\mathcal{D}$ be a tensor derivation on M and suppose that in a local chart we have $\mathcal{D}(\frac{\partial}{\partial x^i}) = \sum D^j_i \frac{\partial}{\partial x^j}$ for smooth functions D^j_i. Show that $\mathcal{D}(dx^j) = -\sum D^j_i dx^i$. Let X be a fixed vector field with components X^i in our chart. Find the D^j_i in the case that $\mathcal{D} = \mathcal{L}_X$.

6. Let $\tau \in \mathfrak{T}^2_0(M)$. Show that the component form of the Lie derivative with respect to a chart is given as
$$(\mathcal{L}_X\tau)^{ab} = \frac{\partial \tau^{ab}}{\partial x^h}X^h - \frac{\partial X^a}{\partial x^h}\tau^{hb} - \frac{\partial X^b}{\partial x^h}\tau^{ah}$$
(where we use the Einstein summation convention). Show that if $\tau \in \mathfrak{T}^0_2(M)$ then the formula becomes
$$(\mathcal{L}_X\tau) = \frac{\partial \tau_{ab}}{\partial x^h}X^h + \frac{\partial X^h}{\partial x^a}\tau_{hb} + \frac{\partial X^h}{\partial x^b}\tau_{ah}$$
Find a formula for $\tau \in \mathfrak{T}^2_2(M)$.

7. Show that our two definitions of the Lie derivative of a tensor field agree with each other.

8. In some chart $(U,(x,y))$ on a two dimensional manifold, let $\tau = x\frac{\partial}{\partial y} \otimes dx \otimes dy + \frac{\partial}{\partial x} \otimes dy \otimes dy$ and let $X = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Compute the coordinate expression for $\mathcal{L}_X\tau$.

9. Suppose the for every chart $(U, \mathtt{x})$ in an atlas for a smooth $n-$manifold M we have assigned n^3 smooth functions Γ^k_{ij} which we call Christoffel symbols. Suppose that rather than obeying the transformation law expected for a tensor we have instead, the following horrible formula relating the Christoffel symbols Γ'^k_{ij} on a chart $(U', \mathtt{y})$ to the symbols Γ^k_{ij} :
$$\Gamma'^k_{ij} = \frac{\partial^2 x^l}{\partial y^i \partial y^j}\frac{\partial y^k}{\partial x^l} + \Gamma^t_{rs}\frac{\partial x^r}{\partial y^i}\frac{\partial x^s}{\partial y^j}\frac{\partial y^k}{\partial x^t}$$

Assuming that such a transformation law hold between the symbol functions for all pairs of intersecting charts. For any pair of vector fields $X, Y \in \mathfrak{X}(M)$ consider the functions $(D_X Y)^k$ given in every chart by the formula $(D_X Y)^k := \frac{\partial Y^k}{\partial x^h} X^h + \Gamma^k_{ij} X^i Y^j$. Show that the local vector fields of the form $(D_X Y)^k \frac{\partial}{\partial x^k}$ defined on each chart, are the restrictions of a single global vector field $D_X Y$. Show that by $D_X : Y \mapsto D_X Y$ is a derivation of $\mathfrak{X}(M)$ and that with $D_X f := Xf$ for smooth functions, we may extent to a tensor derivation. D_X is called a covariant derivative with respect to X. There are many possible covariant derivatives.

10. Continuing on the last problem, show that $D_{fX+gY}\Upsilon = fD_X\Upsilon + gD_Y\Upsilon$ for all $f, g \in C^\infty(M)$ and $X, Y \in \mathfrak{X}(M)$ and $\Upsilon \in \mathfrak{T}^r_s(M)$.

11. Show that if $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}$ are coordinate vector fields then $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] \equiv 0$. Consider the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ arising from standard coordinates on $\mathbb{R}^2$ and also the $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ from polar coordinates. Show that $[\frac{\partial}{\partial x}, \frac{\partial}{\partial r}]$ is not identically zero by explicit computation.

Chapter 7

Differential forms

In one guise, a differential form is nothing but an anti-symmetric tensor field. What is new is that we will introduce an antisymmetrized version of the tensor product and also a very special differential operator called the exterior derivative. Actually, one can approach differential forms without mentioning tensors per se. We will not take this approach but we can at least give a rough and ready description of differential forms in $\mathbb{R}^n$ without mentioning tensors that is sufficient to a reformulation of vector calculus in terms of differential forms. We will do this now as a warm up.

7.1 Differential forms on $\mathbb{R}^n$

First of all, on $\mathbb{R}^n$ we have 0-forms, $1-$forms, $2-$forms and so on until we get to $n-$forms. There are no nonzero k-forms for $k > n$. Let $(x^1, ..., x^n)$ be standard coordinates on $\mathbb{R}^n$. We already know what $1-$forms are and we know that every $1-$form on $\mathbb{R}^n$ can be written $\alpha = \alpha_i dx^i$ (summation). By definition a $0-$form is a smooth function. We now include expressions like $dx^i \wedge dx^j$ for $1 \leq i, j \leq n$. These are our basic $2-$forms. A generic $2-$form on $\mathbb{R}^n$ is an expression of the form $\omega = w_{ij} dx^i \wedge dx^j$ (summation) where w_{ij} are smooth functions. Our first rule is that $dx^i \wedge dx^j = -dx^j \wedge dx^i$ which implies that $dx^i \wedge dx^i = 0$. This means that we may as well assume that $w_{ij} = -w_{ji}$ and also the summation may be taken just over ij such that $i < j$. The $\wedge$ will become a type of product which is bilinear over functions and is called the exterior product or the wedge product that we explain only by example at this point.

Example 7.1 $\mathbb{R}^3$

$$
\begin{aligned}
&(xydx + zdy + dz) \wedge (xdy + zdz) \\
&= xydx \wedge xdy + xydx \wedge zdz + zdy \wedge xdy \\
&+ zdy \wedge zdz + dz \wedge xdy + dz \wedge zdz \\
&= x^2 ydx \wedge dy + xyzdx \wedge dz + z^2 dy \wedge dz \\
&+ xdz \wedge dy \\
&= x^2 ydx \wedge dy + xyzdx \wedge dz + (z^2 - x)dy \wedge dz
\end{aligned}
$$

Example 7.2

$$
\begin{aligned}
&(xyz^2 dx \wedge dy + dy \wedge dz) \wedge (dx + xdy + zdz) \\
&= xyz^2 dx \wedge dy \wedge dx + dy \wedge dz \wedge dx + xyz^2 dx \wedge dy \wedge xdy \\
&+ xyz^2 dx \wedge dy \wedge zdz \\
&+ dy \wedge dz \wedge dz + dy \wedge dz \wedge zdz \\
&= dy \wedge dz \wedge dx + xyz^3 dx \wedge dy \wedge dz = (xyz^3 + 1)dx \wedge dy \wedge dz
\end{aligned}
$$

where we have used $dy \wedge dz \wedge dx = -dy \wedge dx \wedge dz = dx \wedge dy \wedge dz$

Notice that all $n-$forms in $\mathbb{R}^n$ can be written $fdx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ for some $f \in C^\infty$ by using the (anti) commutativity of the wedge product and collecting like terms.

Now we know that the differential of a $0-$form is a 1-form: $d : f \mapsto \frac{\partial f}{\partial x^i} dx^i$. We can inductively extend the definition of d to an operator that takes $k-$forms to $k+1$ forms. This operator will be introduced again in more generality; it is called **exterior differentiation** operator. Every k-form is a sum of terms of the form $fdx^{i_1} \wedge \cdots \wedge dx^{i_k}$. We declare d to be linear over real numbers and then define $d(fdx^{i_1} \wedge \cdots \wedge dx^{i_k}) = (df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k})$. For example, if in $\mathbb{R}^2$ we have a 1-form $\alpha = x^2 dx + xydy$ then

$$
\begin{aligned}
d\alpha &= d(x^2 dx + xydy) \\
&= d\left(x^2\right) \wedge dx + d\left(xy\right) \wedge dy \\
&= 2xdx \wedge dx + (ydx + xdy) \wedge dy \\
&= ydx \wedge dy
\end{aligned}
$$

We are not surprised that the answer is a multiple of $dx \wedge dy$ since in $\mathbb{R}^2$ all $2-$forms have the form $f(x,y)dx \wedge dy$ for some function f.

Exercise 7.1 *Show that for a k-form α on $\mathbb{R}^n$ and any other form β on $\mathbb{R}^n$ we have $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k d\beta$.*

Exercise 7.2 *Verify that $d \circ d = 0$.*

All of what we have said works if we consider only an open set in $\mathbb{R}^n$. Also, $k-$forms can be written in other coordinate systems. Since $k-$forms are built up out of 1-forms using the wedge product we just need to know how to write a $1-$form in new coordinates; but we already know how to do this! For example, in $\mathbb{R}^2$, $dx \wedge dy = rdr \wedge d\theta$.

7.2 Vector Analysis on $\mathbb{R}^3$

In $\mathbb{R}^3$ the space of $0-$forms on some open domain U is, as we have said, just the smooth functions $C^\infty(U)$. In this setting we have a new notation for this space: $\Omega^0(U) := C^\infty(U)$. The one forms may all be written (even globally) in the form $\theta = f_1 dx + f_2 dy + f_3 dz$ for some smooth functions f_1 , f_2 and f_2. As we already know, a 1-form is exactly the kind of thing that can be integrated along an oriented path given by a (piecewise) smooth curve.

Now in $\mathbb{R}^3$ all $2-$forms β may be written $\beta = g_1 dy \wedge dz + g_2 dz \wedge dx + g_3 dx \wedge dy$. The forms $dy \wedge dz, dz \wedge dx$, $dx \wedge dy$ form a basis (in the module sense) for the space of 2-forms on $\mathbb{R}^3$ just as dx, dy, dz form a basis for the $1-$forms. The single form $dx \wedge dy \wedge dz$ provides a basis for the 3-forms in $\mathbb{R}^3$. The funny order for the basis of $2-$forms is purposeful as we shall see. A 2-form is exactly the kind of thing that wants to be integrated over a two dimensional subset of $\mathbb{R}^3$. Suppose that $\mathbf{x}(u, v)$ parameterizes a surface $S \subset \mathbb{R}^3$ so that we have a map $\mathbf{x} : U \to \mathbb{R}^3$. Then the surface is oriented by this parameterization and the oriented integral of β over S is

$$\begin{aligned}\int_S \beta &= \int_S g_1 dy \wedge dz + g_2 dz \wedge dx + g_3 dx \wedge dy \\ &= \int_U \left[g_1(\mathbf{x}(u, v)) \frac{dy \wedge dz}{du \wedge dv} + g_2\left(\mathbf{x}(u, v)\right) \frac{dz \wedge dx}{du \wedge dv} + g_3\left(\mathbf{x}(u, v)\right) \frac{dx \wedge dy}{du \wedge dv} \right] dudv\end{aligned}$$

where, for example, $\frac{dy \wedge dz}{du \wedge dv}$ is the determinant of the matrix $\begin{bmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$. Technically speaking, we should indicate the orientation for S which we have taken to be provided by the parameterization.

Exercise 7.3 *Find the integral of $\beta = x dy \wedge dz + 1 dz \wedge dx + xz dx \wedge dy$ over the sphere oriented by the parameterization given by the usual spherical coordinates ϕ, θ, ρ.*

All 3-forms in $\mathbb{R}^3$ are of the form $\omega = h dx \wedge dy \wedge dz$ for some function h. With these we may integrate over any (say bounded) open subset U which we may take to be given the usual orientation implied by the rectangular coordinates x, y, z. In this case

$$\begin{aligned}\int_U \omega &= \int_U h dx \wedge dy \wedge dz \\ &= \int_U h dx dy dz.\end{aligned}$$

It is important to notice that $\int_U h dy \wedge dx \wedge dz = -\int_U h dy dx dz$ since $\int_U h dy dx dz = \int_U h dx dy dz$ but $dy \wedge dx \wedge dz = -dx \wedge dy \wedge dz$.

In order to relate all this to vector calculus on $\mathbb{R}^3$ we will need some ways to relate $1-$forms to vector fields. To a one form $\theta = f_1 dx + f_2 dy + f_3 dz$ we can obviously associate the vector field $\theta^\natural = f_1 \mathbf{i} + f_2 \mathbf{j} + f_3 \mathbf{k}$ and this works fine but we must be careful. The association depends on the notion of orthonormality provided by the dot product. As mentioned before, differential forms can be expressed in other coordinate systems. If θ is expressed in say spherical coordinates $\theta = \widetilde{f}_1 d\rho + \widetilde{f}_2 d\theta + \widetilde{f}_3 d\phi$ then it is **not** true that $\theta^\natural = \widetilde{f}_1\mathbf{i} + \widetilde{f}_2\mathbf{j} + \widetilde{f}_3\mathbf{k}$. Neither is it generally true that $\theta^\natural = \widetilde{f}_1\widehat{\rho} + \widetilde{f}_2\widehat{\theta} + \widetilde{f}_3\widehat{\phi}$ where $\widehat{\rho}, \widehat{\theta}, \widehat{\phi}$ are unit vectors fields in the coordinate directions[1] and the $\widetilde{f}_i$ are just the f_i expressed in polar coordinates. Rather, we have

$$\theta^\natural = \widetilde{f}_1\widehat{\rho} + \widetilde{f}_2\frac{1}{\rho}\widehat{\theta} + \widetilde{f}_3\frac{1}{\rho\sin\theta}\widehat{\phi}$$

In other words, the rule for $\natural$ is only easy when our basis of 1-forms is orthonormal with respect to the dot product. Since differential forms have no intrinsic need of an inner product, the current project of relating differential forms to vector field calculus is a bit unnatural in some ways. This is an important point. The sharping operator in $\mathbb{R}^3$ is defined on $1-$forms by

$$\begin{array}{lccc} \natural: & dx & \to & \mathbf{i} \\ \natural: & dy & \to & \mathbf{j} \\ \natural: & dz & \to & \mathbf{k} \end{array}$$

and extended bilinearly. A more general formula: Let u^1, u^2, u^3 be general curvilinear coordinates on (some open set in) $\mathbb{R}^3$ and $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^3}$ the corresponding basis vector fields. We then define the Euclidean metric components in these coordinates to be $g_{ij} = \langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \rangle$. We also need (g^{ij}) which, as a matrix, is the inverse of (g_{ij}) so that $g_{ik}g^{kj} = \delta^i_j$. Using the summation convention we have

$$\natural : \theta = f_j du^j \mapsto f^i\frac{\partial}{\partial u^i} := g^{ij}\ f_j\frac{\partial}{\partial u^i}.$$

In good old fashioned tensor notation, **sharping** (also called index raising) is $f_j \mapsto f^i := g^{ij}\ f_j$ where again the Einstein summation convention is applied. The inverse operation taking vector fields to $1-$forms is called **flatting** and is

$$\flat : f^i\frac{\partial}{\partial u^i} \mapsto f_j du^j := g_{ij} f^i du^j$$

but in rectangular coordinates we simply have the expected

$$\begin{array}{lccc} \flat: & \mathbf{i} & \mapsto & dx \\ \flat: & \mathbf{j} & \mapsto & dy \\ \flat: & \mathbf{k} & \mapsto & dz \end{array}.$$

We now come to our first connection with traditional vector calculus. If f is a smooth function then $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$ is a 1-form and the associated vector field is $\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$ which is none other than the gradient $\operatorname{grad} f$. In

[1] Here θ is the polar angle ranging from 0 to π.

spherical coordinates we have

$$\begin{array}{lccc} \natural: & d\rho & \mapsto & \widehat{\rho} \\ \natural: & \rho d\theta & \mapsto & \widehat{\theta} \\ \natural: & \rho\sin\theta d\phi & \mapsto & \widehat{\phi} \end{array}.$$

Sharping is always just the inverse of flatting.

As an example, we can derive the familiar formula for the gradient in spherical coordinate by first just writing f in the new coordinates $f(\rho,\theta,\phi) := f(x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi))$ and then sharping the differential:

$$df = \frac{\partial f}{\partial \rho}d\rho + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi$$

to get

$$\operatorname{grad} f = (df)^{\natural} = \frac{\partial f}{\partial \rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta} + \frac{1}{\rho\sin\theta}\frac{\partial f}{\partial \phi}\frac{\partial}{\partial \phi}$$

where we have used

$$(g^{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho\sin\theta \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho} & 0 \\ 0 & 0 & \frac{1}{\rho\sin\theta} \end{bmatrix}$$

7.2.1 Star Operator in $\mathbb{R}^3$

In order to proceed to the point of including the curl and divergence we need a way to relate 2-forms with vector fields. This part definitely depends on the fact that we are talking about forms in $\mathbb{R}^3$. We associate to a $2-$form $\eta = g_1 dy\wedge dz + g_2 dz\wedge dx + g_3 dx\wedge dy$ the vector field $X = g_1\mathbf{i} + g_2\mathbf{j} + g_3\mathbf{k}$. This should be thought of as first applying the so called star operator to η to get the one form $g_1 dx + g_2 dy + g_3 dz$ and then applying the sharping operator to get the resulting vector field. Again things are more complicated in curvilinear coordinates because of the hidden role of the dot product and the lack of orthonormality of general curvilinear frame fields. The star operator $*$ works on any form and is based on the following prescription valid in standard rectangular coordinates in $\mathbb{R}^3$:

$$\begin{array}{ccc} f & \mapsto & f dx\wedge dy\wedge dz \\ f_1 dx + f_2 dy + f_3 dz & \mapsto & f_1 dy\wedge dz + f_2 dz\wedge dx + f_3 dx\wedge dy \\ g_1 dy\wedge dz + g_2 dz\wedge dx + g_3 dx\wedge dy & \mapsto & g_1 dx + g_2 dy + g_3 dz \\ f dx\wedge dy\wedge dz & \mapsto & f \end{array}$$

So $*$ is a map that takes $k-$forms to $(3-k)-$ forms[2]. It is easy to check that in our current rather simple context we have $*(*\beta) = \beta$ for any form on $U \subset \mathbb{R}^3$. This is true more generally up to a possible sign.

[2]As defined more generally on a Riemannian manifold of dimension n the star operator maps $\Omega^k(M)$ to $\Omega^{n-k}(M)$.

Now we can see how the divergence of a vector field comes about. First flat the vector field, say $X = f_1\mathbf{i}+f_2\mathbf{j}+f_3\mathbf{k}$, to obtain $X^\flat = f_1dx + f_2dy + f_3dz$ and then apply the star operator to obtain $f_1dy \wedge dz + f_2dz \wedge dx + f_3dx \wedge dy$ and then finally apply exterior differentiation! We then obtain

$$\begin{aligned}
&d\,(f_1dy \wedge dz + f_2dz \wedge dx + f_3dx \wedge dy)\\
&= df_1 \wedge dy \wedge dz + df_2 \wedge dz \wedge dx + df_3 \wedge dx \wedge dy\\
&= \left(\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz\right) \wedge dy \wedge dz + \text{the obvious other two terms}\\
&= \frac{\partial f_1}{\partial x}dx \wedge dy \wedge dz + \frac{\partial f_2}{\partial x}dx \wedge dy \wedge dz + \frac{\partial f_3}{\partial x}dx \wedge dy \wedge dz\\
&= \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x} + \frac{\partial f_3}{\partial x}\right) dx \wedge dy \wedge dz.
\end{aligned}$$

Now we see the divergence appearing. In fact, if we apply the star operator one more time we get the function $\operatorname{div} X = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial x} + \frac{\partial f_3}{\partial x}$. We are thus led to the formula $*d * X^\flat = \operatorname{div} X$.

What about the curl? For this we just take $dX^\flat$ to get

$$\begin{aligned}
&d\,(f_1dx + f_2dy + f_3dz)\\
&= df_1 \wedge dx + df_2 \wedge dy + df_3 \wedge dz\\
&= \left(\frac{\partial f_1}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_3}{\partial z}dz\right) \wedge dx + \text{the obvious other two terms}\\
&= \frac{\partial f_2}{\partial y}dy \wedge dx + \frac{\partial f_3}{\partial z}dz \wedge dx + \frac{\partial f_1}{\partial x}dx \wedge dy + \frac{\partial f_3}{\partial z}dz \wedge dy\\
&+ \frac{\partial f_1}{\partial x}dx \wedge dz + \frac{\partial f_2}{\partial y}dy \wedge dz\\
&= \left(\frac{\partial f_2}{\partial y} - \frac{\partial f_3}{\partial z}\right) dy \wedge dz + \left(\frac{\partial f_3}{\partial z} - \frac{\partial f_1}{\partial x}\right) dz \wedge dx + \left(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y}\right) dx \wedge dy
\end{aligned}$$

and then apply the star operator to get back to vector fields obtaining $\left(\frac{\partial f_2}{\partial y} - \frac{\partial f_3}{\partial z}\right)\mathbf{i}+\left(\frac{\partial f_3}{\partial z} - \frac{\partial f_1}{\partial x}\right)\mathbf{j}+\left(\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y}\right)\mathbf{k} = \operatorname{curl} X$. In short we have

$$*dX^\flat = \operatorname{curl} X$$

Exercise 7.4 *Show that the fact that $dd = 0$ leads to both of the following the familiar facts:*

$$\begin{aligned}
\operatorname{curl}(\operatorname{grad} f) &= 0\\
\operatorname{div}(\operatorname{curl} X) &= 0
\end{aligned}$$

Let g denote the determinant of the matrix $[g_{ij}]$. The 3-form $dx \wedge dy \wedge dz$ is called the (oriented) volume element of $\mathbb{R}^3$. Every 3-form is a function times this volume form and integration of a 3−form over sufficiently nice subset (say open) is given by $\int_D \omega = \int_D f dx \wedge dy \wedge dz = \int_D f dxdydz$ (usual Riemann integral). Let us denote $dx \wedge dy \wedge dz$ this by $\mathrm{d}V$. Of course, $\mathrm{d}V$ is not to be considered as the exterior derivative of some object V. Now we will show that in curvilinear coordinates $\mathrm{d}V = \sqrt{g} du^1 \wedge du^2 \wedge du^3$. In any case there must be some function f such that $\mathrm{d}V = f du^1 \wedge du^2 \wedge du^3$. Let's discover this function.

$$\begin{aligned} dV &= dx \wedge dy \wedge dz = \frac{\partial x}{\partial u^i} du^i \wedge \frac{\partial y}{\partial u^j} du^j \wedge \frac{\partial z}{\partial u^k} du^k \\ &= du^1 \wedge du^2 \wedge du^3 \\ &\frac{\partial x}{\partial u^i} \frac{\partial y}{\partial u^j} \frac{\partial z}{\partial u^k} \epsilon_{ijk}^{123} du^1 \wedge du^2 \wedge du^3 \end{aligned}$$

where ϵ_{123}^{ijk} is the sign of the permutation $123 \mapsto ijk$. Now we see that $\frac{\partial x}{\partial u^i} \frac{\partial y}{\partial u^j} \frac{\partial z}{\partial u^k} \epsilon_{123}^{ijk}$ is $\det(\frac{\partial x^i}{\partial u^j})$. On the other hand, $g_{ij} = \frac{\partial x^k}{\partial u^i} \frac{\partial x^k}{\partial u^j}$ or $(g_{ij}) = (\frac{\partial x^i}{\partial u^i})(\frac{\partial x^i}{\partial u^j})$. Thus $g :=$ $\det(g_{ij}) = \det(\frac{\partial x^i}{\partial u^i}) \det(\frac{\partial x^i}{\partial u^j}) = \left(\det(\frac{\partial x^i}{\partial u^i})\right)^2$. From this we get $\det(\frac{\partial x^i}{\partial u^i}) = \sqrt{g}$ and so $\mathrm{d}V = \sqrt{g} du^1 \wedge du^2 \wedge du^3$. A familiar example is the case when (u^1, u^2, u^3) is spherical coordinates ρ, θ, ϕ then

$$\mathrm{d}V = \rho^2 \sin\theta d\rho \wedge d\theta \wedge d\phi$$

and if an open set $U \subset \mathbb{R}^3$ is parameterized by these coordinates then

$$\begin{aligned} \int_U f \mathrm{d}V &= \int_U f(\rho, \theta, \phi) \rho^2 \sin\theta d\rho \wedge d\theta \wedge d\phi \\ &= \int_U f(\rho, \theta, \phi) \rho^2 \sin\theta d\rho d\theta d\phi \text{ (now a Riemann integral)} \\ &\text{which is equal to } \int_U f(x, y, z) dxdydz. \end{aligned}$$

Exercise 7.5 *Show that if* $\mathbf{B} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ *is given in general curvilinear coordinates by*

$$\mathbf{B} = b^i \frac{\partial}{\partial u^i}$$

then $\operatorname{div} \mathbf{B} = \frac{1}{\sqrt{g}} \frac{\partial(\sqrt{g} b^i)}{\partial u^i}$ *(summation implied).*

Exercise 7.6 *Discover the local expression for* $\nabla^2 f = \operatorname{div}(\operatorname{grad} f)$ *in any coordinates by first writing* $\operatorname{div}(\operatorname{grad} f)$. *Notice that* $\nabla^2 f = *d*(df)^\natural$.

The reader should have notices that (for the case of $\mathbb{R}^3$) we have explained how to integrate 1-forms along curves, 2−forms over surfaces, and 3 forms over open domains in $\mathbb{R}^3$. To round things out let us define the "integral" of a function over an ordered pair of points (p_1, p_2) as $f(p_2) - f(p_1)$. Let M denote one of the following: (1) A curve oriented by a parameterization, (2) a surface oriented by a parameterization and having a smooth curve as boundary, (3) a domain in $\mathbb{R}^3$ oriented by the usual orientation on $\mathbb{R}^3$ and having a smooth surface as a boundary. Now let ∂M denote in the first case the ordered set $(c(a), c(b))$ of beginning and ending points of the curve in the first case, the counterclockwise traversed boundary curve of M if M is a surface as in the second case, or finally the surface which is the boundary of M (assumed to be nice and smooth) when M is a domain of $\mathbb{R}^3$. Finally, let ω be a 0-form in case (1), a $1-$form in case (2) and a 2 form in case (3).As a special case of Stokes' theorem on manifolds we have the following.

$$\int_M d\omega = \int_{\partial M} d\omega$$

The three cases become

$$\begin{aligned}
\int_{M=c} df &= \int_{\partial M=\{c(a),c(b)\}} f \text{ (which is} = \int_a^b f(c(t))dt = f(c(b)) - f(c(a))) \\
\int_S d\alpha &= \int_{\partial M=c} \alpha \\
\int_D d\omega &= \int_{S=\partial D} \omega
\end{aligned}$$

If we go to the trouble of writing these in terms of vector fields associated to the forms in an appropriate way we get the following familiar theorems (using standard notation):

$$\int_c \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$\iint_S curl(\mathbf{X}) \times \mathbf{dS} = \oint_c X \cdot d\mathbf{r} \text{ (Stokes' theorem)}$$

$$\iiint_D div\mathbf{X}\, dV = \iint_S \mathbf{X} \cdot \mathbf{dS} \text{ (Divergence theorem)}$$

Similar and simpler things can be done in $\mathbb{R}^2$ leading for example to the following version of Green's theorem for a planar domain D with (oriented) boundary $c = \partial D$.

$$\int_D \left(\frac{\partial M}{\partial y} - \frac{\partial M}{\partial y} \right) dx \wedge dy = \int_D d(M dx + N dy) = \int_c M\, dx + N\, dy$$

All of the standard integral theorems from vector calculus are special cases of the general Stokes' theorem that we introduce later in this chapter.

7.3 Differential forms on a general smooth manifold

Now let us give a presentation of the general situation of differential forms on a manifold. Since we treat differential forms as completely antisymmetric tensor fields we can take either the bottom up or top down veiw point as explained. We start of with some more multilinear algebra on abstract modules.

Definition 7.1 *Let* V *and* F *be modules over a ring* R *or an* $\mathbb{F}-$*algebra* R *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$. *A* k*-multilinear map* $\alpha : \mathrm{V} \times \cdots \times \mathrm{V} \to \mathrm{W}$ *is called* ***alternating*** *if* $\alpha(\mathrm{v}_1, ..., \mathrm{v}_k) = 0$ *whenever* $\mathrm{v}_i = \mathrm{v}_j$ *for some* $i \neq j$. *The space of all* ***alternating*** k*-multilinear maps into* W *will be denoted by* $L^k_{alt}(\mathrm{V}; \mathrm{W})$ *or by* $L^k_{alt}(\mathrm{V})$ *if the* $\mathrm{W} = \mathsf{R}$.

Since we are dealing in the cases where R is either one of the fields $\mathbb{R}$ and $\mathbb{C}$ (which have characteristic zero) or an algebra of real or complex valued functions, it is easy to see that **alternating** k-multilinear maps are the same as (completely) **antisymmetric** k-multilinear maps which are defined by the property that for any permutation σ of the letters $1, 2, ..., k$ we have

$$\omega(\mathrm{w}_1, \mathrm{w}_2, .., \mathrm{w}_k) = sgn(\sigma)\omega(\mathrm{w}_{\sigma(1)}, \mathrm{w}_{\sigma(2)}, .., \mathrm{w}_{\sigma(k)}).$$

Let us denote the group of permutations of the k letter $1, 2, ..., k$ by S_k. In what follows we will occasionally write σ_i in place of $\sigma(i)$ etc.

Definition 7.2 *The* ***antisymmetrization map*** $Alt^k : {T^0}_k(\mathrm{V}) \to L^k_{alt}(\mathrm{V})$ *is defined by*

$$Alt^k(\omega)(v_1, v_2, .., v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma)\omega(v_{\sigma_1}, v_{\sigma_2}, .., v_{\sigma_k}).$$

Lemma 7.1 *For* $\alpha \in {T^0}_{k_1}(\mathrm{V})$ *and* $\beta \in {T^0}_{k_2}(\mathrm{V})$ *we have*

$$Alt^{k_1+k_2}(Alt^{k_1}\alpha \otimes \beta) = Alt^{k_1+k_2}(\alpha \otimes \beta)$$

and

$$Alt^{k_1+k_2}(Alt^{k_1}\alpha \otimes Alt^{k_2}\beta) = Alt^{k_1+k_2}(\alpha \otimes \beta)$$

Proof. For a permutation $\sigma \in S_k$ and any $T \in {T^0}_k(\mathrm{V})$, let σT denote the element of ${T^0}_k(\mathrm{V})$ given by $\sigma T(v_1, ..., v_k) := \tau(v_{\sigma(1)}, ..., v_{\sigma(k)})$. We then have $Alt^k(\sigma T) = \mathrm{sgn}(\sigma)\, Alt^k(T)$ as may easily be checked. Also, by definition $Alt^k(T) = \sum \mathrm{sgn}(\sigma)\sigma T$. We have

$$\begin{aligned}
&Alt^{k_1+k_2}(Alt^{k_1}\alpha \otimes \beta) \\
&= Alt^{k_1+k_2}\left(\left(\frac{1}{k_1!}\sum_{\rho \in S_{k_1}} \mathrm{sgn}\,\rho\,(\rho\alpha)\right) \otimes \beta\right) \\
&= \left(\frac{1}{k_1!}\sum_{\rho \in S_{k_1}} \mathrm{sgn}\,\rho\,(\rho\alpha \otimes \beta)\right) \\
&= \frac{1}{k_1!}\sum_{\rho \in S_{k_1}} \mathrm{sgn}\,\rho\;Alt^{k_1+k_2}(\rho\alpha \otimes \beta)
\end{aligned}$$

Let us now examine the expression $\mathrm{sgn}\,\rho\;Alt^{k_1+k_2}(\rho\alpha \otimes \beta)$. If we extend each $\rho \in S_{k_1}$ to a corresponding element $\rho' \in S_{k_1+k_2}$ by letting $\rho'(i) = \rho(i)$ for

$i \leq k_1$ and $\rho'(i) = i$ for $i > k_1$ then we have $\rho\alpha \otimes \beta = \rho'(\alpha \otimes \beta)$ and also $\operatorname{sgn}(\rho) = \operatorname{sgn}(\rho')$. Thus $\operatorname{sgn}\rho \, Alt^{k_1+k_2}(\rho\alpha \otimes \beta) = \operatorname{sgn}\rho' \, Alt^{k_1+k_2}\rho'(\alpha \otimes \beta)$ and so

$$\begin{aligned}
&Alt^{k_1+k_2}(Alt^{k_1}\alpha \otimes \beta) \\
&= \frac{1}{k_1!} \sum_{\rho \in S_{k_1}} \operatorname{sgn}\rho' Alt^{k_1+k_2}\rho'(\alpha \otimes \beta) \\
&= \frac{1}{k_1!} \sum_{\rho \in S_{k_1}} \operatorname{sgn}\rho' \operatorname{sgn}\rho' Alt^{k_1+k_2}(\alpha \otimes \beta) \\
&= Alt^{k_1+k_2}(\alpha \otimes \beta) \frac{1}{k_1!} \sum_{\rho \in S_{k_1}} = Alt^{k_1+k_2}(\alpha \otimes \beta)
\end{aligned}$$

We arrive at $Alt^{k_1+k_2}(Alt^{k_1}\alpha \otimes \beta) = Alt^{k_1+k_2}(\alpha \otimes \beta)$. ■

Now given $\omega \in L^{k_1}_{alt}(\mathrm{V})$ and $\eta \in L^{k_2}_{alt}(\mathrm{V})$ we define their **exterior product** or **wedge product** $\omega \wedge \eta \in L^{k_1+k_2}_{alt}(\mathrm{V})$ by the formula

$$\omega \wedge \eta := \frac{(k_1+k_2)!}{k_1!k_2!} Alt^{k_1+k_2}(\omega \otimes \eta)$$

Written out this is

$$\omega \wedge \eta(\mathrm{v}_1, ..., \mathrm{v}_{k_1}, \mathrm{v}_{k_1+1}, ..., \mathrm{v}_{k_1+k_2}) := \frac{1}{k_1!k_2!} \sum_{\sigma \in S_{k_1+k_2}} \operatorname{sgn}(\sigma)\omega(\mathrm{v}_{\sigma_1}, ..., \mathrm{v}_{\sigma_{k_1}})\eta(\mathrm{v}_{\sigma_{k_1+1}}, ..., \mathrm{v}_{\sigma_{k_1+k_2}})$$

It is an exercise in combinatorics that we also have

$$\omega \wedge \eta(\mathrm{v}_1, ..., \mathrm{v}_{k_1}, \mathrm{v}_{k_1+1}, ..., \mathrm{v}_{k_1+k_2}) := \sum_{k_1,k_2-\text{shuffles } \sigma} \operatorname{sgn}(\sigma)\omega(\mathrm{v}_{\sigma_1}, ..., \mathrm{v}_{\sigma_{k_1}})\eta(\mathrm{v}_{\sigma_{k_1+1}}, ..., \mathrm{v}_{\sigma_{k_1+k_2}}).$$

In the latter formula we sum over all permutations such that $\sigma(1) < \sigma(2) < ... < \sigma(k_1)$ and $\sigma(k_1+1) < \sigma(k_1+2) < .. < \sigma(k_1+k_2)$. This kind of permutation is called a k_1, k_2-**shuffle** as indicated in the summation. The most important case is for $\omega, \eta \in L^1_{alt}(\mathrm{V})$ in which case

$$(\omega \wedge \eta)(v, w) = \omega(v)\eta(w) - \omega(w)\eta(v)$$

This is clearly an antisymmetric multilinear map, which is just what we call antisymmetric in the case of two 2 variables.

Proposition 7.1 *For $\alpha \in L^{k_1}_{alt}(\mathrm{V})$, $\beta \in L^{k_2}_{alt}(\mathrm{V})$ and $\gamma \in L^{k_3}_{alt}(\mathrm{V})$ we have*

(i) $\wedge : L^{k_1}_{alt}(\mathrm{V}) \times L^{k_2}_{alt}(\mathrm{V}) \to L^{k_1+k_2}_{alt}(\mathrm{V})$ *is* R*-bilinear*

(ii) $\alpha \wedge \beta = (-1)^{k_1k_2}\beta \wedge \alpha$

(iii) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$

Proof. We leave the proof of (i) as an easy exercise.
For (iii) we consider the special permutation f given by $(f(1), f(2), ..., f(k_1 + k_2)) = (k_1 + 1, ...k_1 + k_2, 1, ..., k\ _1)$. We have that $\alpha \otimes \beta = f(\alpha \otimes \beta)$. Also $\operatorname{sgn}(f) = (-1)^{k_1 k_2}$. So we have $Alt^{k_1+k_2}(\alpha \otimes \beta) = Alt^{k_1+k_2}(f(\beta \otimes \alpha)) = (-1)^{k_1 k_2} Alt^{k_1+k_2}(\beta \otimes \alpha)$ which gives (ii).

For (iii) we compute

$$\begin{aligned}
\alpha \wedge (\beta \wedge \gamma) &= \frac{(k_1 + k_2 + k_3)!}{k_1!\,(k_2 + k_3)!} Alt(\alpha \otimes (\beta \wedge \gamma)) \\
&= \frac{(k_1 + k_2 + k_3)!}{k_1!\,(k_2 + k_3)!} \frac{(k_2 + k_3)!}{k_2! k_3!} Alt(\alpha \otimes Alt\,(\beta \otimes \gamma)) \\
&= \frac{(k_1 + k_2 + k_3)!}{k_1! k_2! k_3!} Alt(\alpha \otimes Alt\,(\beta \otimes \gamma))
\end{aligned}$$

By Lemma 7.1 we know that $Alt(\alpha \otimes Alt\,(\beta \otimes \gamma)) = Alt(\alpha \otimes (\beta \otimes \gamma))$ and so we arrive $\alpha \wedge (\beta \wedge \gamma) = Alt(\alpha \otimes (\beta \otimes \gamma))$. By a symmetric computation, we also have $(\alpha \wedge \beta) \wedge \gamma = Alt((\alpha \otimes \beta) \otimes \gamma)$ and so by using the associativity of the tensor product we obtain the result. ■

Lemma 7.2 *Let $\alpha^1, ..., \alpha^k$ be elements of $L^k_{alt}(\mathrm{V})$ and let $v_1, ..., v_k$ be elements of. Then we have*

$$\alpha^1 \wedge \cdots \wedge \alpha^k (v_1, ..., v_k) = \det A$$

where $A = (a^i_j)$ is the matrix whose $ij - th$ entry is $a^i_j = \alpha^i(v_j)$.

Proof. From the proof of the last theorem we $\alpha \wedge (\beta \wedge \gamma) = \frac{(k_1+k_2+k_3)!}{k_1!(k_2+k_3)!} Alt(\alpha \otimes (\beta \wedge \gamma))$. By inductive application of this we have

$$\alpha^1 \wedge \cdots \wedge \alpha^k = k! Alt(\alpha^1 \otimes \cdots \otimes \alpha^k)$$

Thus

$$\begin{aligned}
\alpha^1 \wedge \cdots \wedge \alpha^k (v_1, ..., v_k) &= \sum_\sigma \operatorname{sgn}(\sigma) \alpha^1(v_{\sigma 1}) \cdots \alpha^k(v_{\sigma k}) \\
&= \det(A)
\end{aligned}$$

■

Let $\{\varepsilon^1, \varepsilon^2, ..., \varepsilon^n\}$ be (an ordered) basis for V^* which is dual to a basis $\{e_1, ..., e_n\}$ for V. Since any $\alpha \in L^k_{alt}$ is also a member of $T^0{}_k(\mathrm{V})$ we may write $\alpha = \sum \alpha_{i_1 \ldots i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}$ where $\alpha_{i_1 \ldots i_k} = \alpha(e_{i_1}, ..., e_{i_k})$. We have $\alpha = \sum \alpha_{i_1 \ldots i_k} Alt\left(\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}\right) = \frac{1}{k!} \sum \alpha_{i_1 \ldots i_k} \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}$. We conclude that the set of elements of the form $\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}$ spans $L^{k_1}_{alt}(\mathrm{V})$. Furthermore, if we use the fact that both $\alpha_{i_{\sigma 1} \ldots i_{\sigma k}} = \operatorname{sgn} \sigma\, \alpha_{i_1 \ldots i_k}$ and $\varepsilon^{i_{\sigma 1}} \wedge \cdots \wedge \varepsilon^{i_{\sigma k}} = \operatorname{sgn} \sigma\, \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}$ for any permutation $\sigma \in S_k$ we see that we can permute the indices into increasing order and collect terms to get

$$\begin{aligned}
\alpha &= \frac{1}{k!} \sum \alpha_{i_1 \ldots i_k} \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} \\
&= \sum_{i_1 < i_2 < \ldots < i_k} a_{i_1 i_2, .., i_k} \varepsilon^{i_1} \wedge \varepsilon^{i_2} \wedge \cdots \wedge \varepsilon^{i_k}
\end{aligned}$$

where in last expression we sum only over the set of strictly increasing indices. We can check that the set of $\frac{n!}{k!(n-k)!}$ elements of the form $\varepsilon^{i_1} \wedge \varepsilon^{i_2} \wedge \cdots \wedge \varepsilon^{i_k}$ with $i_1 < i_2 < ... < i_k$ is linearly independent as follows: Suppose $\sum_{i_1<i_2<...<i_k} a_{i_1 i_2,..,i_k} \varepsilon^{i_1} \wedge \varepsilon^{i_2} \wedge \cdots \wedge \varepsilon^{i_k} = 0$. Fix arbitrary $j_1 < j_2 < ... < j_k$. Then we have

$$
\begin{aligned}
0 &= \left(\sum_{i_1<i_2<...<i_k} a_{i_1 i_2,..,i_k} \varepsilon^{i_1} \wedge \varepsilon^{i_2} \wedge \cdots \wedge \varepsilon^{i_k} \right) (e_{j_1}, ..., e_{j_k}) \\
&= \sum_{i_1<i_2<...<i_k} a_{i_1 i_2,..,i_k} \varepsilon^{i_1} \wedge \varepsilon^{i_2} \wedge \cdots \wedge \varepsilon^{i_k} (e_{j_1}, ..., e_{j_k})
\end{aligned}
$$

But $\varepsilon^{i_1} \wedge \varepsilon^{i_2} \wedge \cdots \wedge \varepsilon^{i_k}(e_{j_1}, ..., e_{j_k})$ is zero unless $i_r = j_r$ for $r = 1, ..., k$ since otherwise it would be the determinant of a matrix with at least one row of zeros. Thus we get $0 = a_{j_1 j_2,..,j_k}$ and since the choice of j's was arbitrary we have shown independence.

Remark 7.1 *In order to facilitate notation we will abbreviate a sequence of k integers, say $i_1, i_2, ..., i_k$, from the set $\{1, 2, ..., \dim(\mathrm{V})\}$ as I and $\varepsilon^{i_1} \wedge \varepsilon^{i_2} \wedge \cdots \wedge \varepsilon^{i_k}$ is written as ε^I. Also, if we require that $i_1 < i_2 < ... < i_k$ we will write $\vec{I}$. We will freely use similar self explanatory notation as we go along without further comment. For example, we may write*

$$\alpha = \sum a_{\vec{I}} \varepsilon^{\vec{I}}$$

to mean $\alpha = \sum_{i_1<i_2<...<i_k} a_{i_1 i_2,..,i_k} \varepsilon^{i_1} \wedge \varepsilon^{i_2} \wedge \cdots \wedge \varepsilon^{i_k}$

A corollary of the above discussion is that the dimension of $L^k_{alt}(\mathrm{V})$ is $\frac{n!}{k!(n-k)!}$ where $n = \dim(\mathrm{V})$. In particular, $L^k_{alt}(\mathrm{V}) = 0$ for $k > n$.

If one defines $L^0_{alt}(\mathrm{V})$ to be the scalar ring R and recalling that $L^1_{alt}(\mathrm{V}) = \mathrm{V}^*$ then the sum

$$L_{alt}(\mathrm{V}) = \bigoplus_{k=0}^{\dim(\mathrm{V})} L^k_{alt}(\mathrm{V})$$

is made into a graded algebra via the wedge product just defined.

The Abstract Grassmann Algebra

We wish to construct a space that is universal with respect to alternating multilinear maps. To this end, consider the tensor space $T^k(\mathrm{V}) := \mathrm{V}^{k\otimes}$ and let A be the submodule of $T^k(\mathrm{V})$ generated by elements of the form

$$\mathrm{v}_1 \otimes \cdots \mathrm{v}_i \otimes \cdots \otimes \mathrm{v}_i \cdots \otimes \mathrm{v}_k.$$

In other words, A is generated by decomposable tensors with two (or more) equal factors. We define the space of k-vectors to be

$$\mathrm{V} \wedge \cdots \wedge \mathrm{V} := \bigwedge^k \mathrm{V} := T^k(\mathrm{V})/\mathrm{A}.$$

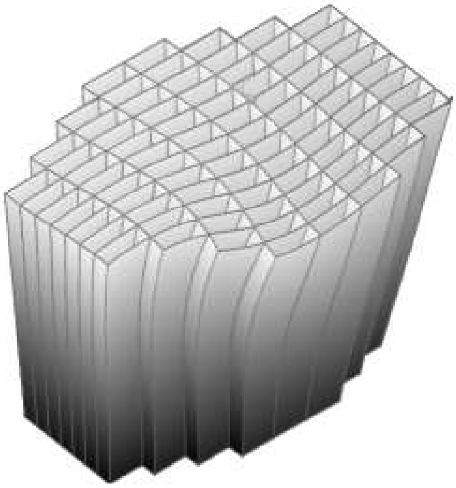

Figure 7.1: 2-form as "flux tubes".

Let $\mathrm{A}_k : \mathrm{V} \times \cdots \times \mathrm{V} \to T^k(\mathrm{V}) \to \not\supseteq^k \mathrm{V}$ be the canonical map composed with projection onto $\bigwedge^k \mathrm{V}$. This map turns out to be an alternating multilinear map. We will denote $\mathrm{A}_k(\mathrm{v}_1, ..., \mathrm{v}_k)$ by $\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_k$. The pair $(\bigwedge^k \mathrm{V},\ \mathrm{A}_k)$ is universal with respect to alternating k-multilinear maps: Given any alternating k-multilinear map $\alpha : \mathrm{V} \times \cdots \times \mathrm{V} \to \mathrm{F}$, there is a unique linear map $\alpha_\wedge : \bigwedge^k \mathrm{V} \to \mathrm{F}$ such that $\alpha = \alpha_\wedge \circ \mathrm{A}_k$; that is$\bigwedge^k$

$$\begin{array}{ccc} \mathrm{V} \times \cdots \times \mathrm{V} & \xrightarrow{\alpha} & \mathrm{F} \\ \mathrm{A}_k \downarrow & \nearrow_{\alpha_\wedge} & \\ \bigwedge^k \mathrm{V} & & \end{array}$$

commutes. Notice that we also have that $\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_k$ is the image of $\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_k$ under the quotient map. Next we define $\bigwedge \mathrm{V} := \sum_{k=0}^{\infty} \bigwedge^k \mathrm{V}$ and impose the multiplication generated by the rule

$$(\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_i) \times (\mathrm{v}_1' \wedge \cdots \wedge \mathrm{v}_j') \mapsto \mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_i \wedge \mathrm{v}_1' \wedge \cdots \wedge \mathrm{v}_j' \in \bigwedge^{i+j} \mathrm{V}.$$

The resulting algebra is called the Grassmann algebra or exterior algebra. If we need to have a $\mathbb{Z}$ grading rather than a $\mathbb{Z}^+$ grading we may define $\bigwedge^k \mathrm{V} := 0$ for $k < 0$ and extend the multiplication in the obvious way.

Notice that since $(\mathrm{v} + \mathrm{w}) \wedge (\mathrm{w} + \mathrm{v}) = 0$, it follows that $\mathrm{v} \wedge \mathrm{w} = -\mathrm{w} \wedge \mathrm{v}$. In fact, any odd permutation of the factors in a decomposable element such as

$\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_k$, introduces a change of sign:

$$\begin{aligned}&\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_i \wedge \cdots \wedge \mathrm{v}_j \wedge \cdots \wedge \mathrm{v}_k \\ &= -\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_j \wedge \cdots \wedge \mathrm{v}_i \wedge \cdots \wedge \mathrm{v}_k\end{aligned}$$

Lemma 7.3 *If* V *is has rank* n, *then* $\bigwedge^k \mathrm{V} = 0$ *for* $k \geq n$. *If* $f_1, ..., f_n$ *is a basis for* V *then the set*

$$\{f_{i_1} \wedge \cdots \wedge f_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

is a basis for $\bigwedge^k \mathrm{V}$ *where we agree that* $f_{i_1} \wedge \cdots \wedge f_{i_k} = 1$ *if* $k = 0$.

The following lemma follows easily from the universal property of $\alpha_\wedge : \bigwedge^k \mathrm{V} \to \mathrm{F}$:

Lemma 7.4 *There is a natural isomorphism*

$$L^k_{alt}(\mathrm{V}; \mathrm{F}) \cong L(\bigwedge^k \mathrm{V}; \mathrm{F})$$

In particular,

$$L^k_{alt}(\mathrm{V}) \cong \left(\bigwedge^k \mathrm{V}\right)^*.$$

Remark 7.2 (Convention) *Let* $\alpha \in L^k_{alt}(\mathrm{V}; \mathrm{F})$. *Because the above isomorphism is so natural it may be taken as an identification and so we sometimes write* $\alpha(v_1, ..., v_k)$ *as* $\alpha(v_1 \wedge \cdots \wedge v_k)$.

We would now like to embedd $\bigwedge^k \mathrm{V}^*$ into $\bigotimes^k \mathrm{V}^*$ and this involves a choice. Hor each k let $A_k : \mathrm{V}^* \times \cdots \times \mathrm{V}^* \to \bigotimes^k \mathrm{V}^*$ be defined by

$$A_k(\alpha_1, ..., \alpha_k) := \sum_\sigma sgn(\sigma)\alpha_{\sigma 1} \otimes \cdots \otimes \alpha_{\sigma k}$$

By the universal property of $\bigwedge^k \mathrm{V}^*$ we obtain an induced map

$$\widetilde{A_k} : \bigwedge^k \mathrm{V}^* \to \bigotimes^k \mathrm{V}^*$$

If we identify $\bigotimes^k \mathrm{V}^*$ with $T_k(\mathrm{V})$ then we get a map $\widetilde{A_k} : \bigwedge^k \mathrm{V}^* \to T_k(\mathrm{V})$.

Proposition 7.2 *If* V *is a finitely generated free module then the map* $\widetilde{A} : \bigwedge^k \mathrm{V}^* \to T_k(\mathrm{V})$ *is a linear isomorphism with image equal to* $L^k_{alt}(\mathrm{V})$ *such that*

$$\widetilde{A_k}(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1, ..., v_k) = \det(\alpha_i\,(v_j)).$$

Next we combine these maps to obtain a module isomorphsm

$$\widetilde{A} : \bigwedge \mathrm{V}^* \to L_{alt}(\mathrm{V})$$

Now both $L_{alt}(\mathrm{V})$ and $\bigwedge \mathrm{V}^*$ have already idependently been given the exterior algebra structures. One may check that $\widetilde{A}$ has been defined in such a way as to be an isomorphism of these algebras:

$$\bigwedge \mathrm{V}^* \cong L_{alt}(\mathrm{V}) \text{ (as exterior algebras)}$$

Also notice that

$$\bigwedge^k \mathrm{V}^* \cong L^k_{alt}(\mathrm{V}) \cong \left(\bigwedge^k \mathrm{V}\right)^*$$

which allows us to thing of $\bigwedge^k \mathrm{V}^*$ as dual to $\bigwedge^k \mathrm{V}$ in such a way that

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(v_1 \wedge \cdots \wedge v_k) = \det\left[\alpha_i\left(v_j\right)\right]$$

Differential forms

Let M be a smooth n dimensional manifold. We now bundle together the various spaces $L^k_{alt}(T_pM)$. That is we form the natural bundle $L^k_{alt}(TM)$ that has as its fiber at p the space $L^k_{alt}(T_pM)$. Thus $L^k_{alt}(TM) = \bigsqcup_{p\in M} L^k_{alt}(T_pM)$.

Exercise 7.7 *Exhibit, simultaneously the smooth structure and vector bundle structure on* $L^k_{alt}(TM) = \bigsqcup_{p\in M} L^k_{alt}(T_pM)$.

Let the smooth sections of this bundle be denoted by

$$\Omega^k(M) = \Gamma(M; L^k_{alt}(TM)). \tag{7.1}$$

and sections over $U \subset M$ by $\Omega^k_M(U)$.

Definition 7.3 *Sections of* $\Omega^k(M)$ *are called differential* $k-$*forms or just* k-*forms.*

The space $\Omega^k(M)$ is a module over the algebra of smooth functions $C^\infty(M) = \mathcal{F}(M)$. The assignment $U \mapsto \Omega^k_M(U)$ is a (pre)sheaf of modules over the sheaf of smooth functions C^∞_M. We have the *direct* sum

$$\Omega_M(U) = \textstyle\sum_{n=0}^{\dim M} \Omega^k_M(U) = \Gamma\left(U, \sum_{n=0}^{\dim M} L^k_{alt}(TM)\right).$$

Definition 7.4 *For* $\omega \in \Omega^{k_1}_M(U)$ *and* $\eta \in \Omega^{k_2}_M(U)$ *we define the* ***exterior product*** $\omega \wedge \eta \in \Omega^{k_1+k_2}_M(U)$ *by*

$$(\omega \wedge \eta)(p) := \omega(p) \wedge \eta(p).$$

It is clear that $\omega \wedge \eta = (-1)^{k_1 k_2} \eta \wedge \omega$ and that we can extend the exterior product to a map $\Omega_M(U) \times \Omega_M(U) \to \Omega_M(U)$.

Definition 7.5 *The sections of the bundle $\Omega_M(U)$ are called* ***differential forms*** *on U. We identify $\Omega^k_M(U)$ with the obvious subspace of $\Omega_M(U) = \sum_{n=0}^{\dim M} \Omega^k_M(U)$. In this context we consider $\Omega^k_M(U)$ subspace of $\Omega_M(U)$. A differential form in $\Omega^k_M(U)$ is said to be* ***homogeneous*** *of* ***degree*** *k. If $U = M$ we write $\Omega^k(M)$.*

The reason for the subscript M in $\Omega^k_M(U)$ is to suggest the sheaf viewpoint and to keep things clear when there is more than one manifold involved. In cases where there is little chance of confusion we will drop this subscript. Whenever convenient we may extend this to a sum over all $n \in \mathbb{Z}$ by defining (as before) $\Omega^k_M(U) := 0$ for $n < 0$ and $\Omega^k_M(U) := 0$ if $n > \dim(M)$. Of course, we have made the trivial extension of $\wedge$ to the $\mathbb{Z}$-**graded algebra** by declaring that $\omega \wedge \eta = 0$ if either η or ω is homogeneous of negative degree.

Just as a tangent vector is the infinitesimal version of a (parameterized) curve through a point $p \in M$ so a covector at $p \in M$ is the infinitesimal version of a function defined near p. At this point one must be careful. It is true that for any single covector $\alpha_p \in T_pM$ there always exists a function f such $df_p = \alpha_p$. But as we saw in chapter 2, if $\alpha \in \Omega^1_M(U)$ then it is not necessarily true that there is a function $f \in C^\infty(U)$ such that $df = \alpha$. Now if $f_1, f_2, ..., f_k$ are smooth functions then one way to picture the situation is by think of the intersecting family of level sets of the functions $f_1, f_2, ..., f_k$ which in some cases can be pictured as a sort of "egg crate" structure. For a 2-form in a three dimensional manifold one may picture flux tubes as in figure 7.1. The infinitesimal version of this is a sort of straightened out "linear egg crate structure" which may be thought of as existing in the tangent space at a point. This is the rough intuition for $df_1|_p \wedge ... \wedge df_k|_p$ and the $k-$form $df_1 \wedge ... \wedge df_k$ is a field of such structures which somehow fit the level sets of the family $f_1, f_2, ..., f_k$. Of course, $df_1 \wedge ... \wedge df_k$ is a very special kind of $k-$form. In general a $k-$form over U may not arise from a family of functions.

Pull-back of a differential form.

Since we treat differential forms as alternating covariant tensor fields, we have a notion of pull-back already defined. It is easy to see that the pull-back of an alternating tensor field is also and alternating tensor field and so given any smooth map $f : M \to N$ we get a map $f^* : \Omega^k(N) \to \Omega^k(M)$. For convenience we recall here the definition:

$$(f^*\eta)(p)(v_1, ..., v_k) = \eta_{f(p)}(T_pf \cdot v_1, ..., T_pf \cdot v_k)$$

for tangent vectors $v_1, ..., v_k \in T_pM$. The pull-back extends in the obvious way to a map $f^* : \Omega(N) \to \Omega(M)$.

The pull-back is a very natural operation as exhibited by the following propositions.

Proposition 7.3 *Let $f : M \to N$ smooth map and $\eta_1, \eta_2 \in \Omega(N)$ we have*

$$f^*(\eta_1 \wedge \eta_2) = f^*\eta_1 \wedge f^*\eta_2$$

Proof: Exercise

Proposition 7.4 *Let $f_1 : M \to N$ and $f_2 : N \to P$ be smooth maps. Then for any smooth differential form $\eta \in \Omega(P)$ we have $(f_1 \circ f_2)^* \eta = f_2^*(f_1^* \eta)$. Thus $(f_1 \circ f_2)^* = f_2^* \circ f_1^*$.*

In case S is a regular submanifold of M then we have the inclusion map $\iota : S \hookrightarrow M$ which maps $p \in S$ to the very same point $p \in M$. As mentioned before, it natural to identify T_pS with $T_p\iota(T_pS)$ for any $p \in S$. In other words, we normally do not distinguish between a vector $v_p \in T_pS$ and $T_p\iota(v_p)$. Both are then written as v_p. Thus we we view the tangent bundle of S as subset (in fact, a subbundle) of TM. With this in mind we must realize that for any $\alpha \in \Omega^k(M)$ the form $\iota^*\alpha$ is just the restriction of α to vectors tangent to S. In particular, if $U \subset M$ is open and $\iota : U \hookrightarrow M$ then $\iota^*\alpha = \alpha|_U$.

The local expression for the pull-back is described as follows. First we define

$$
\begin{aligned}
\epsilon^I_L &= \epsilon^{i_1 \ldots i_k}_{l_1 \ldots l_k} \\
&= \begin{cases} 0 & \text{if} \quad I \text{ is not a permutation of } J \\ 1 & \text{if} \quad I \text{ is an even permutation of } J \\ -1 & \text{if} \quad I \text{ is an odd permutation of } J \end{cases}
\end{aligned}
$$

Let $(U, \mathtt{x})$ be a chart on M and $(V, \mathtt{y})$ a chart on N with $\mathtt{x}(U) \subset V$ then writing $\eta = \sum b_{\vec{J}} dy^{\vec{J}}$ and abbreviating $\frac{\partial\left(y^{j_1} \circ f\right)}{\partial x^{i_1}}$ to simply $\frac{\partial y^{j_1}}{\partial x^{i_1}}$ etc., we have

$$
\begin{aligned}
f^*\eta &= \sum b_{\vec{J}} \circ f \ \ d\left(y^{\vec{J}} \circ f\right) \\
&= \sum_{\vec{J}} \left(b_{\vec{J}} \circ f\right) \left(\sum_{i_1} \frac{\partial y^{j_1}}{\partial x^{i_1}} dx^{i_1} \right) \wedge \cdots \wedge \left(\sum_{i_k} \frac{\partial y^{j_1}}{\partial x^{i_1}} dx^{i_k} \right) \\
&= \sum_{\vec{J}} \sum_{I} \left(b_{\vec{J}} \circ f\right) \epsilon^I_L \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_1}}{\partial x^{i_1}} dx^{l_1} \wedge \cdots \wedge dx^{l_k} \\
&= \sum b_{\vec{J}} \circ f \ \frac{\partial y^{\vec{J}}}{\partial x^{\vec{L}}} dx^{\vec{L}}
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\partial y^{\vec{J}}}{\partial x^{\vec{L}}} &= \frac{\partial(y^{j_1} \cdots y^{j_k})}{\partial\left(x^{l_1} \cdots x^{l_k}\right)} \\
&= \det \begin{bmatrix} \frac{\partial y^{j_1}}{\partial x^{l_1}} & \cdots & \frac{\partial y^{j_1}}{\partial x^{l_k}} \\ \vdots & & \vdots \\ \frac{\partial y^{j_k}}{\partial x^{l_1}} & \cdots & \frac{\partial y^{j_k}}{\partial x^{l_k}} \end{bmatrix}
\end{aligned}
$$

Since this highly combinatorial notation is a bit intimidating at first sight we work out the case where $\dim M = 2$, $\dim N = 3$ and $k = 2$. As a warm up notice that since $dx^1 \wedge dx^1 = 0$ we have

$$\begin{aligned} d\left(y^2 \circ f\right) \wedge d\left(y^3 \circ f\right) &= \frac{\partial y^2}{\partial x^i} dx^i \wedge \frac{\partial y^3}{\partial x^j} dx^j \\ &= \frac{\partial y^2}{\partial x^i} \frac{\partial y^3}{\partial x^j} dx^i \wedge dx^j \\ &= \frac{\partial y^2}{\partial x^1} \frac{\partial y^3}{\partial x^2} dx^1 \wedge dx^2 + \frac{\partial y^2}{\partial x^2} \frac{\partial y^3}{\partial x^1} dx^2 \wedge dx^1 \\ &= \left(\frac{\partial y^2}{\partial x^1} \frac{\partial y^3}{\partial x^2} - \frac{\partial y^2}{\partial x^2} \frac{\partial y^3}{\partial x^1} \right) dx^2 \wedge dx^3 \\ \text{or } & \frac{\partial\left(y^2, y^3\right)}{\partial\left(x^2, x^3\right)} dx^2 \wedge dx^3 \end{aligned}$$

Using similar expressions, we have

$$\begin{aligned} f^*\eta &= f^*\left(b_{23} dy^2 \wedge dy^3 + b_{13} dy^1 \wedge dy^3 + b_{12} dy^1 \wedge dy^2\right) \\ &= b_{23} \circ f \sum \frac{\partial y^2}{\partial x^i} dx^i \wedge \frac{\partial y^3}{\partial x^j} dx^j + b_{13} \circ f \sum \frac{\partial y^1}{\partial x^i} dx^i \wedge \frac{\partial y^3}{\partial x^j} dx^j \\ &+ b_{12} \circ f \sum \frac{\partial y^1}{\partial x^i} dx^i \wedge \frac{\partial y^2}{\partial x^j} dx^j \\ &= \left(b_{23} \circ f \ \frac{\partial\left(y^2, y^3\right)}{\partial\left(x^1, x^2\right)} + b_{13} \circ f \ \frac{\partial\left(y^1, y^3\right)}{\partial\left(x^1, x^2\right)} + b_{12} \circ f \ \frac{\partial\left(y^1, y^2\right)}{\partial\left(x^1, x^2\right)} \right) dx^1 \wedge dx^2. \end{aligned}$$

Remark 7.3 *Notice the space $\Omega^0_M(U)$ is just the space of smooth functions $C^\infty(U)$ and so unfortunately we now have several notations for the same set: $C^\infty(U) = \Omega^0_M(U) = \mathfrak{T}^0_0(U)$. The subscript M will be omitted where confusion is unlikely to result.*

All that follows and much of what we have done so far works well for $\Omega_M(U)$ whether $U = M$ or not and will also respect restriction maps. Thus we will simply write Ω_M instead of $\Omega_M(U)$ or $\Omega(M)$ and $\mathfrak{X}_M$ instead of $\mathfrak{X}(U)$ so forth (sheaf notation). In fact, the exterior derivative d defined below commutes with restrictions and so is really a presheaf map.

The algebra of smooth differential forms $\Omega(U)$ is an example of a $\mathbb{Z}$ graded algebra over the ring $C^\infty(U)$ and is also a graded vector space over $\mathbb{R}$**.** We have for each $U \subset M$

1) The direct sum decomposition

$$\Omega(U) = \cdots \oplus \Omega^{-1}(U) \oplus \Omega^0(U) \oplus \Omega^1(U) \oplus \Omega^2(U) \cdots$$

where $\Omega^k(U) = 0$ if $k < 0$ or if $k > \dim(U)$;

2) The exterior product is a graded product:

$$\alpha \wedge \beta \in \Omega^{k+l}(U) \text{ for } \alpha \in \Omega^k(U) \text{ and } \beta \in \Omega^l(U)$$

that is

3) graded commutative: $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ for $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^l(U)$.

7.3.1 Exterior Derivative

Here we will define and study the exterior derivative d. First a useful general definition:

Definition 7.6 *A (natural)* ***graded derivation*** *of degree r on $\Omega := \Omega_M$ is a family of maps, one for each open set $U \subset M$, denoted' $\mathcal{D}_U : \Omega_M(U) \to \Omega_M(U)$ such that for each $U \subset M$,*

$$\mathcal{D}_U : \Omega^k_M(U) \to \Omega^{k+r}_M(U)$$

such that

1. *$\mathcal{D}_U$ is $\mathbb{R}$ linear;*
2. *$\mathcal{D}_U(\alpha \wedge \beta) = \mathcal{D}_U\alpha \wedge \beta + (-1)^{kr}\alpha \wedge \mathcal{D}_U\beta$; for $\alpha \in \Omega^k(U)$ and $\beta \in \Omega(U)$*
3. *$\mathcal{D}_U$ is natural with respect to restriction:*

$$\begin{array}{ccc} \Omega^k(U) & \overset{\mathcal{D}_U}{\to} & \Omega^{k+r}(U) \\ \downarrow & & \downarrow \\ \Omega^k(V) & \overset{\mathcal{D}_V}{\to} & \Omega^{k+r}(V) \end{array}$$

As usual we will denote all of the maps by a single symbol $\mathcal{D}$. In summary, we have a map of (pre)sheaves $\mathcal{D} : \Omega_M \to \Omega_M$. Along lines similar to our study of tensor derivations one can show that a graded derivation of Ω_M is completely determined by, and can be defined by its action on 0-forms (functions) and 1-forms. In fact, since every form can be locally built out of functions and exact one forms, i.e. differentials, we only need to know the action on 0-forms and exact one forms to determine the graded derivation.

Lemma 7.5 *Suppose $\mathcal{D}_1 : \Omega^k_M(U) \to \Omega^{k+r}_M(U)$ and $\mathcal{D}_2 : \Omega^k_M(U) \to \Omega^{k+r}_M(U)$ are defined for each open set $U \subset M$ which both satisfy 1,2 and 3 of definition 7.6 above. Then if $\mathcal{D}_1 = \mathcal{D}_2$ on Ω^0 and Ω^1 and if Ω^1 is generated by $\mathcal{D}_1(\Omega^0)$ then $\mathcal{D}_1 = \mathcal{D}_2$.*

Exercise 7.8 *Prove Lemma 7.5 above.*

The differential d defined by

$$df(X) = Xf \text{ for } X \in \mathfrak{X}_M(U) \text{ and } f \in C^\infty_M(U) \tag{7.2}$$

gives a map $\Omega^0_M \to \Omega^1_M$. Next we show that this map can be extended to a degree one graded derivation.

Theorem 7.1 *Let M a finite dimensional smooth manifold. There is a unique degree 1 graded derivation $d : \Omega_M \to \Omega_M$ such that for each $f \in C^\infty(U) = \Omega^0_M(U)$ we have that df coincides with the usual differential and such that $d \circ d = 0$.*

Proof. Let $(U, \mathtt{x})$ be a coordinate system. For a 0-form on U (i.e. a smooth function) we just define $d_{\mathtt{x}} f$ to be the usual differential given by $df = \sum \frac{\partial f}{\partial x^i} dx^i$. Now for $\alpha \in \Omega^k_M(U)$ we have $\alpha = \sum a_{\vec{I}} dx^{\vec{I}}$ we define $d_{\mathtt{x}}\alpha = \sum d_{\mathtt{x}} a_{\vec{I}} \wedge dx^{\vec{I}}$. To show the graded commutativity consider $\alpha = \sum a_{\vec{I}} dx^{\vec{I}} \in \Omega^k_M(U)$ and $\beta = \sum b_{\vec{J}} dx^{\vec{J}} \in \Omega^l_M(U)$. Then

$$
\begin{aligned}
d_{\mathtt{x}} (\alpha \wedge \beta) &= d_{\mathtt{x}} \left(\sum a_{\vec{I}} dx^{\vec{I}} \wedge \sum b_{\vec{J}} dx^{\vec{J}} \right) \\
&= d_{\mathtt{x}} \left(\sum a_{\vec{I}} b_{\vec{J}} dx^{\vec{I}} \wedge dx^{\vec{J}} \right) \\
&= \sum \left(\left(da_{\vec{I}} \right) b_{\vec{J}} + a_{\vec{I}} \left(db_{\vec{J}} \right) \right) dx^{\vec{I}} \wedge dx^{\vec{J}} \\
&= \left(\sum_{\vec{I}} da_{\vec{I}} \wedge dx^{\vec{I}} \right) \wedge \sum_{\vec{J}} b_{\vec{J}} dx^{\vec{J}} \\
&\quad + \sum_{\vec{I}} a_{\vec{I}} dx^{\vec{I}} \wedge \left((-1)^k \sum_{\vec{J}} db_{\vec{J}} \wedge dx^{\vec{J}} \right)
\end{aligned}
$$

since $db_{\vec{J}} \wedge dx^{\vec{I}} = (-1)^k dx^{\vec{I}} \wedge db_{\vec{J}}$ due to the k interchanges of the basic differentials dx^i. This means that the commutation rule holds at least in local coordinates. Also, easily verified in local coordinates is that for any function f we have $d_{\mathtt{x}} d_{\mathtt{x}} f = d_{\mathtt{x}} df = \sum_{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \right) dx^i \wedge dx^j = 0$ since $\frac{\partial^2 f}{\partial x^i \partial x^j}$ is symmetric in ij and $dx^i \wedge dx^j$ is antisymmetric in i, j. More generally, for any functions $f, g \in C^\infty(U)$ we have $d_{\mathtt{x}}(df \wedge dg) = 0$ because of the graded commutativity. Inductively we get $d_{\mathtt{x}}(df_1 \wedge df_2 \wedge \cdots \wedge df_k) = 0$ for any functions $f_i \in C^\infty(U)$. From this it easily follows that for any $\alpha = \sum a_{\vec{I}} dx^{\vec{I}} \in \Omega^k_M(U)$ we have $d_{\mathtt{x}} d_{\mathtt{x}} \sum a_{\vec{I}} dx^{\vec{I}} = d_{\mathtt{x}} \sum d_{\mathtt{x}} a_{\vec{I}} \wedge dx^{\vec{I}} = 0$. We have now defined an operator $d_{\mathtt{x}}$ for each coordinate chart $(U, \mathtt{x})$, that clearly has the desired properties on that chart. Consider two different charts $(U, \mathtt{x})$ and $(V, \mathtt{y})$ such that $U \cap V \neq \emptyset$. We need to show that $d_{\mathtt{x}}$ restricted to $U \cap V$ coincides with $d_{\mathtt{y}}$ restricted to $U \cap V$ but is clear that these restrictions of $d_{\mathtt{x}}$ and $d_{\mathtt{y}}$ satisfy the hypothesis of Lemma 7.5 and so they must agree on $U \cap V$.

It is now clear that the individual operators on coordinate charts fit together to give a well defined operator with the desired properties. ∎

Definition 7.7 *The degree one graded derivation just introduced is called the* ***exterior derivative****.*

Another approach to the existence of the exterior derivative is to exhibit a global coordinate free formula. This approach is given in G.2. Let $\omega \in \Omega^k(U)$ and view ω as an alternating multilinear map on $\mathfrak{X}_M(U)$. Then for then for

$X_0, X_1, ..., X_k \in \mathfrak{X}_M(U)$ define

$$d\omega(X_0, X_1, ..., X_k) = \sum_{0 \leq i \leq k} (-1)^i X_i(\omega(X_0, ..., \widehat{X_i}, ..., X_k)) + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([X_i, X_j], X_0, ..., \widehat{X_i}, ..., \widehat{X_j}, ..., X_k).$$

One can check that $d\omega$ is an alternating multilinear map on $\mathfrak{X}_M(U)$ and so defines a differential form of degree $k+1$. It can be shown that this formula give the same operator defined previously (One can then verify that the global formula reproduces the correct local formula).

Lemma 7.6 *Given any smooth map $f : M \to N$ we have that d is natural with respect to the pull-back:*

$$f^*(d\eta) = d(f^*\eta)$$

Definition 7.8 *A smooth differential form α is called* ***closed*** *if $d\alpha = 0$ and* ***exact*** *if $\alpha = d\beta$ for some differential form β.*

Notice that since $d \circ d = 0$, every exact form is closed. In general, the converse is not true. The extent to which it fails is a topological property of the manifold. This is the point of De Rham cohomology to be studied in detail in chapter 10. Here we just give the following basic definitions. The set of closed forms of degree k on a smooth manifold M is the kernel of $d : \Omega^k(M) \to \Omega^{k+1}(M)$ and is denoted $Z^k(M)$. The set of exact forms is the image of this map and is denoted $B^k(M)$. Since, $d \circ d = 0$ we have $B^k(M) \subset Z^k(M)$.

Definition 7.9 *The $k-th$* ***de Rham cohomology group*** *(actually a vector space) given by*

$$H^i(M) = \frac{Z^k(M)}{B^k(M)}. \tag{7.3}$$

In other words, we look at closed forms and identify any two whose difference is an exact form.

7.3.2 Vector Valued and Algebra Valued Forms.

For $\alpha \in L^k_{alt}(\mathrm{V};\mathrm{W})$ and $\beta \in L^k_{alt}(\mathrm{V};\mathrm{W})$ we define the wedge product using the same formula as before except that we use the tensor product so that $\alpha \wedge \beta \in L^k_{skew}(\mathrm{V};\mathrm{W} \otimes \mathrm{W})$:

$$(\omega \wedge \eta)(v_1, v_2, .., v_r, v_{r+1}, v_{r+2}, .., v_{r+s}) = \sum_{r,s-\text{shuffles } \sigma} \operatorname{sgn}(\sigma)\omega(v_{\sigma_1}, v_{\sigma_2}, .., v_{\sigma_r}) \otimes \eta(v_{\sigma_{r+1}}, v_{\sigma_{r+2}}, .., v_{\sigma_{r+s}})$$

We want to globalize this algebra. Let M be a smooth $n-$manifold and consider the set

$$L^k_{alt}(TM;\mathrm{W}) = \bigsqcup_{p \in M} L^k_{alt}(T_pM;\mathrm{W})$$

This set can easily be given a rather obvious vector bundle structure. In this setting it is convenience to identify $L^k_{alt}(T_pM; \mathrm{W})$ with $\mathrm{W} \otimes (\bigwedge^k T^*_pM)$ so that our bundle is indentified with the vector bundle $\mathrm{W} \otimes (\bigwedge^k T^*M)$ whose fiber at p is $\mathrm{W} \otimes (\bigwedge^k T^*_pM)$. The $C^\infty(M)-$module of sections of this bundle are denoted $\Omega^k(M, \mathrm{W})$. We obtain an exterior product $\Omega^k(M, \mathrm{W}) \times \Omega^l(M, \mathrm{W}) \to \Omega^{k+l}(M, \mathrm{W} \otimes \mathrm{W})$ as usual by $(\alpha \wedge \beta)(p) := \alpha(p) \wedge \beta(p)$. We still have a pull-back operation defined as before and also a natural exterior derivative

$$d : \Omega^k(M, \mathrm{W}) \to \Omega^{k+1}(M, \mathrm{W})$$

defined either in local charts or by the formula

$$\begin{aligned}
&d\omega(X_0, ..., X_k) \\
&= \sum_{1 \le i \le k} (-1)^i X_i(\omega(X_0, ..., \widehat{X_i}, ..., X_k)) \\
&+ \sum_{1 \le i < j \le k} (-1)^{i+j} \omega(X_0, ..., [X_i, X_j], ..., X_k)
\end{aligned}$$

where now $\omega(X_0, ..., \widehat{X_i}, ..., X_k)$ is a W-valued function. We know that the differential of a vector space valued function is defined so in the above formula

$$\begin{aligned}
&X_i(\omega(X_0, ..., \widehat{X_i}, ..., X_k))(p) \\
&= d(\omega(X_0, ..., \widehat{X_i}, ..., X_k))(X_i(p))
\end{aligned}$$

which is an element of W under the usual identification of W with any of its tangent spaces.

To give a local formula valid for finite dimensional M, we let $f_1, ..., f_n$ be a basis of W and $(x^1, ..., x^n)$ local coordinates defined on U. For $\omega = \sum a^j_{\vec{I}}, f_j \otimes dx^{\vec{I}}$ we have

$$\begin{aligned}
d\omega &= d(f_j \otimes \sum a_{\vec{I},j} dx^{\vec{I}}) \\
&= \sum (f_j \otimes da_{\vec{I},j} \wedge dx^{\vec{I}}).
\end{aligned}$$

The elements $f_j \otimes dx^{\vec{I}}_p$ form a basis for the vector space $\mathrm{W} \otimes (\bigwedge^k T^*_pM)$ for every $p \in U$.

Now if W happens to be an algebra then the algebra product $\mathrm{W} \times \mathrm{W} \to \mathrm{W}$ is bilinear and so gives rise to a linear map $m : \mathrm{W} \otimes \mathrm{W} \to \mathrm{W}$. We compose the exterior product with this map to get a wedge product $\overset{m}{\wedge} : \Omega^k(M, \mathrm{W}) \times \Omega^l(M, \mathrm{W}) \to \Omega^{k+l}(M, \mathrm{W})$

$$\begin{aligned}
&(\omega \overset{m}{\wedge} \eta)(X_1, X_2, .., X_r, X_{r+1}, X_{r+2}, .., X_{r+s}) \\
&= \sum_{r,s-\text{shuffles } \sigma} \operatorname{sgn}(\sigma) m \left[\omega(X_{\sigma_1}, X_{\sigma_2}, .., X_{\sigma_r}) \otimes \eta(X_{\sigma_{r+1}}, X_{\sigma_{r+2}}, .., X_{\sigma_{r+s}})\right] \\
&= \sum_{r,s-\text{shuffles } \sigma} \operatorname{sgn}(\sigma) \omega(X_{\sigma_1}, X_{\sigma_2}, .., X_{\sigma_r}) \cdot \eta(X_{\sigma_{r+1}}, X_{\sigma_{r+2}}, .., X_{\sigma_{r+s}})
\end{aligned}$$

A particularly important case is when W is a Lie algebra $\mathfrak{g}$ with bracket $[.,.]$. Then we write the resulting product $\overset{m}{\wedge}$ as $[.,.]_\wedge$ or just $[.,.]$ when there is no risk of confusion. Thus if $\omega, \eta \in \Omega^1(U, \mathfrak{g})$ are Lie algebra valued 1-forms then

$$[\omega, \eta]_\wedge(X) = [\omega(X), \eta(Y)] + [\eta(X), \omega(Y)].$$

In particular, $\frac{1}{2}[\omega, \omega]_\wedge(X, Y) = [\omega(X), \omega(Y)]$ which might not be zero in general!

7.3.3 Vector Bundle Valued Forms.

It will be convenient in several contexts to have on hand the notion of a differential form with values in a vector bundle. Let $\xi = (E, \pi, M)$ be a smooth vector bundle. We can consider the vector bundle $L^k_{alt}(TM, E)$ over M whose fiber at p is $L^k_{alt}(T_pM, E_p)$. We identify $L^k_{alt}(T_pM, E_p)$ with $E_p \otimes \wedge^k T^*_p M$ and thus the bundle is identified with $E \otimes \wedge^k T^* M$.

Definition 7.10 *Let $\xi = (E, \pi, M,)$ be a smooth vector bundle. A differential $k-$form with values in ξ (or values in E) is a smooth section of the bundle $E \otimes \wedge^k T^* M$. These are denoted by $\Omega^k(M; E)$.*

Remark 7.4 *The reader should avoid confusion between $\Omega^k(M; E)$ and the space of sections $\Gamma(M, \wedge^k E)$.*

In order to get a grip on the meaning of the bundle let exhibit transition functions. For a vector bundle, knowing the transition functions is tantamount to knowing how local expressions with respect to a frame transform as we change frame. A frame for $E \otimes \wedge^p T^* M$ is given by combining a local frame for E with a local frame for $\wedge^p TM$. Of course we must choose a common refinement of the VB-charts in order to do this but this is obviously no problem. Let $(e_1, ..., e_k)$ be a frame field defined on an open set U. We may as well take U to also be a chart domain for the manifold M. Then any local section of $\Omega^p(\xi)$ defined on U has the form

$$\sigma = \sum a^j_{\vec{I}} e_j \otimes dx^I$$

for some smooth functions $a^j_{\vec{I}} = a^j_{i_1 \ldots i_p}$ defined in U. Then for a new local set up with frames $(f_1, ..., f_k)$ and $dy^{\vec{I}} = dy^{i_1} \wedge \cdots \wedge dy^{i_p}$ $(i_1 < ... < i_p)$ we have

$$\sigma = \sum \acute{a}^j_{\vec{I}} f_j \otimes dy^{\vec{I}}$$

and the transformation law

$$\acute{a}^j_{\vec{I}} = a^i_{\vec{J}} C^j_i \frac{\partial x^{\vec{J}}}{\partial y^{\vec{I}}}$$

and where C^j_i is defined by $f_s C^s_j = e_j$.

Exercise 7.9 *Derive the above transformation law.*

A more elegant way of describing the transition functions is just to recall that anytime we have two vector bundles over the same base space and respective typical fibers V and W then the respective transition functions $g_{\alpha\beta}$ and $h_{\alpha\beta}$ (on a common cover) combine to give $g_{\alpha\beta} \otimes h_{\alpha\beta}$ where for a given $x \in U_{\alpha\beta}$

$$g_{\alpha\beta}(x) \otimes h_{\alpha\beta}(x) : \mathrm{V} \otimes \mathrm{W} \to \mathrm{V} \otimes \mathrm{W}$$
$$g_{\alpha\beta}(x) \otimes h_{\alpha\beta}(x)(v, w) = g_{\alpha\beta}(x)v \otimes h_{\alpha\beta}(x)w.$$

At any rate, these transformation laws fade into the background since if all our expressions are manifestly invariant (or invariantly defined in the first place) then we don't have to bring them up.

Now we want to define an important graded module structure on $\Omega(M; E) = \sum_{p=0}^{n} \Omega^p(M; E)$. This will be a module over the graded algebra $\Omega(M)$. The action of $\Omega(M)$ on $\Omega(M; E)$ is given by maps $\overset{\wedge}{\otimes} : \Omega^k(M) \times \Omega^l(M; E) \to \Omega^{k+l}(M; E)$ which in turn are defined by extending the following rule linearly:

$$\mu^1 \overset{\wedge}{\otimes} (\sigma \otimes \mu^2) := \sigma \otimes \mu^1 \wedge \mu^2$$

If the vector bundle is actually an algebra bundle then (naming the bundle $\mathcal{A} \to M$ now for "algebra") we may turn $\mathcal{A} \otimes \wedge T^*M := \sum_{p=0}^{n} \mathcal{A} \otimes \wedge^p T^*M$ into an algebra bundle by defining

$$(v_1 \otimes \mu^1) \wedge (v_2 \otimes \mu^2) := v_1 v_2 \otimes \mu^1 \wedge \mu^2$$

and then extending linearly:

$$(a_j^i v_i \otimes \mu^j) \wedge (b_l^k v_k \otimes \mu^l) := v_i v_j \otimes \mu^j \wedge \mu^l$$

From this the sections $\Omega(M, \mathcal{A}) = \Gamma(M, \mathcal{A} \otimes \wedge T^*M)$ become an algebra over the ring of smooth functions. For us the most important example is where $\mathcal{A} = \mathrm{End}(E)$. Locally, say on U, sections σ_1 and σ_2 of $\Omega(M, \mathrm{End}(E))$ take the form $\sigma_1 = A_i \otimes \alpha^i$ and $\sigma_2 = B_i \otimes \beta^i$ where A_i and B_i are maps $U \to \mathrm{End}(E)$. Thus for each $x \in U$, the A_i and B_i evaluate to give $A_i(x), B_i(x) \in \mathrm{End}(E_x)$. The multiplication is then

$$(A_i \otimes \alpha^i) \wedge (B_j \otimes \beta^j) = A_i B_j \otimes \alpha^i \wedge \beta^j$$

where the $A_i B_j : U \to \mathrm{End}(E)$ are local sections given by composition:

$$A_i B_j : x \mapsto A_i(x) \circ B_j(x).$$

Exercise 7.10 *Show that $\Omega(M, \mathrm{End}(E))$ acts on $\Omega(M, E)$ making $\Omega(M, E)$ a bundle of modules over the bundle of algebras $\Omega(M, \mathrm{End}(E))$.*

If this seems all to abstract to the newcomer perhaps it would help to think of things this way: We have a cover of a manifold M by open sets $\{U_\alpha\}$ that simultaneously trivialize both E and TM. Then these give also trivializations on these open sets of the bundles $\mathrm{Hom}(E,E)$ and $\wedge TM$. Associated with each is a frame field for $E \to M$ say $(e_1, ..., e_k)$ which allows us to associate with each section $\sigma \in \Omega^p(M,E)$ a $k-$tuple of $p-$forms $\sigma_U = (\sigma_U^i)$ for each U. Similarly, a section $A \in \Omega^q(M, \mathrm{End}(E))$ is equivalent to assigning to each open set $U \in \{U_\alpha\}$ a matrix of $q-$forms A_U. The algebra structure on $\Omega(M, \mathrm{End}(E))$ is then just matrix multiplication were the entries are multiplied using the wedge product $A_U \wedge B_U$ where

$$(A_U \wedge B_U)^i_j = A^i_k \wedge B^k_j$$

The module structure is given locally by $\sigma_U \mapsto A_U \wedge \sigma_U$. Where did the bundle go? The global topology is now encoded in the transformation laws which tell us what the same section looks like when we change to a new frame field on an overlap $U_\alpha \cap U_\beta$. In this sense the bundle is a combinatorial recipe for pasting together local objects.

7.4 Lie derivative, interior product and exterior derivative.

The Lie derivative acts on differential forms since the latter are, from one viewpoint, tensors. When we apply the Lie derivative to a differential form we get a differential form so we should think about the Lie derivative in the context of differential forms.

Lemma 7.7 *For any $X \in \mathfrak{X}(M)$ and any $f \in \Omega^0(M)$ we have $\mathcal{L}_X df = d\mathcal{L}_X f$.*

Proof. For a function f we compute as

$$\begin{aligned}
&(\mathcal{L}_X df)(Y) \\
&= (\frac{d}{dt}(\varphi_t^X)^* df)(Y) = \frac{d}{dt} df(T\varphi_t^X \cdot Y) \\
&= \frac{d}{dt} Y((\varphi_t^X)^* f) = Y(\frac{d}{dt}(\varphi_t^X)^* f) \\
&= Y(\mathcal{L}_X f) = d(\mathcal{L}_X f)(Y)
\end{aligned}$$

where $Y \in \mathfrak{X}(M)$ is arbitrary. ■

Exercise 7.11 *Show that $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X \alpha \wedge \beta + \alpha \wedge \mathcal{L}_X \beta$.*

We now have two ways to differentiate sections in $\Omega(M)$. Once again we write Ω_M instead of $\Omega(U)$ or $\Omega(M)$ since every thing works equally well in either case. In other words we are thinking of the presheaf $\Omega_M : U \mapsto \Omega(U)$. First, there is the Lie derivative which turns out to be a graded derivation of degree zero;

$$\mathcal{L}_X : \Omega^i_M \to \Omega^i_M. \tag{7.4}$$

Second, there is the exterior derivative d which is a graded derivation of degree 1. In order to relate the two operations we need a third map which, like the Lie derivative, is taken with respect to a given field $X \in \Gamma(U; TM)$. This map is a degree -1 graded derivation and is defined by

$$\iota_X \omega(X_1, \ldots, X_{i-1}) = \omega(X, X_1, \ldots, X_{i-1}) \tag{7.5}$$

where we view $\omega \in \Omega^i_M$ as a skew-symmetric multi-linear map from $\mathfrak{X}_M \times \cdots \times \mathfrak{X}_M$ to $\mathcal{C}^\infty_M$. We could also define ι_X as that unique operator that satisfies

$$\iota_X \theta = \theta(X) \text{ for } \theta \in \Omega^1_M \text{ and } X \in \mathfrak{X}_M$$

$$\iota_X(\alpha \wedge \beta) = (\iota_X \alpha) \wedge \beta + (-1)^k \wedge \alpha \wedge (\iota_X \beta) \text{ for } \alpha \in \Omega^k_M.$$

In other word, ι_X is the graded derivation of Ω_M of degree -1 determined by the above formulas.

In any case, we will call this operator the **interior product** or **contraction operator**.

Notation 7.1 *Other notations for $\iota_X \omega$ include $X \lrcorner \omega = \langle X, \omega \rangle$. These notations make the following theorem look more natural:*

Theorem 7.2 *The Lie derivative is a derivation with respect to the pairing $\langle X, \omega \rangle$. That is*

$$\mathcal{L}_X \langle X, \omega \rangle = \langle \mathcal{L}_X X, \omega \rangle + \langle X, \mathcal{L}_X \omega \rangle$$

or

$$\mathcal{L}_X (X \lrcorner \omega) = (\mathcal{L}_X X) \lrcorner \omega + X \lrcorner (\mathcal{L}_X \omega)$$

Using the "ι_X" notation: $\mathcal{L}_X(\iota_X \omega) = \iota_{\mathcal{L}_X X} \omega + \iota_X \mathcal{L}_X \omega$.

Proof. Exercise. ■

Now we can relate the Lie derivative, the exterior derivative and the contraction operator.

Theorem 7.3 *Let $X \in \mathfrak{X}_M$. Then we have Cartan's homotopy formula;*

$$\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d \tag{7.6}$$

Proof. One can check that both sides define derivations and so we just have to check that they agree on functions and exact 1-forms. On functions we have $\iota_X f = 0$ and $\iota_X df = Xf = \mathcal{L}_X f$ so the formula holds. On differentials of functions we have

$$(d \circ \iota_X + \iota_X \circ d) df = (d \circ \iota_X) df = d\mathcal{L}_X f = \mathcal{L}_X df$$

where we have used lemma 7.7 in the last step. ■

As a corollary can now extend lemma 7.7:

Corollary 7.1 $d \circ \mathcal{L}_X = \mathcal{L}_X \circ d$

Proof.

$$d\mathcal{L}_X\alpha = d(d\iota_X + \iota_X d)(\alpha)$$
$$= d\iota_X d\alpha = d\iota_X d\alpha + \iota_X dd\alpha = \mathcal{L}_X \circ d$$

■

Corollary 7.2 *We have the following formulas:*
1) $\iota_{[X,Y]} = \mathcal{L}_X \circ \iota_Y - \iota_Y \circ \mathcal{L}_X$
2) $\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge \iota_X\omega$ *for all* $\omega \in \Omega(M)$.

Proof. Exercise. ■

7.5 Orientation

A rank n vector bundle $E \to M$ is called oriented if every fiber E_p is given a smooth choice of orientation. There are several equivalent ways to make a rigorous definition:

1. A vector bundle is orientable if and only if it has an atlas of bundle charts (local trivializations) such that the corresponding transition maps take values in $Gl^+(n,\mathbb{R})$ the group of positive determinant matrices. If the vector bundle is orientable then this divides the set of all bundle charts into two classes. Two bundle charts are in the same orientation class the transition map takes values in $Gl^+(n,\mathbb{R})$. If the bundle is not orientable there is only one class.

2. If there is a smooth global section s on the bundle $\bigwedge^n E \to M$ then we say that this determines an orientation on E. A frame $(f_1, ..., f_n)$ of fiber E_p is positively oriented with respect to s if and only if $f_1 \wedge ... \wedge f_n = as(p)$ for a positive real number $a > 0$.

3. If there is a smooth global section ω on the bundle $\bigwedge^n E^* \cong L^k_{alt}(E) \to M$ then we say that this determines an orientation on E. A frame $(f_1, ..., f_n)$ of fiber E_p is positively oriented with respect to ω if and only if $\omega(p)(f_1, ..., f_n) > 0$.

Exercise 7.12 *Show that each of these three approaches are equivalent.*

Exercise 7.13 *If* $\pi_1 : E_1 \to M$ *and* $\pi_2 : E_2 \to M$ *are orientable then so is the Whitney sum* $\pi_1 \oplus \pi_2 : E_1 \oplus E_2 \to M$.

Now let M be an n-dimensional manifold. Let U be some open subset of M which may be all of M. Consider a **top form**, i.e. an n-form $\varpi \in \Omega^n_M(U)$ where $n = \dim(M)$ and assume that ϖ is never zero on U. In this case we will say that ϖ is nonzero or that ϖ is a **volume form**. Every other top form μ is of the form $\mu = f\varpi$ for some smooth function f. This latter fact follows easily from $\dim(\bigwedge^n T_pM) = 1$ for all p. If $\varphi : U \to U$ is a diffeomorphism then we must have that $\varphi^*\varpi = \delta\varpi$ for some $\delta \in C^\infty(U)$ that we will call the **Jacobian determinant** of φ with respect to the volume element ϖ:

$$\varphi^*\varpi = J_\varpi(\varphi)\varpi$$

Proposition 7.5 *The sign of $J_\varpi(\varphi)$ is independent of the choice of volume form ϖ.*

Proof. Let $\varpi' \in \Omega^n_M(U)$. We have

$$\varpi = a\varpi'$$

for some function a that is never zero on U. We have

$$\begin{aligned} J(\varphi)\varpi &= (\varphi^*\varpi) = (a \circ \varphi)(\varphi^*\varpi') \\ &= (a \circ \varphi)J_{\varpi'}(\varphi)\varpi' = \frac{a \circ \varphi}{a}\varpi \end{aligned}$$

and since $\frac{a\circ\varphi}{a} > 0$ and ϖ is nonzero the conclusion follows. ∎

Let us consider a very important special case of this: Suppose that $\varphi : U \to U$ is a diffeomorphism and $U \subset \mathbb{R}^n$. Then letting $\varpi_0 = du^1 \wedge \cdots \wedge du^n$ we have

$$\begin{aligned} \varphi^*\varpi_0(x) &= \varphi^*du^1 \wedge \cdots \wedge \varphi^*du^n(x) \\ &= \left(\sum \left.\frac{\partial(u^1 \circ \varphi)}{\partial u^{i_1}}\right|_x du^{i_1}\right) \wedge \cdots \wedge \left(\sum \left.\frac{\partial(u^n \circ \varphi)}{\partial u^{i_n}}\right|_x du^{i_n}\right) \\ &= \det\left(\frac{\partial(u^i \circ \varphi)}{\partial u^j}(x)\right) = J\varphi(x). \end{aligned}$$

so in this case $J_{\varpi_0}(\varphi)$ is just the usual Jacobian determinant of φ.

Definition 7.11 *A diffeomorphism $\varphi : U \to U \subset \mathbb{R}^n$ is said to be positive or orientation preserving if* $\det(T\varphi) > 0$.

More generally, let a nonzero top form ϖ be defined on $U \subset M$ and let ϖ' be another defined on $U' \subset N$. Then we say that a diffeomorphism $\varphi : U \to U'$ is orientation preserving (or positive) with respect to the pair ϖ, ϖ' if the unique function $J_{\varpi,\varpi'}$ such that $\varphi^*\varpi' = J_{\varpi,\varpi'}\varpi$ is strictly positive on U.

Definition 7.12 *A smooth manifold M is said to be* ***orientable*** *if and only if there is an atlas of admissible charts such that for any pair of charts ψ_α, U_α and ψ_β, U_β from the atlas with $U_\alpha \cap U_\beta \neq \emptyset$, the transition map $\psi_\beta \circ \psi_\alpha^{-1}$ is orientation preserving. Such an atlas is called an orienting atlas.*

Exercise 7.14 *The tangent bundle is a vector bundle. Show that this last definition agrees with our definition of an orientable vector bundle in that M is an orientable manifold in the current sense if and only if TM is an orientable vector bundle.*

Let $\mathcal{A}_M$ be the maximal atlas for an orientable smooth manifold M. Then there are two subatlases $\mathcal{A}$ and $\mathcal{A}'$ with $\mathcal{A} \cup \mathcal{A}' = \mathcal{A}_M$, $\mathcal{A} \cap \mathcal{A}' = \emptyset$ and such that the transition maps for charts from $\mathcal{A}$ are all positive and similarly the transition maps of $\mathcal{A}'$ are all positive.. Furthermore if $\psi_\alpha, U_\alpha \in \mathcal{A}$ and $\psi_\beta, U_\beta \in \mathcal{A}'$ then $\psi_\beta \circ \psi_\alpha^{-1}$ is negative (orientation reversing). A choice of one these two atlases is called an **orientation on** M. Every orienting atlas is a subatlas of exactly one of $\mathcal{A}$ or $\mathcal{A}'$. If such a choice is made then we say that M is **oriented**. Alternatively, we can use the following proposition to specify an orientation on M:

Proposition 7.6 *Let $\varpi \in \Omega^n(M)$ be a volume form on M, i.e. ϖ is a nonzero top form. Then ϖ determines an orientation by determining an (orienting) atlas $\mathcal{A}$ by the rule*

$$\psi_\alpha, U_\alpha \in \mathcal{A} \Longleftrightarrow \psi_\alpha \text{ is orientation preserving resp. } \varpi, \varpi_0$$

where ϖ_0 is the standard volume form on $\mathbb{R}^n$ introduced above.

Exercise 7.15 *Prove the last proposition and then prove that we can use an orienting atlas to construct a volume form on an orientable manifold that gives the same orientation as the orienting atlas.*

We now construct a two fold covering manifold $\mathrm{Or}(M)$ for any finite dimensional manifold called the orientation cover. The orientation cover will itself always be orientable. Consider the vector bundle $\bigwedge^n T^*M$ and remove the zero section to obtain

$$(\bigwedge^n T^*M)^\times := \bigwedge^n T^*M - \{\text{zero section}\}$$

Define an equivalence relation on $(\bigwedge^n T^*M)^\times$ by declaring $\nu_1 \sim \nu_2$ if and only if ν_1 and ν_2 are in the same fiber and if $\nu_1 = a\nu_2$ with $a > 0$. The space of equivalence classes is denoted $\mathrm{Or}(M)$. There is a unique map π_{Or} making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Or}(M) & \rightarrow & (\bigwedge^n T^*M)^\times \\ & & \downarrow \\ \pi_{Or} & \searrow & \\ & & M \end{array}$$

$\mathrm{Or}(M) \to M$ is a two fold covering space with the quotient topology and in fact is a smooth manifold.

7.5.1 Orientation of manifolds with boundary

Recall that a **half space chart** ψ_α for a manifold with boundary M is a bijection (actually diffeomorphism) of an open subset U_α of M onto an open subset of

$\mathbb{H}^n_-$. A C^r **half space atlas** is a collection ψ_α, U_α of such charts such that for any two; ψ_α, U_α and ψ_β, U_β, the map $\psi_\alpha \circ \psi_\beta^{-1}$ is a C^r diffeomorphism on its natural domain (if non-empty). Note: "Diffeomorphism" means in the extended sense of a being homeomorphism such that both $\psi_\alpha \circ \psi_\beta^{-1} :: \mathbb{H}^n_- \to \mathsf{M}$ and its inverse are C^r in the sense of definition 1.3.

Let us consider the case of finite dimensional manifolds. Then letting $\mathsf{M} = \mathbb{R}^n$ and $\lambda = u^1 : \mathbb{R}^n \to \mathbb{R}$ we have the half space $\mathbb{H}^n_- = \mathbb{R}^n_{u^1 \le 0}$. The funny choice of sign is to make $\mathbb{H}^n_- = \mathbb{R}^n_{u^1 \le 0}$ rather than $\mathbb{R}^n_{u^1 \ge 0}$. The reason we do this is to be able to get the right induced orientation on ∂M without introducing a minus sign into our Stoke's formula proved below. The reader may wish to re-read remark 1.6 at this time.

Now, imitating our previous definition we define an oriented (or orienting) atlas for a finite dimensional manifold with boundary to be an atlas of half-space charts such that the overlap maps $\psi_\alpha \circ \psi_\beta^{-1} :: \mathbb{R}^n_{u^1 \le 0} \to \mathbb{R}^n_{u^1 \le 0}$ are orientation preserving. A manifold with boundary together a choice of (maximal) oriented atlas is called an **oriented manifold with boundary**. If there exists an orienting atlas for M then we say that M is **orientable** just as the case of a manifold without boundary.

Now if $\mathcal{A} = \{(\psi_\alpha, U_\alpha)\}_{\alpha \in A}$ is an orienting atlas for M as above with domains in $\mathbb{R}^n_{u^1 \le 0}$ then the induced atlas $\{(\psi_\alpha|_{U_\alpha \cap \partial M}, U_\alpha \cap \partial M)\}_{\alpha \in A}$ is an orienting atlas for the manifold ∂M and the resulting choice of orientation is called the induced orientation on ∂M. If M is oriented we will always assume that ∂M is given this induced orientation.

Definition 7.13 *A basis $f_1, f_2, ..., f_n$ for the tangent space at a point p on an oriented manifold (with or without boundary) is called positive if whenever $\psi_\alpha = (x^1, ..., x^n)$ is an oriented chart on a neighborhood of p then $(dx^1 \wedge ... \wedge dx^n)(f_1, f_2, ..., f_n) > 0$.*

Recall that vector v in T_pM for a point p on the boundary ∂M is called outward pointing if $T_p\psi_\alpha \cdot v \in \mathbb{H}^n_-$ is outward pointing in the sense that $\lambda(T_p\psi_\alpha \cdot v) < 0$.

Since we have chosen $\lambda = u^1$ and hence $\mathbb{H}^n_- = \mathbb{R}^n_{u^1 \le 0}$ for our definition in choosing the orientation on the boundary we have that in this case v is outward pointing if and only if $T_p\psi_\alpha \cdot v \in \mathbb{R}^n_{u^1 \le 0}$.

Definition 7.14 *A **nice chart** on a smooth manifold (possibly with boundary) is a chart ψ_α, U_α where ψ_α is a diffeomorphism onto $\mathbb{R}^n_{u^1 \le 0}$ if $U_\alpha \cap \partial M \ne \emptyset$ and a diffeomorphism onto the interior $\mathbb{R}^n_{u^1 < 0}$ if $U_\alpha \cap \partial M = \emptyset$.*

Lemma 7.8 *Every (oriented) smooth manifold has an (oriented) atlas consisting of nice charts.*

Proof. If ψ_α, U_α is an oriented chart with range in the interior of the left half space $\mathbb{R}^n_{u^1 \le 0}$ then we can find a ball B inside $\psi_\alpha(U_\alpha)$ in $\mathbb{R}^n_{u^1 < 0}$ and then we form a new chart on $\psi_\alpha^{-1}(B)$ with range B. But a ball is diffeomorphic to $\mathbb{R}^n_{u^1 < 0}$. So composing with such a diffeomorphism we obtain the nice chart. If

ψ_α, U_α is an oriented chart with range meeting the boundary of the left half space $\mathbb{R}^n_{u^1 \leq 0}$ then we can find a half ball B_- in $\mathbb{R}^n_{u^1 \leq 0}$ with center on $\mathbb{R}^n_{u^1=0}$. Reduce the chart domain as before to have range equal to this half ball. But every half ball is diffeomorphic to the half space $\mathbb{R}^n_{u^1 \leq 0}$ so we can proceed by composition as before. ■

Exercise 7.16 *If $\psi_\alpha = (x^1, ..., x^n)$ is an oriented chart on a neighborhood of p on the boundary of an oriented manifold with boundary then the vectors $\frac{\partial}{\partial x^2}, ..., \frac{\partial}{\partial x^n}$ form a positive basis for $T_p \partial M$ with respect to the induced orientation on ∂M. More generally, if f_1 is outward pointing and $f_1, f_2, ..., f_n$ is positive on M at p, then $f_2, ..., f_n$ will be positive for ∂M at p.*

Exercise 7.17 *Show that if M is simply connected then it must be orientable.*

7.6 Problems

1. Show that if $\theta_1, ..., \theta_k$ covectors on a vector space V then $\theta_1 \wedge .... \wedge \theta_k \neq 0$ if and only if $\theta_1, ..., \theta_k$ are linearly independent.

2. Show that if $f_1, ..., f_n$ are smooth functions on an open set in an $n-$manifold. Let p be in their common domain. Then there is an open set U conntaning p such that $f_1, ..., f_n$ are coordinate functions of chart on some neighborhood of p if and only if $df_1 \wedge .... \wedge df_n$ is nonzero at p.

3. Show that the sphere is orientable.

4. Show that $d \circ L_X = L_X \circ d$.

5. Prove (i) of Proposition 7.1.

6. Prove Proposition 7.3.

7. Prove Corollary 7.2.

8. Prove Cartan's Lemma: Let $k \leq n = \dim M$ and $\omega_1, ..., \omega_k$ be 1-forms on M which are linearly independent at each point. Suppose that there are 1-forms $\theta_1, ..., \theta_k$ such that

$$\sum_{i=1}^{k} \theta_i \wedge \omega_i = 0 \text{ (identically)}.$$

Then there exists a symmetric $k \times k$ matrix of smooth functions (A_{ij}) such

$$\theta_i = \sum_{j=1}^{k} A_{ij} \omega_j \text{ for } i = 1, ..., k$$

9. Let $M = \mathbb{R}^3 \backslash \{0\}$ and let

$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

Find $d\omega$ and determine whether ω is closed and if so, whether it is exact. Find the expression for ω in spherical polar coordinates.

10. Show that every simply connected manifold is orientable.

Chapter 8

Integration and Stokes' Theorem

In this chapter we explore a fundamental aspect of differential forms that has not been mentioned yet. Namely, a differential $n-$form on and n-dimensional *oriented* manifold M "wants" to be integrated over the manifold to give number. Actually, we will first want to make sense of integrating a $k-$form on M over certain smooth maps $\phi : U \to M$ where $U \subset \mathbb{R}^k$.

First we will talk about integrating 1-forms over a parameterized curve. The idea is already familiar from the calculus of several variables and is none other than the familiar line integral. For example, consider the curve $\gamma : [0, \pi] \to \mathbb{R}^3$ given by $\gamma(t) := (\cos t, \sin t, t)$. If $\omega = xydx + dy + xzdz$ then the integral is

$$\begin{aligned}
\int_\gamma \omega &:= \int_\gamma xydx + dy + xzdz \\
&= \int_{[0,\pi]} \gamma^* \omega \text{ (this step is to be a definition)} \\
&\int_0^\pi (x(t)y(t)dx/dt + dydt + x(t)z(t)dz/dt)dt \\
&= \int_0^\pi (\cos t \sin t \sin t - \cos t + t \cos t)dt = -2
\end{aligned}$$

Notice the analogy in the fact that a $1-$form at a point takes a vector (infinitesimal curve) and gives a number while the $1-$form globally takes a curve and via integration also yields a number.

Since we want to be precise about what we are doing and also since the curves we integrate over may have singularities we now put forth various definitions regarding curves.

The Let O be an open set in $\mathbb{R}$. A continuous map $c : O \to M$ is C^k if the $k-$th derivative $(\mathtt{x} \circ c)^{(k)}$ exist and is continuous on $c^{-1}(U)$ for every coordinate chart $(U, \mathtt{x})$ such that $U \cap c(I) \neq \emptyset$. If I is a subset of $\mathbb{R}$ then a map $c : I \to M$

is said to be C^k if there exists a C^k extension $\widetilde{c} : O \to M$ for some open set O containing I. We are particularly interested in the case where I is an interval. This interval may be open, closed, half open etc. We also allow intervals for which one of the "end points" a or b is $\pm\infty$.

Definition 8.1 *Let $O \subset \mathbb{R}$ be open. A continuous map $c : O \to M$ is said to be piecewise C^k if there exists a discrete sequence of points $\{t_n\} \subset O$ with $t_i < t_{i+1}$ such that c restricted to each $(t_i, t_{i+1}) \cap O$ is C^k. If I is subset of $\mathbb{R}$, then a map $c : I \to M$ is said to be piecewise C^k if there exists a piecewise extension of c to some open set containing I.*

Definition 8.2 *A **parametric curve** in M is a piecewise differentiable map $c : I \to M$ where I is either an interval or finite union of intervals. If c is an interval then we say that c is a **connected parametric curve**.*

Definition 8.3 *A elementary parametric curve is a regular curve $c : I \to \mathbb{R}^n$ such that I is an open connected interval.*

Definition 8.4 *If $c : I_1 \to M$ and $b : I_2 \to M$ are curves then we say that b is a positive (resp. negative) reparametrization of c if there exists a bijection $h : I_2 \to I_1$ with $c \circ h = b$ such that h is smooth and $h'(t) > 0$ (resp. $h'(t) > 0$) for all $t \in I_2$.*

We distinguish between a C^k map $c : I \to M$ and its image (or trace) $c(I)$ as a subset of M. The geometric curve is the set $c(I)$ itself while the parameterized curve is the map in question. The set of all parameterization of a curve fall into two classes according to whether they are positive reparametrizations of each other or not. A choice of one of these classes gives an orientation to the geometric curve. Once the orientation is fixed then we may integrate a $1-$form over this oriented geometric curve using any parameterization in the chosen class.

A 1-form on an interval $I = [a, b]$ may always be given as $f\,dt$ for some smooth function on I. The integral of $f\ dt$ over $[a, b]$ is just the usual Riemann integral $\int_{[a,b]} f(t)\,dt$. If α is a $1-$form on M and $\gamma : [a, b] \to M$ is a parameterized curve whose image happens to be contained in a coordinate chart U,then the line integral of α along γ is defined as

$$\int_\gamma \alpha := \int_{[a,b]} \gamma^* \alpha$$

If γ is continuous but merely piecewise smooth (C^1 is enough) then we just integrate α along each smooth piece and add the results:

$$\int_\gamma \alpha := \sum_i \int_{[t_i, t_{i+1}]} \gamma^* \alpha$$

Next we move to 2 dimensions. We start we another simple example from calculus. Let $\sigma(u, v) = (\sin u \cos v, \cos u \cos v, \cos u)$ for $(\phi, \theta) \in (0, \pi] \times (0, 2\pi)$.

This gives a parameterization of the sphere S^2. There are a few places where our parameterization is not quite perfect but those are measure zero sets so it will not make an difference when we integrate. Now we need something to integrate; say $\omega := zdy \wedge dz + xdx \wedge dy$. The integration is done by analogy with what we did before. We pull the form back to the uv space and then integrate just as one normally would in calculus of several variables:

$$
\begin{aligned}
\int_\sigma \omega &:= \int_{(0,\pi]\times(0,2\pi)} \sigma^*\omega \\
&= \int_{(0,\pi]\times(0,2\pi)} \left(z(u,v)\frac{dy \wedge dz}{du \wedge dv} + x(u,v)\frac{dx \wedge dy}{du \wedge dv} \right) du \wedge dv \\
&= \int_0^\theta \int_0^{2\pi} \big[\cos u(\sin^2 u \cos v - \cos u \sin v \sin u) \\
&+ \sin u \cos v \; (\cos^2 u \cos^2 v - \sin^2 u \sin v \cos v) \big] \, dudv
\end{aligned}
$$

Rather than finishing the above calculation, let us apply a powerful theorem that we will prove below (Stokes' theorem). The reason this will work depends on the fact that our map σ gives a good enough parameterization of the sphere that we are justified in interpreting the integral $\int_\sigma \omega$ as being an integral over the sphere $\int_{S^2} \omega$. This is a special case of integrating an $n-$form over a compact n-dimensional manifold. We will get back to this shortly. As far as the integral above goes, the Stokes' theorem is really a reformulation of the Stoke's theorem from calculus of 3 variables. The theorem instructs us that if we take the exterior derivative of ω and integrate that over the ball which is the interior of the sphere we should get the same thing. It is important to realize that this only works because our map σ parameterizes the whole sphere and the sphere is the boundary of the ball. Now a quick calculation of gives $d\omega = 0$ and so no matter what the integral would be zero and so we get zero for the above complicated integral too! In summary, what Stokes' theorem gives us is

$$
\int_{S^2} \omega \overset{Stokes}{=} \int_B d\omega = \int_B 0 = 0
$$

Let us now make sense out the more general situation. A smooth $k-$form on an open subset $O \subset \mathbb{R}^k$ can always be written in the form

$$
f \; dx^1 \wedge \cdots \wedge dx^k
$$

for some smooth function. It is implicit that $\mathbb{R}^k$ is oriented by $dx^1 \wedge \cdots \wedge dx^k$. We then simply define $\int_S f \, dx^1 \wedge \cdots \wedge dx^k := \int_S f \, dx^1 \cdots dx^k$ where the integral is the Riemann integral and S is any reasonable set on which such an integration makes sense. We could also take the integral to be the Lebesgue integral and S a Borel set. So integration of a k-form over a set in Euclidean space of the same dimension k turns out to be just ordinary integration as long as we write our k-form as a function times the basic "volume form" $dx^1 \wedge \cdots \wedge dx^k$.

Let O be an bounded open subset of $\mathbb{R}^k$. A parameterized patch in M is a smooth map $\phi : O \to M$. If α is a smooth $k-$form then we define

$$\int_\phi \alpha := \int_O \phi^* \alpha$$

Notice that what we have done is integrate over a map. However, it was important that the map had as its domain an open set in a Euclidean space of a dimension that matches the degree of the form. Notice that we have implicitly used the standard orientation of $\mathbb{R}^k$.

A variation on the idea of integration over a map is the following:

Definition 8.5 *A smooth map $\sigma : S \to M$ is called a singular simplex if $S = \{x \in \mathbb{R}^k : x^i \geq 0$ and $\sum x^i \leq 1\}$. A smooth map $\sigma : S \to M$ is called a singular cube if $S = \{x \in \mathbb{R}^k : 0 \leq x^i \leq 1\}$. In either case, we define the integral of a $k-$form α as*

$$\int_\sigma \alpha := \int_S \phi^* \alpha$$

Now lets integrate over manifolds (rather than maps). Let M be a smooth n-manifold possibly with boundary ∂M and assume that M is oriented and that ∂M has the induced orientation. From our discussion on orientation of manifolds with boundary and by general principles it should be clear that we may assume that all the charts in our orienting atlas have range in the left half space $\mathbb{R}^n_{u^1 \leq 0}$. If $\partial M = \emptyset$ then the ranges will be in the interior $\mathbb{R}^n_{u^1 < 0} \subset \mathbb{R}^n_{u^1 \leq 0}$.

Definition 8.6 *A the* ***support*** *of a differential form $\alpha \in \Omega(M)$ is the* closure *of the set $\{p \in M : \alpha(p) \neq 0\}$ and is denoted by* supp(α). *The set of all k-forms $\alpha^{(k)}$ that have compact support contained in $U \subset M$ is denoted by $\Omega^k_c(U)$.*

Let us return to the case of a k-form $\alpha^{(k)}$ on an open subset U of $\mathbb{R}^k$. If $\alpha^{(k)}$ has compact support in U we may define the integral $\int_U \alpha^{(k)}$ by

$$\begin{aligned}\int_U \alpha^{(k)} &= \int_U a(u) du^1 \wedge \cdots \wedge du^k \\ &:= \int_U a(u) \left|du^1 \cdots du^k\right|\end{aligned}$$

where this latter integral is the Riemann (or Lebesgue) integral of $a(u)$. We have written $\left|du^1 \cdots du^k\right|$ instead of $du^1 \cdots du^k$ to emphasize that the order of the du^i does not matter as it does for $du^1 \wedge \cdots \wedge du^k$. Of course, we would get the wrong answer if the order of the $u's$ did not give the standard orientation on $\mathbb{R}^k$.

Now consider an oriented $n-$dimensional manifold M and let $\alpha \in \Omega^n_M$. If α has compact support inside U_α for some chart $(U_\alpha, \mathtt{x}_\alpha)$ compatible with the orientation then $\mathtt{x}_\alpha^{-1} : \mathtt{x}_\alpha(U_\alpha) \to U_\alpha$ and $(\mathtt{x}_\alpha^{-1})^* \alpha$ has compact support in $\mathtt{x}_\alpha(U_\alpha) \subset \mathbb{R}^n_{u^1 \leq 0}$. We define

$$\int \alpha := \int_{\mathtt{x}_\alpha(U_\alpha)} (\mathtt{x}_\alpha^{-1})^* \alpha.$$

The standard change of variables formula show that this definition is independent of the oriented chart chosen. Now if $\alpha \in \Omega^n(M)$ does not have support contained in some chart domain then we choose a locally finite cover of M by oriented charts $(\mathtt{x}_i, U_i)$ and a smooth partition of unity (ρ_i, U_i) , $\mathrm{supp}(\rho_i) \subset U_i$. Then we define

$$\int \alpha := \sum_i \int_{\mathtt{x}_i(U_i)} (\mathtt{x}_i^{-1})^*(\rho_i \alpha)$$

Proposition 8.1 *The above definition is independent of the choice of the charts $\mathtt{x}_i, U_i$ and smooth partition of unity ρ_i, U_i.*

Proof. Let $(\bar{\mathtt{x}}_i, V_i)$, and $\overline{\rho}_i$ be another such choice. Then we have

$$\begin{aligned}
\int \alpha &:= \sum_i \int_{\mathtt{x}_i(U_i)} (\mathtt{x}_i^{-1})^*(\rho_i \alpha) \\
&= \sum_i \int_{\mathtt{x}_i(U_i)} (\mathtt{x}_i^{-1})^*(\rho_i \sum_j \overline{\rho}_j \alpha) \\
&\sum_i \sum_j \int_{\mathtt{x}_i(U_i \cap U_j)} (\mathtt{x}_i^{-1})^*(\rho_i \overline{\rho}_j \alpha) \\
&= \sum_i \sum_j \int_{\bar{\mathtt{x}}_j(U_i \cap U_j)} (\bar{\mathtt{x}}_j^{-1})^*(\rho_i \overline{\rho}_j \alpha) \\
&= \sum_j \int_{\bar{\mathtt{x}}_i(U_i)} (\bar{\mathtt{x}}_j^{-1})^*(\overline{\rho}_j \alpha)
\end{aligned}$$

■

8.1 Stokes' Theorem

Let us start with a couple special cases .

Case 8.1 (1) *Let $\omega_j = f du^1 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^n$ be a smooth $n-1$ form with compact support contained in the interior of $\mathbb{R}^n_{u^1 \leq 0}$ where the hat symbol over the du^j means this j-th factor is omitted. All $n-1$ forms on $\mathbb{R}^n_{u^1 \leq 0}$ are sums* *of forms of this type. Then we have*

$$\begin{aligned}
\int_{\mathbb{R}^n_{u^1\le 0}} d\omega_j &= \int_{\mathbb{R}^n_{u^1\le 0}} d(f du^1 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^n) \\
&= \int_{\mathbb{R}^n_{u^1\le 0}} (df \wedge du^1 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^n) \\
&= \int_{\mathbb{R}^n_{u^1\le 0}} (\sum_k \frac{\partial f}{\partial u^k} du^k \wedge du^1 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^n) \\
&= \int_{\mathbb{R}^n_{u^1\le 0}} (-1)^{j-1} \frac{\partial f}{\partial u^j} du^1 \wedge \cdots \wedge du^n = \int_{\mathbb{R}^n} (-1)^{j-1} \frac{\partial f}{\partial u^j} du^1 \cdots du^n \\
&= 0
\end{aligned}$$

by the fundamental theorem of calculus and the fact that f has compact support.

Case 8.2 (2) *Let $\omega_j = f du^1 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^n$ be a smooth $n-1$ form with compact support meeting $\partial \mathbb{R}^n_{u^1\le 0} = \mathbb{R}^n_{u^1=0} = 0 \times \mathbb{R}^{n-1}$ then*

$$\begin{aligned}
\int_{\mathbb{R}^n_{u^1\le 0}} d\omega_j &= \int_{\mathbb{R}^n_{u^1\le 0}} d(f du^1 \wedge \cdots \wedge \widehat{du^j} \wedge \cdots \wedge du^n) \\
&= \int_{\mathbb{R}^{n-1}} (-1)^{j-1} \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial u^j} du^j \right) du^1 \cdots \widehat{du^j} \cdots du^n = \\
&= 0 \text{ if } j \ne 1 \text{ and if } j = 1 \text{ we have} \int_{\mathbb{R}^n_{u^1\le 0}} d\omega_1 = \\
&= \int_{\mathbb{R}^{n-1}} (-1)^{j-1} \left(\int_{-\infty}^{0} \frac{\partial f}{\partial u^1} du^1 \right) du^2 \wedge \cdots \wedge du^n \\
&= \int_{\mathbb{R}^{n-1}} f(0, u^2, ..., u^n) du^2 \cdots du^n \\
&= \int_{\partial \mathbb{R}^n_{u^1\le 0}} f(0, u^2, ..., u^n) du^2 \wedge \cdots \wedge du^n = \int_{\partial \mathbb{R}^n_{u^1\le 0}} \omega_1
\end{aligned}$$

Now since clearly $\int_{\partial \mathbb{R}^n_{u^1\le 0}} \omega_j = 0$ if $j \ne 1$ or if ω_j has support that doesn't meet $\partial \mathbb{R}^n_{u^1\le 0}$ we see that in any case $\int_{\mathbb{R}^n_{u^1\le 0}} d\omega_j = \int_{\partial \mathbb{R}^n_{u^1\le 0}} \omega_j$. Now as we said all $n-1$ forms on $\mathbb{R}^n_{u^1\le 0}$ are sums of forms of this type and so summing such we have for any smooth $n-1$ form on $\mathbb{R}^n_{u^1\le 0}$.

$$\int_{\mathbb{R}^n_{u^1\le 0}} d\omega = \int_{\partial \mathbb{R}^n_{u^1\le 0}} \omega.$$

Now we define integration on a manifold (possibly with boundary). Let $\mathcal{A}_M = (U_\alpha, \mathtt{x}_\alpha)_{\alpha \in A}$ be an oriented atlas for a smooth orientable n-manifold

M consisting of nice charts so either $\mathtt{x}_\alpha : U_\alpha \cong \mathbb{R}^n$ or $\mathtt{x}_\alpha : U_\alpha \cong \mathbb{R}^n_{u^1 \leq 0}$. Now let $\{\rho_\alpha\}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$. Notice that $\{\rho_\alpha|_{U_\alpha \cap \partial M}\}$ is a partition of unity for the cover $\{U_\alpha \cap \partial M\}$ of ∂M. Then for $\omega \in \Omega^{n-1}(M)$ we have that

$$\begin{aligned}
\int_M d\omega &= \int_{U_\alpha} \sum_\alpha d(\rho_\alpha \omega) = \sum_\alpha \int_{U_\alpha} d(\rho_\alpha \omega) \\
&= \sum_\alpha \int_{\mathtt{x}_\alpha(U_\alpha)} (\mathtt{x}_\alpha^{-1})^* d(\rho_\alpha \omega) = \sum_\alpha \int_{\mathtt{x}_\alpha(U_\alpha)} d((\mathtt{x}_\alpha^{-1})^* \rho_\alpha \omega) \\
&= \sum_\alpha \int_{\mathtt{x}_\alpha(U_\alpha)} d((\mathtt{x}_\alpha^{-1})^* \rho_\alpha \omega) = \sum_\alpha \int_{\partial\{\mathtt{x}_\alpha(U_\alpha)\}} ((\mathtt{x}_\alpha^{-1})^*_\alpha \rho \omega) \\
&= \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha \omega = \int_{\partial M} \omega
\end{aligned}$$

so we have proved

Theorem 8.1 (Stokes' Theorem) *Let M be an oriented manifold with boundary (possibly empty) and give ∂M the induced orientation. Then for any $\omega \in \Omega^{n-1}(M)$ we have*

$$\int_M d\omega = \int_{\partial M} \omega$$

8.2 Differentiating integral expressions

Suppose that $S \subset M$ is a regular submanifold with boundary ∂S (possibly empty) and Φ_t is the flow of some vector field $X \in \mathfrak{X}(M)$. In this case $S_t := \Phi_t(S)$ is also an regular submanifold with boundary. We then consider $\frac{d}{dt} \int_S \Phi_t^* \eta$. We have

$$\begin{aligned}
\frac{d}{dt} \int_S \Phi_t^* \eta &= \lim_{h \to 0} \frac{1}{h} \left[\int_S \Phi_{t+h}^* \eta - \int_S \Phi_t^* \eta \right] \\
&= \lim_{h \to 0} \frac{1}{h} \left[\Phi_t^* \int_S (\Phi_h^* \eta - \eta) \right] \\
&= \lim_{h \to 0} \left[\Phi_t^* \int_S \frac{1}{h} (\Phi_h^* \eta - \eta) \right] \\
&= \left[\int_{\Phi_t S} \lim_{h \to 0} \frac{1}{h} (\Phi_h^* \eta - \eta) \right] \\
&= \int_{S_t} \mathcal{L}_X \eta
\end{aligned}$$

But also $\int_S \Phi_t^* \eta = \int_{S_t} \eta$ and the resulting formula $\frac{d}{dt} \int_{S_t} \eta = \int_{\Phi_t S} \mathcal{L}_X \eta$ is quite useful. As a special case ($t = 0$) we have

$$\left. \frac{d}{dt} \right|_{t=0} \int_{S_t} \eta = \int_S \mathcal{L}_X \eta$$

We can go farther using Cartan's formula $\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X$. We get

$$\begin{aligned}\frac{d}{dt}\int_{S_t}\eta = \frac{d}{dt}\int_S \Phi_t^*\eta &= \int_{\Phi_t S}\iota_X d\eta + \int_{\Phi_t S} d\iota_X\eta \\ &= \int_{\Phi_t S}\iota_X d\eta + \int_{\partial S_t}\iota_X\eta.\end{aligned}$$

This becomes particularly interesting in the case that $S = \Omega$ is an open submanifold of M with compact closure and smooth boundary and *vol* is a volume form on M. We then have $\frac{d}{dt}\int_{\Omega_t} vol = \int_{\partial\Omega_t}\iota_X vol$ and then

$$\left.\frac{d}{dt}\right|_{t=0}\int_{\Omega_t} vol = \int_{\partial\Omega}\iota_X vol$$

Definition 8.7 *If vol is a volume form orienting a manifold M then $\mathcal{L}_X vol = (\operatorname{div} X)\, vol$ for a unique function $\operatorname{div} X$ called the divergence of X with respect to the volume form vol.*

We have

$$\left.\frac{d}{dt}\right|_{t=0}\int_{\Omega_t} vol = \int_{\partial\Omega}\iota_X vol$$

and

$$\left.\frac{d}{dt}\right|_{t=0}\int_{\Omega_t} vol = \int_{\Omega}\mathcal{L}_X vol = \int_{\Omega}(\operatorname{div} X)\,\mathrm{d}\,V.$$

Now let $E_1, ..., E_n$ be a local frame field on $U \subset M$ and $\varepsilon^1, ..., \varepsilon^n$ the dual frame field. Then for some smooth function ρ we have

$$vol = \rho\varepsilon^1 \wedge \cdots \wedge \varepsilon^n$$

and so

$$
\begin{aligned}
\mathcal{L}_X vol &= \mathcal{L}_X \left(\rho\varepsilon^1 \wedge \cdots \wedge \varepsilon^n\right) = d\iota_X \left(\rho\varepsilon^1 \wedge \cdots \wedge \varepsilon^n\right) \\
&= d\sum_{j=1}^{n}(-1)^{j-1}\rho\varepsilon^1 \wedge \cdots \iota_X \varepsilon^k \wedge \cdots \wedge \varepsilon^n \\
&= d\sum_{j=1}^{n}(-1)^{j-1}\rho\varepsilon^1 \wedge \cdots \wedge \varepsilon^k(X) \wedge \cdots \wedge \varepsilon^n \\
&= d\sum_{j=1}^{n}(-1)^{j-1}\rho\varepsilon^1 \wedge \cdots \wedge \varepsilon^k(\sum_{r=1}^{n} X^r \varepsilon_r) \wedge \cdots \wedge \varepsilon^n \\
&= d\sum_{k=1}^{n}(-1)^{j-1}\rho X^k \varepsilon^1 \wedge \cdots \wedge \widehat{\varepsilon^k} \wedge \cdots \wedge \varepsilon^n \\
&= \sum_{k=1}^{n}(-1)^{j-1} d(\rho X^k) \wedge \varepsilon^1 \wedge \cdots \wedge \widehat{\varepsilon^k} \wedge \cdots \wedge \varepsilon^n \\
&= \sum_{k=1}^{n}(-1)^{j-1} \sum_{i=1}^{n} (\rho X^k)_i \varepsilon^i \wedge \varepsilon^1 \wedge \cdots \wedge \widehat{\varepsilon^k} \wedge \cdots \wedge \varepsilon^n \\
&= \sum_{k=1}^{n} \left(\frac{1}{\rho}(\rho X^k)_k\right) \rho\varepsilon^1 \wedge \cdots \wedge \varepsilon^n \\
&\sum_{k=1}^{n} \left(\frac{1}{\rho}(\rho X^k)_k\right) vol
\end{aligned}
$$

where $(\rho X^k)_k := d(\rho X^k)(E_k)$. Thus

$$
\operatorname{div} X = \sum_{k=1}^{n} \frac{1}{\rho}(\rho X^k)_k
$$

In particular, if $E_k = \frac{\partial}{\partial x^k}$ for some chart $(U, \mathtt{x}) = (x^1, ..., x^n)$ then

$$
\operatorname{div} X = \sum_{k=1}^{n} \frac{1}{\rho}\frac{\partial}{\partial x^k}(\rho X^k)
$$

Now if we were to replace the volume form vol by $-vol$ then divergence with respect to that volume form would be given locally by $\sum_{k=1}^{n} \frac{1}{-\rho}\frac{\partial}{\partial x^k}(-\rho X^k) = \sum_{k=1}^{n} \frac{1}{\rho}\frac{\partial}{\partial x^k}(\rho X^k)$ and so the orientation seems superfluous! What if M isn't even orientable? In fact, since divergence is a local concept and orientation a global concept it seems that we should replace the volume form in the definition by something else that makes sense even on a nonorientable manifold. But what? This brings us to our next topic.

8.3 Pseudo-forms

Let M be a smooth manifold and $\{(U_\alpha, \mathtt{x}_\alpha)\}$ be an atlas for M. The flat orientation line bundle $O_{flat}(TM)$ is the vector bundle constructed from the local bundles $U_\alpha \times \mathbb{R}$ by the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl(1, \mathbb{R})$ defined by

$$g_{\alpha\beta}(p) := \frac{\det\left(\mathtt{x}_\alpha \circ \mathtt{x}_\beta^{-1}\right)}{\left|\det\left(\mathtt{x}_\alpha \circ \mathtt{x}_\beta^{-1}\right)\right|} = \pm 1$$

For every chart $(U_\alpha, \mathtt{x}_\alpha)$ there is a frame field for $O_{flat}(TM)$ over U_α consisting of a single section $o_{\mathtt{x}_\alpha}$. A pseudo-k-form is a cross section of $O_{flat}(TM) \otimes \wedge^k T^*M$. The set of all pseudo-$k$-forms will be denoted by $\Omega_o^k(M)$. Now we can extend the exterior product to maps $\wedge : \Omega_o^k(M)\times\ \Omega^l(M) \to \Omega_o^{k+l}(M)$ by the rule

$$(o_1 \otimes \theta_1) \wedge \theta_2 = o_1 \otimes \theta_1 \wedge \theta_2$$

with a similar and obvious map $\wedge : \Omega^k(M)\times\ \Omega_o^l(M) \to \Omega_o^{k+l}(M)$. Similarly, we have a map $\wedge : \Omega_o^k(M)\times\ \Omega_o^l(M) \to \Omega^{k+l}(M)$ given by

$$(o_1 \otimes \theta_1) \wedge (o_2 \otimes \theta_2) = (o_1 o_2) \otimes \theta_1 \wedge \theta_2$$

and where $(o_1 o_2)$ is the locally constant function equal to $+1$ wherever $o_1 = o_2$ and -1 otherwise. Now we can extend the exterior algebra to $\sum_{k,l=0}^n \left(\Omega^k(M) \oplus \Omega_o^l(M)\right)$. If $\omega \in \Omega_o^k(M)$ then with respect to the chart $(U_\alpha, \mathtt{x}_\alpha)$, α has the local expression

$$\omega = o_{\mathtt{x}_\alpha} \otimes a_{\vec{I}}^\alpha dx_\alpha^{\vec{I}}$$

and if $\omega = o_{\mathtt{x}_\beta} \otimes a_{\vec{J}}^\beta dx_\beta^{\vec{J}}$ for some other chart $(U_\beta, \mathtt{x}_\beta)$ then $a_{\vec{I}}^\alpha = \frac{\det\left(\mathtt{x}_\alpha \circ \mathtt{x}_\beta^{-1}\right)}{\left|\det\left(\mathtt{x}_\alpha \circ \mathtt{x}_\beta^{-1}\right)\right|} \frac{dx^{\vec{J}}}{dx_\beta^{\vec{I}}} a_{\vec{J}}^\beta$. In particular if ω is a pseudo-n-form (a volume pseudo-form) then $\vec{I} = (1, 2, ..., n) = \vec{J}$ and $\frac{dx^{\vec{J}}}{dx_\beta^{\vec{I}}} = \det\left(\mathtt{x}_\alpha \circ \mathtt{x}_\beta^{-1}\right)$ and so that rule becomes

$$a_{12..n}^\alpha = \frac{\det\left(\mathtt{x}_\alpha \circ \mathtt{x}_\beta^{-1}\right)}{\left|\det\left(\mathtt{x}_\alpha \circ \mathtt{x}_\beta^{-1}\right)\right|} \det\left(\mathtt{x}_\alpha \circ \mathtt{x}_\beta^{-1}\right) a_{12..n}^\beta$$

or

$$a_{12..n}^\alpha = \left|\det\left(\mathtt{x}_\alpha \circ \mathtt{x}_\beta^{-1}\right)\right| a_{12..n}^\beta.$$

There is another way to think about pseudo-forms that has the advantage of having a clearer global description. Recall that the set of frames $\{e_1, ..., e_n\}$ in T_pM are divided into two equivalence classes called orientations. Equivalently, an orientation at p is an equivalence class of elements of $\wedge^n T_p^*M$. For each p there are exactly two such orientations and the set of all orientations $\mathrm{Or}(M)$ at all points $p \in M$ form a set with a differentiable structure and the map π_{Or} that takes both orientations at p the point p is a two fold covering map.

The group $\mathbb{Z}/2\mathbb{Z}$ =$\{1, -1\}$ action as deck transformations on $\mathrm{Or}(M)$ so that -1 sends each orientation to its opposite. Denote the action of $g \in \mathbb{Z}/2\mathbb{Z}$ by $l_g : \mathrm{Or}(M) \to \mathrm{Or}(M)$. Now we think of a pseudo-k-form as being nothing more than a k-form η on the manifold $\mathrm{Or}(M)$ with the property that $l_{-1}^*\eta = -\eta$. Now we would like to be able to integrate a k-form over a map $h : N \to M$ where N is a $k-$dimensional manifold. By definition h is said to be orientable if there is a lift $\widetilde{h} : \mathrm{Or}(N) \to \mathrm{Or}(M)$

$$\begin{array}{ccc} \mathrm{Or}(N) & \stackrel{\widetilde{h}}{\to} & \mathrm{Or}(M) \\ \downarrow \pi_{Or_N} & & \downarrow \pi_{Or_M} \\ N & \stackrel{h}{\to} & M \end{array}$$

We will say that $\widetilde{h}$ is said to orient the map. In this case we define the integral of a pseudo-k-form η over h to be

$$\int_h \eta := \frac{1}{2} \int_{\mathrm{Or}(N)} \widetilde{h}^* \eta$$

Now there is clearly another lift $\widetilde{h_-}$ which sends each $\widetilde{n} \in \mathrm{Or}(N)$ to the opposite orientation of $\widetilde{h}(\widetilde{n})$. This is nothing more that saying $\widetilde{h_-} = l_{-1} \circ \widetilde{h} = \widetilde{h} \circ l_{-1}$.

Exercise 8.1 *Assume M is connected. Show that there are at most two such lifts $\widetilde{h}$.*

Now

$$\int_{\mathrm{Or}(N)} \widetilde{h}_-^* \eta = \int_{\mathrm{Or}(N)} \widetilde{h}^* l_g^* \eta = \int_{\mathrm{Or}(N)} \widetilde{h}^* \eta$$

and so the definition of $\int_h \eta$ is independent of the lift $\widetilde{h}$.

If $S \subset M$ is a regular k-submanifold and if the inclusion map $\iota_S : S \hookrightarrow M$ map is orientable than we say that S has a transverse orientation in M. In this case we define the integral of a pseudo-k-form η over S to be

$$\int_S \eta := \int_{\iota_S} \eta = \frac{1}{2} \int_{\mathrm{Or}(N)} \widetilde{\iota_S}^* \eta$$

Exercise 8.2 *Show that the identity map $id_M : M \to M$ is orientable:*

$$\begin{array}{ccc} \mathrm{Or}(M) & \stackrel{\widetilde{id}}{\to} & \mathrm{Or}(M) \\ \downarrow \wp_{Or_M} & & \downarrow \wp_{Or_M} \\ M & \stackrel{id}{\to} & M \end{array}$$

$\widetilde{id} = \pm id_{Or(M)}$.

Now finally, if ω is a pseudo-n-form on M then by definition

$$\int_M \omega := \frac{1}{2} \int_{\mathrm{Or}(M)} \widetilde{id}^* \omega = \frac{1}{2} \int_{\mathrm{Or}(M)} \omega$$

If $(U, \mathtt{x})$ is a chart then the map $\sigma_{\mathtt{x}} : p \mapsto [dx^1 \wedge \cdots \wedge dx^n]$ is a local cross section of the covering $\wp_{Or} : \mathrm{Or}(M) \to M$ meaning that the following diagram commutes

$$\begin{array}{ccc} & & \mathrm{Or}(M) \\ & \overset{\sigma_{\mathtt{x}}}{\nearrow} & \downarrow \wp_{Or} \\ U & \hookrightarrow & M \end{array}$$

and we can define the integral of ω locally using a partition of unity $\{\rho_\alpha\}$ subordinate to a cover U_α, ρ_α :

$$\int_M j\omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \sigma_{\mathtt{x}_\alpha}^* \omega.$$

Now suppose we have a vector field $X \in \mathfrak{X}(M)$. Since $\wp_{Or} : \mathrm{Or}(M) \to M$ is a surjective local diffeomorphism there is a vector field $\widetilde{X} \in \mathfrak{X}(\widetilde{M})$ such that $T\wp \cdot \widetilde{X}_{\widetilde{p}} = X_p$ (where $\wp_{\mathrm{Or}}(\widetilde{p}) = p$) for all p. Similarly, if vol is a volume form on M then there is a volume pseudo-form $\widetilde{vol}$ on M, i.e. a $\mathbb{Z}/2\mathbb{Z}$ anti-invariant $n-$form $\widetilde{vol}$ on $\mathrm{Or}(M)$ such $\widetilde{vol} = \wp_{\mathrm{Or}}^* vol$. In this case, it is easy to show that the divergence $\operatorname{div} \widetilde{X}$ of $\widetilde{X}$ with respect to $\widetilde{vol}$ is the lift of $\operatorname{div} X$ (with respect to vol). Thus if M is not orientable and so has no volume form we may still define $\operatorname{div} X$ (with respect to the pseudo-volume form $\widetilde{vol}$) to be the unique vector field on M which is $\wp_{\mathrm{Or}}-$related to $\operatorname{div} \widetilde{X}$ (with respect to volume form $\widetilde{vol}$ on $\mathrm{Or}(M)$).

8.3.1 Twisted Forms

Consider the orientation cover $\pi_{Or} : \mathrm{Or}(M) \to M$ of a manifold M. A smooth local section of $\mathrm{Or}(M)$ over an open set U is a smooth map $\sigma : p \mapsto o_p$ such that $\pi_{Or} \circ \sigma = id_U$. Thus a local section is smooth choice of orientation for each of the tangents spaces at point in U. For every p in M there is a smooth local section on some open U containing p. For each p, let $\mathrm{Or}_p(M)$ the the discrete fiber of $\mathrm{Or}(M)$ over p consisting of the two orientations of T_pM. A twisted $k-$form at $p \in M$ is a map $\alpha_p : T_pM \times \cdots \times T_pM \times \mathrm{Or}_p(M) \to \mathbb{R}$ such that for each fixed $o_p \in \mathrm{Or}_p(M)$ with o_p an orientation of T_pM, the map $\alpha_p(., o_p) : T_pM \times \cdots \times T_pM \to \mathbb{R}$ given by $(v_1, ..., v_k) \mapsto \alpha_p(v_1, ..., v_k, o)$ is an element of $L^k_{alt}(T_pM)$ and such that for all $(v_1, ..., v_k, o_p) \in T_pM \times \cdots \times T_pM \times \mathrm{Or}_p(M)$ we have

$$\alpha_p(v_1, ..., v_k, -o_p) = -\alpha_p(v_1, ..., v_k, o_p)$$

where $-o$ denotes the orientation opposite to o_p. A twisted differential k-form on M is an assignment $p \mapsto \alpha_p$of a twisted $k-$form at p for every $p \in M$ and such that for any smooth vector fields $X_1, ...X_k$ and local section $\sigma : p \mapsto o_p$ of the cover $\mathrm{Or}(M) \to M$ simultaneously defined over an open U the function $p \to (X_1(p), ..., X_k(p), o_p)$ is a smooth function. We will denote the set of twisted $k-$forms by $\Omega^k_{tw}(M)$. If α is such a twisted k form then it is easy to see that for any smooth section $\sigma : p \mapsto o_p$ of $\mathrm{Or}(M)$ over an open set U, the map

$p \to \alpha_p(., o_p)$ is a smooth $k-$form over U. Let $\alpha \in \Omega^k_{tw}(M)$. In order to define the pull-back of a twisted form we need more that just a smooth map.

Definition 8.8 *Given a map $f : N \to M$ we call a smooth map $\widetilde{f} : Or(N) \to Or(M)$ an orientation for f if the following diagram commutes:*

$$\begin{array}{ccc} Or(N) & \xrightarrow{\widetilde{f}} & Or(M) \\ \downarrow & & \downarrow \\ N & \xrightarrow{f} & M \end{array}$$

We call $\widetilde{f}$ an orienting map for f.

In general, an orienting map may or may not exist.

Definition 8.9 *Let $\widetilde{f} : Or(N) \to Or(M)$ be an orienting map for $f : N \to M$. The for any twisted $k-$form α we define the pull-back $\widetilde{f}^*\alpha$ of α by $\widetilde{f}$ to be defined by*

$$\widetilde{f}^*\alpha(u_1, ...u_k, o_x) := \alpha(T_x f \cdot u_1, ...T_x f \cdot u_k, \widetilde{f}\,(o_x))$$

It is not hard to see that if α is a twisted $k-$form then so also is $\widetilde{f}^*\alpha$ for any orienting map $\widetilde{f}$.

Now we come to a crucial point. Namely, if $f : N \to M$ is a diffeomorphism then it has a natural orienting map defined in the obvious way: If $o_x \in Or_x(N)$ is given by the equivalence class of a basis $(u_1, ..., u_n)$ of T_xN then $\widetilde{f}\,(o_x)$ is the orientation defined by the basis $(T_x f \cdot u_1, ...T_x f \cdot u_n)$ which we just denote by $f\,(o_x)$. In this case we just indicate the pull-back by $f^*\alpha$.

Let U, V be open set in $\mathbb{R}^n$ and $\varphi : U \to V$ a diffeomorphism. Let us compare $\varphi^*\alpha(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^1}, o)$ with $\alpha(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^1}, o)$ where we identify all the tangent spaces and o is the standard orientation of $\mathbb{R}^n$. We have

$$\begin{aligned} &\varphi^*\alpha(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^1}, o) \\ &= \alpha(T\varphi\frac{\partial}{\partial x^1}, ..., T\varphi\frac{\partial}{\partial x^1}, \varphi\,(o)) \\ &= J(\varphi)\alpha(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^1}, \varphi o) \end{aligned}$$

but $\varphi o = \operatorname{sgn} J(\varphi) \cdot o$ and so $\varphi^*\alpha = |J(\varphi)|\,\alpha$. This is an interesting result because it allows us to define integration of twisted $n-$forms on $n-$manifolds.

First we have

Definition 8.10 *Let $\alpha \in \Omega^n_{tw}(U)$ where $V \subset \mathbb{R}^n$. Define the integral of α over U by*

$$\int_V \alpha := \int_V \alpha(\frac{\partial}{\partial v^1}, ..., \frac{\partial}{\partial v^1}, o)dv^1 \cdots dv^n$$

Lemma 8.1 *Let U, V be open set in $\mathbb{R}^n$ and $\varphi : U \to V$ a diffeomorphism which is not necessarily orientation preserving. Then for $\alpha \in \Omega^n_{tw}(U)$ we have $\int_V \alpha = \int_U \varphi^* \alpha$.*

Proof. $\int_V \alpha = \int_V \alpha(\frac{\partial}{\partial u^1}, ..., \frac{\partial}{\partial u^1}, o) du^1 \cdots du^n = \int_U \alpha(\frac{\partial}{\partial u^1}, ..., \frac{\partial}{\partial u^1}, o) J(\varphi) du^1 \cdots du^n$
$\int_U \alpha(\frac{\partial}{\partial u^1}, ..., \frac{\partial}{\partial u^1}, o) \operatorname{sgn} J(\varphi) \, |J(\varphi)| \, du^1 \cdots du^n$
$= \int_U \alpha(\frac{\partial}{\partial u^1}, ..., \frac{\partial}{\partial u^1}, \operatorname{sgn} J(\varphi) o) \, |J(\varphi)| \, du^1 \cdots du^n$
$= \int_U \varphi^* \alpha(\frac{\partial}{\partial u^1}, ..., \frac{\partial}{\partial u^1}, o) du^1 \cdots du^n = \int_U \varphi^* \alpha$ ■

Now let $(U, \mathtt{x})$ be a chart and suppose that α is a twisted n form with compact support inside U. Then $(\mathtt{x}^{-1})^* \alpha$ is a twisted $n-$form on $V = \mathtt{x}(U)$. We can now unambiguously define $\int_U \alpha$ to be $\int_{\mathtt{x}U} (\mathtt{x}^{-1})^* \alpha$ and by the discussion above this is independent of the coordinate chart regardless of orientation considerations. Now if $\alpha \in \Omega^n_{tw}(M)$ we may define the integral $\int_M \alpha$ by using a partition of unity as we did for ordinary forms.

8.4 Problems

1. Let $\iota : S^2 \hookrightarrow \mathbb{R}^3 \backslash \{0\}$ be the inclusion map. Let $\tau = \iota^* \omega$ where

$$\omega = \frac{x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

Compute $\int_{S^2} \tau$ where S^2 given the orientation induced by τ itself.

2. Let M be an oriented smooth compact manifold with boundary ∂M and suppose that ∂M has two connected components N_0 and N_1. Let $\imath_i : N_0 \hookrightarrow M$ be the inclusion map for $i = 0, 1$. Suppose that there α is a $p-$form with $\imath_0^* \alpha = 0$ and β an $n - p - 1$ form with $\imath_1^* \beta = 0$. Prove that in this case

$$\int_M d\alpha \wedge \beta = (-1)^{p+1} \int_M \alpha \wedge d\beta$$

Chapter 9

Distributions and Frobenius' Theorem

9.1 Definitions

In this section we take M to be a smooth n dimensional manifold modeled. Roughly speaking, a smooth distribution is an assignment $\triangle$ of a subspace $\triangle_p \subset T_pM$ to each $p \in M$ such that for each fixed $p_0 \in M$ there is a family of smooth vector fields $X_1, ..., X_k$ defined on some neighborhood U_{p_0} of p_0 and such that $\triangle_x = span\{X_1(x), ..., X_k(x)\}$ for each $x \in U_{p_0}$. We call the distribution regular if we can always choose these vector fields to be linearly independent on each tangent space T_xM for $x \in U_{p_0}$ and each U_{p_0}. It follows that in this case k is locally constant. For a regular distribution k is called the rank of the distribution. A rank k regular distribution is the same thing as a rank k subbundle of the tangent bundle.

Definition 9.1 *A (smooth) regular distribution on M is a smooth vector subbundle of the tangent bundle TM.*

9.2 Integrability of Regular Distributions

By definition a regular distribution $\triangle$ is just another name for a subbundle $\triangle \subset TM$ of the tangent bundle and we write $\triangle_p \subset T_pM$ for the fiber of the subbundle at p. So what we have is a smooth assignment of a subspace $\triangle_p$ at every point. The subbundle definition guarantees that the spaces $\triangle_p$ all have the same dimension. (we could let the dimension vary from connected component to connected component but this would not really give any advantage). This dimension is called the rank of the distribution (it is just the rank of the the subbundle). There is a more general notion of distribution that we call a singular distribution and that is defined in the same way except that the requirement of constancy of dimension is dropped. We shall study singular distributions later.

Definition 9.2 *Let X locally defined vector field. We say that X **lies in** the distribution $\triangle$ if $X(p) \in \triangle_p$ for each p in the domain of X. In this case, we write $X \in \triangle$ (a slight abuse of notation).*

Note that in the case of a regular distribution we can say that for X to lie in the distribution $\triangle$ means that X takes values in the subbundle $\triangle \subset TM$.

Definition 9.3 *We say that a locally defined differential j-form ω **vanishes on** $\triangle$ if for every choice of vector fields $X_1, ..., X_j$ defined on the domain of ω that lie in $\triangle$ the function $\omega(X_1, ..., X_j)$ is identically zero.*

For a regular distribution $\triangle$ consider the following two conditions.

Fro1 For every pair of locally defined vector fields X and Y with common domain that lie in the distribution $\triangle$ the bracket $[X, Y]$ also lies in the distribution.

Fro2 For each locally defined smooth 1-form ω that vanishes on $\triangle$ the 2-form $d\omega$ also vanishes on $\triangle$.

Lemma 9.1 *Conditions (1) and (2) above are equivalent.*

Proof. The proof that these two conditions are equivalent follows easily from the formula

$$d\omega(X, Y) = X(\omega(Y)) - Y\omega(X) - \omega([X, Y]).$$

Suppose that (1) holds. If ω vanishes on $\triangle$ and X, Y lie in $\triangle$ then the above formula becomes

$$d\omega(X, Y) = -\omega([X, Y])$$

which shows that $d\omega$ vanishes on $\triangle$ since $[X, Y] \in \triangle$ by condition (1). Conversely, suppose that (2) holds and that $X, Y \in \triangle$. Then $d\omega(X, Y) = -\omega([X, Y])$ again and a local argument using the Hahn-Banach theorem shows that $[X, Y] = 0$. ∎

Definition 9.4 *If either of the two equivalent conditions introduced above holds for a distribution $\triangle$ then we say that $\triangle$ is **involutive**.*

Exercise 9.1 *Suppose that $\mathfrak{X}$ is a family of locally defined vector fields of M such that for each $p \in M$ and each local section X of the subbundle $\triangle$ defined in a neighborhood of p, there is a finite set of local fields $\{X_i\} \subset \mathfrak{X}$ such that $X = \sum a^i X_i$ on some possible smaller neighborhood of p. Show that if $\mathfrak{X}$ is closed under bracketing then $\triangle$ is involutive.*

There is a very natural way for distributions to arise. For instance, consider the punctured 3-space $M = \mathbb{R}^3 - \{0\}$. The level sets of the function $\varepsilon : (x, y, x) \mapsto x^2 + y^2 + x^2$ are spheres whose union is all of $\mathbb{R}^3 - \{0\}$. Now define a distribution by the rule that $\triangle_p$ is the tangent space at p to the sphere containing p. Dually, we can define this distribution to be the given by the rule

$$\triangle_p = \{v \in T_pM : d\varepsilon(v) = 0\}.$$

The main point is that each p contains a submanifold S such that $\triangle_x = T_xS$ for all $x \in S \cap U$ for some sufficiently small open set $U \subset M$. On the other hand, not all distributions arise in this way.

Definition 9.5 *A distribution $\triangle$ on M is called **integrable at** $p \in M$ there is a submanifold S_p containing p such that $\triangle_x = T_xS_p$ for all $x \in S$. (Warning: S_p is locally closed but not necessarily a closed subset and may only be defined very near p.) We call such submanifold a **local integral submanifold** of $\triangle$.*

Definition 9.6 *A regular distribution $\triangle$ on M is called **(completely) integrable** if for every $p \in M$ there is a (local) integral submanifold of $\triangle$ containing p.*

If one considers a distribution on a finite dimensional manifold there is a nice picture of the structure of an integrable distribution. Our analysis will eventually allow us to see that a regular distribution $\triangle$ of rank k on an n-manifold M is (completely) integrable if and only if there is a cover of M by charts ψ_a, U_a such that if $\psi_a = (y^1, ..., y^n)$ then for each $p \in U_a$ the submanifold $S_{\alpha,p}$ defined by $S_{\alpha,p} := \{x \in U_a : y^i(x) = y^i(p) \text{ for } k+1 \le i \le n\}$ has

$$\triangle_x = T_xS_{\alpha,p} \text{ for all } x \in S_p.$$

Some authors use this as the definition of integrable distribution but this definition would be inconvenient to generalize to the infinite dimensional case. A main goal of this section is to prove the theorem of Frobenius which says that a regular distribution is integrable if and only if it is involutive.

9.3 The local version Frobenius' theorem

Here we study regular distributions; also known as tangent subbundles. The presentation draws heavily on that given in [L1]. Since in the regular case a distribution is a subbundle of the tangent bundle it will be useful to consider such subbundles a little more carefully. Recall that if $E \to M$ is a subbundle of TM then $E \subset TM$ and there is an atlas of adapted VB-charts for TM; that is, charts $\phi : \pi_M^{-1}(U) \to U \times \mathsf{M} = U \times \mathsf{E} \times \mathsf{F}$ where $\mathsf{E} \times \mathsf{F}$ is a fixed splitting of M. Thus M is modeled on the split space $\mathsf{E} \times \mathsf{F} = \mathsf{M}$. Now for all local questions we may assume that in fact the tangent bundle is a trivial bundle of the form $(U_1 \times U_2) \times (\mathsf{E} \times \mathsf{F})$ where $U_1 \times U_2 \subset \mathsf{E} \times \mathsf{F}$. It is easy to see that our subbundle must now consist of a choice of subspace $\mathsf{E}_1(x, y)$ of $(\mathsf{E} \times \mathsf{F})$ for every $(x, y) \in U_1 \times U_2$. In fact, the inverse of our trivialization gives a map

$$\phi^{-1} : (U_1 \times U_2) \times (\mathsf{E} \times \mathsf{F}) \to (U_1 \times U_2) \times (\mathsf{E} \times \mathsf{F})$$

such that the image under ϕ^{-1} of $\{(x,y)\} \times \mathsf{E} \times \{0\}$ is exactly $\{(x,y)\} \times \mathsf{E}_1(x,y)$. The map ϕ^{-1} must have the form

$$\phi((x,y),v,w) = ((x,y), f_{(x,y)}(v,w), g_{(x,y)}(v,w))$$

for where $f_{(x,y)} : \mathsf{E} \times \mathsf{F} \to \mathsf{E}$ and $g_{(x,y)} : \mathsf{E} \times \mathsf{F} \to \mathsf{E}$ are linear maps depending smoothly on (x,y). Furthermore, for all (x,y) the map $f_{(x,y)}$ takes $\mathsf{E} \times \{0\}$ isomorphically onto $\{(x,y)\} \times \mathsf{E}_1(x,y)$. Now the composition

$$\kappa : (U_1 \times U_2) \times \mathsf{E} \hookrightarrow (U_1 \times U_2) \times (\mathsf{E} \times \{0\}) \overset{\phi^{-1}}{\to} (U_1 \times U_2) \times (\mathsf{E} \times \mathsf{F})$$

maps $\{(x,y)\} \times \mathsf{E}$ isomorphically onto $\{(x,y)\} \times \mathsf{E}_1(x,y)$ and must have the form

$$\kappa(x,y,v) = (x,y,\lambda(x,y) \cdot v, \ell(x,y) \cdot v)$$

for some smooth maps $(x,y) \mapsto \lambda(x,y) \in L(\mathsf{E},\mathsf{E})$ and $(x,y) \mapsto \ell(x,y) \in L(\mathsf{E},\mathsf{F})$. By a suitable "rotation" of the space $\mathsf{E} \times \mathsf{F}$ for each (x,y) we may assume that $\lambda_{(x,y)} = \mathrm{id}_{\mathsf{E}}$. Now for fixed $v \in \mathsf{E}$ the map $X_v : (x,y) \mapsto (x,y,v,\ell_{(x,y)}v)$ is (a local representation of) a vector field with values in the subbundle E. The principal part is $\mathsf{X}_v(x,y) = (v, \ell_{(x,y)} \cdot v)$.

Now $\ell(x,y) \in L(\mathsf{E},\mathsf{F})$ and so $D\ell(x,y) \in L(\mathsf{E} \times \mathsf{F}, L(\mathsf{E},\mathsf{F}))$. In general for a smooth family of linear maps Λ_u and a smooth map $v : (x,y) \mapsto v(x,y)$ we have

$$D(\Lambda_u \cdot v)(w) = D\Lambda_u(w) \cdot v + \Lambda_u \cdot (Dv)(w)$$

and so in the case at hand

$$\begin{aligned} &D(\ell(x,y) \cdot v)(w_1,w_2) \\ &= (D\ell(x,y)(w_1,w_2)) \cdot v + \ell(x,y) \cdot (Dv)(w_1,w_2). \end{aligned}$$

For any two choices of smooth maps v_1 and v_2 as above we have

$$\begin{aligned} [\mathsf{X}_{v_1},\mathsf{X}_{v_2}]_{(x,y)} &= (D\mathsf{X}_{v_2})_{(x,y)}\mathsf{X}_{v_1}(x,y) - (D\mathsf{X}_{v_1})_{(x,y)}\mathsf{X}_{v_2}(x,y) \\ &= ((Dv_2)(v_1,\ell_{(x,y)}v_1) - (Dv_1)(v_2,\ell_{(x,y)}v_2), D\ell(x,y)(v_1,\ell_{(x,y)}v_1) \cdot v_2 \\ &+ \ell(x,y) \cdot (Dv_2)(v_1,\ell_{(x,y)}v_1) - D\ell(x,y)(v_2,\ell_{(x,y)}v_2) \cdot v_1 \\ &- \ell(x,y) \cdot (Dv_1)(v_2,\ell_{(x,y)}v_2)) \\ &= (\xi, D\ell(x,y)(v_1,\ell_{(x,y)}v_1) \cdot v_2 - D\ell(x,y)(v_2,\ell_{(x,y)}v_2) \cdot v_1 + \ell(x,y) \cdot \xi). \end{aligned}$$

where $\xi = (Dv_2)(v_1,\ell_{(x,y)}v_1) - (Dv_1)(v_2,\ell_{(x,y)}v_2)$. Thus $[\mathsf{X}_{v_1},\mathsf{X}_{v_2}]_{(x,y)}$ is in the subbundle if and only if

$$D\ell(x,y)(v_1,\ell_{(x,y)}v_1) \cdot v_2 - D\ell(x,y)(v_2,\ell_{(x,y)}v_2) \cdot v_1.$$

We thus arrive at the following characterization of involutivity:

Lemma 9.2 *Let Δ be a subbundle of TM. For every $p \in M$ there is a tangent bundle chart containing T_pM of the form described above so that any vector field* *vector field taking values in the subbundle is represented as a map* $\mathsf{X}_v : U_1 \times U_2 \to \mathsf{E} \times \mathsf{F}$ *of the form* $(x, y) \mapsto (v(x, y), \ell_{(x,y)} v(x, y))$*. Then* Δ *is involutive (near* p*) if and only if for any two smooth maps* $v_1 : U_1 \times U_2 \to \mathsf{E}$ *and* $v_2 : U_1 \times U_2 \to \mathsf{E}$ *we have*

$$D\ell(x, y)(v_1, \ell_{(x,y)} v_1) \cdot v_2 - D\ell(x, y)(v_2, \ell_{(x,y)} v_2) \cdot v_1.$$

Theorem 9.1 *A regular distribution* Δ *on* M *is integrable if and only if it is involutive.*

Proof. First suppose that Δ is integrable. Let X and Y be local vector fields that lie in Δ. Pick a point x in the common domain of X and Y. Our choice of x being arbitrary we just need to show that $[X, Y](x) \in \Delta$. Let $S \subset M$ be a local integral submanifold of Δ containing the point x. The restrictions $X|_S$ and $Y|_S$ are related to X and Y by an inclusion map and so by the result on related vector fields we have that $[X|_S, Y|_S] = [X, Y]|_S$ on some neighborhood of x. Since S is a manifold and $[X|_S, Y|_S]$ a local vector field on S we see that $[X, Y]|_S (x) = [X, Y](x)$ is tangent to S and so $[X, Y](x) \in \Delta$. Suppose now that Δ is involutive. Since this is a local question we may assume that our tangent bundle is a trivial bundle $(U_1 \times U_2) \times (\mathsf{E} \times \mathsf{F})$ and by our previous lemma we know that for any two smooth maps $v_1 : U_1 \times U_2 \to \mathsf{E}$ and $v_2 : U_1 \times U_2 \to \mathsf{E}$ we have

$$D\ell(x, y)(v_1, \ell_{(x,y)} v_1) \cdot v_2 - D\ell(x, y)(v_2, \ell_{(x,y)} v_2) \cdot v_1.$$

Claim 9.1 *For any* $(x_0, y_0) \in U_1 \times U_2$ *there exists possibly smaller open product* $U_1' \times U_2' \subset U_1 \times U_2$ *containing* (x_0, y_0) *and a unique smooth map* $\alpha : U_1' \times U_2' \to U_2$ *such that* $\alpha(x_0, y) = y$ *for all* $y \in U_2'$ *and*

$$D_1 \alpha(x, y) = \ell(x, \alpha(x, y))$$

for all $(x, y) \in U_1' \times U_2'$*.*

Before we prove this claim we show how the result follows from it. For any $y \in U_2'$ we have the partial map $\alpha_y(x) := \alpha(x, y)$ and equation **??** above reads $D\alpha_y(x, y) = \ell(x, \alpha_y(x))$. Now if we define the map $\phi : U_1' \times U_2' \to U_1 \times U_2$ by $\phi(x, y) := (x, \alpha_y(x))$ then using this last version of equation **??** and the condition $\alpha(x_0, y) = y$ from the claim we see that

$$\begin{aligned} D_2 \alpha(x_0, y_0) &= D\alpha(x_0, .)(y_0) \\ &= D \operatorname{id}_{U_2'} = \operatorname{id}. \end{aligned}$$

Thus the Jacobian of ϕ at (x_0, y_0) has the block form

$$\begin{pmatrix} \operatorname{id} & 0 \\ * & \operatorname{id} \end{pmatrix}.$$

By the inverse function theorem ϕ is a local diffeomorphism in a neighborhood of (x_0, y_0). We also have that

$$\begin{aligned} (D_1 \phi)(x, y) \cdot (v, w) &= (v, D\alpha_y(x) \cdot w) \\ &= (v, \ell(x, \alpha_y(x)) \cdot v). \end{aligned}$$

Which represents an elements of the subbundle but is also of the form of tangents to the submanifolds that are the images of $U_1' \times \{y\}$ under the diffeomorphism ϕ for various choices of $y \in U_2$. This clearly saying that the subbundle is integrable.

Proof of the claim: By translation we may assume that $(x_0, y_0) = (0, 0)$. We use theorem H.2 from appendix B. With the notation of that theorem we let $f(t, x, y) := \ell(tz, y) \cdot z$ where $y \in U_2$ and z is an element of some ball $B(0, \epsilon)$ in E. Thus the theorem provides us with a smooth map $\beta : J_0 \times B(0, \epsilon) \times U_2$ satisfying $\beta(0, z, y) = y$ and

$$\frac{\partial}{\partial t}\beta(t, z, y) = \ell(tz, \beta(t, z, y)) \cdot z.$$

We will assume that $1 \in J$ since we can always arrange for this by making a change of variables of the type $t = as, z = x/a$ for a sufficiently small positive number a (we may do this at the expense of having to choose a smaller ϵ for the ball $B(0, \epsilon)$. We claim that if we define

$$\alpha(x, y) := \beta(1, x, y)$$

then for sufficiently small $|x|$ we have the required result. In fact we shall show that

$$D_2\beta(t, z, y) = t\ell(tz, \beta(t, z, y))$$

from which it follows that

$$D_1\alpha(x, y) = D_2\beta(1, x, y) = \ell(x, \alpha(x, y))$$

with the correct initial conditions (recall that we translated to (x_0, y_0)). Thus it remains to show that equation **??** holds. ¿From (3) of theorem H.2 we know that $D_2\beta(t, z, y)$ satisfies the following equation for any $v \in \mathsf{E}$:

$$\begin{aligned}\frac{\partial}{\partial t}D_2\beta(t, z, y) = {} & t\frac{\partial}{\partial t}\ell(tz, \beta(t, z, y)) \cdot v \cdot z \\ & + D_2\ell(tz, \beta(t, z, y)) \cdot D_2\beta(t, z, y) \cdot v \cdot z \\ & + \ell(tz, \beta(t, z, y)) \cdot v.\end{aligned}$$

Now we fix everything but t and define a function of one variable:

$$\Phi(t) := D_2\beta(t, z, y) \cdot v - t\ell(tz, \beta(t, z, y).$$

Clearly, $\Phi(0) = 0$. Now we use two fixed vectors v,z and construct the fields $\mathsf{X}_v(x, y) = (v, \ell_{(x,y)} \cdot v)$ and $\mathsf{X}_z(x, y) = (z, \ell_{(x,y)} \cdot z)$. In this special case, the equation of lemma 9.2 becomes

$$D\ell(x, y)(v, \ell_{(x,y)}v) \cdot z - D\ell(x, y)(z, \ell_{(x,y)}z) \cdot v.$$

Now with this in mind we compute $\frac{d}{dt}\Phi(t)$:

$$
\begin{aligned}
\frac{d}{dt}\Phi(t) &= \frac{\partial}{\partial t}(D_2\beta(t,z,y)\cdot v - t\ell(tz,\beta(t,z,y)) \\
&= \frac{\partial}{\partial t}D_2\beta(t,z,y)\cdot v - t\frac{d}{dt}\ell(tz,\beta(t,z,y)) - \ell(tz,\beta(t,z,y)) \\
&= \frac{\partial}{\partial t}D_2\beta(t,z,y)\cdot v - t\{D_1\ell(tz,\beta(t,z,y))\cdot z \\
&+ D_2\ell(tz,\beta(t,z,y))\cdot\frac{\partial}{\partial t}\beta(t,z,y) - \ell(tz,\beta(t,z,y)) \\
&= D_2\ell(tz,\beta(t,z,y))\cdot\{D_2\beta(t,z,y)\cdot v - t\ell(tz,\beta(t,z,y)\}\cdot z \text{ (use 9.3)} \\
&= D_2\ell(tz,\beta(t,z,y))\cdot\Phi(t)\cdot z.
\end{aligned}
$$

So we arrive at $\frac{d}{dt}\Phi(t) = D_2\ell(tz,\beta(t,z,y))\cdot\Phi(t)\cdot z$ with initial condition $\Phi(0) = 0$ which implies that $\Phi(t) \equiv 0$. This latter identity is none other than $D_2\beta(t,z,y)\cdot v = t\ell(tz,\beta(t,z,y)$. ■

It will be useful to introduce the notion of a codistribution and then explore the dual relationship existing between distributions and codistributions.

Definition 9.7 *A (regular) codistribution Ω on a manifold M is a subbundle of the cotangent bundle. Thus a smooth assignment of a subspace $\Omega_x \subset T_x^*M$ for every $x \in M$. If $\dim\Omega_x = l < \infty$ we call this a rank l codistribution.*

Using the definition of vector bundle chart adapted to a subbundle it is not hard to show , as indicated in the first paragraph of this section, that a (smooth) distribution of rank $k < \infty$ can be described in the following way:

Claim 9.2 *For a smooth distribution $\triangle$ of rank on M we have that for every $p \in M$ there exists a family of smooth vector fields $X_1, ..., X_k$ defined near p such that $\triangle_x = \operatorname{span}\{X_1(x), ..., X_k(x)\}$ for all x near p.*

Similarly, we have

Claim 9.3 *For a smooth codistribution Ω of rank k on M we have that for every $p \in M$ there exists a family of smooth 1-forms fields $\omega_1, ..., \omega_k$ defined near p such that $\Omega_x = \operatorname{span}\{\omega_1(x), ..., \omega_k(x)\}$ for all x near p.*

On the other hand we can use a codistribution to define a distribution and visa-versa. For example, for a regular codistribution Ω on M we can define a distribution $\triangle^{\perp\Omega}$ by

$$\triangle_x^{\perp\Omega} := \{v \in T_xM : \omega_x(v) = 0 \text{ for all } \omega_x \in \Omega_x\}.$$

Similarly, if $\triangle$ is a regular distribution on M then we can define a codistribution $\Omega^{\perp\triangle}$ by

$$\Omega_x^{\perp\triangle} := \{\omega_x \in T_x^*M : \omega_x(v) = 0 \text{ for all } v \in \triangle_x\}.$$

Notice that if $\triangle_1 \subset \triangle_2$ then $\triangle_2^{\perp\Omega} \subset \triangle_1^{\perp\Omega}$ and $(\triangle_1 \cap \triangle_2)^{\perp\Omega} = \triangle_1^{\perp\Omega} + \triangle_2^{\perp\Omega}$ etc.

9.4 Foliations

Definition 9.8 *Let M be a smooth manifold modeled on M and assume that $\mathsf{M} = \mathsf{E} \times \mathsf{F}$. A **foliation** $\mathcal{F}_M$ of M (or on M) is a partition of M into a family of disjoint subsets connected $\{\mathcal{L}_\alpha\}_{\alpha \in A}$ such that for every $p \in M$, there is a chart centered at p of the form $\varphi : U \to V \times W \subset \mathsf{E} \times \mathsf{F}$ with the property that for each $\mathcal{L}_\alpha$ the connected components $(U \cap \mathcal{L}_\alpha)_\beta$ of $U \cap \mathcal{L}_\alpha$ are given by*

$$\varphi((U \cap \mathcal{L}_\alpha)_\beta) = V \times \{c_{\alpha,\beta}\}$$

*where $c_{\alpha,\beta} \in W \subset \mathsf{F}$ are constants. These charts are called distinguished charts for the foliation or **foliation charts**. The connected sets $\mathcal{L}_\alpha$ are called the **leaves** of the foliation while for a given chart as above the connected components $(U \cap \mathcal{L}_\alpha)_\beta$ are called **plaques**.*

Recall that the connected components $(U \cap \mathcal{L}_\alpha)_\beta$ of $U \cap \mathcal{L}_\alpha$ are of the form $C_x(U \cap \mathcal{L}_\alpha)$ for some $x \in \mathcal{L}_\alpha$. An important point is that a fixed leaf $\mathcal{L}_\alpha$ may intersect a given chart domain U in many, even an infinite number of disjoint connected pieces no matter how small U is taken to be. In fact, it may be that $C_x(U \cap \mathcal{L}_\alpha)$ is dense in U. On the other hand, each $\mathcal{L}_\alpha$ is connected by definition. The usual first example of this behavior is given by the irrationally foliated torus. Here we take $M = T^2 := S^1 \times S^1$ and let the leaves be given as the image of the immersions $\iota_a : t \mapsto (e^{\mathrm{i}at}, e^{\mathrm{i}t})$ where a is a real numbers. If a is irrational then the image $\iota_a(\mathbb{R})$ is a (connected) dense subset of $S^1 \times S^1$. On the other hand, even in this case there are an infinite number of distinct leaves.

It may seem that a foliated manifold is just a special manifold but from one point of view a foliation is a generalization of a manifold. For instance, we can think of a manifold M as foliation where the points are the leaves. This is called the **discrete foliation** on M. At the other extreme a manifold may be thought of as a foliation with a single leaf $\mathcal{L} = M$ (the **trivial foliation**). We also have handy many examples of 1-dimensional foliations since given any global flow the orbits (maximal integral curves) are the leaves of a foliation. We also have the following special cases:

Example 9.1 *On a product manifold say $M \times N$ we have two complementary foliations:*

$$\{\{p\} \times N\}_{p \in M}$$

and

$$\{M \times \{q\}\}_{q \in N}$$

Example 9.2 *Given any submersion $f : M \to N$ the level sets $\{f^{-1}(q)\}_{q \in N}$ form the leaves of a foliation. The reader will soon realize that any foliation is given locally by submersions. The global picture for a general foliation can be very different from what can occur with a single submersion.*

Example 9.3 *The fibers of any vector bundle foliate the total space.*

Example 9.4 *(Reeb foliation) Consider the strip in the plane given by* $\{(x,y) : |x| \leq 1\}$. *For* $a \in \mathbb{R} \cup \{\pm\infty\}$ *we form leaves* $\mathcal{L}_a$ *as follows:*

$$\mathcal{L}_a := \{(x, a + f(x)) : |x| \leq 1\} \text{ for } a \in \mathbb{R}$$
$$\mathcal{L}_{\pm\infty} := \{(\pm 1, y) : |y| \leq 1\}$$

where $f(x) := \exp\left(\frac{x^2}{1-x^2}\right) - 1$. *By rotating this symmetric foliation about the* y *axis we obtain a foliation of the solid cylinder. This foliation is such that translation of the solid cylinder* C *in the* y *direction maps leaves diffeomorphically onto leaves and so we may let* $\mathbb{Z}$ *act on* C *by* $(x,y,z) \mapsto (x, y+n, z)$ *and then* $C/\mathbb{Z}$ *is a solid torus with a foliation called the Reeb foliation.*

Example 9.5 *The one point compactification of* $\mathbb{R}^3$ *is homeomorphic to* $S^3 \subset \mathbb{R}^4$. *Thus* $S^3 - \{p\} \cong \mathbb{R}^3$ *and so there is a copy of the Reeb foliated solid torus inside* S^3. *The complement of a solid torus in* S^3 *is another solid torus. It is possible to put another Reeb foliation on this complement and thus foliate all of* S^3. *The only compact leaf is the torus that is the common boundary of the two complementary solid tori.*

Exercise 9.2 *Show that the set of all* $v \in TM$ *such that* $v = T\varphi^{-1}(v, 0)$ *for some* $v \in \mathsf{E}$ *and some foliated chart* φ *is a (smooth) subbundle of* TM *that is also equal to* $\{v \in TM : v \text{ is tangent to a leaf}\}$.

Definition 9.9 *The* ***tangent bundle of a foliation*** $\mathcal{F}_M$ *with structure pseudogroup* $\mathcal{G}_{\mathsf{M,F}}$ *is the subbundle* $T\mathcal{F}_M$ *of* TM *defined by*

$$\begin{aligned} T\mathcal{F}_M &:= \{v \in TM : v \text{ is tangent to a leaf}\} \\ &= \{v \in TM : v = T\varphi^{-1}(v,0) \text{ for some } v \in \mathsf{E} \text{ and some foliated chart } \varphi\} \end{aligned}$$

9.5 The Global Frobenius Theorem

The first step is to show that the (local) integral submanifolds of an integrable regular distribution can be glued together to form maximal integral submanifolds. These will form the leaves of a distribution.

Exercise 9.3 *If* Δ *is an integrable regular distribution of* TM, *then for any two local integral submanifolds* S_1 *and* S_2 *of* Δ *that both contain a point* x_0, *there is an open neighborhood* U *of* x_0 *such that*

$$S_1 \cap U = S_2 \cap U$$

Theorem 9.2 *If* Δ *is a subbundle of* TM *(i.e. a regular distribution) then the following are equivalent:*
1) Δ *is involutive.*
2) Δ *is integrable.*
3) There is a foliation $\mathcal{F}_M$ *on* M *such that* $T\mathcal{F}_M = \Delta$.

Proof. The equivalence of (1) and (2) is the local Frobenius theorem already proven. Also, the fact that (3) implies (2) is follows from 9.2. Finally, assume that (2) holds so that Δ is integrable. Recall that each (local) integral submanifold is an immersed submanifold which carries a submanifold topology generated by the connected components of the intersections of the integral submanifolds with chart domains. Any integral submanifold S has a smooth structure given by restricting charts U, ψ on M to connected components of $S \cap U$ (not on all of $S \cap U$!). Recall that a local integral submanifold is a regular submanifold (we are not talking about maximal immersed integral submanifolds!). Thus we may take U small enough that $S \cap U$ is connected. Now if we take two (local) integral submanifolds S_1 and S_2 of Δ and any point $x_0 \in S_1 \cap S_2$ (assuming this is nonempty) then a small enough chart U, ψ with $x_0 \in U$ induces a chart $U \cap S_1, \psi|_{U \cap S_1}$ on S_1 and a chart $C_{x_0}(U \cap S_2), \psi|_{C_{x_0}(U \cap S_2)}$ on S_2. But as we know $S_1 \cap U = S_2 \cap U$ and the overlap is smooth. Thus the union $S_1 \cup S_2$ is a smooth manifold with charts given by $U \cap (S_1 \cup S_2), \psi|_{U \cap (S_1 \cup S_2)}$ and the overlap maps are $U \cap (S_1 \cap S_2), \psi|_{U \cap (S_1 \cap S_2)}$. We may extend to a maximal connected integral submanifold using Zorn's lemma be we can see the existence more directly. Let $\mathcal{L}_a(x_0)$ be the set of all points that can be connected to x_0 by a smooth path $c : [0,1] \to M$ with the property that for any $t_0 \in [0,1]$, the image $c(t)$ is contained inside a (local) integral submanifold for all t sufficiently near t_0. Using what we have seen about gluing intersecting integral submanifold together and the local uniqueness of such integral submanifolds we see that $\mathcal{L}_a(x_0)$ is a smooth connected immersed integral submanifold that must be the maximal connected integral submanifold containing x_0. Now since x_0 was arbitrary there is a maximal connected integral submanifold containing any point of M. By construction we have that the foliation $\mathcal{L}$ given by the union of all these leaves satisfies (3). ∎

There is an analogy between the notion of a foliation on a manifold and a differentiable structure on a manifold.

From this point of view we think of a foliation as being given by a maximal **foliation atlas** that is defined to be a cover of M by foliated charts. The compatibility condition on such charts is that when the domains of two foliation charts, say $\varphi_1 : U_1 \to V_1 \times W_1$ and $\varphi_2 : U_2 \to V_2 \times W_2$, then the overlap map has the form

$$\varphi_2 \circ \varphi_1^{-1}(x, y) = (f(x, y), g(y)).$$

A plaque in a chart $\varphi_1 : U_1 \to V_1 \times W_1$ is a connected component of a set of the form $\varphi_1^{-1}\{(x, y) : y = \text{constant}\}$.

9.6 Singular Distributions

Lemma 9.3 *Let $X_1, ..., X_n$ be vector fields defined in a neighborhood of $x \in M$ such that $X_1(x), ..., X_n(x)$ are a basis for T_xM and such that $[X_i, X_j] = 0$ in a neighborhood of x. Then there is an open chart $U, \psi = (y^1, ..., y^n)$ containing x such that $X_i|_U = \frac{\partial}{\partial y^i}$.*

Proof. For a sufficiently small ball $B(0,\epsilon) \subset \mathbb{R}^n$ and $t = (t_1, ..., t_n) \in B(0,\epsilon)$ we define

$$f(t_1, ..., t_n) := \varphi_{t_1}^{X_1} \circ \cdots \circ \varphi_{t_n}^{X_n}(x).$$

By theorem 2.9 the order that we compose the flows does not change the value of $f(t_1, ..., t_n)$. Thus

$$\begin{aligned}
&\frac{\partial}{\partial t_i} f(t_1, ..., t_n) \\
&= \frac{\partial}{\partial t_i} \varphi_{t_1}^{X_1} \circ \cdots \circ \varphi_{t_n}^{X_n}(x) \\
&= \frac{\partial}{\partial t_i} \varphi_{t_i}^{X_i} \circ \varphi_{t_1}^{X_1} \circ \cdots \circ \varphi_{t_n}^{X_n}(x) \text{ (put the } i\text{-th flow first)} \\
&X_i(\varphi_{t_1}^{X_1} \circ \cdots \circ \varphi_{t_n}^{X_n}(x)).
\end{aligned}$$

Evaluating at $t = 0$ shows that $T_0 f$ is nonsingular and so $(t_1, ..., t_n) \mapsto f(t_1, ..., t_n)$ is a diffeomorphism on some small open set containing 0. The inverse of this map is the coordinate chart we are looking for (check this!). ∎

Definition 9.10 *Let $\mathfrak{X}_{loc}(M)$ denote the set of all sections of the presheaf $\mathfrak{X}_M$. That is*

$$\mathfrak{X}_{loc}(M) := \bigcup_{open\ U \subset M} \mathfrak{X}_M(U).$$

Also, for a distribution Δ let $\mathfrak{X}_\Delta(M)$ denote the subset of $\mathfrak{X}_{loc}(M)$ consisting of local fields X with the property that $X(x) \in \Delta_x$ for every x in the domain of X.

Definition 9.11 *We say that a subset of local vector fields $\mathcal{X} \subset \mathfrak{X}_\Delta(M)$* ***spans*** *a distribution Δ if for each $x \in M$ the subspace Δ_x is spanned by $\{X(x) : X \in \mathcal{X}\}$.*

If Δ is a smooth distribution (and this is all we shall consider) then $\mathfrak{X}_\Delta(M)$ spans Δ. On the other hand, as long as we make the convention that the empty set spans the set $\{0\}$ for every vector space we are considering, then any $\mathcal{X} \subset \mathfrak{X}_\Delta(M)$ spans some smooth distribution which we denote by $\Delta(\mathcal{X})$.

Definition 9.12 *An immersed integral submanifold of a distribution Δ is an injective immersion $\iota : S \to M$ such that $T_s\iota(T_sS) = \Delta_{\iota(s)}$ for all $s \in S$. An immersed integral submanifold is called* ***maximal*** *its image is not properly contained in the image of any other immersed integral submanifold.*

Since an immersed integral submanifold is an injective map we can think of S as a subset of M. In fact, it will also turn out that an immersed integral submanifold is automatically smoothly universal so that the image $\iota(S)$ is an initial submanifold. Thus in the end, we may as well assume that $S \subset M$ and that $\iota : S \to M$ is the inclusion map. Let us now specialize to the finite dimensional case. Note however that we do *not* assume that the rank of the distribution is constant.

Now we proceed with our analysis. If $\iota : S \to M$ is an immersed integral submanifold and of a distribution $\triangle$ then if $X \in \mathfrak{X}_\Delta(M)$ we can make sense of $\iota^* X$ as a local vector field on S. To see this let U be the domain of X and take $s \in S$ with $\iota(s) \in U$. Now $X(\iota(s)) \in T_s\iota(T_s S)$ we can define

$$\iota^* X(s) := (T_s\iota)^{-1} X(\iota(s)).$$

$\iota^* X(s)$ is defined on some open set in S and is easily seen to be smooth by considering the local properties of immersions. Also, by construction $\iota^* X$ is ι related to X.

Next we consider what happens if we have two immersed integral submanifolds $\iota_1 : S_1 \to M$ and $\iota_2 : S_2 \to M$ such that $\iota_1(S_1) \cap \iota_2(S_2) \neq \emptyset$. By proposition 2.7 we have

$$\iota_i \circ \varphi_t^{\iota_i^* X} = \varphi_t^X \circ \iota_i \text{ for } i = 1, 2.$$

Now if $x_0 \in \iota_1(S_1) \cap \iota_2(S_2)$ then we choose s_1 and s_2 such that $\iota_1(s_1) = \iota_2(s_2) = x_0$ and pick local vector fields $X_1, ..., X_k$ such that $(X_1(x_0), ..., X_k(x_0))$ is a basis for $\triangle_{x_0}$. For $i = 1$ and 2 we define

$$f_i(t^1, ..., t^k) := (\varphi_{t^1}^{\iota_i^* X_1} \circ \cdots \circ \varphi_{t^k}^{\iota_i^* X_k})$$

and since $\frac{\partial}{\partial t^j}\big|_0 f_i = \iota_i^* X_j$ for $i = 1, 2$ and $j = 1, ..., k$ we conclude that f_i, $i = 1, 2$ are diffeomorphisms when suitable restricted to a neighborhood of $0 \in \mathbb{R}^k$. Now we compute:

$$\begin{aligned}
(\iota_2^{-1} \circ \iota_1 \circ f_1)(t^1, ..., t^k) &= (\iota_2^{-1} \circ \iota_1 \circ \varphi_{t^1}^{\iota_1^* X_1} \circ \cdots \circ \varphi_{t^k}^{\iota_1^* X_k})(x_1) \\
&= (\iota_2^{-1} \varphi_{t^1}^{X_1} \circ \cdots \circ \varphi_{t^k}^{X_k} \circ \iota_1)(x_1) \\
&= (\varphi_{t^1}^{\iota_2^* X_1} \circ \cdots \circ \varphi_{t^k}^{\iota_2^* X_k} \circ \iota_2^{-1} \circ \iota_1)(x_1) \\
&= f_2(t^1, ..., t^k).
\end{aligned}$$

Now we can see that $\iota_2^{-1} \circ \iota_1$ is a diffeomorphism. This allows us to glue together the all the integral manifolds that pass through a fixed x in M to obtain a unique maximal integral submanifold through x. We have prove the following result:

Proposition 9.1 *For a smooth distribution Δ on M and any $x \in M$ there is a unique maximal integral manifold L_x containing x called the* ***leaf*** *through x.*

Definition 9.13 *Let $\mathcal{X} \subset \mathfrak{X}_{loc}(M)$. We call X a* ***stable*** *family of local vector fields if for any $X, Y \in \mathcal{X}$ we have*

$$(\varphi_t^X)^* Y \in \mathcal{X}$$

whenever $(\varphi_t^X)^* Y$ is defined. Given an arbitrary subset of local fields $\mathcal{X} \subset \mathfrak{X}_{loc}(M)$ let $\mathcal{S}(\mathcal{X})$ denote the set of all local fields of the form

$$(\varphi_{t^1}^{X_1} \circ \varphi_{t^2}^{X_2} \circ \cdots \circ \varphi_{t^t}^{X_k})^* Y$$

where $X_i, Y \in \mathcal{X}$ and where $t = (t^1, ..., t^k)$ varies over all k-tuples such that the above expression is defined.

Exercise 9.4 *Show that $\mathcal{S}(\mathcal{X})$ is the smallest stable family of local vector fields containing $\mathcal{X}$.*

Definition 9.14 *If a diffeomorphism ϕ of a manifold M with a distribution Δ is such that $T_x\phi(\Delta_x) \subset \Delta_{\phi(x)}$ for all $x \in M$ then we call ϕ an **automorphism of** Δ. If $\phi : U \to \phi(U)$ is such that $T_x\phi(\Delta_x) \subset \Delta_{\phi(x)}$ for all $x \in U$ we call ϕ a **local automorphism of** Δ.*

Definition 9.15 *If $X \in \mathfrak{X}_{loc}(M)$ is such that $T_x\varphi_t^X(\Delta_x) \subset \Delta_{\varphi_t^X(x)}$ we call X a (local) **infinitesimal automorphism** of Δ. The set of all such is denoted $\mathrm{aut}_{loc}(\Delta)$.*

Example 9.6 *Convince yourself that $\mathrm{aut}_{loc}(\Delta)$ is stable.*

For the next theorem recall the definition of $\mathfrak{X}_\Delta$.

Theorem 9.3 *Let Δ be a smooth singular distribution on M. Then the following are equivalent:*
1) Δ is integrable.
2) $\mathfrak{X}_\Delta$ is stable.
3) $\mathrm{aut}_{loc}(\Delta) \cap \mathfrak{X}_\Delta$ spans Δ.
4) There exists a family $\mathcal{X} \subset \mathfrak{X}_{loc}(M)$ such that $\mathcal{S}(\mathcal{X})$ spans Δ.

Proof. Assume (1) and let $X \in \mathfrak{X}_\Delta$. If $\mathcal{L}_x$ is the leaf through $x \in M$ then by proposition 2.7

$$\varphi_{-t}^X \circ \iota = \iota \circ \varphi_{-t}^{\iota^* X}$$

where $\iota : \mathcal{L}_x \hookrightarrow M$ is inclusion. Thus

$$\begin{aligned} T_x(\varphi_{-t}^X)(\Delta_x) &= T(\varphi_{-t}^X) \cdot T_x\iota \cdot (T_x\mathcal{L}_x) \\ &= T(\iota \circ \varphi_{-t}^{\iota^* X}) \cdot (T_x\mathcal{L}_x) \\ &= T\iota T_x(\varphi_{-t}^{\iota^* X}) \cdot (T_x\mathcal{L}_x) \\ &= T\iota T_{\varphi_{-t}^{\iota^* X}(x)}\mathcal{L}_x = \Delta_{\varphi_{-t}^{\iota^* X}(x)}. \end{aligned}$$

Now if Y is in $\mathfrak{X}_\Delta$ then at an arbitrary x we have $Y(x) \in \Delta_x$ and so the above shows that $((\varphi_t^X)^*Y)(x) \in \Delta$ so $(\varphi_t^X)^*Y)$ is in $\mathfrak{X}_\Delta$. We conclude that $\mathfrak{X}_\Delta$ is stable and have shown that (1) $\Rightarrow$ (2).

Next, if (2) hold then $\mathfrak{X}_\Delta \subset \mathrm{aut}_{loc}(\Delta)$ and so we have (3).

If (3) holds then we let $\mathcal{X} := \mathrm{aut}_{loc}(\Delta) \cap \mathfrak{X}_\Delta$. Then for $Y, Y \in \mathcal{X}$ we have $(\varphi_t^X)^*Y \in \mathfrak{X}_\Delta$ and so $\mathcal{X} \subset \mathcal{S}(\mathcal{X}) \subset \mathfrak{X}_\Delta$. from this we see that since $\mathcal{X}$ and $\mathfrak{X}_\Delta$ both span Δ so does $\mathcal{S}(\mathcal{X})$.

Finally, we show that (4) implies (1). Let $x \in M$. Since $\mathcal{S}(\mathcal{X})$ spans the distribution and is also stable by construction we have

$$T(\varphi_t^X)\Delta_x = \Delta_{\varphi_t^X(x)}$$

for all fields X from $\mathcal{S}(\mathcal{X})$. Let the dimension Δ_x be k and choose fields $X_1, ..., X_k \in \mathcal{S}(\mathcal{X})$ such that $X_1(x), ..., X_k(x)$ is a basis for Δ_x. Define a map $f :: \mathbb{R}^k \to M$ by

$$f(t^1, ..., t^n) := (\varphi_{t^1}^{X_1} \varphi_{t^2}^{X_2} \circ \cdots \circ \varphi_{t^k}^{X_k})(x)$$

which is defined (and smooth) near $0 \in \mathbb{R}^k$. As in lemma 9.3 we know that the rank of f at 0 is k and the image of a small enough open neighborhood of 0 is a submanifold. In fact, this image, say $S = f(U)$ is an integral submanifold of Δ through x. To see this just notice that the $T_x S$ is spanned by $\frac{\partial f}{\partial t^j}(0)$ for $j = 1, 2, ..., k$ and

$$\begin{aligned}
\frac{\partial f}{\partial t^j}(0) &= \left.\frac{\partial}{\partial t^j}\right|_0 (\varphi_{t^1}^{X_1} \varphi_{t^2}^{X_2} \circ \cdots \circ \varphi_{t^k}^{X_k})(x) \\
&= T(\varphi_{t^1}^{X_1} \varphi_{t^2}^{X_2} \circ \cdots \circ \varphi_{t^{j-1}}^{X_{j-1}}) X_j((\varphi_{t^j}^{X_j} \varphi_{t^{j+1}}^{X_{j+1}} \circ \cdots \circ \varphi_{t^k}^{X_k})(x)) \\
&= ((\varphi_{-t^1}^{X_1})^* (\varphi_{-t^2}^{X_2})^* \circ \cdots \circ (\varphi_{-t^{j-1}}^{X_{j-1}})^* X_j)(f(t^1, ..., t^n)).
\end{aligned}$$

But $\mathcal{S}(\mathcal{X})$ is stable so each $\frac{\partial f}{\partial t^j}(0)$ lies in $\Delta_{f(t)}$. From the construction of f and remembering **??** we see that $\operatorname{span}\{\frac{\partial f}{\partial t^j}(0)\} = T_{f(t)} S = \Delta_{f(t)}$ and we are done. ∎

9.7 Problems

1. Let H be the Heisenberg group consisting of all matrices of the form

$$A = \begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

   That is, the upper-diagonal 3×3 real matrices with 1's on the diagonal. Let V_{12}, V_{13}, V_{23} be the left-invariant vector-fields on H that have values at the identity $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively. Let $\Delta_{\{V_{12}, V_{13}\}}$ and $\Delta_{\{V_{12}, V_{23}\}}$ be the 2−dimensional distributions generated by the indicated pairs of vector fields. Show that $\Delta_{\{V_{12}, V_{13}\}}$ is integrable and $\Delta_{\{V_{12}, V_{23}\}}$ is not.

Chapter 10

De Rham Cohomology

For a given manifold M of dimension n we have the sequence of maps

$$C^{\infty}(M) = \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

and we have defined the de Rham cohomology groups (actually vector spaces) as the quotients

$$H^i(M) = \frac{Z^i(M)}{B^i(M)}$$

where $Z^i(M) := \ker(d : \Omega^i(M) \to \Omega^{i+1}(M))$ and $B^i(M) := \mathrm{Im}(d : \Omega^{i-1}(M) \to \Omega^i(M))$. The elements of $Z^i(M)$ are called **closed** i-forms or i-**cocycles** and the elements of $B^i(M)$ are called **exact** i-forms or i-**coboundaries.**

Let us immediately consider a simple situation that will help the reader see what these cohomologies are all about. Let $M = \mathbb{R}^2 - \{0\}$ and consider the 1-form

$$\vartheta := \frac{xdy - ydx}{x^2 + y^2}.$$

We got this 1-form by taking the exterior derivative of $\theta = \arctan(x)$. This function is not defined as a single valued smooth function on all of $M = \mathbb{R}^2 - \{0\}$ but it turns out that the result $\frac{ydx - xdy}{x^2+y^2}$ is well defined on *all* of M. One may also check that $d\vartheta = 0$ and so ϑ is closed. We have the following situation:

1. $\vartheta := \frac{xdy - ydx}{x^2+y^2}$ is a smooth 1-form M with $d\vartheta = 0$.
2. There is no function f defined on (all of) M such that $\vartheta = df$.
3. For any small ball $B(p, \varepsilon)$ in M there is a function $f \in C^{\infty}(B(p, \varepsilon))$ such that $\vartheta|_{B(p,\varepsilon)} = df$.

   (1) says that ϑ is globally well defined and closed while (2) says that ϑ is not exact. (3) says that ϑ is what we might call locally exact. What prevents us from finding a (global) function with $\vartheta = df$? Could the same kind of situation occur if $M = \mathbb{R}^2$? The answer is no and this difference between $\mathbb{R}^2$ and $\mathbb{R}^2 - \{0\}$ is that $H^1(\mathbb{R}^2) = 0$ while $H^1(\mathbb{R}^2 - \{0\}) \neq 0$.

Exercise 10.1 *Verify (1) and (3) above.*

The reader may be aware that this example has something to do with path independence. In fact, if we could show that for a given 1-form α, the path integral $\int_c \alpha$ only depended on the beginning and ending points of the curve $c(0)$ then we could define a function $f(x) := \int_{x_0}^x \alpha$ where $\int_{x_0}^x \alpha$ is just the path integral for any path beginning at a fixed x_0 and ending at x. With this definition one can show that $df = \alpha$ and so α would be exact. In our example the form ϑ is not exact and so there must be a failure of path independence.

Exercise 10.2 *A smooth fixed endpoint homotopy between a path* $c_0 : [x_0, x] \to M$ *and* $c_1 : [x_0, x] \to M$ *is a one parameter family of paths* h_s *such that* $h_0 = c_0$ *and* $h_1 = c_0$ *and such that the map* $H(s,t) := h_s(t)$ *is smooth on* $[0,1] \times [x_0, x]$. *Show that if* α *is an exact* 1*-form then* $\frac{d}{ds}\int_{h_s} \alpha = 0$.

Since we have found a closed $1-$form on $\mathbb{R}^2\backslash\{0\}$ that is not exact we know that $H^1(\mathbb{R}^2\backslash\{0\}) \neq 0$. We are not yet in a position to determine $H^1(\mathbb{R}^2\backslash\{0\})$ completely. We will start out with even simpler spaces and eventually, develop the machinery to bootstrap our way up to more complicated situations.

First, let $M = \{p\}$. That is, M consists of a single point and is hence a 0-dimensional manifold. In this case,

$$\Omega^k(\{p\}) = Z^k(\{p\}) = \begin{cases} \mathbb{R} & \text{for } k = 0 \\ \\ 0 & \text{for } k > 0 \end{cases}$$

Furthermore, $B^k(\{p\}) = 0$ and so

$$H^k(\{p\}) = \begin{cases} \mathbb{R} & \text{for } k = 0 \\ \\ 0 & \text{for } k > 0 \end{cases}.$$

Next we consider the case $M = \mathbb{R}$. Here, $Z^0(\mathbb{R})$ is clearly just the constant functions and so is (isomorphic to) $\mathbb{R}$. On the other hand, $B^0(\mathbb{R}) = 0$ and so

$$H^0(\mathbb{R}) = \mathbb{R}.$$

Now since $d : \Omega^1(\mathbb{R}) \to \Omega^2(\mathbb{R}) = 0$ we see that $Z^1(\mathbb{R}) = \Omega^1(\mathbb{R})$. If $g(x)dx \in \Omega^1(\mathbb{R})$ then letting

$$f(x) := \int_0^x g(x)dx$$

we get $df = g(x)dx$. Thus, every $\Omega^1(\mathbb{R})$ is exact; $B^1(\mathbb{R}) = \Omega^1(\mathbb{R})$. We are led to

$$H^1(\mathbb{R}) = 0.$$

From this modest beginning we will be able to compute the de Rham cohomology for a large class of manifolds. Our first goal is to compute $H^k(\mathbb{R})$ for all k. In order to accomplish this we will need a good bit of preparation. The methods are largely algebraic and so will need to introduce a small portion of "homological algebra".

Definition 10.1 *Let R be a commutative ring. A* ***differential*** *$R-$**complex** is a direct sum of modules $C = \bigoplus_{k \in \mathbb{Z}} C^k$ together with a linear map $d : C \to C$ such that $d \circ d = 0$ and such that $d(C^k) \subset C^{k+1}$. Thus we have a sequence of linear maps*

$$\cdots C^{k-1} \xrightarrow{d} C^k \xrightarrow{d} C^{k+1}$$

where we have denoted the restrictions $d_k = d|\, C^k$ all simply by the single letter d.

Let $A = \bigoplus_{k \in \mathbb{Z}} A^k$ and $B = \bigoplus_{k \in \mathbb{Z}} B^k$ be differential complexes. A map $f : A \to B$ is called a **chain map** if f is a (degree 0) graded map such that $d \circ f = f \circ g$. In other words, if we let $f|\, A^k := f_k$ then we require that $f_k(A^k) \subset B^k$ and that the following diagram commutes for all k:

$$\begin{array}{ccccccccc}
\xrightarrow{d} & A^{k-1} & \xrightarrow{d} & A^k & \xrightarrow{d} & A^{k+1} & \xrightarrow{d} \\
 & f_{k-1} \downarrow & & f_k \downarrow & & f_{k+1} \downarrow & & . \\
\xrightarrow{d} & B^{k-1} & \xrightarrow{d} & B^k & \xrightarrow{d} & B^{k+1} & \xrightarrow{d}
\end{array}$$

Notice that if $f : A \to B$ is a chain map then $\ker(f)$ and $\operatorname{im}(f)$ are complexes with $\ker(f) = \bigoplus_{k \in \mathbb{Z}} \ker(f_k)$ and $\operatorname{im}(f) = \bigoplus_{k \in \mathbb{Z}} \operatorname{im}(f_k)$. Thus the notion of **exact sequence of chain maps** may be defined in the obvious way.

Definition 10.2 *The k-th cohomology of the complex $C = \bigoplus_{k \in \mathbb{Z}} C^k$ is*

$$H^k(C) := \frac{\ker(\,d|\, C^k)}{\operatorname{im}(\,d|\, C^{k-1})}$$

*The elements of $\ker(\,d|\, C^k)$ (also denoted $Z^k(C)$) are called $k-$**cocycles** while the elements of $\operatorname{im}(\,d|\, C^{k-1})$ (also denoted $B^k(C)$) are called $k-$**coboundaries**.*

We already have an example since by letting $\Omega^k(M) := 0$ for $k < 0$ we have a differential complex $d : \Omega(M) \to \Omega(M)$ where d is the exterior derivative. In this case, $H^k(\Omega(M)) = H^k(M)$ by definition.

Remark 10.1 *In fact, $\Omega(M)$ is an algebra under the exterior product (recall that $\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{l+k}(M)$). This algebra structure actually remains active at the level of cohomology: If $\alpha \in Z^k(M)$ and $\beta \in Z^l(M)$ then for any $\alpha', \beta' \in \Omega^{k-1}(M)$ and any $\beta' \in \Omega^{l-1}(M)$ we have*

$$\begin{aligned}
(\alpha + d\alpha') \wedge \beta &= \alpha \wedge \beta + d\alpha' \wedge \beta \\
&= \alpha \wedge \beta + d(\alpha' \wedge \beta) - (-1)^{k-1} \alpha' \wedge d\beta \\
&= \alpha \wedge \beta + d(\alpha' \wedge \beta)
\end{aligned}$$

and similarly $\alpha \wedge (\beta + d\beta') = \alpha \wedge \beta + d(\alpha \wedge \beta')$. Thus we may define a product $H^k(M) \times H^l(M) \to H^{k+l}(M)$ by $[\alpha] \wedge [\beta] := [\alpha \wedge \beta]$.

Returning to the general algebraic case, if $f : A \to B$ is a chain map then it is easy to see that there is a natural (degree 0) graded map $f^* : H \to H$ defined by

$$f^*([x]) := [f(x)] \text{ for } x \in C^k.$$

Definition 10.3 *An exact sequence of chain maps of the form*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called a ***short exact sequence****.*

Associated to every short exact sequence of chain maps there is a long exact sequence of cohomology groups:

$$\begin{tikzcd}
 & & \\
H^{k+1}(A) \arrow[r, "f^*"] & H^{k+1}(B) \arrow[r, "g^*"] & H^{k+1}(C) \arrow[ul, bend right, "d^*"'] \\
H^k(A) \arrow[r, "f^*"] & H^k(B) \arrow[r, "g^*"] & H^k(C) \arrow[ull, bend right, "d^*"'] \\
 & \arrow[ul, bend left, "d^*"] &
\end{tikzcd}$$

The maps f^* and g^* are the maps induced by f and g and the "coboundary map" $d^* : H^k(C) \to H^{k+1}(A)$ is defined as follows: Let $c \in Z^k(C) \subset C^k$ represent the class $[c] \in H^k(C)$ so that $dc = 0$. Starting with this we hope to end up with a well defined element of $H^{k+1}(A)$ but we must refer to the following diagram to explain how we arrive at a choice of representative of our class $d^*([c])$:

$$\begin{array}{ccccccc}
0 \longrightarrow & A^{k+1} & \xrightarrow{f} & B^{k+1} & \xrightarrow{g} & C^{k+1} & \longrightarrow 0 \\
 & d\uparrow & & d\uparrow & & d\uparrow & \\
0 \longrightarrow & A^k & \xrightarrow{f} & B^k & \xrightarrow{g} & C^k & \longrightarrow 0
\end{array}$$

By the surjectivity of g there is an $b \in B^k$ with $g(b) = c$. Also, since $g(db) = d(g(b)) = dc = 0$, it must be that $db = f(a)$ for some $a \in A^{k+1}$. The scheme of the process is

$$c \dashrightarrow b \dashrightarrow a.$$

Certainly $f(da) = d(f(a)) = ddb = 0$ and so since f is 1-1 we must have $da = 0$ which means that $a \in Z^{k+1}(C)$. We would like to define $d^*([c])$ to be $[a]$ but we must show that this is well defined. Suppose that we repeat this process starting with $c' = c + dc_{k-1}$ for some $c_{k-1} \in C^{k-1}$. In our first step we find $b' \in B^k$

with $g(b') = c'$ and then a' with $f(a') = db'$. We wish to show that $[a] = [a']$. We have $g(b - b') = c - c = 0$ and so there is an $a_k \in A^k$ with $f(a_k) = b - b'$. By commutativity we have

$$\begin{aligned} f(d(a_k)) &= d(f(a_k)) = d(b - b') \\ &= db - db' = f(a) - f(a') = f(a - a') \end{aligned}$$

and then since f is 1-1 we have $d(a_k) = a - a'$ which means that $[a] = [a']$. Thus our definition $\delta([c]) := [a]$ is independent of the choices. We leave it to the reader to check (if there is any doubt) that d^*, so defined, is linear.

We now return to the de Rham cohomology. If $f : M \to N$ is a C^∞ map then we have $f^* : \Omega(N) \to \Omega(M)$. Since pull-back commutes with exterior differentiation and preserves the degree of differential forms, f^* is a chain map. Thus we have the induced map on the cohomology that we will also denote by f^* :

$$\begin{aligned} &f^* : H^*(M) \to H^*(M) \\ &f^* : [\alpha] \mapsto [f^*\alpha] \end{aligned}$$

where we have used $H^*(M)$ to denote the direct sum $\bigoplus_i H^i(M)$. Notice that $f \mapsto f^*$ together with $M \mapsto H^*(M)$ is a contravariant functor which means that for $f : M \to N$ and $g : N \to P$ we have

$$(g \circ f)^* = f^* \circ g^*.$$

In particular if $\iota_U : U \to M$ is inclusion of an open set U then $\iota_U^* \alpha$ is the same as restriction of the form α to U. If $[\alpha] \in H^*(M)$ then $\iota_U{}^*([\alpha]) \in H^*(U)$;

$$\iota_U{}^* : H^*(M) \to H^*(U).$$

10.1 The Meyer Vietoris Sequence

Suppose that $M = U_0 \cup U_1$ for open sets U. Let $U_0 \sqcup U_1$ denote the disjoint union of U and V. We then have inclusions $\iota_1 : U_1 \to M$ and $\iota_2 : U_2 \to M$ as well as the inclusions

$$\partial_0 : U_0 \cap U_1 \to U_1 \hookrightarrow U_0 \sqcup U_1$$

and

$$\partial_1 : U_0 \cap U_1 \to U_0 \hookrightarrow U_0 \sqcup U_1$$

that we indicate (following [Bo Tu]) by writing

$$M \underset{\iota_1}{\overset{\iota_0}{\leftleftarrows}} \;\; U_0 \sqcup U_1 \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} U_0 \cap U_1 \;\; .$$

This gives rise to the **Mayer-Vietoris short exact sequence**

$$0 \to \Omega(M) \xrightarrow{\iota^*} \Omega(U_0) \oplus \Omega(U_1) \xrightarrow{\partial^*} \Omega(U_0 \cap U_1) \to 0$$

where $\iota(\omega) := (\iota_0^*\omega, \iota_1^*\omega)$ and $\partial^*(\alpha, \beta) := (\partial_0^*(\beta) - \partial_1^*(\alpha))$. Notice that $\iota_0^*\omega \in \Omega(U_0)$ while $\iota_1^*\omega \in \Omega(U_1)$. Also, $\partial_0^*(\beta) = \beta|_{U_0 \cap U_1}$ and $\partial_1^*(\alpha) = \alpha|_{U_0 \cap U_1}$ and so live in $\Omega(U_0 \cap U_1)$.

Let us show that this sequence is exact. First if $\iota(\omega) := (\iota_1^*\omega, \iota_0^*\omega) = (0, 0)$ then $\omega|_{U_0} = \omega|_{U_1} = 0$ and so $\omega = 0$ on $M = U_0 \cup U_1$ thus ι^* is 1-1 and exactness at $\Omega(M)$ is demonstrated.

Next, if $\eta \in \Omega(U_0 \cap U_1)$ then we take a smooth partition of unity $\{\rho_0, \rho_1\}$ subordinate to the cover $\{U_0, U_1\}$ and then let $\omega := (-(\rho_1\eta)^{U_0}, (\rho_0\eta)^{U_1})$ where we have extended $\rho_1|_{U_0 \cap U_1}\eta$ by zero to a smooth function $(\rho_1\eta)^{U_0}$ on U_0 and $\rho_0|_{U_0 \cap U_1}\eta$ to a function $(\rho_0\eta)^{U_1}$ on U_1 (think about this). Now we have

$$\begin{aligned}
&\partial^*(-(\rho_1\eta)^{U_0}, (\rho_0\eta)^{U_1}) \\
&= ((\rho_0\eta)^{U_1}|_{U_0 \cap U_1} + (\rho_1\eta)^{U_0}|_{U_0 \cap U_1}) \\
&= \rho_0\eta|_{U_0 \cap U_1} + \rho_1\eta|_{U_0 \cap U_1} \\
&= (\rho_0 + \rho_1)\eta = \eta.
\end{aligned}$$

Perhaps the notation is too pedantic. If we let the restrictions and extensions by zero take care of themselves, so to speak, then the idea is expressed by saying that ∂^* maps $(-\rho_1\eta, \rho_0\eta) \in \Omega(U_0) \oplus \Omega(U_1)$ to $\rho_0\eta - (-\rho_1\eta) = \eta \in \Omega(U_0 \cap U_1)$. Thus we see that ∂^* is surjective.

It is easy to see that $\partial^* \circ \iota^* = 0$ so that $\mathrm{im}(\partial^*) \subset \ker(\iota^*)$. Finally, let $(\alpha, \beta) \in \Omega(U_0) \oplus \Omega(U_1)$ and suppose that $\partial^*(\alpha, \beta) = (0, 0)$. This translates to $\alpha|_{U_0 \cap U_1} = \beta|_{U_0 \cap U_1}$ which means that there is a form $\omega \in \Omega(U_0 \cup U_1) = \Omega(M)$ such that ω coincides with α on U_0 and with β on U_0. Thus

$$\begin{aligned}
\iota^*\omega &= (\iota_0^*\omega, \iota_1^*\omega) \\
&= (\alpha, \beta)
\end{aligned}$$

so that $\ker(\iota^*) \subset \mathrm{im}(\partial^*)$ that together with the reverse inclusion gives $\mathrm{im}(\partial^*) = \ker(\iota^*)$.

Following the general algebraic pattern, the Mayer-Vietoris short exact sequence gives rise to the **Mayer-Vietoris long exact sequence**:

$$\begin{tikzcd}
H^{k+1}(M) \arrow[r] & H^{k+1}(U) \oplus H^{k+1}(V) \arrow[r, "\partial"] & H^{k+1}(U \cap V) \\
H^k(M) \arrow[r] & H^k(U) \oplus H^k(V) \arrow[r, "\partial"] & H^k(U \cap V) \arrow[ull, bend right, "d^*"']
\end{tikzcd}$$

Since our description of the coboundary map in the algebraic case was rather abstract we will do well to take a closer look at d^* in the present context. Referring to the diagram below, $\omega \in \Omega^k(U \cap V)$ represents a cohomology class $[\omega] \in H^k(U \cap V)$ so that in particular $d\omega = 0$.

$$\begin{array}{ccccccccc} & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \Omega^{k+1}(M) & \to & \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) & \to & \Omega^{k+1}(U \cap V) & \to 0 \\ & & d\uparrow & & d\uparrow & & d\uparrow & & \\ 0 & \to & \Omega^k(M) & \to & \Omega^k(U) \oplus \Omega^k(V) & \to & \Omega^k(U \cap V) & \to 0 \end{array}$$

By exactness of the rows we can find a form $\xi \in \Omega^k(U) \oplus \Omega^k(V)$ which maps to ω. In fact, we may take $\xi = (-\rho_V \omega, \rho_U \omega)$ as before. Now since $d\omega = 0$ and the diagram commutes, we see that $d\xi$ must map to 0 in $\Omega^{k+1}(U \cap V)$. This just tells us that $-d(\rho_V \omega)$ and $d(\rho_U \omega)$ agree on the intersection $U \cap V$. Thus there is a well defined form in $\Omega^{k+1}(M)$ which maps to $d\xi$. This global form η is given by

$$\eta = \begin{cases} -d(\rho_V \omega) & \text{on } U \\ \\ d(\rho_U \omega) & \text{on } V \end{cases}$$

and then by definition $d^*[\omega] = [\eta] \in H^{k+1}(M)$.

Exercise 10.3 *Let the circle S^1 be parameterized by angle θ in the usual way. Let U be that part of a circle with $-\pi/4 < \theta < \pi/4$ and let V be given by $3\pi/4 < \theta < 5\pi/4$.*

(a) Show that $H(U) \cong H(V) \cong \mathbb{R}$

(b) Show that the "difference map" $H(U) \oplus H(V) \xrightarrow{\partial} H(U \cap V)$ has 1-dimensional image.

Now we come to an important lemma which provides the leverage needed to compute the cohomology of some higher dimensional manifolds based on that of lower dimensional manifolds. We start we a smooth manifold M and also form the product manifold $M \times \mathbb{R}$. We then have the obvious projection $pr_1 : M \times \mathbb{R} \to M$ and the "zero section" $s_0 : M \to M \times \mathbb{R}$ given by $x \mapsto (x, 0)$. We can apply the cohomology functor to this pair of maps. First

$$\begin{array}{ccc} M \times \mathbb{R} & & \Omega(M \times \mathbb{R}) \\ pr_1 \downarrow\uparrow s_0 & \rightsquigarrow & pr_1^* \downarrow\uparrow s_0^* \\ M & & \Omega(M) \end{array}$$

and then the maps pr_1^* and s_0^* induce maps on cohomology:

$$\begin{array}{c} H^*(M \times \mathbb{R}) \\ pr_1^* \downarrow\uparrow s_0^* \\ H^*(M) \end{array}$$

Theorem 10.1 (Poincaré Lemma) *Given M and the maps defined above we have that $pr_1^* : H^*(M \times \mathbb{R}) \to H^*(M)$ and $s_0^* : H^*(M) \to H^*(M \times \mathbb{R})$ are mutual inverses. In particular,*

$$H^*(M \times \mathbb{R}) \cong H^*(M).$$

Proof. The main idea of the proof is the use of a so called homotopy operator which in the present case is a degree -1 map $K : \Omega(M \times \mathbb{R}) \to \Omega(M \times \mathbb{R})$ with the property that $id_M - pr_1^* \circ s_0^* = \pm(d \circ K - K \circ d)$. The point is that such a map must send closed forms to closed forms and exact forms to exact forms. Thus on the level of cohomology $\pm(d \circ K - K \circ d) = 0$ and hence $id_M - pr_1^* \circ s_0^*$ must be the zero map so that in fact $id = pr_1^* \circ s_0^*$ on $H^*(M)$.

Our task is to construct K. First notice that a function $\phi(x,t)$ on $M \times \mathbb{R}$ which happens to be constant with respect to the second variable must be of the form $pr_1^* f$ for some $f \in C^\infty(M)$. Similarly, for $\alpha \in \Omega^k(M)$ we think of the form $pr_1^*\alpha$ as not depending on $t \in \mathbb{R}$. For any $\omega \in \Omega^k(M \times \mathbb{R})$ we can find a pair of functions $f_1(x,t)$ and $f_2(x,t)$ such that

$$\omega = f_1(x,t) pr_1^*\alpha + f_2(x,t) pr_1^*\beta \wedge dt$$

for some forms $\alpha, \beta \in \Omega^k(M)$. This decomposition is unique in the sense that if $f_1(x,t)pr_1^*\alpha + f_2(x,t)pr_1^*\beta \wedge dt = 0$ then $f_1(x,t)pr_1^*\alpha = 0$ and $f_2(x,t)pr_1^*\beta \wedge dt = 0$. Using this decomposition we have a well defined map

$$\Omega^k(M \times \mathbb{R}) \ni \omega \mapsto \int_0^t f_2(x,\tau) d\tau \times pr_1^*\beta \in \Omega^{k-1}(M \times \mathbb{R})$$

This map is our proposed homotopy operator K.

Let us now check that K has the required properties. It is clear from what we have said that we may check the action of K separately on forms of the type and $f_1(x,t)pr_1^*\alpha$ and the type $f_2(x,t)pr_1^*\beta \wedge dt$.

Case I (type $f_1(x,t)pr_1^*\alpha$).If $\omega = f_1(x,t)pr_1^*\alpha$ then $K\omega = 0$ and so $(d \circ K - K \circ d)\omega = -K(d\omega)$ and

$$\begin{aligned}
K(d\omega) &= K(d(f_1(x,t)pr_1^*\alpha)) \\
&= K(df_1(x,t) \wedge pr_1^*\alpha + f_1(x,t)pr_1^* d\alpha) \\
&= K(df_1(x,t) \wedge pr_1^*\alpha) = \pm K(\frac{\partial f_1}{\partial t}(x,t) pr_1^*\alpha \wedge dt) \\
&= \pm \int_0^t \frac{\partial f_1}{\partial t}(x,\tau) d\tau \times pr_1^*\alpha = \pm\{f_1(x,t) - f(x,0)\} pr_1^*\alpha
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(id_M - pr_1^* \circ s_0^*)\omega = \\
(id_M - pr_1^* \circ s_0^*) f_1(x,t) pr_1^*\alpha &= f_1(x,t)pr_1^*\alpha - f_1(x,0)pr_1^*\alpha \\
&= \pm\{f_1(x,t) - f(x,0)\} pr_1^*\alpha
\end{aligned}$$

as above. So in the present case we get $(d \circ K - K \circ d)\omega = \pm(id_M - pr_1^* \circ s_0^*)\omega$.

Case II. (type $\omega = f(x,t)dt \wedge pr_1^*\beta$) In this case

$$
\begin{aligned}
&(d \circ K - K \circ d)\omega \\
&= dK\{f(x,t)dt \wedge pr_1^*\beta\} - Kd\{f(x,t)dt \wedge pr_1^*\beta\} \\
&= d\{\int_0^t f(x,\tau)d\tau \times pr_1^*\beta\} - Kd(f(x,t)dt \wedge pr_1^*\beta) + 0 \\
&= d\{\int_0^t f(x,\tau)d\tau \times pr_1^*\beta\} - K\left\{\frac{\partial f}{\partial t}(x,t)dt \wedge pr_1^*\beta + f(x,t)pr_1^*d\beta\right\} \\
&= f(x,t)dt \wedge pr_1^*\beta + \int_0^t f(x,\tau)d\tau \times pr_1^*d\beta - pr_1^*d\beta \times \int_0^t f(x,\tau)d\tau - \int_0^t \frac{\partial f}{\partial t}dt \times pr_1^*\beta \\
&= f(x,t)pr_1^*\beta \wedge dt = \omega
\end{aligned}
$$

On the other hand, we also have $(id_M - pr_1^* \circ s_0^*)\omega = \omega$ since $s_0^*dt = 0$.
This concludes the proof. ■

Corollary 10.1

$$H^k(point) = H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & if\ k = 0 \\ 0 & otherwise \end{cases}$$

Proof. One first verifies that the statement is true for $H^k(point)$. Then the remainder of the proof is a simple induction:

$$
\begin{aligned}
H^k(point) &\cong H^k(point \times \mathbb{R}) = H^k(\mathbb{R}) \\
&\cong H^k(\mathbb{R} \times \mathbb{R}) = H^k(\mathbb{R}^2) \\
&\cong ..... \\
&\cong H^k(\mathbb{R}^{n-1} \times \mathbb{R}) = H^k(\mathbb{R}^n)
\end{aligned}
$$

■

Corollary 10.2 (Homotopy invariance) *If $f : M \to N$ and $g : M \to N$ are homotopic then the induced maps $f^* : H^*(N) \to H^*(M)$ and $g^* : H^*(N) \to H^*(M)$ are equal.*

Proof. By extending the homotopies in a trivial way we may assume that we have a map $F : M \times \mathbb{R} \to N$ such that

$$
\begin{aligned}
F(x,t) &= f(x) \text{ for } t \geq 1 \\
F(x,t) &= g(x) \text{ for } t \leq 0.
\end{aligned}
$$

If $s_1(x) := (x,1)$ and $s_0(x) := (x,0)$ then $f = F \circ s_1$ and $g = F \circ s_0$ and so

$$
\begin{aligned}
f^* &= s_1^* \circ F^* \\
g^* &= s_0^* \circ F^*.
\end{aligned}
$$

It is easy to check that s_1^* and s_0^* are one sided inverses of pr_1^*. But we have shown that pr_1^* is an isomorphism. It follows that $s_1^* = s_0^*$ and then from above we have $f^* = g^*$. ∎

Homotopy plays a central role in algebraic topology and so this last result is very important. Recall that if there exist maps $f : M \to N$ and $g : N \to M$ such that both $f \circ g$ and $g \circ f$ are defined and homotopic to id_N and id_M respectively then f (or g) is called a homotopy equivalence and M and N are said to have the same homotopy type. In particular, if a topological space has the same homotopy type as a single point then we say that the space is contractible. If we are dealing with smooth manifolds we may take the maps to be smooth. In fact, any continuous map between two manifolds is continuously homotopic to a smooth map. We shall use this fact often without mention. The following corollary follows easily:

Corollary 10.3 *If M and N are smooth manifolds which are of the same homotopy type then $H^*(M) \cong H^*(N)$.*

Next consider the situation where A is a subset of M and $i : A \hookrightarrow M$ is the inclusion map. If there exist a map $r : M \to A$ such that $r \circ i = id_A$ then we say that r is a retraction of M onto A. If A is a submanifold of a smooth manifold M then if there is a retraction r of M onto A we may assume that r is smooth. If we can find a retraction r such that $i \circ r$ is homotopic to the identity id_M then we say that r is a deformation retraction. The following exercises show the usefulness of these ideas.

Exercise 10.4 *Let U_+ and U_- be open subsets of the sphere $S^n \subset \mathbb{R}^n$ given by*

$$\begin{aligned} U_+ &:= \{(x^i) \in S^n : -\varepsilon < x^{n+1} \leq 1 \\ U_- &:= \{(x^i) \in S^n : -1 \leq x^{n+1} < \varepsilon \end{aligned}$$

where $0 < \varepsilon < 1/2$. Show that there is a deformation retraction of $U_+ \cap U_-$ onto the equator $x^{n+1} = 0$ in S^n. Notice that the equator is a two point set in case $n = 0$.

Exercise 10.5 *Use the last exercise and the long Meyer-Vietoris sequence to show that*

$$H^k(S^n) = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } n \\ 0 & \text{otherwise} \end{cases}$$

10.1.1 Compactly Supported Cohomology

Let $\Omega_c(M)$ denote the algebra of compactly supported differential forms on a manifold M. Obviously, if M is compact then $\Omega_c(M) = \Omega(M)$ so our main interest here is the case where M is not compact. We now have a slightly different complex

$$\cdots \xrightarrow{d} \Omega_c^k(M) \xrightarrow{d} \Omega_c^{k+1}(M) \xrightarrow{d} \cdots$$

which has a corresponding cohomology $H_c^*(M)$ called the **de Rham cohomology with compact support**. Now if we look at the behavior of differential forms under the operation of pull-back we immediately realize that the pull-back of a differential form with compact support may not have compact support. In order to get desired functorial properties we consider the class of smooth maps called proper maps Recall that a smooth map $f : P \to M$ is called a **proper map** if $f^{-1}(K)$ is compact whenever K is compact.

It is easy to verify that the set of all smooth manifolds together with proper smooth maps is a category and the assignments $M \mapsto \Omega_c(M)$ and $f \mapsto \{\alpha \mapsto f^*\alpha\}$ is a contravariant functor. In plain language this means that if $f : P \to M$ is a proper map then $f^* : \Omega_c(M) \to \Omega_c(P)$ and for two such maps we have $(f \circ g)^* = g^* \circ f^*$ as before but the assumption that f and g are proper maps is essential.

We can also achieve functorial behavior by using the assignment $M \mapsto \Omega_c(M)$. The first thing we need is a new category (which is fortunately easy to describe). The category we have in mind has a objects the set of all open subsets of a fixed manifold M. The morphisms are inclusion maps $V \overset{j_{VU}}{\hookrightarrow} U$ which are only defined in case $V \subset U$. Now for any such inclusion j_{VU} we define a map $(j_{V,U})_* : \Omega_c(V) \to \Omega_c(U)$ according to the following simple prescription: For any $\alpha \in \Omega_c(V)$ let $(j_{V,U})_* \alpha$ be the form in $\Omega_c(U)$ which is equal to α at all points in V and equal to zero otherwise (this is referred to as extension by zero). Since the support of α is neatly inside the open set V we can see that the extension $(j_{V,U})_* \alpha$ is perfectly smooth.

We warn the reader that what we are about to describe is much simpler than the notation which describes it and a little doodling while reading might be helpful. If U and V are open subsets of M with nonempty intersection then we have inclusions $j_{V\cap U,U} : V \cap U \to U$ and $j_{V\cap U,V} : V \cap U \to V$ as well as the inclusions $j_{V,M} : V \to M$ and $j_{U,M} : U \to M$. Now if $U \sqcup V$ is the disjoint union of U and V, then we have inclusions $V \to U \sqcup V$ and $U \to U \sqcup V$ and after composing in the obvious way we get two different maps $j_1 : V \cap U \hookrightarrow U \sqcup V$ and $j_2 : V \cap U \hookrightarrow U \sqcup V$. Also, using $j_{V,M} : V \to M$ and $j_{U,M} : U \to M$ we get another obvious map $U \sqcup V \to M$ (which is not exactly an inclusion). Following [Bo Tu] we denote this situation as

$$M \longleftarrow U \sqcup V \leftleftarrows V \cap U. \tag{10.1}$$

Now let us define a map $\delta : \Omega_c(V\cap U) \to \Omega_c(V) \oplus \Omega_c(U)$ by $\alpha \mapsto (-j_{V\cap U,U*}\alpha, j_{V\cap U,V*}\alpha)$ which we can see as arising from $U \sqcup V \leftleftarrows V \cap U$. If we also consider the map $\Omega_c(V) \oplus \Omega_c(U) \to \Omega_c(M)$ which sums: $(\alpha, \beta) \mapsto \alpha + \beta$ then we can associate to the sequence 10.1 above, the new sequence

$$0 \leftarrow \Omega_c(M) \overset{sum}{\longleftarrow} \Omega_c(V) \oplus \Omega_c(U) \overset{\delta}{\longleftarrow} \Omega_c(V \cap U) \leftarrow 0$$

This is the (short) Mayer-Vietoris sequence for differential forms with compact support.

Theorem 10.2 *The sequence 10.1.1 above is exact.*

The proof is not hard but in the interest of saving space we simply refer the reader to the book [Bo Tu].

Corollary 10.4 *There is a long exact sequence*

$$\begin{tikzcd}
H_c^{k+1}(M) \arrow[r] & H_c^{k+1}(U) \oplus H_c^{k+1}(V) \arrow[r, "\partial"] & H_c^{k+1}(U \cap V) \\
H_c^{k}(M) \arrow[r] & H_c^{k}(U) \oplus H_c^{k}(V) \arrow[r, "\partial"] & H_c^{k}(U \cap V) \arrow[ull, bend right, "d^*"']
\end{tikzcd}$$

which is called the (long) Mayer-Vietoris sequence for cohomology with compact supports.

Chapter 11

Complex Manifolds

11.1 Some complex linear algebra

The set of all n-tuples of complex $\mathbb{C}^n$ numbers is a complex vector space and by choice of a basis, every complex vector space of finite dimension (over $\mathbb{C}$) is linearly isomorphic to $\mathbb{C}^n$ for some n. Now multiplication by $i := \sqrt{-1}$ is a complex linear map $\mathbb{C}^n \to \mathbb{C}^n$ and since $\mathbb{C}^n$ is also a real vector space $\mathbb{R}^{2n}$ under the identification

$$(x^1 + iy^1, ..., x^n + iy^n) \rightleftharpoons (x^1, y^1, ..., x^n, y^n)$$

we obtain multiplication by i as a real linear map $J_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given by the matrix

$$\begin{bmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{bmatrix}.$$

Conversely, if V is a real vector space of dimension $2n$ and there is a map $J : \mathrm{V} \to \mathrm{V}$ with $J^2 = -1$ then we can define the structure of a complex vector space on V by defining the scalar multiplication by complex numbers via the formula

$$(x + \mathrm{i}y)v := xv + yJv \text{ for } v \in \mathrm{V}.$$

Denote this complex vector space by V_J. Now if $e_1, ...e_n$ is a basis for V_J (over $\mathbb{C}$) then we claim that $e_1, ..., e_n, Je_1, ..., Je_n$ is a basis for V over $\mathbb{R}$. We only need to show that $e_1, ..., e_n, Je_1, ..., Je_n$ span. For this let $v \in \mathrm{V}$. Then for some complex numbers $c^i = a^i + \mathrm{i}b^j$ we have $v = \sum c^i e_i = \sum (a^j + \mathrm{i}b^j) e_j = \sum a^j e_j + \sum b^j J e_j$ which is what we want.

Next we consider the complexification of V which is $\mathrm{V}_{\mathbb{C}} := \mathbb{C}\otimes\mathrm{V}$. Now any real basis $\{f_j\}$ of V is also a basis for $\mathrm{V}_{\mathbb{C}}$ if we identify f_j with $1\otimes f_j$. Furthermore, the linear map $J : \mathrm{V} \to \mathrm{V}$ extends to a complex linear map $J : \mathrm{V}_{\mathbb{C}} \to \mathrm{V}_{\mathbb{C}}$ and still satisfies $J^2 = -1$. Thus this extension has eigenvalues i and $-$i. Let $\mathrm{V}^{1,0}$ be the eigenspace for i and let $\mathrm{V}^{0,1}$ be the $-$i eigenspace. Of course we must have $\mathrm{V}_{\mathbb{C}} = \mathrm{V}^{1,0} \oplus \mathrm{V}^{0,1}$. The reader may check that the set of vectors $\{e_1 - \mathrm{i}Je_1, ..., e_n - \mathrm{i}Je_n\}$ span $\mathrm{V}^{1,0}$ while $\{e_1 + \mathrm{i}Je_1, ..., e_n + \mathrm{i}Je_n\}$ span $\mathrm{V}^{0,1}$. Thus we have a convenient basis for $\mathrm{V}_{\mathbb{C}} = \mathrm{V}^{1,0} \oplus \mathrm{V}^{0,1}$.

Lemma 11.1 *There is a natural complex linear isomorphism* $\mathrm{V}_J \cong \mathrm{V}^{1,0}$ *given by* $e_i \mapsto e_i - \mathrm{i}Je_i$. *Furthermore, the conjugation map on* $\mathrm{V}_{\mathbb{C}}$ *interchanges the spaces* $\mathrm{V}^{1,0}$ *and* $\mathrm{V}^{0,1}$.

Let us apply these considerations to the simple case of the complex plane $\mathbb{C}$. The realification is $\mathbb{R}^2$ and the map J is

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

If we identify the tangent space of $\mathbb{R}^{2n}$ at 0 with $\mathbb{R}^{2n}$ itself then $\{ \frac{\partial}{\partial x^i}\big|_0, \frac{\partial}{\partial y^i}\big|_0 \}_{1\leq i\leq n}$ is a basis for $\mathbb{R}^{2n}$. A complex basis for $\mathbb{R}^2_J \cong \mathbb{C}$ is $e_1 = \frac{\partial}{\partial x}\big|_0$ and so $\frac{\partial}{\partial x}\big|_0, J\frac{\partial}{\partial x}\big|_0$ provides a basis for $\mathbb{R}^2$. This is clear anyway since $J\frac{\partial}{\partial x}\big|_0 = \frac{\partial}{\partial y}\big|_0$. Now the complexification of $\mathbb{R}^2$ is $\mathbb{R}^2_{\mathbb{C}}$ which has basis consisting of $e_1 - \mathrm{i}Je_1 = \frac{\partial}{\partial x}\big|_0 - \mathrm{i}\frac{\partial}{\partial y}\big|_0$ and $e_1 + \mathrm{i}Je_1 = \frac{\partial}{\partial x}\big|_0 + \mathrm{i}\frac{\partial}{\partial y}\big|_0$. Multiplying these basis vector by 1/2 gives basis vectors which are usually denoted by $\frac{\partial}{\partial z}\big|_0$ and $\frac{\partial}{\partial \bar{z}}\big|_0$. More generally, we see that if $\mathbb{C}^n$ is realified to $\mathbb{R}^{2n}$ which is then complexified to $\mathbb{R}^{2n}_{\mathbb{C}} := \mathbb{C}\otimes\mathbb{R}^{2n}$ then a basis for $\mathbb{R}^{2n}_{\mathbb{C}}$ is given by

$$\{ \frac{\partial}{\partial z^1}\bigg|_0, ..., \frac{\partial}{\partial z^n}\bigg|_0, \frac{\partial}{\partial \bar{z}^1}\bigg|_0 ..., \frac{\partial}{\partial \bar{z}^n}\bigg|_0 \}$$

where

$$\frac{\partial}{\partial z^i}\bigg|_0 := \frac{1}{2}\left(\frac{\partial}{\partial x^i}\bigg|_0 - \mathrm{i}\,\frac{\partial}{\partial y^i}\bigg|_0 \right)$$

and

$$\frac{\partial}{\partial \bar{z}^i}\bigg|_0 := \frac{1}{2}\left(\frac{\partial}{\partial x^i}\bigg|_0 + \mathrm{i}\,\frac{\partial}{\partial y^i}\bigg|_0 \right).$$

Now if we consider the tangent bundle $U \times \mathbb{R}^{2n}$ of an open set $U \subset \mathbb{R}^{2n}$ then we have the basis vector fields $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}$. We can complexify the tangent bundle of $U \times \mathbb{R}^{2n}$ to get $U \times \mathbb{R}^{2n}_{\mathbb{C}}$ and then following the ideas above we have that the fields $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}$ also span each complexified tangent space $T_pU := \{p\} \times \mathbb{R}^{2n}_{\mathbb{C}}$. On the other hand, so do the fields $\{\frac{\partial}{\partial z^1}, .., \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1}, .., \frac{\partial}{\partial \bar{z}^n}\}$. Now if $\mathbb{R}^{2n}$ had some complex structure, say $\mathbb{C}^n \cong (\mathbb{R}^{2n}, J_0)$, then J_0 defines a bundle map given $J_0 : TU \to TU$ given by $(p, v) \mapsto (p, J_0 v)$. This can be extended to a complex bundle map $J_0 : TU_{\mathbb{C}} = \mathbb{C}\otimes TU \to TU_{\mathbb{C}} = \mathbb{C}\otimes TU$ and we get a bundle decomposition

$$TU_{\mathbb{C}} = T^{1.0}U \oplus T^{0.1}U$$

where $\frac{\partial}{\partial z^1}, .., \frac{\partial}{\partial z^n}$ spans $T^{1.0}U$ at each point and $\frac{\partial}{\partial \bar{z}^1}, .., \frac{\partial}{\partial \bar{z}^n}$ spans $T^{0.1}U$.

Now the symbols $\frac{\partial}{\partial z^1}$ etc., already have meaning as differential operators. Let us now show that this view is at least consistent with what we have done above. For a smooth complex valued function $f : U \subset \mathbb{C}^n \to \mathbb{C}$ we have for $p = (z_1, ..., z_n) \in U$

$$\begin{aligned}
\left.\frac{\partial}{\partial z^i}\right|_p f &= \frac{1}{2}\left(\left.\frac{\partial}{\partial x^i}\right|_p f - \mathrm{i} \left.\frac{\partial}{\partial y^i}\right|_p f\right) \\
&= \frac{1}{2}\left(\left.\frac{\partial}{\partial x^i}\right|_p u - \mathrm{i} \left.\frac{\partial}{\partial y^i}\right|_p u - \left.\frac{\partial}{\partial x^i}\right|_p \mathrm{i}v - \mathrm{i} \left.\frac{\partial}{\partial y^i}\right|_p \mathrm{i}v\right) \\
&\frac{1}{2}\left(\left.\frac{\partial u}{\partial x^i}\right|_p + \left.\frac{\partial v}{\partial y^i}\right|_p\right) + \frac{\mathrm{i}}{2}\left(\left.\frac{\partial u}{\partial y^i}\right|_p - \left.\frac{\partial v}{\partial x^i}\right|_p\right).
\end{aligned}$$

and

$$\begin{aligned}
\left.\frac{\partial}{\partial \bar{z}^i}\right|_p f &= \frac{1}{2}\left(\left.\frac{\partial}{\partial x^i}\right|_p f + \mathrm{i} \left.\frac{\partial}{\partial y^i}\right|_p f\right) \\
&= \frac{1}{2}\left(\left.\frac{\partial}{\partial x^i}\right|_p u + \mathrm{i} \left.\frac{\partial}{\partial y^i}\right|_p u + \left.\frac{\partial}{\partial x^i}\right|_p \mathrm{i}v + \mathrm{i} \left.\frac{\partial}{\partial y^i}\right|_p \mathrm{i}v\right) \\
&= \frac{1}{2}\left(\left.\frac{\partial u}{\partial x^i}\right|_p - \left.\frac{\partial v}{\partial y^i}\right|_p\right) + \frac{\mathrm{i}}{2}\left(\left.\frac{\partial u}{\partial y^i}\right|_p + \left.\frac{\partial v}{\partial x^i}\right|_p\right).
\end{aligned}$$

Definition 11.1 *A function $f : U \subset \mathbb{C}^n \to \mathbb{C}$ is called **holomorphic** if*

$$\frac{\partial}{\partial \bar{z}^i} f \equiv 0 \qquad \text{(all } i\text{)}$$

*on U. A function f is called **antiholomorphic** if*

$$\frac{\partial}{\partial z^i} f \equiv 0 \qquad \text{(all } i\text{).}$$

Definition 11.2 *A map $f : U \subset \mathbb{C}^n \to \mathbb{C}^m$ given by functions $f_1, ..., f_m$ is called **holomorphic** (resp. **antiholomorphic**) if each component function $f_1, ..., f_m$ is holomorphic (resp. antiholomorphic).*

Now if $f : U \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic then by definition $\left.\frac{\partial}{\partial \bar{z}^i}\right|_p f \equiv 0$ for all $p \in U$ and so we have the **Cauchy-Riemann equations**

$$\begin{aligned}
\frac{\partial u}{\partial x^i} &= \frac{\partial v}{\partial y^i} \\
\frac{\partial v}{\partial x^i} &= -\frac{\partial u}{\partial y^i}
\end{aligned} \qquad \text{(Cauchy-Riemann)}$$

and from this we see that for holomorphic f

$$\begin{aligned} &\frac{\partial f}{\partial z^i} \\ &= \frac{\partial u}{\partial x^i} + \mathrm{i}\frac{\partial v}{\partial x^i} \\ &= \frac{\partial f}{\partial x^i} \end{aligned}$$

which means that as derivations on the sheaf $\mathcal{O}$ of locally defined holomorphic functions on $\mathbb{C}^n$, the operators $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial x^i}$ are equal. This corresponds to the complex isomorphism $T^{1.0}U \cong (TU, J_0)$ which comes from the isomorphism in lemma **??**. In fact, if one looks at a function $f : \mathbb{R}^{2n} \to \mathbb{C}$ as a differentiable map of real manifolds then with J_0 giving the isomorphism $\mathbb{R}^{2n} \cong \mathbb{C}^n$, our map f is holomorphic if and only if

$$Tf \circ J_0 = J_0 \circ Tf$$

or in other words

$$\begin{pmatrix} \frac{\partial u}{\partial x^1} & \frac{\partial u}{\partial y^1} & \\ \frac{\partial v}{\partial x^1} & \frac{\partial v}{\partial y^1} & \\ & & \ddots \end{pmatrix} \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & \ddots \end{pmatrix} = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & \ddots \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x^1} & \frac{\partial u}{\partial y^1} & \\ \frac{\partial v}{\partial x^1} & \frac{\partial v}{\partial y^1} & \\ & & \ddots \end{pmatrix}.$$

This last matrix equation is just the Cauchy-Riemann equations again.

11.2 Complex structure

Definition 11.3 *A manifold M is said to be an* ***almost complex manifold*** *if there is a smooth bundle map $J : TM \to TM$, called an* ***almost complex structure*** *, having the property that $J^2 = -1$.*

Definition 11.4 *A* ***complex manifold*** *M is a manifold modeled on $\mathbb{C}^n$ for some n, together with an atlas for M such that the transition functions are all holomorphic maps. The charts from this atlas are called* ***holomorphic charts****. We also use the phrase "****holomorphic coordinates****".*

Since as real normed vector spaces we have $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and since homomorphic maps are always smooth we see that underlying every complex manifold of dimension n there is a real manifold of dimension $2n$. Furthermore, we will see that each (real) tangent space of the underlying real manifold of a complex manifold has a natural almost complex structure. Those almost complex structures arising in this way are also called complex structures. Not all almost complex structures arise in this way (there is an integrability condition).

Example 11.1 *Let $S^2(1/2) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1/4\}$ be given coordinates $\psi^+ : (x_1, x_2, x_3) \mapsto \frac{1}{1-x_3}(x_1 + \mathrm{i}x_2) \in \mathbb{C}$ on $U^+ :=$*

$\{(x_1, x_2, x_3) \in S^2 : 1 - x_3 \neq 0\}$ and $\psi^- : (x_1, x_2, x_3) \mapsto \frac{1}{1+x_3}(x_1 + \mathrm{i}x_2) \in \mathbb{C}$ on $U^- := \{(x_1, x_2, x_3) \in S^2 : 1 + x_3 \neq 0\}$. The chart overlap map or transition function is $\psi^- \circ \psi^+(z) = 1/z$. Since on $\psi^+U^+ \cap \psi^-U^-$ the map $z \mapsto 1/z$ is a biholomorphism we see that $S^2(1/2)$ can be given the structure of a complex 1-manifold.

*Another way to get the same complex 1-manifold is given by taking two copies of the complex plane, say $\mathbb{C}_z$ with coordinate z and $\mathbb{C}_w$ with coordinate w and then identify $\mathbb{C}_z$ with $\mathbb{C}_w - \{0\}$ via the map $w = 1/z$. This complex surface is of course topologically a sphere and is also the 1 point compactification of the complex plane. As the reader will no doubt already be aware, this complex 1-manifold is called the **Riemann sphere**.*

Example 11.2 *Let $P_n(\mathbb{C})$ be the set of all complex lines through the origin in $\mathbb{C}^{n+1}$, which is to say, the set of all equivalence classes of nonzero elements of $\mathbb{C}^{n+1}$ under the equivalence relation*

$$(z^1, ..., z^{n+1}) \sim \lambda(z^1, ..., z^{n+1}) \text{ for } \lambda \in \mathbb{C}$$

For each i with $1 \leq i \leq n+1$ define the set

$$U_i := \{[z^1, ..., z^{n+1}] \in P_n(\mathbb{C}) : z^i \neq 0\}$$

and corresponding map $\psi_i : U_i \to \mathbb{C}^n$ by

$$\psi_i([z^1, ..., z^{n+1}]) = \frac{1}{z^i}(z^1, ..., \widehat{z^i}, ..., z^{n+1}) \in \mathbb{C}^n.$$

*One can check that these maps provide a holomorphic atlas for $P_n(\mathbb{C})$ which is therefore a complex manifold (**complex projective n-space**).*

Example 11.3 *Let $M_{m\times n}(\mathbb{C})$ be the space of $m \times n$ complex matrices. This is clearly a complex manifold since we can always "line up" the entries to get a map $M_{m\times n}(\mathbb{C}) \to \mathbb{C}^{mn}$ and so as complex manifolds $M_{m\times n}(\mathbb{C}) \cong \mathbb{C}^{mn}$. A little less trivially we have the complex general linear group $GL(n, \mathbb{C})$ which is an open subset of $M_{m\times n}(\mathbb{C})$ and so is an n^2 dimensional complex manifold.*

Example 11.4 (Grassmann manifold) *To describe this important example we start with the set $(M_{n\times k}(\mathbb{C}))_*$ of $n \times k$ matrices with rank $k < n$ (maximal rank). The columns of each matrix from $(M_{n\times k}(\mathbb{C}))_*$ span a k-dimensional subspace of $\mathbb{C}^n$. Define two matrices from $(M_{n\times k}(\mathbb{C}))_*$ to be equivalent if they span the same k-dimensional subspace. Thus the set $G(k, n)$ of equivalence classes is in one to one correspondence with the set of complex k dimensional subspaces of $\mathbb{C}^n$. Now let U be the set of all $[A] \in G(k, n)$ such that A has its first k rows linearly independent. This property is independent of the representative A of the equivalence class $[A]$ and so U is a well defined set. This last fact is easily proven by a Gaussian reduction argument. Now every element $[A] \in U \subset G(k, n)$ is an equivalence class that has a unique member A_0 of the form*

$$\begin{pmatrix} I_{k\times k} \\ Z \end{pmatrix}.$$

Thus we have a map on U defined by $\Psi : [A] \mapsto Z \in M_{n-k\times k}(\mathbb{C}) \cong \mathbb{C}^{k(n-k)}$. We wish to cover $G(k,n)$ with sets U_σ similar to U and defined similar maps. Let $\sigma_{i_1 \ldots i_k}$ be the shuffle permutation that puts the k columns indexed by $i_1, ..., i_k$ into the positions $1, ..., k$ without changing the relative order of the remaining columns. Now consider the set $U_{i_1 \ldots i_k}$ of all $[A] \in G(k,n)$ such that any representative A has its k rows indexed by $i_1, ..., i_k$ linearly independent. The permutation induces an obvious 1-1 onto map $\widetilde{\sigma_{i_1 \ldots i_k}}$ from $U_{i_1 \ldots i_k}$ onto $U = U_{1\ldots k}$. We now have maps $\Psi_{i_1 \ldots i_k} : U_{i_1 \ldots i_k} \to M_{n-k\times k}(\mathbb{C}) \cong \mathbb{C}^{k(n-k)}$ given by composition $\Psi_{i_1 \ldots i_k} := \Psi \circ \widetilde{\sigma_{i_1 \ldots i_k}}$. These maps form an atlas $\{\Psi_{i_1 \ldots i_k}, U_{i_1 \ldots i_k}\}$ for $G(k,n)$ that turns out to be a holomorphic atlas (biholomorphic transition maps) and so gives $G(k,n)$ the structure of a complex manifold called the ***Grassmann manifold*** *of complex k-planes in $\mathbb{C}^n$.*

Definition 11.5 *A complex 1-manifold (so real dimension is 2) is called a* ***Riemann surface****.*

If S is a subset of a complex manifold M such that near each $p_0 \in S$ there exists a holomorphic chart $U, \psi = (z^1, ..., z^n)$ such that $0 \in S \cap U$ if and only if $z^{k+1}(p) = \cdots = z^n(p) = 0$ then the coordinates $z^1, ..., z^k$ restricted to $U \cap S$ give a chart on the set S and the set of all such charts gives S the structure of a complex manifold. In this case we call S a **complex submanifold** of M.

Definition 11.6 *In the same way as we defined differentiability for real manifolds we define the notion of a* ***holomorphic map*** *(resp.* ***antiholomorphic map****) from one complex manifold to another. Note however, that we must use holomorphic charts for the definition.*

The proof of the following lemma is straightforward.

Lemma 11.2 *Let $\mathsf{z} : U \to \mathbb{C}^n$ be a holomorphic chart with $p \in U$. Writing $\mathsf{z} = (z^1, ..., z^n)$ and $z^k = x^k + \mathrm{i} y^k$ we have that the map $J_p : T_pM \to T_pM$ given by*

$$J_p \left. \frac{\partial}{\partial x^i} \right|_p = \left. \frac{\partial}{\partial y^i} \right|_p$$

$$J_p \left. \frac{\partial}{\partial y^i} \right|_p = - \left. \frac{\partial}{\partial x^i} \right|_p$$

is well defined independently of the choice of coordinates.

The maps J_p combine to give a bundle map $J : TM \to TM$ and so we get an almost complex structure on M called the almost complex structure *induced* by the holomorphic atlas.

Definition 11.7 *An almost complex structure J on M is said to be* ***integrable*** *if there is an holomorphic atlas giving the map J as the induced almost complex structure. That is, if there is a family of admissible charts $\mathtt{x}_\alpha : U_\alpha \to \mathbb{R}^{2n}$ such that after identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ the charts form a holomorphic atlas with J the induced almost complex structure. In this case, we call J a* ***complex structure****.*

11.3 Complex Tangent Structures

Let $\mathcal{F}_p(\mathbb{C})$ denote the algebra of germs of *complex* valued smooth functions at p on a complex n-manifold M thought of as a smooth real $2n-$manifold with real tangent bundle TM. Let $\mathrm{Der}_p(\mathcal{F})$ be the space of derivations this algebra. It is not hard to see that this space is isomorphic to the complexified tangent space $T_pM_{\mathbb{C}} = \mathbb{C}\otimes T_pM$. The (complex) algebra of germs of holomorphic functions at a point p in a complex manifold is denoted $\mathcal{O}_p$ and the set of derivations of this algebra denoted $\mathrm{Der}_p(\mathcal{O})$. We also have the algebra of germs of antiholomorphic functions at p which is $\overline{\mathcal{O}}_p$ and also $\mathrm{Der}_p(\overline{\mathcal{O}})$, the derivations of this algebra.

If $\psi : U \to \mathbb{C}^n$ is a holomorphic chart then writing $\psi = (z^1, ..., z^n)$ and $z^k = x^k + \mathrm{i}y^k$ we have the differential operators at $p \in U$:

$$\left\{ \left.\frac{\partial}{\partial z^i}\right|_p, \left.\frac{\partial}{\partial \bar{z}^i}\right|_p \right\}$$

(now transferred to the manifold). To be pedantic about it, we now denote the standard complex coordinates on $\mathbb{C}^n$ by $w_i = u_i + \mathrm{i}v_i$ and then

$$\begin{aligned} \left.\frac{\partial}{\partial z^i}\right|_p f &:= \left.\frac{\partial f \circ \psi^{-1}}{\partial w^i}\right|_{\psi(p)} \\ \left.\frac{\partial}{\partial \bar{z}^i}\right|_p f &:= \left.\frac{\partial f \circ \psi^{-1}}{\partial \bar{w}^i}\right|_{\psi(p)} \end{aligned}$$

Thought of as derivations, these span $\mathrm{Der}_p(\mathcal{F})$ but we have also seen that they span the complexified tangent space at p. In fact, we have the following:

$$\begin{aligned} T_pM_{\mathbb{C}} &= \mathrm{span}_{\mathbb{C}} \left\{ \left.\frac{\partial}{\partial z^i}\right|_p, \left.\frac{\partial}{\partial \bar{z}^i}\right|_p \right\} = \mathrm{Der}_p(\mathcal{F}) \\ T_pM^{1,0} &= \mathrm{span}_{\mathbb{C}} \left\{ \left.\frac{\partial}{\partial z^i}\right|_p \right\} \\ &= \{v \in \mathrm{Der}_p(\mathcal{F}) : vf = 0 \text{ for all } f \in \overline{\mathcal{O}}_p\} \\ T_pM^{0,1} &= \mathrm{span}_{\mathbb{C}} \left\{ \left.\frac{\partial}{\partial \bar{z}^i}\right|_p \right\} \\ &= \{v \in \mathrm{Der}_p(\mathcal{F}) : vf = 0 \text{ for all } f \in \mathcal{O}_p\} \end{aligned}$$

and of course

$$T_pM = \mathrm{span}_{\mathbb{R}} \left\{ \left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial y^i}\right|_p \right\}.$$

The reader should go back and check that the above statements are consistent with our definitions as long as we view the $\left.\frac{\partial}{\partial z^i}\right|_p, \left.\frac{\partial}{\partial \bar{z}^i}\right|_p$ not only as the algebraic objects constructed above but also as derivations. Also, the definitions of

$T_pM^{1,0}$ and $T_pM^{0,1}$ are independent of the holomorphic coordinates since we also have

$$T_p^{1,0}M = \ker\{J_p : T_pM \to T_pM\}$$

11.4 The holomorphic tangent map.

We leave it to the reader to verify that the constructions that we have at each tangent space globalize to give natural vector bundles $TM_{\mathbb{C}}, TM^{1,0}$ and $TM^{0,1}$ (all with M as base space).

Let M and N be complex manifolds and let $f : M \to N$ be a smooth map. The tangent map (on the underlying real manifolds) extends to a map of the complexified bundles $Tf : TM_{\mathbb{C}} \to TN_{\mathbb{C}}$. Now $TM_{\mathbb{C}} = TM^{1,0} \oplus TM^{0,1}$ and similarly $TN_{\mathbb{C}} = TN^{1,0} \oplus TN^{0,1}$. If f is holomorphic then $Tf(T_p^{1,0}M) \subset T_{f(p)}^{1,0}N$. In fact, it is easily verified that the statement that $Tf(T_pM^{1,0}) \subset T_{f(p)}N^{1,0}$ is equivalent to the statement that the Cauchy-Riemann equations are satisfied by the local representative of f in any holomorphic chart. As a result we have

Proposition 11.1 *$Tf(T_pM^{1,0}) \subset T_{f(p)}N^{1,0}$ for all p if and only if f is a holomorphic map.*

The map given by the restriction $T_pf : T_pM^{1,0} \to T_{f(p)}N^{1,0}$ is called the **holomorphic tangent** map at p. Of course, these maps combine to give a bundle map.

11.5 Dual spaces

Let M be a complex manifold and J the induced complex structure map $TM \to TM$. The dual of $T_pM_{\mathbb{C}}$ is $T_p^*M_{\mathbb{C}} = \mathbb{C} \otimes T_p^*M$. Now the map J has a dual bundle map $J^* : T^*M_{\mathbb{C}} \to T^*M_{\mathbb{C}}$ that must also satisfy $J^* \circ J^* = -1$ and so we have at each $p \in M$, a decomposition by eigenspaces

$$T_p^*M_{\mathbb{C}} = T_p^*M^{1,0} \oplus T_p^*M^{0,1}$$

corresponding to the eigenvalues $\pm\mathrm{i}$.

Definition 11.8 *The space $T_p^*M^{1,0}$ is called the space of holomorphic covectors at p while $T_p^*M^{0,1}$ is the space of antiholomorphic covectors at p.*

We now choose a holomorphic chart $\psi : U \to \mathbb{C}^n$ at p. Writing $\psi = (z^1, ..., z^n)$ and $z^k = x^k + \mathrm{i}y^k$ we have the 1-forms

$$dz^k = dx^k + \mathrm{i}dy^k$$

and

$$d\bar{z}^k = dx^k - \mathrm{i}dy^k.$$

Equivalently, the pointwise definitions are $dz^k\big|_p = dx^k\big|_p + \mathrm{i}\; dy^k\big|_p$ and $d\bar{z}^k\big|_p = dx^k\big|_p - \mathrm{i}\; dy^k\big|_p$. Notice that we have the expected relations:

$$\begin{aligned}
dz^k(\frac{\partial}{\partial z^i}) &= (dx^k + \mathrm{i}dy^k)(\frac{1}{2}\frac{\partial}{\partial x^i} - \mathrm{i}\frac{1}{2}\frac{\partial}{\partial y^i}) \\
&= \frac{1}{2}\delta^k_j + \frac{1}{2}\delta^k_j = \delta^k_j \\
dz^k(\frac{\partial}{\partial \bar{z}^i}) &= (dx^k + \mathrm{i}dy^k)(\frac{1}{2}\frac{\partial}{\partial x^i} + \mathrm{i}\frac{1}{2}\frac{\partial}{\partial y^i}) \\
&= 0 \\
&\text{and similarly} \\
d\bar{z}^k(\frac{\partial}{\partial \bar{z}^i}) &= \delta^k_j \text{ and } d\bar{z}^k(\frac{\partial}{\partial z^i}) = \delta^k_j.
\end{aligned}$$

Let us check the action of J^* on these forms:

$$\begin{aligned}
J^*(dz^k)(\frac{\partial}{\partial z^i}) &= J^*(dx^k + \mathrm{i}dy^k)(\frac{\partial}{\partial z^i}) \\
&= (dx^k + \mathrm{i}dy^k)(J\frac{\partial}{\partial z^i}) \\
&= \mathrm{i}(dx^k + \mathrm{i}dy^k)\frac{\partial}{\partial z^i} \\
&= \mathrm{i}dz^k(\frac{\partial}{\partial z^i})
\end{aligned}$$

and

$$\begin{aligned}
J^*(dz^k)(\frac{\partial}{\partial \bar{z}^i}) &= dz^k(J\frac{\partial}{\partial \bar{z}^i}) \\
&= -\mathrm{i}dz^k(\frac{\partial}{\partial \bar{z}^i}) = 0 = \\
&= \mathrm{i}dz^k(\frac{\partial}{\partial \bar{z}^i}).
\end{aligned}$$

We conclude that $dz^k\big|_p \in T^*_pM^{1,0}$. A similar calculation shows that $d\bar{z}^k\big|_p \in T^*_pM^{0,1}$ and in fact

$$\begin{aligned}
T^*_pM^{1,0} &= \operatorname{span}\left\{ dz^k\big|_p : k = 1, ..., n \right\} \\
T^*_pM^{0,1} &= \operatorname{span}\left\{ d\bar{z}^k\big|_p : k = 1, ..., n \right\}
\end{aligned}$$

and $\{ dz^1\big|_p, ..., dz^n|_p, d\bar{z}^1\big|_p, ..., d\bar{z}^n|_p \}$ is a basis for $T^*_pM_{\mathbb{C}}$.

Remark 11.1 *If we don't specify base points then we are talking about fields (over some open set) that form a basis for each fiber separately. As before these are called frame fields (e.g.* $\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}$*) or coframe fields (e.g.* $dz^k, d\bar{z}^k$*).*

11.6 Examples

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11.7 The holomorphic inverse and implicit functions theorems.

Let $(z^1, ..., z^n)$ and $(w^1, ..., w^m)$ be local coordinates on complex manifolds M and N respectively. Consider a smooth map $f : M \to N$. We suppose that $p \in M$ is in the domain of $(z^1, ..., z^n)$ and that $q = f(p)$ is in the domain of the coordinates $(w^1, ..., w^m)$. Writing $z^i = x^i + \mathrm{i}y^i$ and $w^i = u^i + \mathrm{i}v^i$ we have the following Jacobian matrices:

1. If we consider the underlying real structures then we have the Jacobian given in terms of the frame $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}$ and $\frac{\partial}{\partial u^i}, \frac{\partial}{\partial v^i}$

$$J_p(f) = \begin{bmatrix} \frac{\partial u^1}{\partial x^1}(p) & \frac{\partial u^1}{\partial y^1}(p) & \frac{\partial u^1}{\partial x^2}(p) & \frac{\partial u^1}{\partial y^2}(p) & \cdots \\ \frac{\partial v^1}{\partial x^1}(p) & \frac{\partial v^1}{\partial y^1}(p) & \frac{\partial v^1}{\partial x^2}(p) & \frac{\partial v^1}{\partial y^2}(p) & \cdots \\ \frac{\partial u^2}{\partial x^1}(p) & \frac{\partial u^2}{\partial y^1}(p) & \frac{\partial u^2}{\partial x^2}(p) & \frac{\partial u^2}{\partial y^2}(p) & \cdots \\ \frac{\partial v^2}{\partial x^1}(p) & \frac{\partial v^2}{\partial y^1}(p) & \frac{\partial v^2}{\partial x^2}(p) & \frac{\partial v^2}{\partial y^2}(p) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

2. With respect to the bases $\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}$ and $\frac{\partial}{\partial w^i}, \frac{\partial}{\partial \bar{w}^i}$ we have

$$J_{p,\mathbb{C}}(f) = \begin{bmatrix} J_{11} & J_{12} & \cdots \\ J_{12} & J_{22} & \\ \vdots & & \end{bmatrix}$$

where the J_{ij} are blocks of the form

$$\begin{bmatrix} \frac{\partial w^i}{\partial z^j} & \frac{\partial w^i}{\partial \bar{z}^j} \\ \frac{\partial \bar{w}^i}{\partial z^j} & \frac{\partial \bar{w}^i}{\partial \bar{z}^j} \end{bmatrix}.$$

If f is holomorphic then these block reduce to the form

$$\begin{bmatrix} \frac{\partial w^i}{\partial z^j} & 0 \\ 0 & \frac{\partial \bar{w}^i}{\partial \bar{z}^j} \end{bmatrix}.$$

It is convenient to put the frame fields in the order $\frac{\partial}{\partial z^1}, ..., \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1}, ..., \frac{\partial}{\partial \bar{z}^n}$ and similarly for the $\frac{\partial}{\partial w^i}, \frac{\partial}{\partial \bar{w}^i}$. In this case we have for holomorphic f

$$\mathcal{J}_{p,\mathbb{C}}(f) = \begin{bmatrix} J^{1,0} & 0 \\ 0 & \overline{J^{1,0}} \end{bmatrix}$$

where

$$J^{1,0}(f) = \left[\frac{\partial w^i}{\partial z^j}\right]$$
$$\overline{J^{1,0}}(f) = \left[\frac{\partial \bar{w}^i}{\partial \bar{z}^j}\right].$$

We shall call a basis arising from a holomorphic coordinate system "separated" when arranged this way. Note that $J^{1,0}$ is just the Jacobian of the holomorphic tangent map $T^{1,0}f : T^{1,0}M \to T^{1,0}N$ with respect to this the **holomorphic frame** $\frac{\partial}{\partial z^1}, ..., \frac{\partial}{\partial z^n}$.

We can now formulate the following version of the inverse mapping theorem:

Theorem 11.1 *(1) Let U and V be open sets in $\mathbb{C}^n$ and suppose that the map $f : U \to V$ is holomorphic with $J^{1,0}(f)$ nonsingular at $p \in U$. Then there exists an open set $U_0 \subset U$ containing p such that $f|_{U_0} : U_0 \to f(U_0)$ is a 1-1 holomorphic map with holomorphic inverse. That is, $f|_{U_0}$ is **biholomorphic**.*

(2) Similarly, if $f : U \to V$ is a holomorphic map between open sets of complex manifolds M and N then if $T_p^{1,0}f : T_p^{1,0}M \to T_{fp}^{1,0}N$ is a linear isomorphism then f is a biholomorphic map when restricted to a possibly smaller open set containing p.

We also have a holomorphic version of the implicit mapping theorem.

Theorem 11.2 *(1) Let $f : U \subset \mathbb{C}^n \to V \subset \mathbb{C}^k$ and let the component functions of f be $f_1, ..., f_k$. If $J_p^{1,0}(f)$ has rank k then there are holomorphic functions $g^1, g^2, ..., g^k$ defined near $0 \in \mathbb{C}^{n-k}$ such that*

$$f(z^1, ..., z^n) = p$$
$$\Leftrightarrow$$
$$z^j = g^j(z^{k+1}, ..., z^n) \text{ for } j = 1, .., k$$

(2) If $f : M \to N$ is a holomorphic map of complex manifolds and if for fixed $q \in N$ we have that each $p \in f^{-1}(q)$ is regular in the sense that $T_p^{1,0}f : T_p^{1,0}M \to T_q^{1,0}N$ is surjective, then $S := f^{-1}(q)$ is a complex submanifold of (complex) dimension $n - k$.

Example 11.5 *The map $\varphi : \mathbb{C}^{n+1} \to \mathbb{C}$ given by $(z^1, ..., z^{n+1}) \mapsto (z^1)^2 + \cdots + (z^{n+1})^2$ has Jacobian at any $(z^1, ..., z^{n+1})$ given by*

$$\left[\begin{array}{cccc} 2z^1 & 2z^2 & \cdots & 2z^{n+1} \end{array}\right]$$

*which has rank 1 as long as $(z^1, ..., z^{n+1}) \neq 0$. Thus $\varphi^{-1}(1)$ is a complex submanifold of $\mathbb{C}^{n+1}$ having complex dimension n. **Warning**: This manifold is not the same as the sphere given by $\left|z^1\right|^2 + \cdots + \left|z^{n+1}\right|^2 = 1$ which is a* real *submanifold of $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ of real dimension $2n + 1$.*

Chapter 12

Lie Groups and Lie Algebras

12.1 Lie Algebras

Let $\mathbb{F}$ denote one of the fields $\mathbb{R}$ or $\mathbb{C}$. In definition 2.21 we defined a real Lie algebra $\mathfrak{g}$ as a real algebra with a skew symmetric (bilinear) product called the Lie bracket, usually denoted $(v, w) \mapsto [v, w]$, such that the Jacobi identity holds

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \text{ for all } x, y, z \in \mathfrak{g}. \qquad \text{(Jacobi Identity)}$$

We also have the notion of a *complex* Lie algebra defined analogously.

Remark 12.1 *We will assume that all the Lie algebras we study are finite dimensional unless otherwise indicated.*

Let V be a finite dimensional vector space over $\mathbb{F}$ and recall that $\mathfrak{gl}(\mathrm{V},\mathbb{F})$ is the set of all $\mathbb{F}-$linear maps $\mathrm{V} \to \mathrm{V}$. The space $\mathfrak{gl}(\mathrm{V},\mathbb{F})$ is also denoted $\mathrm{Hom}_{\mathbb{F}}(\mathrm{V}, \mathrm{V})$ or $L_{\mathbb{F}}(\mathrm{V}, \mathrm{V})$ although in the context of Lie algebras we take $\mathfrak{gl}(\mathrm{V},\mathbb{F})$ as the preferred notation. We give $\mathfrak{gl}(\mathrm{V},\mathbb{F})$ its natural Lie algebra structure where the bracket is just the commutator bracket

$$[A, B] := A \circ B - B \circ A.$$

If the field involved is either irrelevant or known from context we will just write $\mathfrak{gl}(\mathrm{V})$. Also, we often identify $\mathfrak{gl}(\mathbb{F}^n)$ with the matrix Lie algebra $\mathbb{M}_{nxn}(\mathbb{F})$ with the bracket $AB - BA$.

For a Lie algebra $\mathfrak{g}$ we can associate to every basis $v_1, ..., v_n$ for $\mathfrak{g}$ the **structure constants** c_{ij}^k which are defined by

$$[v_i, v_j] = \sum_k c_{ij}^k v_k.$$

It then follows from the skew symmetry of the Lie bracket and the Jacobi identity that the structure constants satisfy

$$\begin{aligned} &\text{i)} && c_{ij}^k = -c_{ji}^k \\ &\text{ii)} && \textstyle\sum_k c_{rs}^k c_{kt}^i + c_{st}^k c_{kr}^i + c_{tr}^k c_{ks}^i = 0 \end{aligned} \tag{12.1}$$

Given a real Lie algebra $\mathfrak{g}$ we can extend it to a complex Lie algebra $\mathfrak{g}_\mathbb{C}$ by defining as $\mathfrak{g}_\mathbb{C}$ the complexification $\mathfrak{g}_\mathbb{C} := \mathbb{C} \otimes_\mathbb{R} \mathfrak{g}$ and then extending the bracket by requiring

$$[1 \otimes v, 1 \otimes w] = [v, w] \otimes 1.$$

Then $\mathfrak{g}$ can be identified with its image under the embedding map $v \mapsto 1 \otimes v$. In practice one often omits the symbol $\otimes$ and then the complexification just amounts to formally allowing complex coefficients.

Notation 12.1 *Given two subsets S_1 and S_2 of a Lie algebra $\mathfrak{g}$ we let $[\ S_1, S_2]$ denote the linear span of the set defined by $\{[x, y] : x \in S_1$ and $y \in S_2\}$. Also, let $S_1 + S_2$ denote the vector space of all $x + y : x \in S_1$ and $y \in S_2$.*

It is easy to verify that the following relations hold:

1. $[S_1 + S_2, S] \subset [S_1, S] + [S_2, S]$
2. $[S_1, S_2] = [S_2, S_1]$
3. $[S, [S_1, S_2]] \subset [[S, S_1], S_2] + [S_1, [S, S_2]]$

where S_1, S_2 and S are subsets of a Lie algebra $\mathfrak{g}$.

Definition 12.1 *A vector subspace $\mathfrak{a} \subset \mathfrak{g}$ is called a **subalgebra** if $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$ and an **ideal** if $[\mathfrak{g}, \mathfrak{a}] \subset \mathfrak{a}$.*

If $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ and $v_1, ..., v_k, v_{k+1}, ..., v_n$ is a basis for $\mathfrak{g}$ such that $v_1, ..., v_k$ is a basis for $\mathfrak{a}$ then with respect to this basis the structure constants are such that

$$c_{ij}^s = 0 \text{ if both } i, j \leq k \text{ and } s > k.$$

If $\mathfrak{a}$ is also an ideal then for any j we must have

$$c_{ij}^s = 0 \text{ when both } i \leq k \text{ and } s > k.$$

Remark 12.2 *The numbers c_{ij}^s may be viewed as the components of an element of $T_{1,1}^1(\mathfrak{g})$ (i.e. as an algebraic tensor).*

Example 12.1 *Let $\mathfrak{su}(2)$ denote the set of all traceless and skew-Hermitian 2×2 complex matrices. This is a* real *Lie algebra under the commutator bracket $(AB - BA)$. A commonly used basis for $\mathfrak{su}(2)$ is e_1, e_2, e_3 where*

$$e_1 = \frac{1}{2}\begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad e_2 = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad e_2 = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

The commutation relations satisfied by these matrices are

$$[e_i, e_j] = \epsilon_{ijk} e_k \text{ (no sum)}$$

where ϵ_{ijk} *is the totally antisymmetric symbol given by*

$$\epsilon_{ijk} := \begin{cases} 0 & \text{if } (i,j,k) \text{ is not a permutation of } (1,2,3) \\ 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \end{cases} .$$

Thus in this case the structure constants are $c_{ij}^k = \epsilon_{ijk}$. *In physics it is common to use the Pauli matrices defined by* $\sigma_i := 2\mathrm{i}e_i$ *in terms of which the commutation relations become* $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$ *but now the Lie algebra is the the isomorphic space of* 2×2 *traceless* Hermitian *matrices.*

Example 12.2 *The Weyl basis for* $\mathfrak{gl}(n, \mathbb{R})$ *is given by the* n^2 *matrices* e_{sr} *defined by*

$$(e_{rs})_{ij} := \delta_{ri}\delta_{sj}.$$

Notice that we are now in a situation where "double indices" are convenient. For instance, the commutation relations read

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj}$$

while the structure constants are

$$c_{sm,kr}^{ij} = \delta_s^i \delta_{mk} \delta_r^j - \delta_k^i \delta_{rs} \delta_m^j.$$

12.2 Classical complex Lie algebras

If $\mathfrak{g}$ is a real Lie algebra we have seen that the complexification $\mathfrak{g}_{\mathbb{C}}$ is naturally a complex Lie algebra. As mentioned above, it is convenient to omit the tensor symbol and use the following convention: Every element of $\mathfrak{g}_{\mathbb{C}}$ may be written at $v + \mathrm{i}w$ for $v, w \in \mathfrak{g}$ and then

$$\begin{aligned} &[v_1 + \mathrm{i}w_1, v_2 + \mathrm{i}w_2] \\ &= [v_1, v_2] - [w_1, w_2] + \mathrm{i}([v_1, w_2] + [w_1, v_2]). \end{aligned}$$

We shall now define a series of complex Lie algebras sometimes denoted by A_n, B_n, C_n and D_n for every integer $n > 0$. First of all, notice that the complexification $\mathfrak{gl}(n, \mathbb{R})_{\mathbb{C}}$ of $\mathfrak{gl}(n, \mathbb{R})$ is really just $\mathfrak{gl}(n, \mathbb{C})$; the set of complex $n \times n$ matrices under the commutator bracket.

The algebra A_n The set of all traceless $n \times n$ matrices is denoted A_{n-1} and also by $\mathfrak{sl}(n, \mathbb{C})$.

We call the readers attention to the following general fact: If $b(.,.)$ is a bilinear form on a complex vector space V then the set of all $A \in \mathfrak{gl}(n,\mathbb{C})$ such that $b(Az,w) + b(z,Aw) = 0$ for every $z, w \in V$ is a Lie subalgebra of $\mathfrak{gl}(n,\mathbb{C})$. This follows from the calculation

$$\begin{aligned} b([A,B]z,w) &= b(ABz,w) - b(BAz,w) \\ &= -b(Bz,Aw) + b(Az,Bw) \\ &= b(z,BAw) - b(z,ABw) \\ &= b(z,[B,A]w). \end{aligned}$$

The algebras B_n and D_n Let $m = 2n+1$ and let $b(.,.)$ be a nondegenerate symmetric bilinear form on an m dimensional complex vector space V. Without loss we may assume $V = \mathbb{C}^m$ and we may take $b(z,w) = \sum_{i=1}^m z_i w_i$. We define B_n to be $\mathfrak{o}(m,\mathbb{C})$ where

$$\mathfrak{o}(m,\mathbb{C}) := \{A \in \mathfrak{gl}(m,\mathbb{C}) : b(Az,w) + b(z,Aw) = 0\}.$$

Similarly, for $m = 2n$ we define D_n to be $\mathfrak{o}(m,\mathbb{C})$.

The algebra C_n The algebra associated to a skew-symmetric nondegenerate bilinear form which we may take to be $b(z,w) = \sum_{i=1}^n z_i w_{n+i} - \sum_{i=1}^n z_{n+i} w_i$ on $\mathbb{C}^{2n}$. We obtain the complex symplectic algebra

$$C_n = \mathfrak{sp}(n,\mathbb{C}) := \{A \in \mathfrak{gl}(m,\mathbb{C}) : b(Az,w) + b(z,Aw) = 0\}.$$

12.2.1 Basic Facts and Definitions

The expected theorems hold for homomorphisms; the image $\mathrm{img}(\sigma) := \sigma(\mathfrak{a})$ of a homomorphism $\sigma : \mathfrak{a} \to \mathfrak{b}$ is a subalgebra of $\mathfrak{b}$ and the kernel $\ker(\sigma)$ is an ideal of $\mathfrak{a}$.

Definition 12.2 *Let $\mathfrak{h}$ be an ideal in $\mathfrak{g}$. On the quotient vector space $\mathfrak{g}/\mathfrak{h}$ with quotient map π we can define a Lie bracket in the following way: For $\bar{v}, \bar{w} \in \mathfrak{g}/\mathfrak{h}$ choose $v, w \in \mathfrak{g}$ with $\pi(v) = \bar{v}$ and $\pi(w) = \bar{w}$ we define*

$$[\bar{v}, \bar{w}] := \pi([v,w]).$$

*We call $\mathfrak{g}/\mathfrak{h}$ with this bracket the **quotient Lie algebra**.*

Exercise 12.1 *Show that the bracket defined in the last definition is well defined.*

Given two linear subspaces $\mathfrak{a}$ and $\mathfrak{b}$ of $\mathfrak{g}$ the (not necessarily direct) sum $\mathfrak{a} + \mathfrak{b}$ is just the space of all elements in $\mathfrak{g}$ of the form $a + b$ where $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. It is not hard to see that if $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in $\mathfrak{g}$ then so is $\mathfrak{a} + \mathfrak{b}$.

Exercise 12.2 *Show that for $\mathfrak{a}$ and $\mathfrak{b}$ ideals in $\mathfrak{g}$ we have a natural isomorphism $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b} \cong \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$.*

If $\mathfrak{g}$ is a Lie algebra, $\mathfrak{s}$ a subset of $\mathfrak{g}$, then the **centralizer of $\mathfrak{s}$ in $\mathfrak{g}$** is $\mathfrak{z}(\mathfrak{s}) := \{x \in \mathfrak{g} : [x, y] = 0 \text{ for all } y \in \mathfrak{s}\}$. If $\mathfrak{a}$ is a (Lie) subalgebra of $\mathfrak{g}$ then the **normalizer** of $\mathfrak{a}$ in $\mathfrak{g}$ is $\mathfrak{n}(\mathfrak{a}) := \{v \in \mathfrak{g} : [v, \mathfrak{a}] \subset \mathfrak{a}\}$. One can check that $\mathfrak{n}(\mathfrak{a})$ is an ideal in $\mathfrak{g}$.

There is also a Lie algebra product. Namely, if $\mathfrak{a}$ and $\mathfrak{b}$ are Lie algebras, then we can define a Lie bracket on $\mathfrak{a} \times \mathfrak{b}$ by

$$[(a_1, a_2), (b_1, b_2)] := ([a_1, b_1], [a_2, b_2]).$$

With this bracket, $\mathfrak{a} \times \mathfrak{b}$ is a Lie algebra called the Lie algebra **product** of $\mathfrak{a}$ and $\mathfrak{b}$. The subspaces $\mathfrak{a} \times \{0\}$ and $\{0\} \times \mathfrak{b}$ are ideals in $\mathfrak{a} \times \mathfrak{b}$ that are clearly isomorphic to $\mathfrak{a}$ and $\mathfrak{b}$ respectively. Depending on the context this is also written as $\mathfrak{a} \oplus \mathfrak{b}$ and then referred to as the **direct sum** (external direct sum). If $\mathfrak{a}$ and $\mathfrak{b}$ are subalgebras of a Lie algebra $\mathfrak{g}$ such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{g}$ and $\mathfrak{a} \cap \mathfrak{b} = 0$ then we have the vector space direct sum which, for reasons which will be apparent shortly, we denote by $\mathfrak{a} \dot{+} \mathfrak{b}$. If we have several subalgebras of $\mathfrak{g}$, say $\mathfrak{a}_1, ..., \mathfrak{a}_k$ such that $\mathfrak{a}_i \cap \mathfrak{a}_j = 0$ for $i \neq j$, and if $\mathfrak{g} = \mathfrak{a}_1 \dot{+} \cdots \dot{+} \mathfrak{a}_k$ which is the vector space direct sum. For the Lie algebra direct sum we need to require that $[\mathfrak{a}_i, \mathfrak{a}_j] = 0$ for $i \neq j$. In this case we write $\mathfrak{g} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k$ which is the Lie algebra (internal) direct sum. In this case is is easy to verify that each $\mathfrak{a}_i$ is an ideal in $\mathfrak{g}$. With respect to such a decomposition the Lie product becomes $[\sum a_i, \sum a'_j] = \sum_i [a_i, a'_i]$. Clearly, the internal direct sum $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k$ is isomorphic to $\mathfrak{a}_1 \times \cdots \times \mathfrak{a}_k$ which as we have seen is also denoted as $\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_k$ (the external direct sum this time).

Definition 12.3 *The **center** $\mathfrak{z}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the subspace $\mathfrak{z}(\mathfrak{g}) := \{v \in \mathfrak{g} : [v, y] = 0$ for all $y \in \mathfrak{g}\}$.*

12.3 The Adjoint Representation

In the context of abstract Lie algebras, the **adjoint map** $a \to \mathrm{ad}(a)$ is given by $\mathrm{ad}(a)(b) := [a, b]$. It is easy to see that $\mathfrak{z}(\mathfrak{g}) = \ker(\mathrm{ad})$.

We have $[a^i v_i, b^j v_j] = a^i b^j [v_i, v_j] = a^i b^j c_{ij}^k v_k$ and so the matrix of $\mathrm{ad}(a)$ with respect to the basis $(v_1, ..., v_n)$ is $(ad(a))_j^k = a^i c_{ij}^k$. In particular, $(ad(v_i))_j^k = c_{ij}^k$.

Definition 12.4 *A derivation of a Lie algebra $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ such that*

$$D[v, w] = [Dv, w] + [v, Dw]$$

for all $v, w \in \mathfrak{g}$.

For each $v \in \mathfrak{g}$ the map $\mathrm{ad}(v) : \mathfrak{g} \to \mathfrak{g}$ is actually a derivation of the Lie algebra $\mathfrak{g}$. Indeed, this is exactly the content of the Jacobi identity. Furthermore, it is not hard to check that the space of all derivations of a Lie algebra $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(\mathfrak{g})$. In fact, if D_1 and D_2 are derivations of $\mathfrak{g}$ then so is the commutator $D_1 \circ D_2 - D_2 \circ D_1$. We denote this subalgebra of derivations by $Der(\mathfrak{g})$.

Definition 12.5 *A* ***Lie algebra representation*** *ρ of $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathrm{V})$.*

One can construct Lie algebra representations in various ways from given representations. For example, if $\rho_i : \mathfrak{g} \to \mathfrak{gl}(\mathrm{V}_i)$ $(i = 1, .., k)$ are Lie algebra representations then $\oplus_i \rho_i : \mathfrak{g} \to \mathfrak{gl}(\oplus_i \mathrm{V}_i)$ defined by

$$(\oplus_i \rho_i)(x)(v_1 \oplus \cdots \oplus v_n) = \rho_1(x)v_1 \oplus \cdots \oplus \rho_1(x)v_n \tag{12.2}$$

for $x \in \mathfrak{g}$ is a Lie algebra representation called the **direct sum representation** of the ρ_i. Also, if one defines

$$\begin{aligned}(\otimes_i \rho_i)(x)(v_1 \otimes \cdots \otimes v_k) &:= \rho_1(x)v_1 \otimes v_2 \otimes \cdots \otimes v_k \\ &+ v_1 \otimes \rho_2(x)v_2 \otimes \cdots \otimes v_k + \cdots + v_1 \otimes v_2 \otimes \cdots \otimes \rho_k(x)v_k\end{aligned}$$

(and extend linearly) then $\otimes_i \rho_i$ is a representation $\otimes_i \rho_i : \mathfrak{g} \to \mathfrak{gl}(\otimes_i \mathrm{V}_i)$ is Lie algebra representation called a **tensor product representation**.

Lemma 12.1 *$ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra representation on $\mathfrak{g}$. The image of* ad *is contained in the Lie algebra* $\mathrm{Der}(\mathfrak{g})$ *of all derivation of the Lie algebra $\mathfrak{g}$.*

Proof. This follows from the Jacobi identity (as indicated above) and from the definition of ad. ■

Corollary 12.1 *$\mathfrak{x}(\mathfrak{g})$ is an ideal in $\mathfrak{g}$.*

The image $\mathrm{ad}(\mathfrak{g})$ of ad in $\mathrm{Der}(\mathfrak{g})$ is called the adjoint algebra.

Definition 12.6 *The Killing form for a Lie algebra $\mathfrak{g}$ is the bilinear form given by*

$$K(X, Y) = Tr(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$$

Lemma 12.2 *For any Lie algebra automorphism $\vartheta : \mathfrak{g} \to \mathfrak{g}$ and any $X \in \mathfrak{g}$ we have* $\mathrm{ad}(\vartheta X) = \vartheta \mathrm{ad} X \vartheta^{-1}$

Proof. $\mathrm{ad}(\vartheta X)(Y) = [\vartheta X, Y] = [\vartheta X, \vartheta\vartheta^{-1} Y] = \vartheta[X, \vartheta^{-1} Y] = \vartheta \circ \mathrm{ad} X \circ \vartheta^{-1}(Y)$. ■

Clearly $K(X, Y)$ is symmetric in X and Y but we have more identities:

Proposition 12.1 *The Killing forms satisfies identities:*

1) $K([X, Y], Z) = K([Z, X], Y)$ for all $X, Y, Z \in \mathfrak{g}$

2) $K(\rho X, \rho Y) = K(X, Y)$ for any Lie algebra automorphism $\rho : \mathfrak{g} \to \mathfrak{g}$ and any $X, Y \in \mathfrak{g}$.

Proof. For (1) we calculate

$$\begin{aligned}K([X,Y],Z) &= Tr(\mathrm{ad}([X,Y]) \circ \mathrm{ad}(Z)) \\ &= Tr([\mathrm{ad}X, \mathrm{ad}Y] \circ \mathrm{ad}(Z)) \\ &= Tr(\mathrm{ad}X \circ \mathrm{ad}Y \circ \mathrm{ad}Z - \mathrm{ad}Y \circ \mathrm{ad}X \circ \mathrm{ad}Z) \\ &= Tr(\mathrm{ad}Z \circ \mathrm{ad}X \circ \mathrm{ad}Y - \mathrm{ad}X \circ \mathrm{ad}Z \circ \mathrm{ad}Y) \\ &= Tr([\mathrm{ad}Z, \mathrm{ad}X] \circ \mathrm{ad}Y) \\ &= Tr(\mathrm{ad}[Z,X] \circ \mathrm{ad}Y) = K([Z,X],Y)\end{aligned}$$

where we have used that $Tr(ABC)$ is invariant under cyclic permutations of A, B, C.

For (2) just observe that

$$\begin{aligned}K(\rho X, \rho Y) &= Tr(\mathrm{ad}(\rho X) \circ \mathrm{ad}(\rho Y)) \\ &= Tr(\rho \mathrm{ad}(X)\rho^{-1}\rho \mathrm{ad}(Y)\rho^{-1}) \quad \text{(lemma 12.2)} \\ &= Tr(\rho \mathrm{ad}(X) \circ \mathrm{ad}(Y)\rho^{-1}) \\ &= Tr(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) = K(X,Y).\end{aligned}$$

■

Since $K(X,Y)$ is symmetric in X, Y and so there must be a basis $\{X_i\}_{1\leq i\leq n}$ of $\mathfrak{g}$ for which the matrix (k_{ij}) given by

$$k_{ij} := K(X_i, X_j)$$

is diagonal.

Lemma 12.3 *If $\mathfrak{a}$ is an ideal in $\mathfrak{g}$ then the Killing form of $\mathfrak{a}$ is just the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{a} \times \mathfrak{a}$.*

Proof. Let $\{X_i\}_{1\leq i\leq n}$ be a basis of $\mathfrak{g}$ such that $\{X_i\}_{1\leq i\leq r}$ is a basis for $\mathfrak{a}$. Now since $[\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a}$, the structure constants c^i_{jk} with respect to this basis must have the property that $c^k_{ij} = 0$ for $i \leq r < k$ and all j. Thus for $1 \leq i, j \leq r$ we have

$$\begin{aligned}K_{\mathfrak{a}}(X_i, X_j) &= Tr(\mathrm{ad}(X_i)\mathrm{ad}(X_j)) \\ &= \sum_{k=1}^{r}\sum_{s=1}^{r} c^k_{is}c^s_{jk} = \sum_{k=1}^{n}\sum_{s=1}^{n} c^i_{ik}c^k_{ji} \\ &= K_{\mathfrak{g}}(X_i, X_j).\end{aligned}$$

■

12.4 The Universal Enveloping Algebra

In a Lie algebra $\mathfrak{g}$ the product $[.,.]$ is not associative except in the trivial case that $[.,.] \equiv 0$. On the other hand, associative algebras play an important role in the study of Lie algebras. For one thing, if $\mathfrak{A}$ is an associative algebra then we can introduce the commutator bracket on $\mathfrak{A}$ by

$$[A, B] := AB - BA$$

which gives $\mathfrak{A}$ the structure of Lie algebra. From the other direction, if we start with a Lie algebra $\mathfrak{g}$ then we can construct an associative algebra called the universal enveloping algebra of $\mathfrak{g}$. This is done, for instance, by first forming the full tensor algebra on $\mathfrak{g}$;

$$T(\mathfrak{g}) = \mathbb{F} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots \oplus \mathfrak{g}^{\otimes k} \oplus \cdots$$

and then dividing out by an appropriate ideal:

Definition 12.7 *Associated to every Lie algebra $\mathfrak{g}$ there is an associative algebra $U(\mathfrak{g})$ called the* ***universal enveloping algebra*** *defined by*

$$U(\mathfrak{g}) := T(\mathfrak{g})/J$$

where J is the ideal generated by elements in $T(\mathfrak{g})$ of the form $X \otimes Y - Y \otimes X - [X, Y]$.

There is the natural map of $\mathfrak{g}$ into $U(\mathfrak{g})$ given by the composition $\pi : \mathfrak{g} \hookrightarrow T(\mathfrak{g}) \to T(\mathfrak{g})/J = U(\mathfrak{g})$. For $v \in \mathfrak{g}$, let v^* denote the image of v under this canonical map.

Theorem 12.1 *Let V be a vector space over the field $\mathbb{F}$. For every ρ representation of $\mathfrak{g}$ on V there is a corresponding representation ρ^* of $U(\mathfrak{g})$ on V such that for all $v \in \mathfrak{g}$ we have*

$$\rho(v) = \rho^*(v^*).$$

This correspondence, $\rho \mapsto \rho^*$ is a 1-1 correspondence.

Proof. Given ρ, there is a natural representation $T(\rho)$ on $T(\mathfrak{g})$. The representation $T(\rho)$ vanishes on J since

$$T(\rho)(X \otimes Y - Y \otimes X - [X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X) - \rho([X, Y]) = 0$$

and so $T(\rho)$ descends to a representation ρ^* of $U(\mathfrak{g})$ on $\mathfrak{g}$ satisfying $\rho(v) = \rho^*(v^*)$. Conversely, if σ is a representation of $U(\mathfrak{g})$ on V then we put $\rho(X) = \sigma(X^*)$. The map $\rho(X)$ is linear and a representation since

$$\begin{aligned}
\rho([X, Y]) &= \sigma([X, Y]^*) \\
&= \sigma(\pi(X \otimes Y - Y \otimes X)) \\
&= \sigma(X^* Y^* - Y^* X^*) \\
&= \rho(X)\rho(Y) - \rho(Y)\rho(X)
\end{aligned}$$

for all $X, Y \in \mathfrak{g}$. ∎

Now let $X_1, X_2, ..., X_n$ be a basis for $\mathfrak{g}$ and then form the monomials $X_{i_1}^* X_{i_2}^* \cdots X_{i_r}^*$ in $U(\mathfrak{g})$. The set of all such monomials for a fixed r span a subspace of $U(\mathfrak{g})$, which we denote by $U^r(\mathfrak{g})$. Let c_{ik}^j be the structure constants for the basis $X_1, X_2, ..., X_n$. Then under the map π the structure equations become

$$[X_i^*, X_j^*] = \sum_k c_{ij}^k X_k^*.$$

By using this relation we can replace the spanning set $\mathcal{M}_r = \{X_{i_1}^* X_{i_2}^* \cdots X_{i_r}^*\}$ for $U^r(\mathfrak{g})$ by spanning set $\mathcal{M}_{\leq r}$ for $U^r(\mathfrak{g})$ consisting of all monomials of the form $X_{i_1}^* X_{i_2}^* \cdots X_{i_m}^*$ where $i_1 \leq i_2 \leq \cdots \leq i_m$ and $m \leq r$. In fact one can then concatenate these spanning sets $\mathcal{M}_{\leq r}$ and it turns out that these combine to form a basis for $U(\mathfrak{g})$. We state the result without proof:

Theorem 12.2 (Birchoff-Poincarè-Witt) *Let $e_{i_1 \leq i_2 \leq \cdots \leq i_m} = X_{i_1}^* X_{i_2}^* \cdots X_{i_m}^*$ where $i_1 \leq i_2 \leq \cdots \leq i_m$. The set of all such elements $\{e_{i_1 \leq i_2 \leq \cdots \leq i_m}\}$ for all m is a basis for $U(\mathfrak{g})$ and the set $\{e_{i_1 \leq i_2 \leq \cdots \leq i_m}\}_{m \leq r}$ is a basis for the subspace $U^r(\mathfrak{g})$.*

Lie algebras and Lie algebra representations play an important role in physics and mathematics, and as we shall see below, every Lie group has an associated Lie algebra that, to a surprisingly large extent, determines the structure of the Lie group itself. Let us first explore some of the important abstract properties of Lie algebras. A notion that is useful for constructing Lie algebras with desired properties is that of the **free Lie algebra** $\mathfrak{f}_n$ which is defined to be the quotient of the free algebra on n symbols by the smallest ideal such that we end up with a Lie algebra. Every Lie algebra can be realized as a quotient of one of these free Lie algebras.

Definition 12.8 *The **descending central series** $\{\mathfrak{g}_{(k)}\}$ of a Lie algebra $\mathfrak{g}$ is defined inductively by letting $\mathfrak{g}_{(1)} = \mathfrak{g}$ and then $\mathfrak{g}_{(k+1)} = [\mathfrak{g}_{(k)}, \mathfrak{g}]$.*

The reason for the term “descending” is the that we have the chain of inclusions

$$\mathfrak{g}_{(1)} \supset \cdots \supset \mathfrak{g}_{(k)} \supset \mathfrak{g}_{(k+1)} \supset \cdots$$

From the definition of Lie algebra homomorphism we see that if $\sigma : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism then $\sigma(\mathfrak{g}_{(k)}) \subset \mathfrak{h}_{(k)}$.

Exercise 12.3 (!) *Use the Jacobi identity to prove that for all positive integers i and j, we have $[\mathfrak{g}_{(i)}, \mathfrak{g}_{(i)}] \subset \mathfrak{g}_{(i+j)}$.*

Definition 12.9 *A Lie algebra $\mathfrak{g}$ is called **k-step nilpotent** if and only if $\mathfrak{g}_{(k+1)} = 0$ but $\mathfrak{g}_{(k)} \neq 0$.*

The most studied nontrivial examples are the Heisenberg algebras which are 2-step nilpotent. These are defined as follows:

Example 12.3 *The* $2n+1$ *dimensional* ***Heisenberg algebra*** $\mathfrak{h}_n$ *is the Lie algebra (defined up to isomorphism) with a basis* $\{X_1, ..., X_n, Y_1, ..., Y_n, Z\}$ *subject to the relations*

$$[X_j, Y_j] = Z$$

and all other brackets of elements from this basis being zero. A concrete realization of $\mathfrak{h}_n$ *is given as the set of all* $(n+2) \times (n+2)$ *matrices of the form*

$$\begin{bmatrix} 0 & x_1 & ... & x_n & z \\ 0 & 0 & ... & 0 & y_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & ... & 0 & y_n \\ 0 & 0 & ... & 0 & 0 \end{bmatrix}$$

where x_i, y_i, z *are all real numbers. The bracket is the commutator bracket as is usually the case for matrices. The basis is realized in the obvious way by putting a lone 1 in the various positions corresponding to the potentially nonzero entries. For example,*

$$X_1 = \begin{bmatrix} 0 & 1 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \end{bmatrix}$$

and

$$Z = \begin{bmatrix} 0 & 0 & ... & 0 & 1 \\ 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 \end{bmatrix}.$$

Example 12.4 *The space of all upper triangular* $n \times n$ *matrices* $\mathfrak{n}_n$ *which turns out to be* $n-1$ *step nilpotent.*

We also have the **free k-step nilpotent Lie algebra** given by the quotient $\mathfrak{f}_{n,k} := \mathfrak{f}_n/(\mathfrak{f}_n)_k$ where $\mathfrak{f}_n$ is the free Lie algebra mentioned above. (notice the difference between $\mathfrak{f}_{n,k}$ and $(\mathfrak{f}_n)_k$.

Lemma 12.4 *Every finitely generated* k*-step nilpotent Lie algebra is isomorphic to a quotient of the free* k*-step nilpotent Lie algebra.*

Proof. Suppose that $\mathfrak{g}$ is k-step nilpotent and generated by elements $X_1, ..., X_n$. Let $F_1, ..., F_n$ be the generators of $\mathfrak{f}_n$ and define a map $h : \mathfrak{f}_n \to \mathfrak{g}$ by sending F_i to X_i and extending linearly. This map clearly factors through $\mathfrak{f}_{n,k}$ since $h((\mathfrak{f}_n)_k) = 0$. Then we have a homomorphism $(\mathfrak{f}_n)_k \to \mathfrak{g}$ that is clearly onto and so the result follows. ■

Definition 12.10 *Let $\mathfrak{g}$ be a Lie algebra. We define the commutator series $\{\mathfrak{g}^{(k)}\}$ by letting $\mathfrak{g}^{(1)} = \mathfrak{g}$ and then inductively $\mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$. If $\mathfrak{g}^{(k)} = 0$ for some positive integer k, then we call $\mathfrak{g}$ a **solvable** Lie algebra.*

Clearly, the statement $\mathfrak{g}^{(2)} = 0$ is equivalent to the statement that $\mathfrak{g}$ is abelian. Another simple observation is that $\mathfrak{g}^{(k)} \subset \mathfrak{g}_{(k)}$ so that nilpotency implies solvability.

Exercise 12.4 (!) *Every subalgebra and every quotient algebra of a solvable Lie algebra is solvable. In particular, the homomorphic image of a solvable Lie algebra is solvable. Conversely, if $\mathfrak{a}$ is a solvable ideal in $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{a}$ is solvable, then $\mathfrak{g}$ is solvable. Hint: Use that $(\mathfrak{g}/\mathfrak{a})^{(j)} = \mathfrak{g}^{(j)}/\mathfrak{a}$.*

It follows from this exercise that we have

Corollary 12.2 *Let $h : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra homomorphism. If* $\mathrm{img}(h) := h(\mathfrak{g})$ *and* $\ker(h)$ *are both solvable then $\mathfrak{g}$ is solvable. In particular, if* $\mathrm{img}(\mathrm{ad}) := \mathrm{ad}(\mathfrak{g})$ *is solvable then so is $\mathfrak{g}$.*

Lemma 12.5 *If $\mathfrak{a}$ is a nilpotent ideal in $\mathfrak{g}$ contained in the center $\mathfrak{z}(\mathfrak{g})$ and if $\mathfrak{g}/\mathfrak{a}$ is nilpotent then $\mathfrak{g}$ is nilpotent.*

Proof. First, the reader can verify that $(\mathfrak{g}/\mathfrak{a})_{(j)} = \mathfrak{g}_{(j)}/\mathfrak{a}$. Now if $\mathfrak{g}/\mathfrak{a}$ is nilpotent then $\mathfrak{g}_{(j)}/\mathfrak{a} = 0$ for some j and so $\mathfrak{g}_{(j)} \subset \mathfrak{a}$ and if this is the case then we have $\mathfrak{g}_{(j+1)} = [\mathfrak{g}, \mathfrak{g}_{(j)}] \subset [\mathfrak{g}, \mathfrak{a}] = 0$. (Here we have $[\mathfrak{g}, \mathfrak{a}] = 0$ since $\mathfrak{a} \subset \mathfrak{z}(\mathfrak{g})$.) Thus $\mathfrak{g}$ is nilpotent. ■

Trivially, the center $\mathfrak{z}(\mathfrak{g})$ of a Lie algebra a solvable ideal.

Corollary 12.3 *Let $h : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra homomorphism. If* $\mathrm{img}(\mathrm{ad}) := \mathrm{ad}(\mathfrak{g})$ *is nilpotent then $\mathfrak{g}$ is nilpotent.*

Proof. Just use the fact that $\ker(\mathrm{ad}) = \mathfrak{z}(\mathfrak{g})$. ■

Theorem 12.3 *The sum of any family of solvable ideals in $\mathfrak{g}$ is a solvable ideal. Furthermore, there is a unique maximal solvable ideal that is the sum of all solvable ideals in $\mathfrak{g}$.*

Sketch of proof. The proof is a maximality argument based on the following idea : If $\mathfrak{a}$ and $\mathfrak{b}$ are solvable then $\mathfrak{a} \cap \mathfrak{b}$ is an ideal in the solvable $\mathfrak{a}$ and so is solvable. It is easy to see that $\mathfrak{a} + \mathfrak{b}$ is an ideal. We have by exercise 12.2 $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b} \cong \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$. Since $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$ is a homomorphic image of $\mathfrak{a}$ we see that $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b}) \cong (\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$ is solvable. Thus by our previous result $\mathfrak{a} + \mathfrak{b}$ is solvable. ■

Definition 12.11 *The maximal solvable ideal in $\mathfrak{g}$ whose existence is guaranteed by the last theorem is called the **radical** of $\mathfrak{g}$ and is denoted $rad(\mathfrak{g})$*

Definition 12.12 *A Lie algebra $\mathfrak{g}$ is called* ***simple*** *if it contains no ideals other than $\{0\}$ and $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is called* ***semisimple*** *if it contains no abelian ideals (other than $\{0\}$).*

Theorem 12.4 (Levi decomposition) *Every Lie algebra is the semi-direct sum of its radical and a semisimple Lie algebra.*

Define semi-direct sum before this.

12.5 nnn

The map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) = \mathrm{End}(T_eG)$ is given as the tangent map at the identity of Ad which is a Lie algebra homomorphism. Thus by proposition 4.5 we have obtain

Proposition 12.2 $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is a Lie algebra homomorphism.*

Proof. This follows from our study of abstract Lie algebras and proposition 4.11. ∎

Let's look at what this means. Recall that the Lie bracket for $\mathfrak{gl}(\mathfrak{g})$ is just $A \circ B - B \circ A$. Thus we have

$$\mathrm{ad}([v, w]) = [\mathrm{ad}(v), \mathrm{ad}(w)] = \mathrm{ad}(v) \circ \mathrm{ad}(w) - \mathrm{ad}(w) \circ \mathrm{ad}(v)$$

which when applied to a third vector z gives

$$[[v, w], z] = [v, [w, z]] - [w, [v, z]]$$

which is just a version of the Jacobi identity. Also notice that using the anti-symmetry of the bracket we get $[z, [v, w]] = [w, [z, v]] + [v, [z, w]]$ which in turn is the same as

$$\mathrm{ad}(z)([v, w]) = [\mathrm{ad}(z)v, w] + [v, \mathrm{ad}(z)w]$$

so $\mathrm{ad}(z)$ is actually a derivation of the Lie algebra $\mathfrak{g}$ as explained before.

Proposition 12.3 *The Lie algebra $Der(\mathfrak{g})$ of all derivation of $\mathfrak{g}$ is the Lie algebra of the group of automorphisms* $\mathrm{Aut}(\mathfrak{g})$*. The image* $\mathrm{ad}(\mathfrak{g}) \subset Der(\mathfrak{g})$ *is the Lie algebra of the set of all inner automorphisms* $\mathrm{Int}(\mathfrak{g})$*.*

$$\begin{array}{ccc} \mathrm{ad}(\mathfrak{g}) & \subset & \mathrm{Der}(\mathfrak{g}) \\ \downarrow & & \downarrow \\ \mathrm{Int}(\mathfrak{g}) & \subset & \mathrm{Aut}(\mathfrak{g}) \end{array}$$

Let $\mu : G \times G \to G$ be the multiplication map. Recall that the tangent space $T_{(g,h)}(G \times G)$ is identified with $T_gG \times T_hG$. Under this identification we have

$$T_{(g,h)}\mu(v, w) = T_hL_gw + T_gR_hv$$

where $v \in T_gG$ and $w \in T_hG$. The following diagrams exhibit the relations:

$$
\begin{array}{ccccc}
 & & G \times G, (g,h) & & \\
 & \overset{pr_1}{\swarrow} & \| & \overset{pr_2}{\searrow} & \\
G, g & \rightarrow & G \times G, (g,h) & \leftarrow & G, h \\
 & \underset{R_h}{\searrow} & \downarrow \mu & \underset{L_g}{\swarrow} & \\
 & & G, gh & &
\end{array}
$$

The horizontal maps are the insertions $g \mapsto (g,h)$ and $h \mapsto (g,h)$. Applying the tangent functor to the last diagram gives.

$$
\begin{array}{ccccc}
Tpr_1 & & T_{(g,h)}(G \times G) & & Tpr_2 \\
 & \swarrow & \updownarrow & \searrow & \\
T_gG & \rightarrow & T_gG \times T_hG & \leftarrow & T_hG \\
 & \searrow & \downarrow T\mu & \swarrow & \\
T_gR_h & & T_{gh}G & & T_hL_g
\end{array}
$$

We will omit the proof but the reader should examine the diagrams and try to construct a proof on that basis.

We have another pair of diagrams to consider. Let $\nu : G \rightarrow G$ be the inversion map $\nu : g \mapsto g^{-1}$. We have the following commutative diagrams:

$$
\begin{array}{ccccc}
R_{g^{-1}} & & (G,g) & & L_{g^{-1}} \\
 & \swarrow & & \searrow & \\
(G,e) & & \downarrow \nu & & (G,e) \\
 & \searrow & & \swarrow & \\
L_{g^{-1}} & & \left(G, g^{-1}\right) & & R_{g^{-1}}
\end{array}
$$

Applying the tangent functor we get

$$
\begin{array}{ccccc}
TR_{g^{-1}} & & T_gG & & TL_{g^{-1}} \\
 & \swarrow & & \searrow & \\
T_eG & & \downarrow T\nu & & T_eG \\
 & \searrow & & \swarrow & \\
TL_{g^{-1}} & & T_{g^{-1}}G & & TR_{g^{-1}}
\end{array}
.
$$

The result we wish to express here is that $T_g\nu = TL_{g^{-1}} \circ TR_{g^{-1}} = TR_{g^{-1}} \circ TL_{g^{-1}}$. Again the proof follows readily from the diagrams.

Chapter 13

Lie Group Actions and Homogenous Spaces

Here we set out our conventions regarding (right and left) group actions and the notion of equivariance. There is plenty of room for confusion just from the issues of right as opposed to left if one doesn't make a few observations and set down the conventions carefully from the start. We will make the usual choices but we will note how these usual choices lead to annoyances like the mismatch of homomorphisms with anti-homomorphisms in proposition 13.1 below.

13.1 Our Choices

1. When we write $L_X Y = \frac{d}{dt}\big|_0 \left(\varphi_t^X\right)^* Y$ we are implicitly using a right action of $\mathrm{Diff}(M)$ on $\mathfrak{X}(M)$[1]. Namely, $Y \mapsto f^* Y$.

2. We have chosen to make the bracket of vector fields be defined so that $[X, Y] = XY - YX$ rather than by $YX - XY$. This makes it true that $L_X Y = [X, Y]$ so the first choice seems to influence this second choice.

3. We have chosen to define the bracket in a Lie algebra $\mathfrak{g}$ of a Lie group G to be given by using the identifying linear map $\mathfrak{g} = T_e G \to \mathfrak{X}^L(M)$ where $\mathfrak{X}^L(M)$ is the space of **left** invariant vector fields. What if we had used **right** invariant vector fields? Then we would have $[X_e, Y_e]_{new} = [X', Y']_e$ where $X'_g = TR_g \cdot X_e$ is the right invariant vector field:

$$\begin{aligned} R_h^* X'(g) &= TR_h^{-1} X'(gh) = TR_h^{-1} TR_{gh} \cdot X_e \\ &= TR_h^{-1} \circ T(R_h \circ R_g) \cdot X_e = TR_g \cdot X_e \\ &= X'(g) \end{aligned}$$

[1] At least when X is complete since otherwise Fl_t^X is only a diffeomorphism on relatively compact set open sets and even then only for small enough t).

On the other hand, consider the inversion map $\nu : G \to G$. We have $v \circ R_{g^{-1}} = L_g \circ v$ and also $T\nu = -\operatorname{id}$ at $e \in T_eG$ so

$$\begin{aligned}(\nu^* X')(g) &= T\nu \cdot X'(g^{-1}) = T\nu \cdot TR_{g^{-1}} \cdot X_e \\ &= T(L_g \circ v)X_e = TL_g Tv \cdot X_e \\ &= -TL_g X_e = -X(g)\end{aligned}$$

thus $\nu^*[X', Y'] = [\nu^* X', \nu^* Y'] = [-X, -Y] = [X, Y]$. Now at e we have $(\nu^*[X', Y'])(e) = Tv \circ [X', Y'] \circ \nu(e) = -[X', Y']_e$. So we have $[X, Y]_e = -[X', Y']_e$.

So this choice is different by a sign also.

The source of the problem may just be conventions but it is interesting to note that if we consider $\operatorname{Diff}(M)$ as an infinite dimensional Lie group then the vector fields of that manifold would be maps $\overleftrightarrow{X} : \operatorname{Diff}(M) \to T\operatorname{Diff}(M)$ such $\overleftrightarrow{X}(\phi)$ is a vector field in $\mathfrak{X}(M)$ such that $\varphi_0^{\overleftrightarrow{X}(\phi)} = \phi$. In other words, a field for every diffeomorphism, a "field of fields" so to speak. Then in order to get the usual bracket in $\mathfrak{X}(M)$ we would have to use right invariant (fields of) fields (instead of the conventional left invariant choice) and evaluate them at the identity element of $\operatorname{Diff}(M)$ to get something in $T_{\mathrm{id}} \operatorname{Diff}(M) = \mathfrak{X}(M)$. (THINKTHISTHRUAGAIN) This makes one wonder if right invariant vector fields would have been a better convention to start with. Indeed some authors do make that convention.

13.1.1 Left actions

Definition 13.1 *A left action of a Lie group G on a manifold M is a smooth map $\lambda : G \times M \to M$ such that $\lambda(g_1, \lambda(g_2, m)) = \lambda(g_1 g_2, m))$ for all $g_1, g_2 \in G$. We often write $\lambda(g, m)$ as $g \cdot m$.*

Define the partial map $\lambda_g : M \to M$ by $\lambda_g(m) = \lambda(g, m)$. Then $\widetilde{\lambda} : g \mapsto \lambda_g$ is a group homomorphism $G \to \operatorname{Diff}(M)$.

Definition 13.2 *For a left group action as above, we have for every $v \in \mathfrak{g}$ we define a vector field $v^\lambda \in \mathfrak{X}(M)$ defined by*

$$v^\lambda(m) = \left.\frac{d}{dt}\right|_{t=0} \exp(tv) \cdot m$$

which is called the ***fundamental vector field*** *associated with the action λ.*

Notice that $v^\lambda(m) = T\lambda_{(e,m)} \cdot (v, 0)$.

Proposition 13.1 *Given left action $\lambda : G \times M \to M$ of a Lie group G on a manifold M, the map $\widetilde{\lambda} : g \mapsto \lambda_g$ is a group homomorphism $G \to \operatorname{Diff}(M)$.*

Despite this, the map $X \mapsto X^\lambda$ is a Lie algebra anti-homomorphism *$\mathfrak{g} \to \mathfrak{X}(M)$:*

$$[v, w]^\lambda = -[v^\lambda, w^\lambda]_{\mathfrak{X}(M)}$$

which implies that the bracket for the Lie algebra $\mathfrak{diff}(M)$ of Diff(M) *(as an infinite dimensional Lie group) is in fact $[X, Y]_{\mathfrak{diff}(M)} := -[X, Y]_{\mathfrak{X}(M)}$.*

Proposition 13.2 *If G acts on itself from the* left *by multiplication $L : G \times G \to G$ then the fundamental vector fields are the* right *invariant vector fields!*

13.1.2 Right actions

Definition 13.3 *A right action of a Lie group G on a manifold M is a smooth map $\rho : M \times G \to M$ such that $\rho(\rho(m, g_2), g_1) = \rho(m, g_2 g_1))$ for all $g_1, g_2 \in G$. We often write $\rho(m, g)$ as $m \cdot g$*

Define the partial map $\rho^g : M \to M$ by $\rho^g(m) = \rho(m, g)$. Then $\widetilde{\rho} : g \mapsto \rho^g$ is a group **anti-homomorphism** $G \to \mathrm{Diff}(M)$.

Definition 13.4 *For a right group action as above, we have for every $v \in \mathfrak{g}$ a vector field $v^\rho \in \mathfrak{X}(M)$ defined by*

$$v^\rho(m) = \left.\frac{d}{dt}\right|_{t=0} m \cdot \exp(tv)$$

which is called the ***fundamental vector field*** *associated with the right action ρ.*

Notice that $v^\rho(m) = T\rho_{(m,e)} \cdot (0, v)$.

Proposition 13.3 *Given right action $\rho : M \times G \to M$ of a Lie group G on a manifold M, the map $\widetilde{\rho} : g \mapsto \rho^g$ is a group anti-homomorphism $G \to$* Diff(M) *by definition.* ***However, the map $X \mapsto X^\lambda$ is a true Lie algebra homomorphism*** *$\mathfrak{g} \to \mathfrak{X}(M)$:*

$$[v, w]^\rho = [v^\rho, w^\rho]_{\mathfrak{X}(M)}$$

this disagreement again implies that the Lie algebra $\mathfrak{diff}(M)$ of Diff(M) *(as an infinite dimensional Lie group) is in fact $\mathfrak{X}(M)$, but with the bracket $[X, Y]_{\mathfrak{diff}(M)} := -[X, Y]_{\mathfrak{X}(M)}$.*

Proposition 13.4 *If G acts on itself from the* right *by multiplication $L : G \times G \to G$ then the fundamental vector fields are the* left *invariant vector fields $\mathfrak{X}_L(G)$.*

Proof: Exercise.

13.1.3 Equivariance

Definition 13.5 *Given two left actions* $\lambda_1 : G \times M \to M$ *and* $\lambda_2 : G \times S \to S$ *we say that a map* $f : M \to N$ *is (left)* ***equivariant*** *(with respect to these actions) if*

$$f(g \cdot s) = g \cdot f(s)$$

i.e.

$$f(\lambda_1(g, s)) = \lambda_2(g, f(s))$$

with a similar definition for right actions.

Notice that if $\lambda : G \times M \to M$ is a left action then we have an **associated right action** $\lambda^{-1} : M \times G \to M$ given by

$$\lambda^{-1}(p, g) = \lambda(g^{-1}, p).$$

Similarly, to a right action $\rho : M \times G \to M$ there is an associated left action

$$\rho^{-1}(g, p) = \rho(p, g^{-1})$$

and then we make the follow conventions concerning equivariance when mixing right with left.

Definition 13.6 *Is is often the case that we have a right action on a manifold* P *(such as a principle bundle) and a left action on a manifold* S. *Then equivariance is defined by converting the right action to its associated left action. Thus we have the requirement*

$$f(s \cdot g^{-1}) = g \cdot f(s)$$

or we might do the reverse and define equivariance by

$$f(s \cdot g) = g^{-1} \cdot f(s)$$

13.1.4 The action of $\mathrm{Diff}(M)$ and map-related vector fields.

Given a diffeomorphism $\Phi : M \to N$ define $\Phi_\star : \Gamma(M, TM) \to \Gamma(N, TN)$ by

$$\Phi_* X = T\Phi \circ X \circ \Phi^{-1}$$

and $\Phi^* : \Gamma(M, TN) \to \Gamma(M, TM)$ by

$$\Phi^* X = T\Phi^{-1} \circ X \circ \Phi$$

If $M = N$, this gives a right and left pair of actions of the diffeomorphism group $\mathrm{Diff}(M)$ on the space of vector fields $\mathfrak{X}(M) = \Gamma(M, TM)$.

$$\begin{aligned} \mathrm{Diff}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M) \\ (\Phi, X) &\mapsto \Phi_* X \end{aligned}$$

and

$$\begin{aligned} \mathfrak{X}(M) \times \mathrm{Diff}(M) &\to \mathfrak{X}(M) \\ (X, \Phi) &\mapsto \Phi^* X \end{aligned}$$

13.1.5 Lie derivative for equivariant bundles.

Definition 13.7 *An equivariant left action for a bundle $E \to M$ is a pair of actions $\gamma^E : G \times E \to E$ and $: G \times M \to M$ such that the diagram below commutes*

$$\begin{array}{llll} \gamma^E : & G \times E & \to & E \\ & \downarrow & & \downarrow \\ \gamma : & G \times M & \to & M \end{array}$$

In this case we can define an action on the sections $\Gamma(E)$ via

$$\gamma_g^* \mathbf{s} = (\gamma^E)^{-1} \circ \mathbf{s} \circ \gamma_g$$

and then we get a Lie derivative for $\mathbf{X} \in LG$

$$L_{\mathbf{X}}(\mathbf{s}) = \left.\frac{d}{dt}\right|_0 \gamma^*_{\exp t\mathbf{X}} \mathbf{s}$$

Chapter 14

Homogeneous Spaces and Klein Geometries.

1. Affine spaces: These are the spaces $\mathbf{A}^n$ which are just $\mathbb{R}^n$ acted on the left by the affine group $Aff(n)$ or the proper affine group $Aff^+(n)$.

2. Special affine space: This is again the space $\mathbf{A}^n$ but with the group now restricted to be the special affine group $SAff(n)$.

3. Euclidean spaces: These are the spaces $\mathbf{E}^n$ which are just $\mathbb{R}^n$ acted on the left by the group of Euclidean motions $\mathrm{Euc}(n)$ or the proper affine group $\mathrm{Euc}^+(n)$ (defined by requiring the linear part to be orientation preserving.

4. Projective spaces: These are the spaces $\mathbb{R}P^n = P(\mathbb{R}^{n+1})$ (consisting of lines through the origin in $\mathbb{R}^{n+1}$) acted on the left by the $\mathrm{P}Sl(n)$.

In this chapter we will have actions acting both from the left and from the right and so we will make a distinction in the notation. Recall that the **orbit** of $x \in M$ under a right action by G is denoted $x \cdot G$ or xG and the set of orbits M/G partition M into equivalence classes. For left actions we write $G \cdot x = Gx$ and $G \backslash M$ for the orbit space.

Example 14.1 *If H is a closed subgroup of G then H acts on G from the right by right multiplication.. The space of orbits G/H of this* right *action is just the set of* ***right cosets****. Similarly, we have the space of left cosets $H \backslash G$.*

Warning: We call gH a *right* coset because it comes from a right action but for at least half of the literature the convention is to call gH a left coset and G/H the space of left cosets.

We recall a few more definitions from chapter **??**:

Definition 14.1 *A left (resp. right) action is said to be* ***effective*** *if $g \cdot p = x$ (resp. $x \cdot g = x$) for every $x \in M$ implies that $g = e$ and is said to be* ***free*** *if $g \cdot x = x$ (resp. $x \cdot g = x$) for even one $x \in M$ implies that $g = e$.*

Definition 14.2 *A left (resp. right) action is said to be a* ***transitive action*** *if there is only one orbit in the space of orbits.*

This single orbit would have to be M and transitivity means, by definition, that given pair $x, y \in M$, there is a $g \in G$ with $g \cdot x = y$ (resp. $x \cdot g = y$).

Theorem 14.1 *Let $\lambda : G \times M \to M$ be a left action and fix $x_0 \in M$. Let $H = H_{x_0}$ be the* ***isotropy subgroup*** *of x_0 defined by*

$$H = \{g \in G : g \cdot x_0 = x_0\}.$$

Then we have a natural bijection

$$G \cdot x_0 \cong G/H$$

given by $g \cdot x_0 \mapsto gH$. In particular, if the action is transitive then $G/H \cong M$ and x_0 maps to H.

Exercise 14.1 *Show that the action of $Aff^+(\mathbf{A}^2)$ on $\mathbf{A}^2$ is transitive and effective but not free.*

Exercise 14.2 *Fix a point x_0 (say $(0,0)$) in $\mathbf{A}^2$ and show that $H := \{g \in Aff^+(\mathbf{A}^2) : gx_0 = x_0\}$ is a closed subgroup of $Aff^+(\mathbf{A}^2)$ isomorphic to $Sl(2)$.*

Exercise 14.3 *Let $H \cong Sl(2)$ be as in the previous exercise. Show that there is a natural 1-1 correspondence between the cosets of $Aff^+(\mathbf{A}^2)/H$ and the points of $\mathbf{A}^2$.*

Exercise 14.4 *Show that the bijection of the previous example is a homeomorphism if we give $Aff^+(\mathbf{A}^2)/H$ its quotient topology.*

Exercise 14.5 *Let S^2 be the unit sphere considered a subset of $\mathbb{R}^3$ in the usual way. Let the group $SO(3)$ act on S^2 by rotation. This action is clearly continuous and transitive. Let $n = (0,0,1)$ be the "north pole". Show that if H is the (closed) subgroup of $SO(3)$ consisting of all $g \in \mathrm{SO}(3)$ such that $g \cdot n = n$ then $x = g \cdot n \mapsto g$ gives a well defined bijection $S^2 \cong \mathrm{SO}(3)/H$. Note that $H \cong \mathrm{SO}(2)$ so we may write $S^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2)$.*

Let $\lambda : G \times M \to M$ be a left action and fix $x_0 \in M$. Denote the projection onto cosets by π and also write $r^{x_0} : g \longmapsto gx_0$. Then we have the following equivalence of maps

$$\begin{array}{ccc} G & = & G \\ \pi \downarrow & & \downarrow r^{x_0} \\ G/H & \cong & M \end{array}$$

Thus, in the transitive action case from the last theorem, we may as well assume that $M = G/H$ and then we have the literal equality $r^{x_0} = \pi$ and the left action is just $l_g : g_0H \mapsto hg_0H$ (or $gx \mapsto hx$ where $x \in G/H$). Continuing for now to make a distinction between M and G/H we note that the isotropy H also acts on M or equivalently on G/H. We denote this action by $I^{x_0} : H \times M \to M$ where $I^{x_0} : h \mapsto hx$. The equivalent action on G/H is $(h, gH) \mapsto hgH$. Of course I^{x_0} is just the restriction of $l : G \times M \to M$ to $H \times M \subset G \times M$ and $I_h^{x_0} = l_h$ for any $h \in H$ ($I_h^{x_0} =: I^{x_0}(h)$).

Exercise 14.6 *Let $H_1 := G_{x_1}$ (isotropy of x_1) and $H_2 := G_{x_2}$ (isotropy of x_2) where $x_2 = gx_1$ for some $g \in G$. Show that there is a natural Lie group isomorphisms $H_1 \cong H_2$ and a natural diffeomorphism $G/H_1 \cong G/H_2$ which is an equivalence of actions.*

For each $h \in H$ the map $I_h^{x_0} : M \to M$ fixes the point x_0 and so the differential $T_{x_0} I_h^{x_0}$ maps $T_{x_0}M$ onto itself. Let us abbreviate $T_{x_0} I_h^{x_0}$ to $\iota_{x_0}(h)$. For each h we have a linear automorphism $\iota_{x_0}(h) \in Gl(T_{x_0}M)$ and it is easy to check that $h \mapsto \iota_{x_0}(h)$ is a group representation on the space $T_{x_0}M$. This representation is called the **linear isotropy representation** (at x_0) and the group $I^{x_0}(H) \subset Gl(T_{x_0}M)$ is called the **linear isotropy subgroup**. On the other hand we for each $h \in H$ have another action $C_h : G \to G$ given by $g \longmapsto hgh^{-1}$ which fixes H and whose derivative is the adjoint map $Ad_h : \mathfrak{g} \to \mathfrak{g}$ and $h \longmapsto Ad_h$ is the adjoint representation defined earlier in the book. It is easy to see that the map $Ad_h : \mathfrak{g} \to \mathfrak{g}$ descends to a map $\widetilde{Ad_h} : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$. We are going to show that there is a natural isomorphism $T_{x_0}M \cong \mathfrak{g}/\mathfrak{h}$ such that for each $h \in H$ the following diagram commutes:

$$\begin{array}{ccc} \widetilde{Ad_h}: & \mathfrak{g}/\mathfrak{h} \to & \mathfrak{g}/\mathfrak{h} \\ & \downarrow & \downarrow \\ I_{x_0}: & T_{x_0}M \to & T_{x_0}M \end{array} \tag{14.1}$$

One way to state the meaning of this result is to say that $h \mapsto \widetilde{Ad_h}$ is a representation of H on the vector space $\mathfrak{g}/\mathfrak{h}$ which is equivalent to the linear isotropy representation. The isomorphism $T_{x_0}M \cong \mathfrak{g}/\mathfrak{h}$ is given in the following very natural way: Let $\xi \in \mathfrak{g}$ and consider $T_e\pi(\xi) \in T_{x_0}M$. If $\varsigma \in \mathfrak{h}$ then

$$T_e\pi(\xi + \varsigma) = T_e\pi(\xi) + T_e\pi(\varsigma) = T_e\pi(\xi)$$

and so $\xi \mapsto T_e\pi(\xi)$ induces a map on $\mathfrak{g}/\mathfrak{h}$. Now if $T_e\pi(\xi) = 0 \in T_{x_0}M$ then as you are asked to show in exercise 14.7 below $\xi \in \mathfrak{h}$ which in turn means that the induced map $\mathfrak{g}/\mathfrak{h} \to T_{x_0}M$ has a trivial kernel. As usual this implies that the map is in fact an isomorphism since $\dim(\mathfrak{g}/\mathfrak{h}) = \dim(T_{x_0}M)$. Let us now see why the diagram 14.1 commutes. Let us take a the scenic root to the conclusion since it allows us to see the big picture a bit better. First the following diagram clearly commutes:

$$\begin{array}{ccc} \exp t\xi & \overset{C_h}{\to} & h(\exp t\xi)h^{-1} \\ \pi\downarrow & & \pi\downarrow \\ (\exp t\xi)H & \overset{\tau_h}{\to} & h(\exp t\xi)H \end{array}$$

which under the identification $M = G/H$ is just

$$\begin{array}{ccc} \exp t\xi & \to & h(\exp t\xi)h^{-1} \\ \pi\downarrow & & \pi\downarrow \\ (\exp t\xi)x_0 & \to & h(\exp t\xi)x_0 \end{array}.$$

Applying the tangent functor (looking at the differential) we get the commutative diagram

$$\begin{array}{ccc} \xi & \to & Ad_h\xi \\ \downarrow & & \downarrow \\ T_e\pi(\xi) & \overset{d\tau_h}{\to} & T_e\pi(Ad_h\xi) \end{array}$$

and in turn

$$\begin{array}{ccc} [\xi] & \mapsto & \widetilde{Ad_h}([\xi]) \\ \downarrow & & \downarrow \\ T_e\pi(\xi) & \mapsto & T_e\pi(Ad_h\xi) \end{array}.$$

This latter diagram is in fact the element by element version of 14.1.

Exercise 14.7 *Show that* $T_e\pi(\xi) \in T_{x_0}M$ *implies that* $\xi \in \mathfrak{h}$.

Consider again homogenous space $\mathbf{E}^n$ with the action of $\mathrm{Euc}(n)$. This example has a special property that is shared by many important homogeneous spaces called reductivity.

`Note: More needs to be added here about reductive homogeneous spaces.`

14.1 Geometry of figures in Euclidean space

We take as our first example the case of a Euclidean space. We intend to study figures in $\mathbf{E}^n$ but we first need to set up some machinery using differential forms and moving frames. Let e_1, e_2, e_3 be a moving frame on $\mathbf{E}^3$. Using the identification of $T_x\mathbf{E}^3$ with $\mathbb{R}^3$ we may think of each e_i as a function with values in $\mathbb{R}^3$. If x denotes the identity map then we interpret dx as the map $T\mathbf{E}^3 \to \mathbb{R}^3$ given by composing the identity map on the tangent bundle $T\mathbf{E}^3$

with the canonical projection map $T\mathbf{E}^3 = \mathbf{E}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. If $\theta^1, \theta^2, \theta^3$ is the frame field dual to e_1, e_2, e_3 then we may write

$$dx = \sum e_i \theta^i. \tag{14.2}$$

Also, since we are interpreting each e_i as an $\mathbb{R}^3-$valued function we may take the componentwise exterior derivative to make sense of de_i. Then de_i is a vector valued differential form: If $e_i = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ then $de_i = df_1\mathbf{i} + df_2\mathbf{j} + df_3\mathbf{k}$. We may write

$$de_j = \sum e_i \omega^i_j \tag{14.3}$$

for some set of $1-$forms ω^i_j which we arrange in a matrix $\omega = (\omega^i_j)$. If we take exterior derivative of equations 14.2 and 14.3 For the first one we calculate

$$\begin{aligned} 0 = ddx &= \sum_{i=1}^n e_i \theta^i \\ &= \sum_{i=1}^n de_i \wedge \theta^i + \sum_{i=1}^n e_i \wedge d\theta^i \\ &= \sum_{i=1}^n \left(\sum_{j=1}^n e_j \omega^i_j \right) \wedge \theta^i + \sum_{i=1}^n e_i \wedge d\theta^i. \end{aligned}$$

From this we get the first of the following two structure equations. The second one is obtained similarly from the result of differentiating 14.3.

$$\begin{aligned} d\theta^i &= -\sum \omega^i_j \wedge \theta^j \\ d\omega^i_j &= -\sum \omega^i_k \wedge \omega^k_j \end{aligned} \tag{14.4}$$

Furthermore, if we differentiate $e_i \cdot e_j = \delta_{ij}$ we find out that $\omega^i_j = -\omega^j_i$.

If we make certain innocent identifications and conventions we can relate the above structure equations to the group $\mathrm{Euc}(n)$ and its Lie algebra. We will identify $\mathbf{E}^n$ with the set of column vectors of the form

$$\begin{bmatrix} 1 \\ x \end{bmatrix} \text{ where } x \in \mathbb{R}^n$$

Then the group $\mathrm{Euc}(n)$ is presented as the set of all square matrices of the form

$$\begin{bmatrix} 1 & 0 \\ v & Q \end{bmatrix} \text{ where } Q \in \mathrm{O}(n) \text{ and } v \in \mathbb{R}^n.$$

The action $\mathrm{Euc}(n) \times \mathbf{E}^n \to \mathbf{E}^n$ is then simply given by matrix multiplication (see chapter **??**). One may easily check that the matrix Lie algebra that we identify as the Lie algebra $\mathfrak{euc}(n)$ of $\mathrm{Euc}(n)$ consists of all matrices of the form

$$\begin{bmatrix} 0 & 0 \\ v & A \end{bmatrix} \text{ where } v \in \mathbb{R}^n \text{ and } A \in \mathfrak{so}(n) \text{ (antisymmetric matrices)}$$

The isotropy of the point $o := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is easily seen to be the subgroup G_o consisting of all elements of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} \text{ where } Q \in \mathrm{O}(n)$$

which is clearly isomorphic to $\mathrm{O}(n)$. This isotropy group G_o is just the group of rotations about the origin. The origin is not supposed to special and so we should point out that any point would work fine for what we are about to do. In fact, for any other point $x \sim \begin{bmatrix} 1 \\ x \end{bmatrix}$ we have an isomorphism $G_o \cong G_x$ given by $h \mapsto t_x h t_x$ where

$$t_x = \begin{bmatrix} 1 & 0 \\ x & I \end{bmatrix}.$$

(see exercise 14.6).

For each $x \in \mathbf{E}^n$ tangent space $T_x\mathbf{E}^n$ consists of pairs $x \times v$ where $v \in \mathbb{R}^n$ and so the dot product on $\mathbb{R}^n$ gives an obvious inner product on each $T_x\mathbf{E}^n$: For two tangent vectors in $T_x\mathbf{E}^n$, say $v_x = x \times v$ and $w_x = x \times w$ we have $\langle v_x, w_x \rangle = v \cdot w$.

Remark 14.1 *The existence of an inner product in each tangent space makes* $\mathbf{E}^n$ *a Riemannian manifold (a smoothness condition is also needed). Riemannian geometry (studied in chapter 16) is one possible generalization of Euclidean geometry (See figure* **??** *and the attendant discussion). Riemannian geometry represents an approach to geometry that is initially quite different in spirit from Klein's approach.*

Now think of each element of the frame $e_1, ..., e_n$ as a column vector of functions and form a matrix of functions e. Let x denote the "identity map" given as a column vector of functions $x = (x^1, x^2, x^3) = (x, y, z)$. Next we form the matrix of functions

$$\begin{bmatrix} 1 & 0 \\ x & e \end{bmatrix}$$

This just an element of $\mathrm{Euc}(n)$! Now we can see that the elements of $\mathrm{Euc}(n)$ are in a natural $1-1$ correspondents with the set of all frames on E^n and the matrix we have just introduce corresponds exactly to the moving frame $x \mapsto (e_1(x), ..., e_n(x))$. The differential of this matrix gives a matrix of one forms

$$\varpi = \begin{bmatrix} 0 & 0 \\ \theta & \omega \end{bmatrix}$$

and it is not hard to see that θ is the column consisting of the same θ^i as before and also that $\omega = (\omega^i_j)$. Also, notice that $x \mapsto \varpi(x) = \begin{bmatrix} 0 & 0 \\ \theta & \omega \end{bmatrix}$ takes values in the Lie algebra $\mathfrak{euc}(n)$. This looking like a very natural state of affairs. In fact, the structure equations are encoded as a singe matrix equation

$$d\varpi = \varpi \wedge \varpi.$$

The next amazingly cool fact it that if we pull-back ϖ to $\mathrm{Euc}(n)$ via the projection $\pi : \begin{bmatrix} 0 & 0 \\ x & e \end{bmatrix} \mapsto x \in \mathbf{E}^n$ then we obtain the Maurer-Cartan form of the group $\mathrm{Euc}(n)$ and the equation $d\varpi = \varpi \wedge \varpi$ pulls back to the structure equations for $\mathrm{Euc}(n)$.

Chapter 15

Connections and Covariant Derivatives

15.1 Definitions

The notion of a "connection" on a vector bundle and the closely related notion of "covariant derivative" can be approached in so many different ways that we shall only be able to cover a small portion of the subject. "covariant derivative" and "connection" are sometimes treated as synonymous. In fact, a covariant derivative is sometimes called a Kozsul connection (or even just a connection; the terms are conflated). At first, we will make a distinction but eventually we will use the word connection to refer to both concepts. In any case, the central idea is that of measuring the rate of change of fields on bundles in the direction of a vector or vector field on the base manifold.

A covariant derivative (Koszul connection) can either be defined as a map $\nabla : \mathfrak{X}(M) \times \Gamma(M, E) \to \Gamma(M, E)$ with certain properties from which one deduces a well defined map $\nabla : TM \times \Gamma(M, E) \to \Gamma(M, E)$ with nice properties or the other way around. We also hope that the covariant derivative is natural with respect to restrictions to open sets and so a sheaf theoretic definition could be given. For finite dimensional manifolds the several approaches are essentially equivalent.

Definition 15.1 (I) *A **Koszul connection** or **covariant derivative** on a $C^\infty-$vector bundle $E \to M$ is a map $\nabla : \mathfrak{X}(M) \times \Gamma(M, E) \to \Gamma(M, E)$ (where $\nabla(X, s)$ is written as $\nabla_X s$) satisfying the following four properties*

i) $\nabla_{fX}(s) = f\nabla_X s$ for all $f \in C^\infty$, $X \in \mathfrak{X}(M)$ and $s \in \Gamma(M, E)$

ii) $\nabla_{X_1+X_2} s = \nabla_{X_1} s + \nabla_{X_2} s$ for all $X_1, X_2 \in \mathfrak{X}(M)$ and $s \in \Gamma(M, E)$

iii) $\nabla_X(s_1 + s_2) = \nabla_X s_1 + \nabla_X s_2$ for all $X \in \mathfrak{X}(M)$ and $s_1, s_2 \in \Gamma(M, E)$

iv) $\nabla_X(fs) = (Xf)s + f\nabla_X(s)$ for all $f \in C^\infty$, $X \in \mathfrak{X}(M)$ and $s \in \Gamma(M, E)$

*For a fixed $X \in \mathfrak{X}(M)$ the map $\nabla_X : \Gamma(M, E) \to \Gamma(M, E)$ is called the **covariant derivative with respect to** X.*

As we will see below, for finite dimensional E and M this definition is enough to imply that ∇ induces maps $\nabla^U : \mathfrak{X}(U)\times\Gamma(U,E) \to \Gamma(U,E)$ that are naturally related in the sense we make precise below. Furthermore the value $(\nabla_X s)(p)$ depends only on the value X_p and only on the values of s along any smooth curve c representing X_p. The proof of these facts depends on the existence of cut-off functions. We have already developed the tools to obtain the proof easily in sections **??** and 6.5 and so we leave the verification of this to the reader. In any case we shall take a different route.

We would also like to be able to differentiate sections of a vector bundle along maps $f : N \to M$.

In the case of the tangent bundle of $\mathbb{R}^n$ one can identify vector fields with maps $\mathsf{X} : \mathbb{R}^n \to \mathbb{R}^n$ and thus it makes sense to differentiate a vector field just as we would a function. For instance, if $\mathsf{X} = (f^1, ..., f^n)$ then we can define the directional derivative in the direction of v at $p \in T_p\mathbb{R}^n \cong \mathbb{R}^n$ by $D_v\mathsf{X} = (D_pf^1 \cdot v, ..., D_pf^n \cdot v)$ and we get a vector in $T_p\mathbb{R}^n_\nu$ as an answer. Taking the derivative of a vector field seems to require involve the limit of difference quotient of the type

$$\lim_{t\to 0} \frac{X(p+tv) - X(p)}{t}$$

and yet how can we interpret this in a way that makes sense for a vector field on a general manifold? One problem is that $p + tv$ makes no sense if the manifold isn't a vector space. This problem is easily solve by replacing $p + tv$ by $c(t)$ where $\dot{c}(0) = v$ and $c(0) = p$. We still have the more serious problem that $X(c(t)) \in T_{c(t)}M$ while $X(p) = X(c(0)) \in T_pM$. The difficulty is that $T_{c(t)}M$ is not likely to be the same vector space as T_pM and so what sense does $X(c(t)) - X(p)$ make? In the case of a vector space (like $\mathbb{R}^n$) every tangent space is canonically isomorphic to the vector space itself so there is sense to be made of a difference quotient involving vectors from different tangent spaces. In order to get an idea of how we might define a covariant derivative on a general manifold, let us look again at the case of a submanifold M of $\mathbb{R}^n$. Let $X \in \mathfrak{X}(M)$ and $v \in T_pM$. Form a curve with $\dot{c}(0) = v$ and $c(0) = p$ and consider the composition $X \circ c$. Since every vector tangent to M is also a vector in $\mathbb{R}^n$ we can consider $X \circ c$ to take values in $\mathbb{R}^n$ and then take the derivative

$$\left.\frac{d}{dt}\right|_0 X \circ c.$$

This is well defined but while $X \circ c(t) \in T_{c(t)}M \subset T_{c(t)}\mathbb{R}^n$ we only know that $\left.\frac{d}{dt}\right|_0 X \circ c \in T_p\mathbb{R}^n$. It would be more nature if the result of differentiation of a vector field tangent to M should have been in T_pM. The simple solution is to take the orthogonal projection of $\left.\frac{d}{dt}\right|_0 X \circ c$ onto $T_{c(0)}M$. Denote this orthogonal projection of a vector onto its tangent part by $X \mapsto X^\top$. Our definition of a covariant derivative operator on M induced by D is then

$$\nabla_v X := \left(\left.\frac{d}{dt}\right|_0 X \circ c\right)^\top \in T_pM.$$

This turns out to be a very good definition. In fact, one may easily verify that we have the following properties:

1. (Smoothness) If X and Y are smooth vector fields then the map
$$p \mapsto \nabla_{X_p} Y$$
is also a smooth vector field on M. This vector filed is denoted $\nabla_X Y$.

2. (Linearity over $\mathbb{R}$ in second "slot") For two vector fields X and Y and any $a, b \in \mathbb{R}$ we have
$$\nabla_v(aX_1 + bX_2) = a\nabla_v X_1 + b\nabla_v X_2.$$

3. (Linearity over $C^\infty(M)$ in first "slot") For any three vector fields X, Y and Z and any $f, g \in C^\infty(M)$ we have
$$\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z.$$

4. (Product rule) For $v \in T_pM$, $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ we have
$$\begin{aligned}\nabla_v fX &= f(p)\nabla_v X + (vf)X(p) \\ &= f(p)\nabla_v X + df(v)X(p).\end{aligned}$$

5. $\nabla_v(X \cdot Y) = \nabla_v X \cdot Y + X \cdot \nabla_v Y$ for all v, X, Y.

Notice that if $p \mapsto \nabla_{X_p} Y$ is denoted by $\nabla_X Y$ then $\nabla : X, Y \longmapsto \nabla_X Y$ is a Koszul connection 15.1. But is 15.1 the best abstraction of the properties we need? In finding the correct abstraction we could use the properties of our example on the tangent bundle of a submanifold of $\mathbb{R}^n$ with the aim of defining a so called covariant derivative it is a bit unclear whether we should define $\nabla_X Y$ for a pair of fields X, Y or define $\nabla_v X$ for a tangent vector v and a field X. Different authors take different approaches but it turns out that one can take either approach and, at least in finite dimensions, end up with equivalent notions. We shall make the following our basic definition of a covariant derivative.

Definition 15.2 *A* ***natural covariant derivative*** ∇ *on a smooth vector bundle* $E \to M$ *is an assignment to each open set* $U \subset M$ *of a map* $\nabla^U : \mathfrak{X}(U) \times \Gamma(U, E) \to \Gamma(U, E)$ *written* $\nabla^U : (X, \sigma) \to \nabla^U_X \sigma$ *such that the following hold:*

1. $\nabla^U_X \sigma$ *is* $C^\infty(U)$*-linear in* X,
2. $\nabla^U_X \sigma$ *is* $\mathbb{R}$*-linear in* σ,
3. $\nabla^U_X(f\sigma) = f\nabla^U_X \sigma + (Xf)\sigma$ *for all* $X \in \mathfrak{X}(U)$, $\sigma \in \Gamma(U, E)$ *and all* $f \in C^\infty(U)$.

4. *If $V \subset U$ then $r_V^U(\nabla_X^U \sigma) = \nabla_{r_V^U X}^V r_V^U \sigma$ (naturality with respect to restrictions $r_V^U : \sigma \mapsto \sigma|_V$).*

5. *$(\nabla_X^U Y)(p)$ only depends of the value of X at p (infinitesimal locality).*

Here $\nabla_X^U Y$ is called the covariant derivative of Y with respect to X. We will denote all of the maps ∇^U by the single symbol ∇ when there is no chance of confusion.

We have worked naturality with respect to restrictions and infinitesimal locality into the definition of a natural covariant derivative in order to have a handy definition for infinite dimensional manifolds which circumvents the problem of localization. We will show that on finite dimensional manifolds a Koszul connections induces a natural covariant derivative. We have the following intermediate result still stated for possibly infinite dimensional manifolds.

Lemma 15.1 *Suppose that M admits cut-off functions and $\nabla : \mathfrak{X}(M)\times\Gamma(E) \to \Gamma(E)$ is such that (1), (2), and (3) hold (for $U = M$). Then if on some open U either $X = 0$ or $\sigma = 0$ then*

$$(\nabla_X \sigma)(p) = 0 \text{ for all } p \in U.$$

Proof. We prove the case of $\sigma|_U = 0$ and leave the case of $X|_U = 0$ to the reader.

Let $q \in U$. Then there is some function f that is identically one on a neighborhood $V \subset U$ of q and that is zero outside of U thus $f\sigma \equiv 0$ on M and so since ∇ is linear we have $\nabla(f\sigma) \equiv 0$ on M. Thus since (3) holds for global fields we have

$$\begin{aligned}\nabla(f\sigma)(q) &= f(p)(\nabla_X \sigma)(q) + (X_q f)\sigma(q) \\ &= (\nabla_X \sigma)(q) = 0.\end{aligned}$$

Since $q \in U$ was arbitrary we have the result. ■

In the case of finite dimensional manifolds we have

Proposition 15.1 *Let M be a finite dimensional smooth manifold. Suppose that there exist an operator $\nabla : \mathfrak{X}(M)\times\Gamma(E) \to \Gamma(E)$ such that (1), (2), and (3) hold (for $U = M$) for example a Koszul connection. If we set $\nabla_X^U \sigma := r_U^M(\nabla_{\widetilde{X}}^M \widetilde{\sigma})$ for any extensions $\widetilde{X}$ and $\widetilde{\sigma}$of X and $\sigma \in \Gamma(U, E)$ to global fields then $U \mapsto \nabla^U$ is a natural covariant derivative.*

Proof. By the previous lemma $\nabla_X^U \sigma := r_U^M(\nabla_{\widetilde{X}}^M \widetilde{\sigma})$ is a well defined operator that is easily checked to satisfy (1), (2), (3), and (4) of definition 15.2. We now prove property (5). Let $\alpha \in T^*E$ and fix $\sigma \in \Gamma(U, E)$. define a map $\mathfrak{X}(U) \to C^\infty(U)$ by $X \mapsto \alpha(\nabla_X^U \sigma)$. By theorem J.4 we see that $\alpha(\nabla_X^U \sigma)$ depend only on the value of X at $p \in U$. ■

Since many authors only consider finite dimensional manifolds they define a covariant to be a map $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ satisfying (1), (2), and (3).

It is common to write expressions like $\nabla_{\frac{\partial}{\partial x^i}}\sigma$ where σ is a global field and $\frac{\partial}{\partial x^i}$ is defined only on a coordinate domain U. This still makes sense as a field $p \mapsto \nabla_{\frac{\partial}{\partial x^i}(p)}\sigma$ on U by virtue of (5) or by interpreting $\nabla_{\frac{\partial}{\partial x^i}}\sigma$ as $\nabla_{\frac{\partial}{\partial x^i}}\sigma|_U$ and invoking (4) if necessary.

In the next section we indicate a local condition which guarantees that we will have a natural covariant derivative on $\pi : E \to M$. We now introduce the notion of a system of connection forms.

15.2 Connection Forms

Let $\pi : E \to M$ be a rank r vector bundle with a connection ∇. Recall that a choice of a local frame field over an open set $U \subset M$ is equivalent to a trivialization of the restriction $\pi_U : E|_U \to U$. Namely, if $\phi = (\pi, \Phi)$ is such a trivialization over U then defining $e_i(x) = \phi^{-1}(x, \mathsf{e}_i)$ where (e_i) is the standard basis of $\mathbb{F}^n$ we have that $\sigma = (e_1, ..., e_k)$. We now examine the expression for the connection from the viewpoint of such a local frame. It is not hard to see that for every such frame field there must be a matrix of 1-forms $A_\sigma = (A^i_j)_{1\le i,j\le r}$ such that for $X \in \Gamma(U, E)$ we may write

$$\nabla_X e_i = A^b_i(X)e_b. \text{ (sum over } b)$$

For a section $s = s^i e_i$

$$\begin{aligned}
\nabla_X s &= \nabla_X(s^i e_i)\\
&= (Xs^a)e_a + s^a\nabla_X e_a\\
&= (Xs^i)e_i + s^i A^j_i(X)e_j\\
&= (Xs^i)e_i + s^r A^i_r(X)e_i\\
&= (Xs^i + A^i_r(X)s^r)e_i
\end{aligned}$$

So the $a-$component of $\nabla_X s$ is $(\nabla_X s)^i = Xs^i + A^i_r(X)s^r$. Of course the frames are defined only locally say on some open set U. We may surely choose U small enough that it is also the domain of a moving frame $\{E_\mu\}$ for M. Thus we have $\nabla_{E_\mu} e_j = \Gamma^k_{\mu j}e_k$ where $A^k_{\mu j} = A^j_i(E_\mu)$. We now have the formula

$$\nabla_X s = (X^\mu E_\mu s^i + X^\mu A^i_r(E_\mu)s^r)e_i$$

and in the usual case where $E_\mu = \partial_\mu$ is a holonomic frame associated to a chart then $\nabla_X s = (X^\mu\partial_\mu s^i + X^\mu A^i_{\mu j}s^r)e_i$.

Now suppose that we have two moving frames whose domains overlap, say $\sigma = (e_j)$ and $\sigma' = (e'_j)$. Let us examine how the matrix of forms $A_\sigma = \left(A^j_i\right)$ is related to the forms $A_{\sigma'} = \left(A'^j_i\right)$ on this open set. The change of frame is

$$\sigma'^b = g^b_a\sigma_b$$

which in matrix notation is

$$\sigma' = \sigma g.$$

for some smooth $g : U \cap U' \to Gl(n)$. Differentiating both sides

$$\begin{aligned}
\sigma' &= \sigma g \\
\nabla \sigma' &= \nabla(\sigma g) \\
\sigma' A' &= (\nabla \sigma) g + \sigma dg \\
\sigma' A' &= \sigma g g^{-1} A g + \sigma g g^{-1} dg \\
\sigma' A' &= f \sigma' g^{-1} A g + \sigma' g^{-1} dg \\
A' &= g^{-1} A g + g^{-1} dg
\end{aligned}$$

Conversely, we have the following theorem which we state without proof:

Theorem 15.1 *Let $\pi : E \to M$ be a smooth $\mathbb{F}-$vector bundle of rank k. Suppose that for every moving frame $\sigma = (e_1, ..., e_k)$ we are given a matrix of $1-$forms A_σ so that $A_{\sigma g} = g^{-1} A_\sigma g + g^{-1} dg$. Then there is a unique covariant derivative ∇ on $\pi : E \to M$ such that for a moving frame σ on U_σ*

$$\nabla_X s = (X s^j + A_r^j(X) s^r) e_j$$

for $s = \sum s^j e_j$ and where $A_\sigma = (A_r^j)$.

Definition 15.3 *A family of matrix valued 1-forms related as in the previous theorem will be called a system of* ***connection forms****.*

Sometimes one hears that the A_σ are locally elements of $\Omega(M, \mathrm{End}(E))$ but the transformation law just discovered says that we cannot consider the forms A_σ as coming from a section of $\mathrm{End}(E)$. However, we have the following:

Proposition 15.2 *If A_α, U_α and A_α, U_α are two systems of connection forms for the same cover then the difference forms $\triangle A_\alpha = A_\alpha - A'_\alpha$ are the local expressions of a global $\triangle A \in \Omega(M, \mathrm{End}(E))$.*

$$\begin{aligned}
\triangle A_\alpha &= A_\alpha - A'_\alpha \\
&= g^{-1} A_\alpha g + g^{-1} dg - (g^{-1} A'_\alpha g + g^{-1} dg) \\
&= g^{-1} A_\alpha g - g^{-1} A'_\alpha g \\
&= g^{-1} (\triangle A_\alpha) g
\end{aligned}$$

so that the forms $\triangle A_\alpha$ are local expressions of a global section of $\Omega(M, \mathrm{End}(E))$

Exercise 15.1 *Show that the set of all connections on E is naturally an affine space $C(E)$ whose vector space of differences is* $\mathrm{End}(E)$. *For any fixed connection ∇^0 we have an affine isomorphism* $\mathrm{End}(E) \to C(E)$ *given by $\triangle A \mapsto \nabla^0 + \triangle A$.*

Now in the special case mentioned above for a trivial bundle, the connection form in the defining frame is zero and so in that case $\triangle A_\alpha = A_\alpha$. So in this case A_α determines a section of $\mathrm{End}(E)$. Now any bundle is locally trivial so in this sense A_α is locally in $\mathrm{End}(E)$. But this is can be confusing since we have changed (by force so to speak) the transformation law for A_α among frames defined on the same open set to that of a difference $\triangle A_\alpha$ rather than A_α. The point is that even though $\triangle A_\alpha$ and A_α are equal in the distinguished frame they are not the same after transformation to a new frame.

In the next section we give another way to obtain a covariant derivative which is more geometric and global in character.

15.3 Vertical and Horizontal

One way to get a natural covariant derivative on a vector bundle is through the notion of connection in another sense. To describe this we will need the notion of a vertical bundle. We give the definition not just for vector bundles but also for a general fiber bundle as this will be useful to us later.

Lemma 15.2 *Let $E \xrightarrow{\pi} M$ be a fiber bundle; fix $p \in M$. Set $N := E_p$, and let $\imath : N \hookrightarrow E$ be the inclusion. For all $\xi \in N$,*

$$T_\xi \imath\,(N_\xi) = \ker[T_\xi \imath : T_\xi E \to M_p] = (T_\xi \pi)^{-1}\,(0_p) \subset T_\xi E$$

where $0_p \in M_p$ is the zero vector. If $\varphi : N \to F$ is a diffeomorphism and $\mathtt{x}$ a chart on an open set V in F, then for all $\xi \in \varphi^{-1}(V)$, $d\mathtt{x} \circ T_\xi \varphi$ maps $T_\xi N$ isomorphically onto $\mathbb{R}^m$, where $m := \dim F$.

Proof. $\pi \circ \imath \circ \gamma$ is constant for each C^∞ curve γ in N, so $T\pi \cdot (T\imath \cdot \dot\gamma(0)) = 0_p$; thus $T_\xi \imath\,(N_\xi) \subset (T_\xi \pi)^{-1}\,(0_p)$. On the other hand,

$$\dim((T_\xi \pi)^{-1}\,(0_p)) = \dim E - \dim M = \dim F = \dim N_\xi$$

so$(T_\xi \imath)\,(N) = (T_\xi \pi)^{-1}\,(0_p)$. The rest follows since $d\mathtt{x} = pr_2 \circ T\mathtt{x}$. ∎

Let $\mathcal{V}_p E := (T_\xi \pi)^{-1}\,(0_p)$ and $\mathcal{V}_p E$.

Definition 15.4 *Let $\pi : E \to M$ be a fiber bundle with typical fiber F and $\dim F = m$. The **vertical bundle on** $\pi : E \to M$ is the real vector bundle $\pi_{\mathcal{V}} : \mathcal{V}\,E \to E$ with total space defined by the disjoint union*

$$\mathcal{V}\,E := \bigsqcup_{p \in E} \mathcal{V}_p E \subset TE$$

The projection map is defined by the restriction $\pi_{\mathcal{V}} := T\pi|\,\mathcal{V}\,E$. A vector bundle atlas on $\mathcal{V}E$ consists of the vector bundle charts of the form

$$(\pi_{\mathcal{V}}, d\mathtt{x} \circ T\phi) : \pi_V^{-1}(\pi^{-1}U \cap \phi^{-1}V) \to (\pi^{-1}U) \cap \phi^{-1}V) \times \mathbb{R}^m$$

where $\Phi = (\pi, \phi)$ is a bundle chart on E over U and $\mathtt{x}$ is a chart on V in F.

Exercise 15.2 *Prove: Let $f : N \to M$ be a smooth map and $\pi : E \to M$ a fiber bundle with typical fiber F. Then $\mathcal{V}f^*E \to N$ is bundle isomorphic to $pr_2^*\mathcal{V}E \to N$.* ***Hint:*** *It is enough to prove that $\mathcal{V}f^*E$ is isomorphic to $\mathcal{V}E$ along the map $pr_2 : f^*E \to E$. Also check that $(Tf)^* E = \{(u,v) \in TN \times TE | Tf \cdot u = T\pi \cdot v\}$.*

Suppose now that $\pi : E \to M$ is a vector bundle. The vertical vector bundle $\mathcal{V}E$ is isomorphic to the vector bundle π^*E over E (we say that $\mathcal{V}E$ is isomorphic to E along π). To see this note that if $(\zeta, \xi) \in \pi^*E$, then $\pi(\zeta + t\xi)$ is constant in t. From this we see that the map from π^*E to TE given by $(\zeta, \xi) \mapsto d/dt|_0(\zeta + t\xi)$ maps into $\mathcal{V}E$. It is easy to see that this map (real) vector bundle isomorphism. Let $\mathcal{V}E$ be the vertical bundle on a vector bundle E over M. Denote the isomorphism from π^*E to $\mathcal{V}E$ by j:

$$j : \pi^*E \to \mathcal{V}\mathcal{E} \quad (\zeta, \xi) \mapsto \frac{d}{dt}_0(\zeta + t\xi) =: \xi_\zeta$$

Denote the vector bundle isomorphism from $\mathcal{V}E$ to E along π by pr_2 :

$$pr_2 : \mathcal{V}E \to E \quad \xi_\zeta \mapsto \xi$$

The map $pr_2 : \mathcal{V}E \to E$ pick off the "second component" ξ of ξ_ζ, whereas the projection map $\pi_\mathcal{V} : \mathcal{V}E \to E$ yields the "first component" ζ. The reader should think of ξ_ζ and (ζ, ξ) as the "same object".

Definition 15.5 *A connection on a vector bundle $\pi : E \to M$ is a smooth distribution $\mathcal{H}$ on the total space E such that*

(i) $\mathcal{H}$ is complementary to the vertical bundle:

$$TE = \mathcal{H} \oplus \mathcal{V}E$$

(ii) $\mathcal{H}$ is homogeneous: $T_\xi \mu_r (\mathcal{H}_\xi) = \mathcal{H}_{r\xi}$ for all $\xi \in \mathcal{H}$, $r \in \mathbb{R}$ and where μ_r is the multiplication map $\mu_r : \xi \mapsto r\xi$. The subbundle $\mathcal{H}$ is called the horizontal subbundle (or horizontal distribution).

Any $\xi \in TE$ has a decomposition $\xi = \mathcal{H}v + \mathcal{V}v$. Here, $\mathcal{H} : v \mapsto \mathcal{H}v$ and $\mathcal{V} : v \mapsto \mathcal{V}v$ are the obvious projections. An individual element $v \in T_\xi E$ is **horizontal** if $v \in \mathcal{H}_\xi$ and **vertical** if $v \in \mathcal{V}_\xi E$. A vector field $\widetilde{X} \in \mathfrak{X}(E)$ is said to be a **horizontal vector field** (resp. **vertical vector field**) if $\widetilde{X}(\xi) \in \mathcal{H}_\xi$ (resp. $\widetilde{X}(\xi) \in \mathcal{V}_\xi E$) for all $\xi \in E$.

A map $f : N \to E$ is said to be **horizontal** or parallel if $Tf \cdot v$ is horizontal for all $v \in TN$. The most important case comes when we start with a section of E along a map $f : N \to M$. A horizontal lift of f is a section $\widetilde{f}$ along f such that $\widetilde{f}$ is horizontal; in other words, the following diagram commutes and $T\widetilde{f}$ has image in $\mathcal{H}$:

$$\begin{array}{ccc} & & E \\ & \widetilde{f} \nearrow & \downarrow \\ N & \rightarrow & M \end{array}$$

Now if we have a connection on $\pi : E \to M$ we have a so called connector (connection map) which is the map $\kappa : TE \to E$ defined by $\kappa := pr_2(\mathcal{V}v)$. The connector is a vector bundle homomorphism along the map $\pi : E \to M$:

$$\begin{array}{ccc} TE & \xrightarrow{\kappa} & E \\ \downarrow & & \downarrow \\ E & \xrightarrow{\pi} & M \end{array}$$

Now an interesting fact is that TE is also a vector bundle over TM (via the map $T\pi$) and κ gives a vector bundle homomorphism along $\pi_{TM} : TM \to M$. More precisely, we have the following

Theorem 15.2 *If κ is the connector for a connection on a vector bundle $\pi : E \to M$ then κ gives a vector bundle homomorphism from the bundle $T\pi : TE \to TM$ to the bundle $\pi : E \to M$.*

$$\begin{array}{ccc} TE & \xrightarrow{\kappa} & E \\ \downarrow & & \downarrow \\ TM & \to & M \end{array}$$

Now using this notion of connection with associated connector κ we can get a natural covariant derivative.

Definition 15.6 *If $\mathcal{H}$ is a connection on $E \to M$ with connector κ then we define the* ***covariant derivative*** *of a section $\sigma \in \Gamma_f(E)$ along a map $f : N \to M$ with respect to $v \in T_pN$ by*

$$\nabla_v^f \sigma := \kappa\left(T_p\sigma \cdot v\right).$$

If V is a vector field on N then $\left(\nabla_V \sigma\right)(p) := \nabla_{V(p)}\sigma$.

Now let $\epsilon_1, ..., \epsilon_n$ be a frame field defined over $U \subset M$. Since f is continuous $O = f^{-1}(U)$ is open and on O, the $\epsilon_1 \circ f, ..., \epsilon_n \circ f$ are fields along f and so locally (on O) we may write $\sigma = \sigma^i\left(\epsilon_i \circ f\right)$ for some functions $\sigma^a : U \subset N \to \mathbb{R}$.

For any $p \in O$ and $v \in T_pN$ we have

$$\nabla_v^f \sigma := \left(d\sigma^a \cdot v\right)\epsilon_a(f(p)) + \left.A_r^a\right|_p (Tf \cdot v)\, \sigma^r(p)\epsilon_a(f(p))$$

Exercise 15.3 *Prove this last statement.*

Exercise 15.4 *Prove that if $f = id_M$ then for $v \in TM$, $\nabla_v : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a Koszul connection and that we recover our previous results.*

If all we had were the connection forms we could use the above formula as the definition of $\nabla_v^f \sigma$. Since f might not be even be an immersion this definition would only make sense because of the fact that it is independent of the frame.

In the same way that one extends a derivation to a tensor derivation one may show that a covariant derivative on a vector bundle induces naturally related covariant derivatives on all the multilinear bundles. In particular, $\Pi : E^* \to M$

denotes the dual bundle to $\pi : E \to M$ we may define connections on $\Pi : E^* \to M$ and on $\pi \otimes \Pi : E \otimes E^* \to M$. We do this in such a way that for $s \in \Gamma(M, E)$ and $s^* \in \Gamma(M, E^*)$ we have

$$\nabla_X^{E \otimes E^*}(s \otimes s^*) = \nabla_X s \otimes s^* + s \otimes \nabla_X^{E^*} s^*$$

and

$$(\nabla_X^{E^*} s^*)(s) = X(s^*(s)) - s^*(\nabla_X s).$$

Of course this last formula follows from our insistence that covariant differentiation commutes with contraction:

$$\begin{aligned} X(s^*(s)) &= \\ (\nabla_X C(s \otimes e^*)) &= C(\nabla_X^{E \otimes E^*}(s \otimes s^*)) \\ &= C\left(\nabla_X s \otimes s^* + s \otimes \nabla_X^{E^*} s^*\right) \\ &= s^*(\nabla_X s) + (\nabla_X^{E^*} s^*)(s) \end{aligned}$$

where C denotes the contraction $s \otimes \alpha \mapsto \alpha(s)$. All this works like the tensor derivation extension procedure discussed previously.

Now the bundle $E \otimes E^* \to M$ is naturally isomorphic to $\mathrm{End}(E)$ and by this isomorphism we get a connection on $\mathrm{End}(E)$.

$$(\nabla_X A)(s) = \nabla_X(A(s)) - A(\nabla_X s)$$

Indeed, since $c : s \otimes A \mapsto A(s)$ is a contraction we must have

$$\begin{aligned} \nabla_X(A(s)) &= c\left(\nabla_X s \otimes A + s \otimes \nabla_X A\right) \\ &= A(\nabla_X s) + (\nabla_X A)(s) \end{aligned}$$

15.4 Parallel Transport

Once again let $E \to M$ be a vector bundle with a connection given by a system of connection forms $\{\overset{\alpha}{A}\}_\alpha$. Suppose we are given a curve $c : I \to M$ together with a section along c; that is a map $\sigma : I \to E$ such that the following diagram commutes:

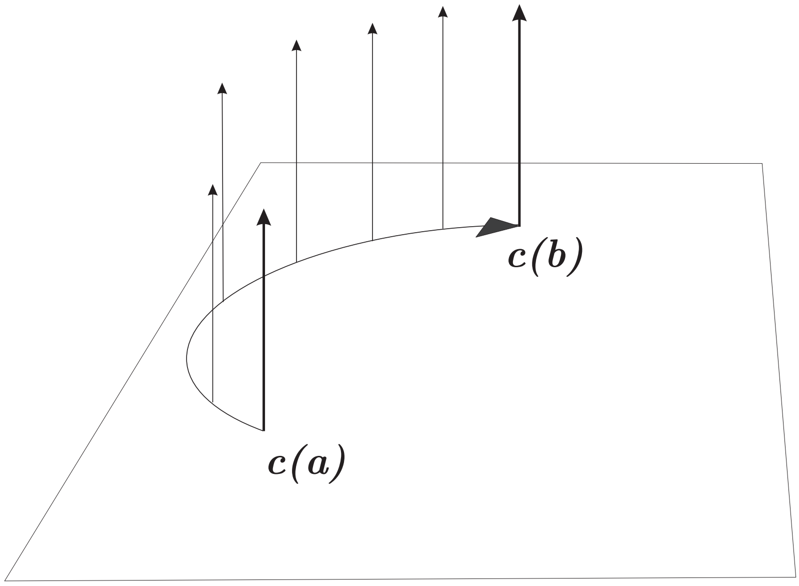
$$\begin{array}{ccc} & & E \\ & \overset{\sigma}{\nearrow} & \downarrow \\ I & \xrightarrow{c} & M \end{array}$$

Motivation: If c is an integral curve of a field X then we have

$$\begin{aligned} (\nabla_X s)(c(t)) &= (X(c(t)).s^a(c(t)) + (\left.A_r^a\right|_{c(t)} X_{c(t)})s^r(c(t)))\epsilon_a(c(t)) \\ &= (\dot{c}(t).s^a(t))\epsilon_a(t) + (\left.A_r^a\right|_{c(t)} \dot{c}(t))s^r(t)\epsilon_a(t) \end{aligned}$$

where we have abbreviated $s^a(t) := s^a(c(t))$ and $\epsilon_a(t) := \epsilon_a(c(t))$. As a special case of covariant differentiation along a map we have

$$\nabla_{\partial_t}\sigma := (\frac{d}{dt}\sigma^a(t))\epsilon_a(t) + (\, A^a_r|_{c(t)}\,\dot{c}(t))\sigma^r(t)\epsilon_a(t)$$



Simple schematic of parallel transport

Now on to the important notion of parallel transport.

Definition 15.7 *Let $c : [a,b] \to M$ be a smooth curve. A section σ along c is said to be* ***parallel along*** *c if*

$$\frac{\nabla\sigma}{dt}(t) := (\nabla_{\partial_t}\sigma)(t) = 0 \textit{ for all } t \in [a,b].$$

Similarly, a section $\sigma \in \Gamma(M,E)$ is said to be ***parallel*** *if $\nabla_X\sigma = 0$ for all $X \in \mathfrak{X}(M)$.*

Exercise 15.5 *Show that if $t \mapsto X(t)$ is a curve in E_p then we can consider X a vector field along the constant map $p : t \mapsto p$ and then $\nabla_{\partial_t}X(t) = X'(t) \in E_p$.*

Exercise 15.6 *Show that $\sigma \in \Gamma(M,E)$ is a parallel section if and only if $X \circ c$ is parallel along c for every curve $c : I \to M$*

Exercise 15.7 *Show that for $f : I \to \mathbb{R}$ and $\sigma : I \to M$ is a section of E along c then $\nabla_{\partial_t}(f\sigma) = \frac{df}{dt}\sigma + f\nabla_{\partial_t}\sigma$.*

Exercise 15.8 *Show that if $\sigma : I \to U \subset M$ where U is the domain of a local frame field $\{\epsilon_1, ..., \epsilon_k\}$ then $\sigma(t) = \sum_{i=1}^{k}\sigma^i(t)\epsilon_i(c(t))$. Write out the local expression in this frame field for $\nabla_v\sigma$ where $v = v^j\epsilon_j$.*

Theorem 15.3 *Given a smooth curve $c : [a, b] \to M$ and numbers $t_0 \in [a, b]$ with $c(t_0) = p$ and vector $v \in E_p$ there is a unique parallel section σ_c along c such that $\sigma_c(t_0) = v$.*

Proof. In local coordinates this reduces to a first order initial value problem that may be shown to have a unique smooth solution. Thus if the image of the curve lies completely inside a coordinate chart then we have the result. The general result follows from patching these together. This is exactly what we do below when we generalize to piecewise smooth curves so we will leave this last part of the proof to the skeptical reader. ■

Under the conditions of this last theorem the value $\sigma_c(t)$ is a vector in the fiber $E_{c(t)}$ and is called the **parallel transport** or **parallel translation** of v along c from $c(t_0)$ to $c(t)$. Let us denote this by $P(c)_{t_0}^{t} v$. Next we suppose that $c : [a, b] \to M$ is a (continuous) piecewise smooth curve. Thus we may find a monotonic sequence $t_0, t_1, ...t_j = t$ such that $c_i := c|_{[t_{i-1}, t_i]}$ (or $c|_{[t_i, t_{i-1}]}$) is smooth[1]. In this case we define

$$P(c)_{t_0}^{t} := P(c)_{t_{j-1}}^{t} \circ \cdots \circ P(c)_{t_0}^{t_1}$$

Now given $v \in E_{c(t_0)}$ as before, the obvious sequence of initial value problems gives a unique piecewise smooth section σ_c along c such that $\sigma_c(t_0) = v$ and the solution must clearly be $P(c)_{t_0}^{t} v$ (why?).

Exercise 15.9 *$P(c)_{t_0}^{t} : E_{c(t_0)} \to E_{c(t)}$ is a linear isomorphism for all t with inverse $P(c)_{t}^{t_0}$ and the map $t \mapsto \sigma_c(t) = P(c)_{t_0}^{t} v$ is a section along c that is smooth wherever c is smooth.*

We may approach the covariant derivative from the direction of parallel transport. Indeed some authors given an axiomatic definition of parallel transport, prove its existence and then use it to define covariant derivative. For us it will suffice to have the following theorem:

Theorem 15.4 *For any smooth section σ of E defined along a smooth curve $c : I \to M$. Then we have*

$$(\nabla_{\partial_t} \sigma)(t) = \lim_{\varepsilon \to 0} \frac{P(c)_{t+\varepsilon}^{t} \sigma(t + \varepsilon) - \sigma(t)}{\varepsilon}$$

Proof. Let $e_1, ..., e_k$ be a basis of $E_{c(t_0)}$ for some fixed $t_0 \in I$. Let $\epsilon_i(t) := P(c)_{t_0}^{t} e_i$. Then $\nabla_{\partial_t} \epsilon_i(t) \equiv 0$ and $\sigma(t) = \sum \sigma^i(t) e_i(t)$. Then

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{P(c)_{t+\varepsilon}^{t} \sigma(t + \varepsilon) - \sigma(t)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\sigma^i(t + \varepsilon) P(c)_{t+\varepsilon}^{t} \epsilon_i(t + \varepsilon) - \sigma(t)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\sigma^i(t + \varepsilon) \epsilon_i(t) - \sigma^i(t) e_i(t)}{\varepsilon} \\
&= \sum \frac{d\sigma^i}{dt}(t) \epsilon_i(t).
\end{aligned}$$

[1] It may be that $t < t_0$.

On the other hand

$$
\begin{aligned}
(\nabla_{\partial_t}\sigma)(t) &= \nabla_{\partial_t}(\sigma^i(t)\epsilon_i(t)) \\
&= \sum \frac{d\sigma^i}{dt}(t)\epsilon_i(t) + \sum \sigma^i(t)\nabla_{\partial_t}\epsilon_i(t) \\
&= \sum \frac{d\sigma^i}{dt}(t)\epsilon_i(t).
\end{aligned}
$$

■

Exercise 15.10 *Let ∇ be a connection on $E \to M$ and let $\alpha : [a,b] \to M$ and $\beta : [a,b] \to E_{\alpha(t_0)}$. If $X(t) := P(\alpha)_{t_0}^t(\beta(t))$ then*

$$
\left(\nabla_{\partial_t}X\right)(t) = P(\alpha)_{t_0}^t(\beta'(t))
$$

Note: $\beta'(t) \in E_{\alpha(t_0)}$ for all t.

An important fact about covariant derivatives is that they don't need to commute. If $\sigma : M \to E$ is a section and $X \in \mathfrak{X}(M)$ then $\nabla_X\sigma$ is a section also and so we may take it's covariant derivative $\nabla_Y\nabla_X\sigma$ with respect to some $Y \in \mathfrak{X}(M)$. In general, $\nabla_Y\nabla_X\sigma \neq \nabla_X\nabla_Y\sigma$ and this fact has an underlying geometric interpretation which we will explore later. A measure of this lack of commutativity is the **curvature operator** that is defined for a pair $X, Y \in \mathfrak{X}(M)$ to be the map $F(X,Y) : \Gamma(E) \to \Gamma(E)$ defined by

$$
\begin{aligned}
F(X,Y)\sigma &:= \nabla_X\nabla_Y\sigma - \nabla_Y\nabla_X\sigma - \nabla_{[X,Y]}\sigma. \\
&\text{i.e.} \\
F(X,Y)\sigma &:= [\nabla_X, \nabla_Y]\sigma - \nabla_{[X,Y]}\sigma
\end{aligned}
$$

Theorem 15.5 *For fixed σ the map $(X,Y) \mapsto F(X,Y)\sigma$ is $C^\infty(M)$ bilinear and antisymmetric.*
$F(X,Y) : \Gamma(E) \to \Gamma(E)$ is a $C^\infty(M)$ module homomorphism; that is it is linear over the smooth functions:

$$
F(X,Y)(f\sigma) = fF(X,Y)(\sigma)
$$

Proof. We leave the proof of the first part as an easy exercise. For the second part we just calculate:

$$
\begin{aligned}
F(X,Y)(f\sigma) &= \nabla_X\nabla_Y f\sigma - \nabla_Y\nabla_X f\sigma - \nabla_{[X,Y]}f\sigma \\
&= \nabla_X(f\nabla_Y\sigma + (Yf)\sigma) - \nabla_Y(f\nabla_X\sigma + (Xf)\sigma) \\
&- f\nabla_{[X,Y]}\sigma - ([X,Y]f)\sigma \\
&= f\nabla_X\nabla_Y\sigma + (Xf)\nabla_Y\sigma + (Yf)\nabla_X\sigma + X(Yf) \\
&- f\nabla_Y\nabla_X\sigma - (Yf)\nabla_X\sigma - (Xf)\nabla_Y\sigma - Y(Xf) \\
&- f\nabla_{[X,Y]}\sigma - ([X,Y]f)\sigma \\
&= f[\nabla_X, \nabla_Y] - f\nabla_{[X,Y]}\sigma = fF(X,Y)\sigma
\end{aligned}
$$

■

Thus we also have F as a map $F : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \Gamma(M, \mathrm{End}(E))$. But then since R is tensorial in the first two slot and antisymmetric we also may think in the following terms

$$F \in \Gamma(M, \mathrm{End}(E) \otimes \wedge^2 M)$$
$$\text{or}$$
$$F \in \Gamma(M, E \otimes E^* \otimes \wedge^2 M).$$

In the current circumstance it is harmless to identify $E \otimes E^* \otimes \wedge^2 M$ with $\wedge^2 M \otimes E \otimes E^*$ the second one seems natural too although when translating into matrix notation the first is more consistent.

We will have many occasions to consider differentiation along maps and so we should also look at the curvature operator for sections along maps:

Let $f : N \to M$ be a smooth map and σ a section of $E \to M$ along f. For $U, V \in \mathfrak{X}(N)$ we define a map $F^f(U, V) : \Gamma_f(E) \to \Gamma_f(E)$ by

$$F^f(U, V)X := \nabla_U \nabla_V X - \nabla_V \nabla_U X - \nabla_{[U,V]} X$$

for all $X \in \Gamma_f(E)$.

Notice that if $X \in \Gamma(E)$ then $X \circ f \in \Gamma_f(E)$ and if $U \in \mathfrak{X}(N)$ then $Tf \circ U \in \Gamma_f(TM) := \mathfrak{X}_f(M)$ is a vector field along f. Now as a matter of notation we let $F(Tf \circ U, Tf \circ V)X$ denote the map $p \mapsto F(Tf \cdot U_p, Tf \cdot V_p)X_p$ which makes sense because the curvature is tensorial. Thus $F(Tf \circ U, Tf \circ V)\ X \in \Gamma_f(E)$. Then we have the following useful fact:

Proposition 15.3 *Let $X \in \Gamma_f(E)$ and $U, V \in \mathfrak{X}(N)$. Then*

$$F^f(U, V)X = F(Tf \circ U, Tf \circ V)X.$$

Proof. Exercise! ■

15.5 The case of the tangent bundle: Linear connections

A connection on the tangent bundle TM of a smooth manifold M is called a linear connection (or affine connection; especially in older literature) on M. A natural linear connection can be given in terms of connection forms but in this case one often sees the concept of Christoffel symbols. This is just another way of presenting the connection forms and so the following is a bit redundant.

Definition 15.8 *A **system of Christoffel symbols** on a smooth manifold M (modeled on M) is an assignment of a differentiable map*

$$\Gamma_\alpha : \psi_\alpha(U_\alpha) \to L(\mathsf{M}, \mathsf{M}; \mathsf{M})$$

to every admissible chart U_α, ψ_α such that if U_α, ψ_α and U_β, ψ_β are two such charts with $U_\alpha \cap U_\beta \neq \emptyset$, then for all $p \in U_\alpha \cap U_\beta$

$$\begin{aligned}&D(\psi_\beta \circ \psi_\alpha^{-1}) \cdot \Gamma_\alpha(x) \\ &= D^2(\psi_\beta \circ \psi_\alpha^{-1}) + \Gamma_\beta(y) \circ (D(\psi_\beta \circ \psi_\alpha^{-1}) \times D(\psi_\beta \circ \psi_\alpha^{-1}))\end{aligned}$$

where $y = \psi_\beta(p)$ and $x = \psi_\alpha(p)$. For finite dimensional manifolds with $\psi_\alpha = (x^1, ..., x^n)$ and $\psi_\beta = (y^1, ..., y^n)$ this last condition reads

$$\frac{\partial y^r}{\partial x^k}(x)\Gamma_{ij}^k(x) = \frac{\partial^2 y^r}{\partial x^i \partial x^j}(x) + \bar{\Gamma}_{pq}^r(y(x))\frac{\partial y^p}{\partial x^i}(x)\frac{\partial y^q}{\partial x^j}(x)$$

where $\Gamma_{ij}^k(x)$ are the components of $\Gamma_\alpha(x)$ and $\bar{\Gamma}_{pq}^r$ the components of $\Gamma_\beta(y(x)) = \Gamma_\beta(\psi_\beta \circ \psi_\alpha^{-1}(x))$ with respect to the standard basis of $L(\mathbb{R}^n, \mathbb{R}^n; \mathbb{R}^n)$.

Proposition 15.4 *Given a system of Christoffel symbols on a smooth manifold M there is a unique natural covariant derivative (a linear connection) ∇ on M such that the local expression $(\nabla_X Y)_U$ of $\nabla_X Y$ with respect to a chart $(U, \mathtt{x})$ is given by $DY_U(x) \cdot X_U(x) + \Gamma_U(x)(X_U(x), Y_U(x))$ for $x \in \mathtt{x}(U)$. Conversely, a natural covariant derivative determines a system of Christoffel symbols.*

Proof. The proof of this is not significantly different from the proof of 15.1.Let a system of Christoffel symbols be given. Now for any open set $U \subset M$ we may let $\{U_a, \mathtt{x}_a\}_a$ be any family of charts such that $\bigcup_a U_a = U$. Given vector fields $X, Y \in \mathfrak{X}(U)$ we define

$$s_{X,Y}(U_a) := \nabla^{U_a}_{r^U_{U_a} X} r^U_{U_a} Y$$

to have principal representation

$$(\nabla_X Y)_{U_\alpha} = DY_{U_\alpha} \cdot X_{U_\alpha} \cdot + \Gamma_{U_\alpha}(X_{U_\alpha}, Y_{U_\alpha}).$$

One should check that the transformation law in the definition of a system of Christoffel symbols implies that $DY_{U_\alpha} \cdot X_{U_\alpha} \cdot + \Gamma_{U_\alpha}(X_{U_\alpha}, Y_{U_\alpha})$ transforms as the principal local representative of a vector. It is straightforward to check that the change of chart formula for Christoffel symbols implies that

$$r^{U_a}_{U_a \cap U_b} s_{X,Y}(U_a) = s_{X,Y}(U_a \cap U_b) = r^{U_b}_{U_a \cap U_b} s_{X,Y}(U_a)$$

and so by there is a unique section

$$\nabla_X Y \in \mathfrak{X}(U)$$

such that

$$r^U_{U_a} \nabla_X Y = s_{X,Y}(U_a)$$

The verification that this defines a natural covariant derivative is now a straightforward (but tedious) verification of (1)-(5) in the definition of a natural covariant derivative.

For the converse, suppose that ∇ is a natural covariant derivative on M. Define the Christoffel symbol for a chart U_a, ψ_α to be in the following way. For fields X and Y one may define $\Theta(X_{U_\alpha}, Y_{U_\alpha}) := (\nabla_X Y)_{U_\alpha} - DY_{U_\alpha} \cdot X_{U_\alpha}$ and then use the properties (1)-(5) to show that $\Theta(X_{U_\alpha}, Y_{U_\alpha})(x)$ depends only on the values of X_{U_α} and Y_{U_α} at the point x. Thus there is a function $\Gamma : U_\alpha \to L(\mathsf{M}, \mathsf{M}; \mathsf{M})$ such that $\Theta(X_{U_\alpha}, Y_{U_\alpha})(x) = \Gamma(x)(X_{U_\alpha}(x), Y_{U_\alpha}(x))$. We wish to show that this defines a system of Christoffel symbols. But this is just an application of the chain rule. ∎

In finite dimensions and using traditional notation

$$\nabla_X Y = \left(\frac{\partial Y^k}{\partial x^j} X^j + \Gamma^k_{ij} X^i Y^j \right) \frac{\partial}{\partial x^k}.$$

where $X = X^j \frac{\partial}{\partial x^j}$ and $Y = Y^j \frac{\partial}{\partial x^j}$. In particular,

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

Let ∇ be a natural covariant derivative on $E \to M$. It is a consequence of proposition 6.5 that for each $X \in \mathfrak{X}(U)$ there is a unique tensor derivation ∇_X on $\mathfrak{T}^r_s(U)$ such that ∇_X commutes with contraction and coincides with the given covariant derivative on $\mathfrak{X}(U)$ (also denoted ∇_X) and with $\mathcal{L}_X f$ on $C^\infty(U)$.

To describe the covariant derivative on tensors more explicitly consider $\Upsilon \in \mathfrak{T}^0_1$ with a 1-form Since we have the contraction $Y \otimes \omega \mapsto C(Y \otimes \omega) = \omega(Y)$ we should have

$$\begin{aligned}
\nabla_X \Upsilon(Y) &= \nabla_X C(Y \otimes \omega) \\
&= C(\nabla_X (Y \otimes \omega)) \\
&= C(\nabla_X Y \otimes \omega + Y \otimes \nabla_X \omega) \\
&= \omega(\nabla_X Y) + (\nabla_X \omega)(Y)
\end{aligned}$$

and so we should define $(\nabla_X \omega)(Y) := \nabla_X(\omega(Y)) - \omega(\nabla_X Y)$. More generally, if $\Upsilon \in \mathfrak{T}^r_s$ then

$$(\nabla_X \Upsilon)(Y_1, ..., Y_s) = \nabla_X(\Upsilon(\omega_1, ..., \omega_r, Y_1, ..., Y_s)) - \sum_{j=1}^{r} \Upsilon(\omega_1, .\nabla_X \omega_j, .., \omega_r, Y_1, ...) \tag{15.1}$$

$$- \sum_{i=1}^{s} \Upsilon(\omega_1, ..., \omega_r, Y_1, ..., \nabla_X Y_i, ...) \tag{15.2}$$

Definition 15.9 *The covariant differential of a tensor field $\Upsilon \in \mathfrak{T}^k_l$ is denoted by $\nabla\Upsilon$ is defined to be the element of $\mathfrak{T}^k_{l+1}$ such that if $\Upsilon \in \mathfrak{T}^k_l$ then*

$$\nabla\Upsilon(\omega^1, ..., \omega^1, X, Y_1, ..., Y_s) := \nabla_X \Upsilon(\omega^1, ..., \omega^1, Y_1, ..., Y_s)$$

For any fixed frame field $E_1, ..., E_n$ we denote the components of $\nabla\Upsilon$ by $\nabla_i \Upsilon^{i_1 \ldots i_r}_{j_1 \ldots j_s}$.

Remark 15.1 *We have place the new variable at the beginning as suggested by our notation $\nabla_i \Upsilon^{i_1}_{j_1 \dots j_s}$ for the for the components of $\nabla \Upsilon$ but in opposition to the equally common $\Upsilon^{i_1}_{j_1 \dots j_s; i}$. This has the advantage of meshing well with exterior differentiation , making the statement of theorem 15.6 as simple as possible.*

The reader is asked in problem 1 to show that $T(X,Y) := \nabla_X Y - \nabla_Y X$ defines a tensor and so $T(X_p, Y_p)$ is well defined for $X_p, Y_p \in T_p M$.

Definition 15.10 *Let ∇ be a connection on M and let $T(X,Y) := \nabla_X Y - \nabla_Y X$. Then T is called the* ***torsion tensor*** *of ∇. If T is identically zero then we say that the connection ∇ is* ***torsion free****.*

15.5.1 Comparing the differential operators

On a smooth manifold we have the Lie derivative $\mathcal{L}_X : \mathfrak{T}^r_s(M) \to \mathfrak{T}^r_s(M)$ and the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ and in case we have a torsion free covariant derivative ∇ then that makes three differential operators which we would like to compare. To this end we restrict attention to purely covariant tensor fields $\mathfrak{T}^0_s(M)$.

The extended map $\nabla_\xi : \mathfrak{T}^0_s(M) \to \mathfrak{T}^0_s(M)$ respects the subspace consisting of alternating tensors and so we have a map

$$\nabla_\xi : L^k_{alt}(M) \to L^k_{alt}(M)$$

which combine to give a degree preserving map

$$\nabla_\xi : L_{alt}(M) \to L_{alt}(M)$$

or in other notation

$$\nabla_\xi : \Omega(M) \to \Omega(M).$$

It is also easily seen that not only do we have $\nabla_\xi(\alpha \otimes \beta) = \nabla_\xi \alpha \otimes \beta + \alpha \otimes \nabla_\xi \beta$ but also

$$\nabla_\xi(\alpha \wedge \beta) = \nabla_\xi \alpha \wedge \beta + \alpha \wedge \nabla_\xi \beta$$

Now as soon as one realizes that $\nabla \omega \in \Omega^k(M)_{C^\infty} \otimes \Omega^1(M)$ instead of $\Omega^{k+1}(M)$ we search for a way to fix things. By antisymmetrizing we get a map $\Omega^k(M) \to \Omega^{k+1}(M)$ which turns out to be none other than our old friend the exterior derivative as will be shown below.

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Now recall 15.1. There is a similar formula for the Lie derivative:

$$(\mathcal{L}_X S)(Y_1, ..., Y_s) = X(S(Y_1, ..., Y_s)) - \sum_{i=1}^{s} S(Y_1, ..., Y_{i-1}, \mathcal{L}_X Y_i, Y_{i+1}, ..., Y_s). \tag{15.3}$$

On the other hand, if ∇ is torsion free so that $\mathcal{L}_X Y_i = [X, Y_i] = \nabla_X Y_i - \nabla_{Y_i} X$ then we obtain the

Proposition 15.5 *For a torsion free connection we have*

$$(\mathcal{L}_X S)(Y_1, ..., Y_s) = (\nabla_X S)(Y_1, ..., Y_s) + \sum_{i=1}^{s} S(Y_1, ..., Y_{i-1}, \nabla_{Y_i} X, Y_{i+1}, ..., Y_s). \tag{15.4}$$

Theorem 15.6 *If ∇ is a torsion free covariant derivative on M then*

$$d = k! Alt \circ \nabla$$

or in other words, if $\omega \in \Omega^k(M)$ then

$$d\omega(X_0, X_1, ..., X_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{X_i} \omega)(X_0, ..., \widehat{X_i}, ..., X_k)$$

Proof. Note that $Alt \circ \nabla \omega(X_0, X_1, ..., X_k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i (\nabla_{X_i} \omega)(X_0, ..., \widehat{X_i}, ..., X_k)$.

The proof is now just a computation

$$
\begin{aligned}
d\omega(X_0, X_1, ..., X_k) &= \sum_{i=0}^{k}(-1)^i X_i(\omega(X_0, ..., \widehat{X_i}, ..., X_k)) \\
&+ \sum_{1\le r<s\le k} (-1)^{r+s}\omega([X_r, X_s], X_0, ..., \widehat{X_r}, ..., \widehat{X_s}, ..., X_k) \\
&= \sum_{i=0}^{k}(-1)^i X_i(\omega(X_0, ..., \widehat{X_i}, ..., X_k)) \\
&+ \sum_{1\le r<s\le k} (-1)^{r+s}\omega(\nabla_{X_r}X_s - \nabla_{X_r}X_s, X_0, ..., \widehat{X_r}, ..., \widehat{X_s}, ..., X_k) \\
&\sum_{i=0}^{k}(-1)^i X_i(\omega(X_0, ..., \widehat{X_i}, ..., X_k)) \\
&+ \sum_{1\le r<s\le k} (-1)^{r+1}\omega(X_0, ..., \widehat{X_r}, ..., \nabla_{X_r}X_s, ..., X_k) \\
&- \sum_{1\le r<s\le k} (-1)^{s}\omega(X_0, ..., \nabla_{X_s}X_r, ..., \widehat{X_s}, ..., X_k) \\
&= \sum_{i=0}^{k}(-1)^i \nabla_{X_i}\omega(X_0, ..., \widehat{X_i}, ..., X_k) \text{ (by using 15.1)}
\end{aligned}
$$

■

15.5.2 Higher covariant derivatives

Now let us suppose that we have a connection ∇^{E_i} on every vector bundle $E_i \to M$ in some family $\{E_i\}_{i\in I}$. We then also have the connections $\nabla^{E_i^*}$ induced on the duals $E_i^* \to M$. By demanding a product formula be satisfied as usual we can form a related family of connections on all bundles formed from tensor products of the bundles in the family $\{E_i, E_i^*\}_{i\in I}$. In this situation it might be convenient to denote any and all of these connections by a single symbol as long as the context makes confusion unlikely. In particular we have the following common situation: By the definition of a connection we have that $X \mapsto \nabla_X^E\sigma$ is $C^\infty(M)$ linear and so $\nabla^E\sigma$ is an section of the bundle $T^*M \otimes E$. We can use the Levi-Civita connection ∇ on M together with ∇^E to define a connection on $E\otimes T^*M$. To get a clear picture of this connection we first notice that a section ξ of the bundle $E \otimes T^*M$ can be written locally in terms of a local frame field $\{\theta^i\}$ on T^*M and a local frame field $\{e_i\}$ on E. Namely, we may write $\xi = \xi^i_j e_i \otimes \theta^j$. Then the connection $\nabla^{E\otimes T^*M}$ on $E \otimes T^*M$ is defined so that a product rule holds:

$$\begin{aligned}\nabla_X^{E\otimes T^*M}\xi &= \nabla_X^E\left(\xi_j^i e_i\right)\otimes\theta^j+\xi_j^i e_i\otimes\nabla_X\theta^j\\ &=\left(X\xi_j^i e_i+\xi_j^i\nabla_X^E e_i\right)\otimes\theta^j+\xi_j^i e_i\otimes\nabla_X\theta^j\\ &=\left(X\xi_l^r e_r+e_r A_i^r(X)\xi_l^i\right)\otimes\theta^l-\omega_l^j(X)\theta^l\otimes\xi_j^r e_r\\ &=\left(X\xi_l^r+A_i^r(X)\xi_l^i-\omega_l^j(X)\xi_j^r\right)e_r\otimes\theta^l\end{aligned}$$

Now let $\xi=\nabla^E\sigma$ for a given $\sigma\in\Gamma(E)$. The map $X\mapsto\nabla_X^{E\otimes T^*M}(\nabla^E\sigma)$ is $C^\infty(M)$ linear and $\nabla^{E\otimes T^*M}(\nabla^E\sigma)$ is an element of $\Gamma\left(E\otimes T^*M\otimes T^*M\right)$ which can again be given the obvious connection. The process continues and denoting all the connections by the same symbol we may consider the $k-$th covariant derivative $\nabla^k\sigma\in\Gamma(E\otimes T^*M^{\otimes k})$ for each $\sigma\in\Gamma(E)$.

It is sometimes convenient to change viewpoints just slightly and define the **covariant derivative operators** $\nabla_{X_1,X_2,\ldots,X_k}:\Gamma(E)\to\Gamma(E)$. The definition is given inductively as

$$\begin{aligned}\nabla_{X_1,X_2}\sigma &:= \nabla_{X_1}\nabla_{X_2}\sigma-\nabla_{\nabla_{X_1}X_2}\sigma\\ \nabla_{X_1,\ldots,X_k}\sigma &:= \nabla_{X_1}\left(\nabla_{X_2,X_3,\ldots,X_k}\sigma\right)\\ &\quad-\nabla_{\nabla_{X_1}X_2,X_3,\ldots,X_k}-\ldots-\nabla_{X_2,X_3,\ldots,\nabla_{X_1}X_k}\end{aligned}$$

Then we have following convenient formula which is true by definition:

$$\nabla^{(k)}\sigma(X_1,\ldots,X_k)=\nabla_{X_1,\ldots,X_k}\sigma$$

Warning. $\nabla_{\partial_\iota}\nabla_{\partial_j}\tau$ is a section of E but not the same section as $\nabla_{\partial_\iota\partial_j}\tau$ since in general

$$\nabla_{\partial_\iota}\nabla_{\partial_j}\tau\neq\nabla_{\partial_i}\nabla_{\partial_j}\sigma-\nabla_{\nabla_{\partial_i}\partial_j}\sigma$$

Now we have that

$$\begin{aligned}\nabla_{X_1,X_2}\sigma-\nabla_{X_2,X_1}\sigma &= \nabla_{X_1}\nabla_{X_2}\sigma-\nabla_{\nabla_{X_1}X_2}\sigma-\left(\nabla_{X_2}\nabla_{X_1}\sigma-\nabla_{\nabla_{X_2}X_1}\sigma\right)\\ &=\nabla_{X_1}\nabla_{X_2}\sigma-\nabla_{X_2}\nabla_{X_1}\sigma-\nabla_{(\nabla_{X_1}X_2-\nabla_{X_2}X_1)}\sigma\\ &=F(X_1,X_2)\sigma-\nabla_{T(X_1,X_2)}\sigma\end{aligned}$$

so if ∇ (the connection on he base M) is torsion free then we recover the curvature

$$\nabla_{X_1,X_2}\sigma-\nabla_{X_2,X_1}\sigma=F(X_1,X_2)\sigma$$

One thing that is quite important to realize is that F depends only on the connection on E while the operators ∇_{X_1,X_2} involve a torsion free connection on the tangent bundle TM.

Clearly there must be some kind of cancellation in the above formula for curvature. This leads naturally to the notion of "exterior covariant derivative".

15.5.3 Exterior covariant derivative

The exterior covariant derivative essentially antisymmetrizes the higher covariant derivatives just defined in such a way that the dependence on the auxiliary torsion free linear connection on the base cancel out. Of course this means that there must be a definition that does not involve this connection on the base at all. We give the definitions below but first we point out a little more algebraic structure. We can give the space of $E-$valued forms $\Omega(M,E)$ the structure of a $(\Omega(M),\Omega(M))-$ "bi-module" over $C^{\infty}(M)$. This means that we define a product $\wedge:\Omega(M)\times\Omega(M,E)\to\Omega(M,E)$ and another product $\wedge:\Omega(M,E)\times\Omega(M)\to\Omega(M,E)$ which are compatible in the sense that

$$(\alpha\wedge\omega)\wedge\beta=\alpha\wedge(\omega\wedge\beta)$$

for $\omega\in\Omega(M,E)$ and $\alpha\in\Omega(M)$. These products are defined by extending linearly the rules

$$\begin{aligned}\alpha\wedge(\sigma\otimes\omega)&:=\sigma\otimes\alpha\wedge\omega\text{ for }\sigma\in\Gamma(E),\alpha,\omega\in\Omega(M)\\(\sigma\otimes\omega)\wedge\alpha&:=\sigma\otimes\omega\wedge\alpha\text{ for }\sigma\in\Gamma(E),\alpha,\omega\in\Omega(M)\\\alpha\wedge\sigma&=\sigma\wedge\alpha=\sigma\otimes\alpha\text{ for }\alpha\in\Omega(M)\text{ and }\sigma\in\Gamma(E)\end{aligned}$$

for. It follows that

$$\alpha\wedge\omega=(-1)^{kl}\omega\wedge\alpha\text{ for }\alpha\in\Omega^k(M)\text{ and }\omega\in\Omega^l(M,E)$$

In the situation where $\sigma\in\Gamma(E):=\Omega^0(M,E)$ and $\omega\in\Omega(M)$ all three of the following conventional equalities:

$$\sigma\omega=\sigma\wedge\omega=\sigma\otimes\omega\text{ (special case)}$$

The proof of the following theorem is analogous the proof of the existence of the exterior derivative

Theorem 15.7 *Given a connection ∇^E on a vector bundle $\pi:E\to M$, there exists a unique operator $d^E:\Omega(M,E)\to\Omega(M,E)$ such that*

(i) $d^E(\Omega^k(M,E))\subset\Omega^{k+1}(M,E)$
(ii) For $\alpha\in\Omega^k(M)$ and $\omega\in\Omega^l(M,E)$ we have

$$\begin{aligned}d^E(\alpha\wedge\omega)&=d\alpha\wedge\omega+(-1)^k\alpha\wedge d^E\omega.\\d^E(\omega\wedge\alpha)&=d^E\omega\wedge\alpha+(-1)^l\omega\wedge d\alpha\end{aligned}$$

(iii) $d^E\alpha=d\alpha$ *for* $\alpha\in\Omega(M)$

(iv) $d^E\sigma=\nabla^E\sigma$ *for* $\sigma\in\Gamma(E)$

In particular, if $\alpha \in \Omega^k(M)$ and $\sigma \in \Omega^0(M,E) = \Gamma(E)$ we have

$$d^E(\sigma \otimes \alpha) = d^E\sigma \wedge \alpha + \sigma \otimes d\alpha.$$

It can be shown that we have the following formula:

$$\begin{aligned} d\omega(X_0,...,X_k) = &\sum_{0\leq i\leq k} (-1)^i \nabla^E_{X_i}(\omega(X_0,...,\widehat{X_i},...,X_k)) \\ &+ \sum_{0\leq i<j\leq k} (-1)^{i+j}\omega([X_i,X_j],X_0,...,\widehat{X_i},...,\widehat{X_j},...,X_k). \end{aligned}$$

Definition 15.11 *The operator whose existence is given by the pervious theorem is called the* ***exterior covariant derivative****.*

Remark 15.2 *In the literature it seems that it isn't always appreciated that there is a difference between $\left(d^E\right)^k$ and $\left(\nabla^E\right)^k$. The higher derivative given by $\left(d^E\right)^k$ are not appropriate for defining $k-th$ order Sobolev spaces since as we shall see $\left(d^E\right)^2$ is zero for any flat connection and $\left(d^E\right)^3 = 0$ is always true (Bianchi identity).*

If ∇^{TM} is a torsion free covariant derivative on M and ∇^E is a connection on the vector bundle $E \to M$ then as before we get a covariant derivative ∇ on all the bundles $E \otimes \wedge^k T^*M$ and as for the ordinary exterior derivative we have the formula

$$d^E = k!Alt \circ \nabla$$

or in other words, if $\omega \in \Omega^k(M,E)$ then

$$d\omega(X_0,X_1,...,X_k) = \sum_{i=0}^{k}(-1)^i(\nabla_{X_i}\omega)(X_0,...,\widehat{X_i},...,X_k)$$

Lets look at the local expressions with respect to a moving frame. In doing so we will rediscover and generalize the local expression obtained above. Let $\{e_i\}$ be a frame field for E on $U \subset M$. Then locally, for $\omega \in \Omega^k(M,E)$ we may write

$$\omega = e_i \otimes \omega^i$$

where $\omega_i \in \Omega^k(M,E)$ and $\sigma_i \in \Omega^0(M,E) = \Gamma(E)$. We have

$$\begin{aligned} d^E\omega = d^E\left(e_i \otimes \omega^i\right) &= e_i \otimes d\omega^i + d^E e_i \wedge \omega^i \\ &= e_j \otimes d\omega^j + A^j_i e_j \wedge \omega^i \\ &= e_j \otimes d\omega^j + e_j \otimes A^j_i \wedge \omega^i \\ &= e_j \otimes \left(d\omega^j + A^j_i \wedge \omega^i\right). \end{aligned}$$

Thus the "$k+1-$form coefficients" of $d^E\omega$ with respect to the frame $\{\sigma_j\}$ are given by $d\omega^j + A_i^j \wedge \omega^i$. Notice that if $k=0$ and we write $s = e_i \otimes s^i$ then we get

$$\begin{aligned} d^E s &= d^E\left(e_i \otimes s^i\right) \\ &= e_j \otimes \left(ds^j + A_i^j s^i\right) \end{aligned}$$

and if we apply this to X we get

$$\begin{aligned} d^E s = \nabla_X^E s &= e_j\left(Xs^j + A_i^j(X)s^i\right) \\ \left(\nabla_X^E s\right)^j &= Xs^j + A_i^j(X)s^i \end{aligned}$$

which is our earlier formula. As before, it is quite convenient to employ matrix notation. If $e_U = (e_1, ..., e_r)$ is the frame over U written as a row vector and $\omega_U = (\omega^1, ..., \omega^r)^t$ is the column vector of $k-$form coefficients then $d^E e_U = e_U A_U$ and the above calculation looks quite nice:

$$\begin{aligned} d^E\left(e_U\omega_U\right) &= e_U d\omega_U + d^E e_U \wedge \omega_U \\ &= e_U\left(d\omega_U + A_U \wedge \omega_U\right) \end{aligned}$$

Now there is another module structure we should bring out. We identify $\mathrm{End}(E)\otimes\wedge^k TM$ with $E\otimes E^*\otimes\wedge^k T^*M$. The space $\Omega(M,\mathrm{End}(E))$ is an algebra over $C^\infty(M)$ where the multiplication is according to $(L_1\otimes\omega_1)\wedge(L_2\otimes\omega_2) = (L_1\circ L_2)\otimes\omega_1\wedge\omega_2$. Now $\Omega(M,E)$ is a module over this algebra because we can multiply as follows

$$(L\otimes\alpha)\wedge(\sigma\otimes\beta) = L\sigma\otimes(\alpha\wedge\beta)$$

To understand what is going on a little better lets consider how the action of $\Omega(M,\mathrm{End}(E))$ on $\Omega(M,E)$ comes about from a slightly different point of view. We can identify $\Omega(M,\mathrm{End}(E))$ with $\Omega(M,E\otimes E^*)$ and then the action of $\Omega(M,E\otimes E^*)$ on $\Omega(M,E)$ is given by tensoring and then contracting: For $\alpha,\beta\in\Omega(M)$, $s,\sigma\in\Gamma(E)$ and $s^*\in\Gamma(E)$ we have

$$\begin{aligned} (s\otimes s^*\otimes\alpha)\wedge(\sigma\otimes\beta) &:= C(s\otimes s^*\otimes\sigma\otimes(\alpha\wedge\beta)) \\ &= s^*(\sigma)s\otimes(\alpha\wedge\beta) \end{aligned}$$

Now from ∇^E get related connections on E^*, $E\otimes E^*$and $E\otimes E^*\otimes E$. The connection on $E\otimes E^*$ is also a connection on $\mathrm{End}(E)$. Of course we have

$$\begin{aligned} &\nabla_X^{\mathrm{End}(E)\otimes E}(L\otimes\sigma) \\ &= (\nabla_X^{\mathrm{End}(E)}L)\otimes\sigma + L\otimes\nabla_X^E\sigma \end{aligned}$$

and after contraction

$$\begin{aligned} &\nabla_X^E(L(\sigma)) \\ &= (\nabla_X^{\mathrm{End}(E)}L)(\sigma) + L(\nabla_X^E\sigma) \end{aligned}$$

The connections on $\mathrm{End}(E)$ and $\mathrm{End}(E)\otimes E$ gives corresponding exterior covariant derivative operators

$$d^{\mathrm{End}(E)} : \Omega^k(M, \mathrm{End}(E)) \to \Omega^{k+1}(M, \mathrm{End}(E))$$
and
$$d^{\mathrm{End}(E)\otimes E} : \Omega^k(M, \mathrm{End}(E)\otimes E) \to \Omega^{k+1}(M, \mathrm{End}(E)\otimes E)$$

Let C denote the contraction $A\otimes\sigma \mapsto A(\sigma)$. For $\alpha \in \Omega^k(M), \beta \in \Omega(M)$, $s \in \Gamma(E)$ and $s^* \in \Gamma(E)$ we have $\nabla_X^E(L(\sigma)) = \nabla_X^E C(L\otimes\sigma) = C(\nabla_X^{\mathrm{End}(E)} L \otimes \sigma + L\otimes\nabla_X^E\sigma) = (\nabla_X^{\mathrm{End}(E)}L)(\sigma) + L(\nabla_X^E\sigma)$. But by definition $X \mapsto L(\nabla_X^E\sigma)$ is just $L\wedge\nabla^E\sigma$. Using this we have

$$\begin{aligned}
d^E((A\otimes\alpha)\wedge(\sigma\otimes\beta)) &:= d^E(A(\sigma)\otimes(\alpha\wedge\beta)) \\
&= \nabla^E(A(\sigma))\wedge(\alpha\wedge\beta) \\
&+ A(\sigma)\otimes d\,(\alpha\wedge\beta) \\
&= (-1)^k\left(\alpha\wedge\nabla^E(A(\sigma))\wedge\beta\right) + A(\sigma)\otimes d\,(\alpha\wedge\beta) \\
&= (-1)^k\left(\alpha\wedge\{\nabla^{\mathrm{End}(E)}A\wedge\sigma + A\wedge\nabla^E\sigma\}\wedge\beta\right) \\
&+ A(\sigma)\otimes d\,(\alpha\wedge\beta) + A(\sigma)\otimes d\alpha\wedge\beta + (-1)^k A(\sigma)\otimes\alpha\wedge d\beta \\
&= (\nabla^{\mathrm{End}(E)}A)\wedge\alpha\wedge\sigma\wedge\beta + (-1)^k A\wedge\alpha\wedge(\nabla^E\sigma)\wedge\beta \\
&+ A(\sigma)\otimes d\alpha\wedge\beta + (-1)^k A(\sigma)\otimes\alpha\wedge d\beta \\
&= (\nabla^{\mathrm{End}(E)}A\wedge\alpha + A\otimes d\alpha)\wedge(\sigma\otimes\beta) \\
&+ (-1)^k\,(A\otimes\alpha)\wedge(\nabla^E\sigma\wedge\beta + \sigma\otimes d\beta) \\
&= (\nabla^{\mathrm{End}(E)}A\wedge\alpha + A\otimes d\alpha)\wedge(\sigma\otimes\beta) \\
&+ (-1)^k\,(A\otimes\alpha)\wedge(\nabla^E\sigma\wedge\beta + \sigma\otimes d\beta) \\
&= d^{\mathrm{End}(E)}(A\otimes\alpha)\wedge(\sigma\otimes\beta) + (-1)^k\,(A\otimes\alpha)\wedge d^E\,(\sigma\otimes\beta)
\end{aligned}$$

By linearity we conclude that for $\Phi \in \Omega^k(M, \mathrm{End}(E))$ and $\omega \in \Omega(M, E)$ we have
$$d^E(\Phi\wedge\omega) = d^{\mathrm{End}(E)}\Phi\wedge\omega + (-1)^k\Phi\wedge d^E\omega$$

Proposition 15.6 *The map $\Omega^k(M,E) \to \Omega^{k+2}(M,E)$ given by $d^E\circ d^E$ given the action of the curvature 2-form of ∇^E.*

Proof. Let us check the case where $k = 0$ first:

$$\begin{aligned}
\left(d^E\circ d^E\sigma\right)(X_1, X_2) &= \left(2!Alt\circ\nabla^E\circ Alt\circ\nabla^E\sigma\right)(X_1, X_2) \\
&= \left(2!Alt\circ\nabla^E\nabla^E\sigma\right)(X_1, X_2) \\
&= \nabla_{X_1,X_2}\sigma - D_{X_2,X_1}\sigma = F(X_1, X_2)\sigma \\
&= (F\wedge\sigma)(X_1, X_2)
\end{aligned}$$

where the last equality is a trivial consequence of the definition of the action of $.\Omega^2(M, \mathrm{End}(E))$ on $\Omega^0(M, E)$.

Now more generally we just check $d^E \circ d^E$ on elements of the form $\sigma \otimes \theta$:

$$\begin{aligned}
d^E \circ d^E \omega &= d^E \circ d^E (\sigma \otimes \theta) \\
&= d^E \left(d^E \sigma \wedge \theta + \sigma \otimes d\theta \right) \\
&= \left(d^E d^E \sigma \right) \wedge \theta - d^E \sigma \wedge d\theta + d^E \sigma \wedge d\theta + 0 \\
&= (F \wedge \sigma) \wedge \theta = F \wedge (\sigma \wedge \theta)
\end{aligned}$$

■

Lets take a look at what things look like in a local frame field. As before restrict to an open set U on which $e_U = (e_1, ..., e_r)$ and then we write a typical element $s \in \Omega^k(M, E)$ as $s = e_U \omega_U$ where $\omega_U = (\omega^1, ..., \omega^r)^t$ is a column vector of smooth $k-$forms

$$\begin{aligned}
d^E d^E s = d^E d^E \left(e_U \omega_U \right) &= d^E \left(e_U d\omega_U + d^E e_U \wedge \omega_U \right) \\
&= d^E \left(e_U \left(d\omega_U + A_U \wedge \omega_U \right) \right) \\
&= d^E e_U \wedge \left(d\omega_U + A_U \wedge \omega_U \right) \\
&+ e_U \wedge d^E \left(d\omega_U + A_U \wedge \omega_U \right) \\
&= e_U A \wedge \left(d\omega_U + A_U \wedge \omega_U \right) \\
&+ e_U \wedge d^E \left(d\omega_U + A_U \wedge \omega_U \right) \\
&= e_U A_U \wedge \left(d\omega_U + A_U \wedge \omega_U \right) \\
&+ e_U \wedge dA_U \wedge \omega_U - e_U \wedge A_U \wedge d\omega_U \\
&= e_U dA_U \wedge \omega_U + e_U A_U \wedge A_U \wedge \omega_U \\
&= e_U \left(dA_U + A_U \wedge A_U \right) \wedge \omega_U
\end{aligned}$$

The matrix $dA_U + A_U \wedge A_U$ represents a section of $End(E)_U \otimes \wedge^2 TU$. In fact, we will now check that these local sections paste together to give a global section of $End(E) \otimes \wedge^2 TM$, i.e., and element of $\Omega^2(M, End(E))$ which is clearly the curvature form: $F : (X, Y) \mapsto F(X, Y) \in \Gamma(End(E))$. Let $F_U = dA_U + A_U \wedge A_U$ and let $F_V = dA_V + A_V \wedge A_V$ be corresponding form for a different moving frame $e_U := e_V g$ where $g : U \cap V \to Gl(\mathbb{F}^r), r = rank(E)$. What we need to verify this is the transformation law

$$F_V = g^{-1} F_U g$$

Recall that $A_V = g^{-1}A_U g + g^{-1}dg$. Using $d\left(g^{-1}\right) = -g^{-1}dgg^{-1}$ we have

$$\begin{aligned}
F_V &= dA_V + A_V \wedge A_V \\
&= d\left(g^{-1}A_U g + g^{-1}dg\right) \\
&+ \left(g^{-1}A_U g + g^{-1}dg\right) \wedge \left(g^{-1}A_U g + g^{-1}dg\right) \\
&= d\left(g^{-1}\right) A_U g + g^{-1}dA_U g - g^{-1}A_U dg \\
& g^{-1}A_U \wedge A_U g + g^{-1}dgg^{-1}A_U g + \\
& g^{-1}A_U dg + g^{-1}dg \wedge g^{-1}dg \\
&= g^{-1}dA_U g + g^{-1}A_U \wedge A_U g = g^{-1}F_U g.
\end{aligned}$$

where we have used $g^{-1}dg \wedge g^{-1}dg = d(g^{-1}g) = 0$.

15.6 Problem Set

1. Let M have a linear connection ∇ and let $T(X,Y) := \nabla_X Y - \nabla_Y X$. Show that T is tensorial ($C^\infty(M)-$bilinear). The resulting tensor is called the torsion tensor for the connection.

   Second order differential equations and sprays

2. A second order differential equation on a smooth manifold M is a vector field on TM, that is a section X of the bundle TTM (second tangent bundle) such that every integral curve α of X is the velocity curve of its projection on M. In other words, $\alpha = \dot{\gamma}$ where $\gamma := \pi_{TM} \circ \alpha$. A solution curve $\gamma : I \to M$ for a second order differential equation X is, by definition, a curve with $\ddot{\alpha}(t) = X(\dot{\alpha}(t))$ for all $\tau \in I$.

   In case $M = \mathbb{R}^n$ show that this concept corresponds to the usual systems of equations of the form

   $$\begin{aligned}
   y' &= v \\
   v' &= f(y, v)
   \end{aligned}$$

   which is the reduction to a first order system of the second order equation $y'' = f(y, y')$. What is the vector field on $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ which corresponds to this system?

   Notation: For a second order differential equation X the maximal integral curve through $v \in TM$ will be denoted by α_v and its projection will be denoted by $\gamma_v := \pi_{TM} \circ \alpha$.

3. A spray on M is a second order differential equation, that is, a section X of TTM as in the previous problem, such that for $v \in TM$ and $s \in \mathbb{R}$, a number $t \in \mathbb{R}$ belongs to the domain of γ_{sv} if and only if st belongs to the domain of γ_v and in this case

   $$\gamma_{sv}(t) = \gamma_v(st)$$

Show that there is an infinite number of sprays on any smooth manifold M.

Hint: (1) Show that a vector field $X \in \mathfrak{X}(TM)$ is a spray if and only if $T\pi \circ X = \mathrm{id}_{TM}$ where $\pi = \pi_{TM}$:

$$\begin{tikzcd} & TTM \arrow[d] \\ TM \arrow[ur, "X"] \arrow[r, "\mathrm{id}_{TM}"] & TM \end{tikzcd}$$

(2) Show that $X \in \mathfrak{X}(TM)$ is a spray if and only if for any $s \in \mathbb{R}$ and any $v \in TM$

$$X_{sv} = T\mu_s(sX_v)$$

where $\mu_s : v \mapsto sv$ is the multiplication map.

(3) Show the existence of a spray on an open ball in a Euclidean space.

(4) Show that X_1 and X_2 both satisfies one of the two characterizations of a spray above then so does any convex combination of X_1 and X_2.

4. Show that if one has a linear connection on M then there is a spray whose solutions are the geodesics of the connection.

5. Show that given a spray on a manifold there is a (not unique) linear connection ∇on the manifold such that $\gamma : I \to M$ is a solution curve of the spray if and only if $\left(\nabla_{\dot\gamma(t)}\dot\gamma\right)(t) = 0$ for all $t \in I$. Note: γ is called a geodesic for the linear connection or the spray. Does the stipulation that the connection be torsion free force uniqueness?

6. Let X be a spray on M. Equivalently, we may start with a linear connection on M which induces a spay. Show that the set $O_X := \{v \in TM : \gamma_v(1) \text{ is defined}\}$ is an open neighborhood of the zero section of TM.

Chapter 16

Riemannian and semi-Riemannian Geometry

The most beautiful thing we can experience is the mysterious. It is the source of all true art and science.

-Albert Einstein

Geometry has a long history indeed and hold a central position in mathematics but also in philosophy both ancient and modern and also in physics. In fact, we might safely find the origin of geometry in the physical universe itself and the fact ever mobile creature must be able to some kind of implicit understanding of geometry. Knowledge of geometry remained largely implicitly even in the human understanding until the intellectual pursuits of the ancient Greeks introduced geometry as a specific and self conscious discipline. One of the earliest figures associated with geometry is the Greek philosopher Thales, born circa 600 BC in Miletus, Asia Minor. He is said to be the founder of philosophy, one of the Seven Wise Men of Greece. Even more famous, at least among mathematicians, is Pythagoras who was instructed in the teachings of Thales. Pythagoras was the founder of religious sect which put mathematics, especially number and geometry, in a central and sacred position. According to the Pythagoreans, the world itself was as bottom nothing but number and proportion. This ontology may seem ridiculous at first glance but modern physics tells a story that is almost exclusively mathematical. Of course, these days we have a deeper appreciation for the role of experimentation and measurement. Philosophers have also become more sensitive to the ways in which our embodiment in the world and our use of metaphor and other tropes influences and constrains our mathematical understanding.

The theorem of Pythagoras relating the sides of right triangles is familiar to every student of mathematics and is of interest in differential geometry (more specifically Riemannian geometry) since the theorem fails to hold in the presence of curvature. Curvature is defined any time one has a connection but the curvature appropriate to semi-Riemannian geometry is of a very special sort since it arises completely out of the metric structure on the manifold. Thus curvature is truly connected to what we might loosely call the shape of the manifold.

The next historical figure we mention is Euclid (300 BC), also a Greek, who changed the rules of the game by introducing the notions of *axiom* and *proof*. Euclid's axioms are also called postulates. The lasting influence on mathematics of Euclid's approach goes without saying but the axiomatic approach to geometric knowledge has also always been a paradigmatic example for philosophy. Theories of a priori knowledge and apodeictic knowledge seem to have Euclid's axiomatics in the background. Euclid's major work is a 13 volume treatise called Elements. A large portion of what is found in this treatise is the axiomatic geometry of planar figures which roughly corresponds to what has been taught in secondary school for many years. What stands out for our purposes is Euclid's fifth axiom. The point is that exactly the same geometries for which the Pythagorean theorem fails are also geometries for which this fifth postulate fails to hold. In modern terms the postulates are equivalent to the following:

1. For every point P and every point $Q \neq P$ there exists a unique line that passes through P and Q.

2. For every pair of segments AB and CD there exists a unique point E such that B is between A and E and segment CD is congruent to segment BE.

3. For every point O and every point $A \neq O$ there exists a circle with center O and radius OA.

4. All right angles are congruent to each other.

5. For every line ℓ and for every point P that is not on ℓ there exists a unique line m through P that is parallel to ℓ.

Of course, "parallel" is a term that must be defined. One possible definition (the one we use) is that two lines are parallel if they do not intersect.

It was originally thought on the basis of intuition that the fifth postulate was a necessary truth and perhaps a logical necessity following from the other 4 postulates. The geometry of a sphere has geodesics as lines (great circles in this case). By identifying antipodal points we get projective space where the lines are the projective great circles. This geometry satisfies the first four axioms above but the fifth does not hold when we make the most useful definition for what it means to be parallel. In fact, on a sphere we find that given a geodesic, and a point not on the geodesic there are no parallel lines since all lines (geodesics) must intersect. After projectivization we get a geometry for which the first four postulates hold but the fifth does not. Thus the fifth cannot be a logical consequence of the first four.

It was Rene Descartes that made geometry susceptible to study via the real number system through the introduction of the modern notion of coordinate system and it is to this origin that the notion of a differentiable manifold can be traced. Of course, Descartes did not define the notion of a differentiable manifold. That task was left to Gauss, Riemann and Whitney. Descartes did make the first systematic connection between the real numbers and the geometry of the line, the plane, and 3-dimensional space. Once the introduction of Descartes' coordinates made many theorems of geometry become significantly easier to prove. In fact, it is said that Descartes was annoyed axiomatic geometry because to much tedious work was required obtain results. Modern differential geometry carries on this tradition by framing everything in the setting of differentiable manifolds where one can always avail oneself of various coordinates.

At first it was not clear that there should be a distinction between the purely differential topological notion of a manifold (such as the plane) which does not include the notion of distance, and the notion of the Riemannian manifold (the rigid plane being an example). A (semi-) Riemannian manifold comes with notions of distance, volume, curvature and so on, all of which are born out of the fundamental tensor called the metric tensor. Today we clearly distinguish between the notions of topological, differential topological and the more rigid notion of (semi-) Riemannian manifold even though it is difficult to achieve a mental representation of even a plane or surface without imposing a metric in imagination. I usually think of a purely differential topological version of, say, a plane or sphere as a wobbling shape shifting surface with the correct topology. Thus the picture is actually of a family of manifolds with various metrics. This is only a mental trick but it is important in (semi-) Riemannian geometry to keep the distinction between a (semi-) Riemannian manifold (defined below) and the underlying differentiable manifold. For example, one may ask questions like 'given a specific differentiable manifold M, what type of metric with what possible curvatures can be imposed on M?'.

The distinction between a Riemannian manifold and the more inclusive notion of a semi-Riemannian manifold is born out of the distinction between a merely nondegenerate symmetric, bilinear form and that of a positive definite, symmetric bilinear form. One special class of semi-Riemannian manifolds called the Lorentz manifold make their appearance in the highly differential geometric theory of gravitation know as Einstein's general theory of relativity. After an exposition of the basics of general semi-Riemannian geometry we will study separately the special cases of Riemannian geometry and Lorentz geometry.

16.1 Metric Tensors

We start out again considering some linear algebra that we wish to globalize. Thus the vector space V that we discuss next should be thought of as tangent space or the fiber of a some vector bundle.

Definition 16.1 *A* ***scalar product*** *on a (real) finite dimensional vector space* V *is a nondegenerate symmetric bilinear form* $\mathrm{g} : \mathrm{V} \times \mathrm{V} \to \mathbb{R}$ *. The scalar product is called*

1. *positive (resp. negative) definite if* $\mathrm{g}(v, v) \geq 0$ *(resp.* $\mathrm{g}(v, v) \leq 0$*) for all* $v \in \mathrm{V}$ *and* $\mathrm{g}(v, v) = 0 \Longrightarrow v = 0$.

2. *positive (resp. negative) semidefinite if* $\mathrm{g}(v, v) \geq 0$ *(resp.*$\mathrm{g}(v, v) \leq 0$*) for all* $v \in \mathrm{V}$.

Nondegenerate means that the map $\mathrm{g} : \mathrm{V} \to \mathrm{V}^*$ given by $v \mapsto \mathrm{g}(v, .)$ is a linear isomorphism or equivalently, if $\mathrm{g}(v, w) = 0$ for all $w \in V$ implies that $v = 0$.

Definition 16.2 *A **scalar product space** is a pair (V, g) where V is a vector space and g is a scalar product. In case the scalar product is positive definite we shall also refer to it as an **inner product** and the pair (V, g) as an inner product space.*

Definition 16.3 *The **index** of a symmetric bilinear g form on V is the largest subspace $W \subset V$ such that the restriction $\mathrm{g}|_W$ is negative definite.*

Given a basis $\mathcal{F} = (f_1, ..., f_n)$ for V we may form the matrix $[\mathrm{g}]_{\mathcal{F}}$ which has as ij-th entry $g_{ij} := \mathrm{g}(f_i, f_j)$. This is the matrix that represents g with respect to the basis $\mathcal{F}$. So if $v = \mathcal{F}\,[v]^{\mathcal{F}} = \sum_{i=1}^n v^i f_i$, $w = \mathcal{F}\,[w]^{\mathcal{F}} = \sum_{i=1}^n w^i f_i$ then

$$\mathrm{g}(v, w) = [v]^{\mathcal{F}}\,[\mathrm{g}]_{\mathcal{F}}\,[w]^{\mathcal{F}} = g_{ij} v^i w^j.$$

It is a standard fact from linear algebra that if g is a scalar product then there exists a basis $e_1, ..., e_n$ for V such that the matrix representative of g with respect to this basis is a diagonal matrix $\mathrm{diag}(-1, ..., 1)$ with ones or minus ones along the diagonal and we may arrange for the minus ones come first. Such a basis is called an orthonormal basis for (V, g). The number of minus ones appearing is the index $\mathrm{ind}(\mathrm{g})$ and so is independent of the orthonormal basis chosen. We sometimes denote the index by the Greek letter ν. It is easy to see that the index $\mathrm{ind}(\mathrm{g})$ is zero if and only if g positive semidefinite. Thus if $e_1, ..., e_n$ is an orthonormal basis for (V, g) then $\mathrm{g}(e_i, e_j) = \varepsilon_i \delta_{ij}$ where $\varepsilon_i = \mathrm{g}(e_i, e_i) = \pm 1$are the entries of the diagonal matrix the first $\mathrm{ind}(\mathrm{g})$ of which are equal to -1 and the remaining are equal to 1. Let us refer to the list of ± 1 given by $(\varepsilon_1, ..., \varepsilon_n)$ as the **signature**.

Remark 16.1 *The convention of putting the minus signs first is not universal and in fact we reserve the right to change the convention to a positive first convention but ample warning will be given. The negative signs first convention is popular in relativity theory but the reverse is usual in quantum field theory. It makes no physical difference in the final analysis as long as one is consistent but it can be confusing when comparing references from the literature.*

Another difference between the theory of positive definite scalar products and indefinite scalar products is the appearance of the ε_i from the signature in formulas that would be familiar in the positive definite case. For example we have the following:

Proposition 16.1 *Let $e_1, ..., e_n$ be an orthonormal basis for (V, g). For any $v \in (V, \mathrm{g})$ we have a unique expansion given by $v = \sum_i \varepsilon_i \langle v, e_i \rangle e_i$.*

Proof. The usual proof works. One just has to notice the appearance of the ε_i. ■

Definition 16.4 *If $v \in V$ then let $|v|$ denote the nonnegative number $|\mathrm{g}(v,v)|^{1/2}$ and call this the (absolute or positive)* ***length*** *of v. Some authors call $\mathrm{g}(v,v)$ or $\mathrm{g}(v,v)^{1/2}$ the length which would make it possible for the length to be complex valued. We will avoid this.*

Definition 16.5 *Let (V,g) be a scalar product space. We say that v and w are mutually orthogonal if and only if $\mathrm{g}(v,w)=0$. Furthermore, given two subspaces W_1 and W_2 of V we say that W_1 is orthogonal to W_2 and write $\mathrm{W}_1 \perp \mathrm{W}_2$ if and only if every element of W_1 is orthogonal to every element of W_2.*

Since in general g is not necessarily positive definite or negative definite it may be that there are elements that are orthogonal to themselves.

Definition 16.6 *Given a subspace W of a scaler product space V we define the orthogonal complement as $\mathrm{W}^{\perp} = \{v \in V : \mathrm{g}(v,w) = 0$ for all $w \in \mathrm{W}\}$.*

We always have $\dim(\mathrm{W}) + \dim(\mathrm{W}^{\perp}) = \dim(\mathrm{V})$ but unless g is definite we may *not* have $\mathrm{W} \cap \mathrm{W}^{\perp} = \emptyset$. Of course by nondegeneracy we will always have $V^{\perp} = 0$.

Definition 16.7 *A subspace W of a scaler product space (V,g) is called* ***nondegenerate*** *if $\mathrm{g}|_{\mathrm{W}}$ is nondegenerate.*

Lemma 16.1 *A subspace $\mathrm{W} \subset (V,\mathrm{g})$ is nondegenerate if and only if $\mathrm{V} = \mathrm{W} \oplus \mathrm{W}^{\perp}$ (inner direct sum).*

Proof. Easy exercise in linear algebra. ■

Just as for inner product spaces we call a linear isomorphism $I : \mathrm{V}_1, \mathrm{g}_1 \to \mathrm{V}_2, \mathrm{g}_2$ from one scalar product space to another to an **isometry** if $\mathrm{g}_1(v,w) = \mathrm{g}_2(Iv, Iw)$. It is not hard to show that if such an isometry exists then g_1 and g_2 have the same index and signature.

16.1.1 Musical Operators and the Star Operator

If we have a scalar product g on a finite dimensional vector space V then there is a natural way to induce a scalar product on the various tensor spaces $T^r_s(\mathrm{V})$ and on the Grassmann algebra. The best way to explain is by way of some examples.

First consider V^*. Since g is nondegenerate there is a linear isomorphism map $\mathrm{g}_\flat : \mathrm{V} \to \mathrm{V}^*$ defined by

$$\mathrm{g}_\flat(v)(w) = \mathrm{g}(v,w).$$

Denote the inverse by $\mathrm{g}^\sharp : \mathrm{V}^* \to \mathrm{V}$. We force this to be an isometry by defining the scalar product on V^* to be

$$\mathrm{g}^*(\alpha,\beta) = \mathrm{g}(\mathrm{g}^\sharp(\alpha), \mathrm{g}^\sharp(\beta)).$$

Under this prescription, the dual basis $e^1, ..., e^n$ corresponding to an orthonormal basis $e_1, ..., e_n$ for V will also be orthonormal. The signature (and hence the index) of g^* and g are the same.

Next consider $T^1_1(\mathrm{V}) = \mathrm{V} \otimes \mathrm{V}^*$. We define the scalar product of two simple tensors $v_1 \otimes \alpha_1$ and $v_2 \otimes \alpha_2 \in \mathrm{V} \otimes \mathrm{V}^*$ by

$$\mathrm{g}^1_1(v_1 \otimes \alpha_1, v_2 \otimes \alpha_2) = \mathrm{g}(v_1, v_2)\mathrm{g}^*(\alpha_1, \alpha_2).$$

One can then see that for orthonormal dual bases $e^1, ..., e^n$ and $e_1, ..., e_n$ we have that

$$\{e_i \otimes e^j\}_{1 \le i,j \le n}$$

is an orthonormal basis for $T^1_1(\mathrm{V}), \mathrm{g}^1_1$. In general one defines g^r_s so that the natural basis for $T^r_s(\mathrm{V})$ formed from the orthonormal basis $\{e^1, ..., e^n\}$ (and its dual $\{e_1, ..., e_n\}$), will also be orthonormal.

Notation 16.1 *In order to reduce notational clutter let us reserve the option to denote all these scalar products coming from* g *by the same letter* g *or, even more conveniently, by* $\langle ., . \rangle$.

Exercise 16.1 *Show that under the natural identification of* $\mathrm{V} \otimes \mathrm{V}^*$ *with* $L(\mathrm{V}, \mathrm{V})$ *the scalar product of linear transformations* A *and* B *is given by* $\langle A, B \rangle = \mathrm{trace}(A^t B)$.

The maps $\mathrm{g}_\flat$ and $\mathrm{g}^\sharp$, and their extensions defined below are called musical isomorphisms . The first one, $\mathrm{g}_\flat$, is called the flatting operator. The effect of this operator is sometime called "index lowering" for reasons which will become clear. The isomorphism $\mathrm{g}^\sharp$ is called the sharping operator and its effect is called"index raising". Next we see how to extend the maps $\mathrm{g}_\flat$ and $\mathrm{g}^\sharp$ to maps on tensors. We give two ways of defining the extensions. In either case, what we want to define is maps $(\mathrm{g}_\flat)^i_\downarrow : T^r{}_s(\mathrm{V}) \to T^{r-1}{}_{s+1}(\mathrm{V})$ and $(\mathrm{g}^\sharp)^\uparrow_j : T^r{}_s(\mathrm{V}) \to T^{r+1}{}_{s-1}(\mathrm{V})$ where $0 \le i \le r$ and $0 \le j \le s$. It is enough to give the definition for simple tensors:

$$\begin{aligned}
&(\mathrm{g}_\flat)^i_\downarrow(w_1 \otimes \cdots \otimes w_r \otimes \omega^1 \otimes \cdots \otimes \omega^s) \\
&:= w_1 \otimes \cdots \otimes \widehat{w_i} \otimes \cdots w_r \otimes \mathrm{g}_\flat(w_i) \otimes \omega^1 \otimes \cdots \otimes \omega^s
\end{aligned}$$

and

$$\begin{aligned}
&(\mathrm{g}^\sharp)^\uparrow_j(w_1 \otimes \cdots \otimes w_r \otimes \omega^1 \otimes \cdots \otimes \omega^s) \\
&:= w_1 \otimes \cdots \otimes w_r \otimes \mathrm{g}^\sharp(\omega^j) \otimes \omega^1 \otimes \cdots \otimes \widehat{\omega^j} \otimes \cdots \otimes \omega^s.
\end{aligned}$$

This definition is extended to all of $T^r_s(\mathrm{V})$ by linearity. For our second, equivalent definition let $\Upsilon \in T^r{}_s(\mathrm{V})$. Then

$$\begin{aligned}
&((\mathrm{g}_\flat)^i_\downarrow \Upsilon)(\alpha^1, ..., \alpha^{r-1}; v_1, ..., v_{s+1}) \\
&:= \Upsilon(\alpha^1, ..., \alpha^{r-1}, \mathrm{g}_\flat(v_1); v_2, ..., v_{s+1})
\end{aligned}$$

and similarly

$$\begin{aligned}&((\mathrm{g}^\sharp)^\uparrow_j \Upsilon)(\alpha^1, ..., \alpha^{r+1}; v_1, ..., v_{s-1}) \\ &:= \Upsilon(\alpha^1, ..., \alpha^r, \mathrm{g}^\sharp(\alpha^{r+1}); v_2, ..., v_{s+1})\end{aligned}$$

Lets us see what this looks like by viewing the components. Let $f_1, ..., f_n$ be an arbitrary basis of V and let $f^1, ..., f^n$ be the dual basis for V^*. Let $\mathrm{g}_{ij} := \mathrm{g}(f_i, f_j)$ and $\mathrm{g}^{ij} = \mathrm{g}^*(f^i, f^j)$. The reader should check that $\sum_k \mathrm{g}^{ik}\mathrm{g}_{kj} = \delta^i_j$. Now let $\tau \in T^r_s(\mathrm{V})$ and write

$$\tau = \tau^{i_1,..,i_r}{}_{j_1,...,j_s} f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes f^{j_1} \otimes \cdots \otimes f^{j_s}.$$

Define

$$\tau^{i_1,...,i_r j}_{j_1,...,j_{b-1},\widehat{k},j_b...,j_{s-1}} := \tau^{i_1,...,i_r}{}_{j_1,...,j_{j-1}}{}^{j}{}_{j_{b+1}...,j_{s-1}} := \sum_m b^{km} \tau^{i_1,..,i_r}_{j_1,...,j_{b-1},m,j_b...,j_{s-1}}$$

then

$$(\mathrm{g}^\sharp)^\uparrow_a \tau = \tau^{i_1,...,i_r j_a}_{j_1,...,j_{a-1},\widehat{j_a},j_{a+1}...,j_{s-1}} f_{i_1} \otimes \cdots \otimes f_{i_r} \otimes f_{j_a} \otimes f^{j_1} \otimes \cdots \otimes f^{j_{s-1}}.$$

Thus the $(\mathrm{g}^\sharp)^\uparrow_a$ visually seems to raise an index out of a-th place and puts it up in the last place above. Similarly, the component version of lowering $(\mathrm{g}_\flat)^a_\downarrow$ takes

$$\tau^{i_1,..,i_r}{}_{j_1,...,j_s}$$

and produces

$$\tau^{i_1,..,\widehat{i_a},...i_r}_{i_a j_1,...,j_s} = \tau^{i_1,..,}{}_{i_a}{}^{,...i_r}{}_{j_1,...,j_s}.$$

How and why would one do this so called index raising and lowering? What motivates the choice of the slots? In practice one applies this type changing only in specific well motivated situations and the choice of slot placement is at least partially conventional. We will comment further when we actually apply these operators. One thing that is useful to know is that if we raise all the lower indices and lower all the upper ones on a tensor then we can "completely contract" against another tensor of the original type with the result being the scalar product. For example, let $\tau = \sum \tau_{ij} f^i \otimes f^j$ and $\chi = \sum \chi_{ij} f^i \otimes f^j$. Then letting the components of $(\mathrm{g}^\sharp)^\uparrow_1 \circ (\mathrm{g}^\sharp)^\uparrow_1(\chi)$ by χ^{ij} we have

$$\chi^{ij} = \mathrm{g}^{ik}\mathrm{g}^{jl}\chi_{kl}$$

and

$$\langle \chi, \tau \rangle = \sum \chi_{ij}\tau^{ij}.$$

In general, unless otherwise indicated, we will preform repeated index raising by raising from the first slot $(\mathrm{g}^\sharp)^\uparrow_1 \circ \cdots \circ (\mathrm{g}^\sharp)^\uparrow_1$ and similarly for repeated lowering $(\mathrm{g}_\flat)^1_\downarrow \circ \cdots \circ (\mathrm{g}_\flat)^1_\downarrow$. For example,

$$A_{ijkl} \mapsto A^i{}_{jkl} = \mathrm{g}^{ia}A_{ajkl} \mapsto A^{ij}{}_{kl} = \mathrm{g}^{ia}\mathrm{g}^{jb}A_{abkl}$$

Exercise 16.2 *Verify the above claim directly from the definition of* $\langle \chi, \tau \rangle$.

Even though elements of $L^k_{alt}(\mathrm{V}) \cong \bigwedge^k(\mathrm{V}^*)$ can be thought of as tensors of type $(0, k)$ that just happen to be anti-symmetric, it is better in most cases to give a scalar product to this space in such a way that the basis

$$\{e^{i_1} \wedge \cdots \wedge e^{i_k}\}_{i_1 < \ldots < i_k} = \{e^{\vec{I}}\}$$

is orthonormal if $e^1, ..., e^n$ is orthonormal. Now given any k-form $\alpha = a_{\vec{I}} e^{\vec{I}}$ where $e^{\vec{I}} = e^{i_1} \wedge \cdots \wedge e^{i_k}$ with $i_1 < ... < i_k$ as explained earlier, we can also write $\alpha = \frac{1}{k!} a_I e^I$ and then as a tensor

$$\begin{aligned} \alpha &= \frac{1}{k!} a_I e^I \\ &= \frac{1}{k!} a_{i_1 \ldots i_k} e^{i_1} \otimes \cdots \otimes e^{i_k}. \end{aligned}$$

Thus if $\alpha = a_{\vec{I}} e^{\vec{I}}$ α and $\beta = b_{\vec{I}} e^{\vec{I}}$ are k-forms considered as covariant tensor fields we have

$$\begin{aligned} \langle \alpha, \beta \rangle &= \frac{1}{(k!)^2} a_{i_1 \ldots i_k} b^{i_1 \ldots i_k} \\ &= a_I b^I \end{aligned}$$

But when $\alpha = a_{\vec{I}} e^{\vec{I}}$ α and $\beta = b_{\vec{I}} e^{\vec{I}}$ are viewed as k-forms then we must have

$$\begin{aligned} \langle \alpha, \beta \rangle &= a_{\vec{I}} b^{\vec{I}} \\ &= \frac{1}{k!} a_I b^I \end{aligned}$$

so the two definitions are different by a factor of $k!$. The reason for the discrepancy might be traced to our normalization when we defined the exterior product. If we had defined the exterior product as $Alt(\alpha \otimes \beta)$ we would not have this problem. Of course the cost would be that some other formulas would become cluttered. The definition for forms can be written succinctly in terms of 1$-$forms as

$$\begin{aligned} &\langle \alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^k, \beta^1 \wedge \beta^2 \wedge \cdots \wedge \beta^k \rangle \\ &= \det(\langle \alpha^i, \beta^j \rangle) \end{aligned}$$

where the α^i and β^i are 1-forms.

Definition 16.8 *We define the scalar product on* $\bigwedge^k \mathrm{V}^* \cong L^k_{alt}(\mathrm{V})$ *by first using the above formula for wedge products of 1-forms and then we extending (bi)linearly to all of* $\bigwedge^k \mathrm{V}^*$. *We can also extend to the whole Grassmann algebra* $\bigwedge \mathrm{V}^* = \bigoplus \bigwedge^k \mathrm{V}^*$ *by declaring forms of different degree to be orthogonal. We also have the obvious similar definition for* $\bigwedge^k \mathrm{V}$ *and* $\bigwedge \mathrm{V}$.

We would now like to exhibit the definition of the very useful star operator. This will be a map from $\bigwedge^k \mathrm{V}^*$ to $\bigwedge^{n-k} \mathrm{V}^*$ for each k, $1 \leq k \leq n$ where $n = \dim(M)$. First of all if we have an orthonormal basis $e^1, ..., e^n$ for V^* then $e^1 \wedge \cdots \wedge e^n \in \bigwedge^n \mathrm{V}^*$. But $\bigwedge^n \mathrm{V}^*$ is one dimensional and if $\ell : \mathrm{V}^* \to \mathrm{V}^*$ is any isometry of V^* then $\ell e^1 \wedge \cdots \wedge \ell e^n = \pm e^1 \wedge \cdots \wedge e^n$. In particular, for any permutation σ of the letters $\{1, 2, ..., n\}$ we have $e^1 \wedge \cdots \wedge e^n = sgn(\sigma) e^{\sigma 1} \wedge \cdots \wedge e^{\sigma n}$.

For a given $\mathcal{E}^* = \{e^1, ..., e^n\}$ for V^* (with dual basis for orthonormal basis $\mathcal{E} = \{e_1, ..., e_n\}$) us denote $e^1 \wedge \cdots \wedge e^n$ by $\varepsilon(\mathcal{E}^*)$. Then we have

$$\langle \varepsilon(\mathcal{E}^*), \varepsilon(\mathcal{E}^*) \rangle = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n = \pm 1$$

and the only elements ω of $\bigwedge^n \mathrm{V}^*$ with $\langle \omega, \omega \rangle = \pm 1$ are $\varepsilon(\mathcal{E}^*)$ and $-\varepsilon(\mathcal{E}^*)$. Given a fixed orthonormal basis $\mathcal{E}^* = \{e^1, ..., e^n\}$, all other orthonormal bases $\mathcal{B}^*$ bases for V^* fall into two classes. Namely, those for which $\varepsilon(\mathcal{B}^*) = \varepsilon(\mathcal{E}^*)$ and those for which $\varepsilon(\mathcal{B}^*) = -\varepsilon(\mathcal{E}^*)$. Each of these two top forms $\pm\varepsilon(\mathcal{E}^*)$ is called a metric volume element for $\mathrm{V}^*, \mathrm{g}^* = \langle , \rangle$. A choice of orthonormal basis determines one of these two volume elements and provides an orientation for V^*. On the other hand, we have seen that any nonzero top form ω determines an orientation. If we have an orientation given by a top form ω then $\mathcal{E} = \{e_1, ..., e_n\}$ determines the same orientation if and only if $\omega(e_1, ..., e_n) > 0$.

Definition 16.9 *Let an orientation be chosen on* V^* *and let* $\mathcal{E}^* = \{e^1, ..., e^n\}$ *be an oriented orthonormal frame so that* $vol := \varepsilon(\mathcal{E}^*)$ *is the corresponding volume element. Then if* $\mathcal{F} = \{f_1, ..., f_n\}$ *is a basis for* V *with dual basis* $\mathcal{F}^* = \{f^1, ..., f^n\}$ *then*

$$vol = \sqrt{|\det(\mathrm{g}_{ij})|} f^1 \wedge \cdots \wedge f^n$$

where $\mathrm{g}_{ij} = \langle f_i, f_j \rangle$.

Proof. Let $e^i = a^i_j f^j$ then

$$\begin{aligned} \varepsilon_i \delta^{ij} = \pm\delta^{ij} = \langle e^i, e^j \rangle &= \langle a^i_k f^k, a^j_m f^m \rangle \\ &= a^i_k a^j_m \langle f^k, f^m \rangle = a^i_k a^j_m \mathrm{g}^{km} \end{aligned}$$

so that $\pm 1 = \det(a^i_k)^2 \det(\mathrm{g}^{km}) = (\det(a^i_k))^2 (\det(\mathrm{g}_{ij}))^{-1}$ and so

$$\sqrt{|\det(\mathrm{g}_{ij})|} = \det(a^i_k).$$

On the other hand,

$$\begin{aligned} \mathrm{vol} &:= \varepsilon(\mathcal{E}^*) = e^1 \wedge \cdots \wedge e^n \\ &= a^1_{k_1} f^{k_1} \wedge \cdots \wedge a^n_{k_1} f^{k_1} = \det(a^i_k) f^1 \wedge \cdots \wedge f^n \end{aligned}$$

and the result follows. ∎

Fix an orientation and let $\mathcal{E}^* = \{e^1, ..., e^n\}$ be an orthonormal basis in that orientation class. Then we have chosen one of the two volume forms, say $\mathrm{vol} = \varepsilon(\mathcal{E}^*)$. Now we define $* : \bigwedge^k \mathrm{V}^* \to \bigwedge^{n-k} \mathrm{V}^*$ by first giving the definition on basis elements and then extending by linearity.

Definition 16.10 *Let* V^* *be oriented and let* $\{e^1, ..., e^n\}$ *be a positively oriented orthonormal basis. Let* σ *be a permutation of* $(1, 2, ..., n)$*. On the basis elements* $e^{\sigma(1)} \wedge \cdots \wedge e^{\sigma(k)}$ *for* $\bigwedge^k \mathrm{V}^*$ *define*

$$*(e^{\sigma(1)} \wedge \cdots \wedge e^{\sigma(k)}) = \varepsilon_{\sigma_1}\varepsilon_{\sigma_2} \cdots \varepsilon_{\sigma_k} sgn(\sigma) e^{\sigma(k+1)} \wedge \cdots \wedge e^{\sigma(n)}.$$

In other words,

$$*(e^{i_1} \wedge \cdots \wedge e^{i_k}) = \pm(\varepsilon_{i_1}\varepsilon_{i_2} \cdots \varepsilon_{i_k}) e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}$$

where we take the $+$ *sign if and only if* $e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge e^{j_1} \wedge \cdots \wedge e^{j_{n-k}} = e^1 \wedge \cdots \wedge e^n$.

Remark 16.2 *In case the scalar product is positive definite* $\varepsilon_1 = \varepsilon_2 \cdots = \varepsilon_n = 1$ *and so the formulas are a bit less cluttered.*

We may develop a formula for the star operator in terms of an arbitrary basis.

Lemma 16.2 *For* $\alpha, \beta \in \bigwedge^k \mathrm{V}^*$ *we have*

$$\langle \alpha, \beta \rangle vol = \alpha \wedge *\beta$$

Proof. It is enough to check this on typical basis elements $e^{i_1} \wedge \cdots \wedge e^{i_k}$ and $e^{m_1} \wedge \cdots \wedge e^{m_k}$. We have

$$\begin{aligned} &(e^{m_1} \wedge \cdots \wedge e^{m_k}) \wedge *(e^{i_1} \wedge \cdots \wedge e^{i_k}) \\ &= e^{m_1} \wedge \cdots \wedge e^{m_k} \wedge (\pm e^{j_1} \wedge \cdots \wedge e^{j_{n-k}}) \end{aligned} \tag{16.1}$$

This latter expression is zero unless $\{m_1, ..., m_k\} \cup \{j_1, ..., j_{n-k}\} = \{1, 2, ..., n\}$ or in other words, unless $\{i_1, ..., i_k\} = \{m_1, ..., m_k\}$. But this is also true for

$$\langle e^{m_1} \wedge \cdots \wedge e^{m_k}, e^{i_1} \wedge \cdots \wedge e^{i_k} \rangle \,\mathrm{vol}. \tag{16.2}$$

On the other hand if $\{i_1, ..., i_k\} = \{m_1, ..., m_k\}$ then both 16.1 and 16.2 give $\pm$ vol. So the lemma is proved up to a sign. We leave it to the reader to show that the definitions are such that the signs match. ∎

Proposition 16.2 *The following identities hold for the star operator:*

1) $*1 = vol$
2) $*vol = (-1)^{\mathrm{ind(g)}}$
3) $**\alpha = (-1)^{\mathrm{ind(g)}}(-1)^{k(n-k)}\alpha$ *for* $\alpha \in \bigwedge^k \mathrm{V}^*$.

Proof. (1) and (2) follow directly from the definitions. For (3) we must first compute $*(e^{j_{k+1}} \wedge \cdots \wedge e^{j_n})$. We must have $*(e^{j_{k+1}} \wedge \cdots \wedge e^{j_n}) = ce^{j_1} \wedge \cdots \wedge e^{j_k}$ for some constant c. On the other hand,

$$\begin{aligned} c\varepsilon_1\varepsilon_2 \cdots \varepsilon_n \,\mathrm{vol} &= \langle e^{j_{k+1}} \wedge \cdots \wedge e^{j_n}, ce^{j_1} \wedge \cdots \wedge e^{j_k} \rangle \\ &= \langle e^{j_{k+1}} \wedge \cdots \wedge e^{j_n}, *(e^{j_{k+1}} \wedge \cdots \wedge e^{j_n}) \rangle \\ &= \varepsilon_{j_{k+1}} \cdots \varepsilon_{j_n} sgn(j_{k+1} \cdots j_n, j_1 \cdots j_k) e^{j_{k+1}} \wedge \cdots \wedge e^{j_n} \wedge e^{j_1} \wedge \cdots \wedge e^{j_k} \\ &\varepsilon_{j_{k+1}} \cdots \varepsilon_{j_n} sgn(j_{k+1} \cdots j_n, j_1 \cdots j_k) \,\mathrm{vol} \\ &= (-1)^{k(n-k)} \varepsilon_{j_{k+1}} \cdots \varepsilon_{j_n} \,\mathrm{vol} \end{aligned}$$

so that $c = \varepsilon_{j_{k+1}} \cdots \varepsilon_{j_n} (-1)^{k(n-k)}$. Using this we have, for any permutation $J = (j_1, ..., j_n)$,

$$\begin{aligned} **\,(e^{j_1} \wedge \cdots \wedge e^{j_k}) &= *\varepsilon_{j_1}\varepsilon_{j_2} \cdots \varepsilon_{j_k} sgn(J) e^{j_{k+1}} \wedge \cdots \wedge e^{j_n} \\ &= \varepsilon_{j_1}\varepsilon_{j_2} \cdots \varepsilon_{j_k}\varepsilon_{j_{k+1}} \cdots \varepsilon_{j_n} sgn(J) e^{j_1} \wedge \cdots \wedge e^{j_k} \\ &= (-1)^{\mathrm{ind(g)}} (-1)^{k(n-k)} e^{j_1} \wedge \cdots \wedge e^{j_k} \end{aligned}$$

which implies the result. ■

16.2 Riemannian and semi-Riemannian Metrics

Consider a regular submanifold M of a Euclidean space, say $\mathbb{R}^n$. Since we identify T_pM as a subspace of $T_p\mathbb{R}^n \cong \mathbb{R}^n$ and the notion of length of tangent vectors makes sense on $\mathbb{R}^n$ it also makes sense for vectors in T_pM. In fact, if $X_p, Y_p \in T_pM$ and c_1, c_2 are some curves with $\dot{c}_1(0) = X_p$, $\dot{c}_2(0) = Y_p$ then c_1 and c_2 are also a curves in $\mathbb{R}^n$. Thus we have an inner product defined $\mathrm{g}_p(X_p, Y_p) = \langle X_p, Y_p \rangle$. For a manifold that is not given as submanifold of some $\mathbb{R}^n$ we must have an inner product assigned to each tangent space as part of an extra structure. The assignment of a nondegenerate symmetric bilinear form $\mathrm{g}_p \in T_pM$ for every p in a smooth way defines a tensor field $\mathrm{g} \in \mathfrak{X}_2^0(M)$ on M called metric tensor.

Definition 16.11 *If* $\mathrm{g} \in \mathfrak{X}_2^0(M)$ *is nondegenerate, symmetric and positive definite at every tangent space we call* g *a* ***Riemannian metric*** *(tensor). If* g *is a Riemannian metric then we call the pair* M, g *a* ***Riemannian manifold*** .

The Riemannian manifold as we have defined it is the notion that best generalizes to manifolds the metric notions from surfaces such as arc length of a curve, area (or volume), curvature and so on. But because of the structure of spacetime as expressed by general relativity we need to allow the metric to be indefinite. In this case, some nonzero vectors v might have zero or negative self scalar product $\langle v, v \rangle$.

Recall that the index $\nu = \mathrm{ind(g)}$ of a bilinear form is the number of negative ones appearing in the signature $(-1, ...1)$.

Definition 16.12 *If* $\mathrm{g} \in \mathfrak{X}_2^0(M)$ *is symmetric nondegenerate and has constant index on* M *then we call* g *a semi-Riemannian metric and* M, g *a* ***semi-Riemannian manifold*** *or* ***pseudo-Riemannian manifold.*** *The index is called the* ***index*** *of* M, g *and denoted* $\mathrm{ind}(M)$*. The signature is also constant and so the manifold has a signature also. If the index of a semi-Riemannian manifold (with* $\dim(M) \geq 2$*) is* $(-1, +1, +1+1, ...)$ *(or according to some conventions* $(1, -1, -1-1, ...)$*) then the manifold is called a* ***Lorentz manifold*** .

The simplest family of semi-Riemannian manifolds are the spaces $\mathbb{R}^{n-v,v}$ which are the Euclidean spaces $\mathbb{R}^n$ endowed with the scalar products given by

$$\langle x,y\rangle_\nu = -\sum_{i=1}^{\nu} x^i y^i + \sum_{i=\nu+1}^{n} x^i y^i.$$

Since ordinary Euclidean geometry does not use indefinite scalar products we shall call the spaces $\mathbb{R}^{\nu,n-\nu}$ **semi-Euclidean** spaces when the index ν is not zero. If we write just $\mathbb{R}^n$ then either we are not concerned with a scalar product at all or the scalar product is assumed to be the usual inner product ($\nu = 0$).Thus a Riemannian metric is just the special case of index 0. Analogous to the groups $S\mathrm{O}(n), \mathrm{O}(n)$ and $\mathrm{Euc}(n)$ are we have, associated to $\mathbb{R}^{\nu,n-\nu}$, the groups $S\mathrm{O}(\nu, n-\nu), \mathrm{O}(\nu, n-\nu)$ (generalizing the Lorentz group) and $\mathrm{Euc}(\nu, n-\nu)$ (generalizing the Poincaré group):

$$\begin{aligned}
\mathrm{O}(\nu, n-\nu) &= \{Q \in End(\mathbb{R}^{\nu,n-\nu}) : \langle Qx, Qy\rangle_\nu = \langle x,y\rangle_\nu \text{ for all } x,y \in \mathbb{R}^{\nu,n-\nu}\} \\
S\mathrm{O}(\nu, n-\nu) &= \{Q \in \mathrm{O}(\nu, n-\nu) : \det Q = \pm 1 \text{ (preserves a choice of orientation)} \\
\mathrm{Euc}(\nu, n-\nu) &= \{L : L(x) = Qx + x_0 \text{ for some } Q \in \mathrm{O}(\nu, n-\nu) \text{ and } x_0 \in \mathbb{R}^{\nu,n-\nu}
\end{aligned}$$

The group $\mathrm{Euc}(\nu, n-\nu)$ is the group of semi-Euclidean motions and we have the expected homogeneous space isomorphism $\mathbb{R}^{\nu,n-\nu} = \mathrm{Euc}(\nu, n-\nu)/\mathrm{O}(\nu, n-\nu)$. To be consistent we should denote this homogeneous space by $\mathbf{E}^{\nu,n-\nu}$.

Remark 16.3 (Notation) *We will usually write $\langle X_p, Y_p\rangle$ or $\mathrm{g}(X_p, Y_p)$ in place of $\mathrm{g}(p)(X_p, X_p)$. Also, just as for any tensor field we define the function $\langle X, Y\rangle$ that, for a pair of vector fields X and Y, is given by $\langle X, Y\rangle(p) = \langle X_p, Y_p\rangle$.*

In local coordinates $(x^1, ..., x^n)$ on $U \subset M$ we have that $\mathrm{g}|_U = \sum \mathrm{g}_{ij} dx^i \otimes dx^j$ where $\mathrm{g}_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\rangle$. Thus if $X = \sum X^i \frac{\partial}{\partial x^i}$ and $Y = \sum Y^i \frac{\partial}{\partial x^i}$ on U then

$$\langle X, Y\rangle = \sum \mathrm{g}_{ij} X^i Y^i \tag{16.3}$$

Remark 16.4 *The expression $\langle X, Y\rangle = \sum \mathrm{g}_{ij} X^i Y^i$ means that for all $p \in U$ we have $\langle X(p), Y(p)\rangle = \sum \mathrm{g}_{ij}(p) X^i(p) Y^i(p)$ where as we know that functions X^i and Y^i are given by $X^i = dx^i(X)$ and $Y^i = dx^i(Y)$.*

The definitions we gave for volume element, star operator and the musical isomorphisms are all defined on each tangent space T_pM of a semi-Riemannian manifold and the concepts globalize to the whole of TM accordingly. We have

1. Let M be oriented. The volume element induced by the metric tensor g is defined to be the $n-$form vol such that vol_p is the metric volume form on T_pM matching the orientation. If $(U, \mathtt{x})$ is a chart on M then we have

$$vol_U = \sqrt{|\det(\mathrm{g}_{ij})|} f^1 \wedge \cdots \wedge f^n$$

If f is a smooth function we may integrate f over M by using the volume form:

$$\int_M f\, vol$$

The volume of a domain $D \subset M$ is defined as

$$vol(D) := \{\int_M f vol : \text{supp}(f) \subset D \text{ and } 0 \leq f \leq 1\}$$

2. The musical isomorphisms are defined in the obvious way from the isomorphisms on each tangent space.

$$((\mathrm{g}^\sharp)^{\uparrow}_a \tau)(p) := (\mathrm{g}^\sharp(p))^{\uparrow}_a \tau(p)$$
$$\left((\mathrm{g}_\flat)^a_{\downarrow}\tau\right)(p) = (\mathrm{g}_\flat(p))^a_{\downarrow}\tau(p)$$

If we choose a local frame field F_i and its dual F^i on $U \subset M$ then in terms of the components of a tensor τ and the metric tensor with respect to this frame we have

$$(\mathrm{g}^\sharp)^{\uparrow}_a \tau_U = \tau^{i_1,\ldots,i_r j_a}_{j_1,\ldots,j_{a-1},\widehat{j_a},j_{a+1}\ldots,j_{s-1}} F_{i_1} \otimes \cdots \otimes F_{i_r} \otimes F_{j_a} \otimes F^{j_1} \otimes \cdots \otimes F^{j_{s-1}}.$$

where

$$\begin{aligned}&\tau^{i_1,\ldots,i_r j}_{j_1,\ldots,j_{b-1},\widehat{k},j_b\ldots,j_{s-1}}(p)\\ &= \tau^{\,i_1,\ldots,i_r}{}_{j_1,\ldots,j_{j-1}}{}^{j}{}_{j_{b+1}\ldots,j_{s-1}}(p) := \sum_m \mathrm{g}^{jm}(p)\tau^{i_1,\ldots,i_r}_{j_1,\ldots,j_{b-1},m,j_b\ldots,j_{s-1}}(p).\end{aligned}$$

A similar formula holds for $(\mathrm{g}_\flat)^a_{\downarrow}\tau$ and we often suppress the dependence on the point p and the reference to the domain U.

3. Once again assume that M is oriented with metric volume form vol. The star operator gives two types of maps which are both denoted by $*$. Namely, the bundle maps $* : \bigwedge^k T^*M \to \bigwedge^{n-k} T^*M$ which has the obvious definition in terms of the star operators on each fiber and the maps on forms $* : \Omega^k(M) \to \Omega^{n-k}(M)$ which is also induce in the obvious way. The definitions are set up so that $*$ is natural with respect to restriction and so that for any oriented orthonormal frame field $\{E_1, ..., E_n\}$ with dual $\{\theta^1, ..., \theta^n\}$ we have $*(\theta^{i_1}\wedge\cdots\wedge\theta^{i_k}) = \pm(\varepsilon_{i_1}\varepsilon_{i_2}\cdots\varepsilon_{i_k})\theta^{j_1}\wedge\cdots\wedge\theta^{j_{n-k}}$ where as before we use the $+$ sign if and only if $\theta^{i_1} \wedge \cdots \wedge \theta^{i_k} \wedge \theta^{j_1} \wedge \cdots \wedge \theta^{j_{n-k}} = \theta^1 \wedge \cdots \wedge \theta^n = vol$

As expected we have $*1 = vol$, $*vol = (-1)^{\mathrm{ind(g)}}$ and $**\alpha = (-1)^{\mathrm{ind(g)}}(-1)^{k(n-k)}\alpha$ for $\alpha \in \Omega^k(M)$.

The inner products induced by $g(p)$ on $\bigwedge^k T_p^*M$ for each p combine to give a pairing $\langle .,. \rangle : \Omega^k(M) \times \Omega^k(M) \to C^\infty(M)$ with $\langle \alpha, \beta\rangle(p) := \langle \alpha(p), \beta(p)\rangle_p = \mathrm{g}_p(\alpha(p), \beta(p))$. We see that $\langle \alpha, \beta\rangle vol = \alpha \wedge *\beta$.

Now we can put an inner product on the space $\Omega(M) = \sum_k \Omega^k(M)$ by letting $(\alpha|\beta) = 0$ for $\alpha \in \Omega^{k_1}(M)$ and $\beta \in \Omega^{k_2}(M)$ with $k_1 \neq k_2$ and

$$(\alpha|\beta) = \int_M \alpha \wedge *\beta = \int_M \langle \alpha, \beta \rangle vol \text{ if } \alpha, \beta \in \Omega^k(M)$$

Definition 16.13 *Let M, g and N, h be two semi-Riemannian manifolds. A diffeomorphism $\Phi : M \to N$ is called an* ***isometry*** *if $\Phi^*\mathrm{h} = \mathrm{g}$. Thus for an isometry $\Phi : M \to N$ we have $\mathrm{g}(v, w) = \mathrm{h}(T\Phi \cdot v, T\Phi \cdot w)$ for all $v, w \in TM$. If $\Phi : M \to N$ is a local diffeomorphism such that $\Phi^*\mathrm{h} = \mathrm{g}$ then Φ is called a* ***local isometry.*** *If there is an isometry $\Phi : M \to N$ then we say that M, g and N, h are* ***isometric****.*

Definition 16.14 *Let $\widetilde{M}$ and M be semi-Riemannian manifolds. If $\wp : \widetilde{M} \to M$ is a covering map such that $\wp$ is a local isometry we call $\wp : \widetilde{M} \to M$ a* ***semi-Riemannian*** *covering.*

Definition 16.15 *The set of all isometries of a semi-Riemannian manifold M to itself is a group called the isometry group. It is denoted by Isom(M).*

The isometry group of a generic manifold is most likely trivial but many examples of manifolds with relatively large isometry groups are easy to find using Lie group theory. This will be taken up at later point in the exposition.

Example 16.1 *We have seen that a regular submanifold of a Euclidean space $\mathbb{R}^n$ is a Riemannian manifold with the metric inherited from $\mathbb{R}^n$. In particular, the sphere $S^{n-1} \subset \mathbb{R}^n$ is a Riemannian manifold. Every isometry of S^{n-1} is the restriction to S^{n-1} of an isometry of $\mathbb{R}^n$ that fixed the origin (and consequently fixes S^{n-1}).*

If we have semi-Riemannian manifolds M, g and N, h then we can consider the product manifold $M \times N$ and the projections $pr_1 : M \times N \to M$ and $pr_2 : M \times N \to N$. The tensor $\mathrm{g} \times \mathrm{h} = pr_1^*\mathrm{g} + pr_2^*\mathrm{h}$ provides a semi-Riemannian metric on the manifold $M_1 \times M_2$ and $M_1 \times M_2, pr_1^*\mathrm{g} + pr_2^*\mathrm{h}$ is called the semi-Riemannian product of M, g and N, h. Let $U_1 \times U_2, (\mathtt{x}, \mathtt{y}) = (x^1, ..., x^{n_1}, y^1, ..., y^{n_2})$ be a natural product chart where we write x^i and y^i instead of the more pedantic $x^i \circ pr_1$ and $y^i \circ pr_2$. We then have coordinate frame fields $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^{n_1}}, \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^{n_2}}$ and the components of $\mathrm{g} \times \mathrm{h} = pr_1^*\mathrm{g} + pr_2^*\mathrm{h}$ in this frame discovered by choosing a point $(p_1, p_2) \in U_1 \times U_2$ and then calculating. However, we will be less likely to be misled if we temporarily return to the notation introduced earlier where

$\widetilde{x}^i = x^i \circ pr_1, \widetilde{y}^i = y^i \circ pr_1$ and $\frac{\partial}{\partial \widetilde{x}^i}$ etc. We then have

$$\begin{aligned}
&\mathrm{g}\times\mathrm{h}(\left.\frac{\partial}{\partial \widetilde{x}^i}\right|_{(p,q)}, \left.\frac{\partial}{\partial \widetilde{y}^j}\right|_{(p,q)}) \\
&= pr_1^*\mathrm{g}(\left.\frac{\partial}{\partial \widetilde{x}^i}\right|_{(p,q)}, \left.\frac{\partial}{\partial \widetilde{y}^j}\right|_{(p,q)}) + pr_2^*\mathrm{h}(\left.\frac{\partial}{\partial \widetilde{x}^i}\right|_{(p,q)}, \left.\frac{\partial}{\partial \widetilde{y}^j}\right|_{(p,q)}) \\
&= \mathrm{g}(Tpr_1 \left.\frac{\partial}{\partial \widetilde{x}^i}\right|_{(p,q)}, Tpr_1 \left.\frac{\partial}{\partial \widetilde{y}^j}\right|_{(p,q)}) + \mathrm{h}(Tpr_2 \left.\frac{\partial}{\partial \widetilde{x}^i}\right|_{(p,q)}, Tpr_2 \left.\frac{\partial}{\partial \widetilde{y}^j}\right|_{(p,q)}) \\
&= \mathrm{g}(\left.\frac{\partial}{\partial x^i}\right|_p, 0_p) + \mathrm{h}(0_q, \left.\frac{\partial}{\partial y^j}\right|_q) \\
&= 0 + 0 = 0
\end{aligned}$$

and (abbreviating a bit)

$$\begin{aligned}
\mathrm{g}\times\mathrm{h}(\left.\frac{\partial}{\partial \widetilde{x}^i}\right|_{(p,q)}, \left.\frac{\partial}{\partial \widetilde{x}^j}\right|_{(p,q)}) &= \mathrm{g}(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^i}\right|_p) + \mathrm{h}(0_q, 0_q) \\
&= \mathrm{g}_{ij}(p)
\end{aligned}$$

Similarly $\mathrm{g}\times\mathrm{h}(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})(p,q) = \mathrm{h}_{ij}(q)$. So with respect to the coordinates $(x^1, ..., x^{n_1}, y^1, ..., y^{n_2})$ the matrix of $\mathrm{g}\times\mathrm{h}$ is of the form

$$\begin{pmatrix} (\mathrm{g}_{ij} \circ pr_1) & 0 \\ 0 & (\mathrm{h}_{ij} \circ pr_2) \end{pmatrix}.$$

Forming semi-Riemannian product manifolds is but one way to get more examples of semi-Riemannian manifolds. Another way to get a variety of examples of semi-Riemannian manifolds is via a group action by isometries (assuming one is lucky enough to have a big supply of isometries). Let us here consider the case of a discrete group that acts smoothly, properly, freely and by isometries. We have already seen that if we have an action $\rho : G \times M \to M$ satisfying the first three conditions then the quotient space M/G has a unique structure as a smooth manifold such that the projection $\kappa : M \to M/G$ is a covering. Now since G acts by isometries $\rho_g^*\langle .,. \rangle = \langle .,. \rangle$ for all $g \in G$. The tangent map $T\kappa : TM \to T(M/G)$ is onto and so for any $\bar{v}_{\kappa(p)} \in T_{\kappa(p)}(M/G)$ there is a vector $v_p \in T_pM$ with $T_p\kappa \cdot v_p = \bar{v}_{\kappa(p)}$. In fact there is more than one such vector in T_pM (except in the trivial case $G = \{e\}$) but if $T_p\kappa \cdot v_p = T_q\kappa \cdot w_q$ then there is a $g \in G$ such that $\rho_g p = q$ and $T_p\rho_g v_p = w_q$. Conversely, if $\rho_g p = q$ then $T_p\kappa \cdot (T_p\rho_g v_p) = T_q\kappa \cdot w_q$. Now for $\bar{v}_1, \bar{v}_2 \in T_pM$ define $h(\bar{v}_1, \bar{v}_2) = \langle v_1, v_2 \rangle$ where v_1 and v_2 are chosen so that $T\kappa \cdot v_i = \bar{v}_i$. From our observations above this is well defined. Indeed, if $T_p\kappa \cdot v_i = T_q\kappa \cdot w_i = \bar{v}_i$ then there is an isometry ρ_g with $\rho_g p = q$ and $T_p\rho_g v_i = w_i$ and so

$$\langle v_1, v_2 \rangle = \langle T_p\rho_g v_1, T_p\rho_g v_2 \rangle = \langle w_1, w_2 \rangle.$$

It is easy to show that $x \mapsto h_x$ defined a metric on M/G with the same signature as that of $\langle .,. \rangle$ and $\kappa^* h = \langle .,. \rangle$. In fact we will use the same notation for either the metric on M/G or on M which is not such an act of violence since κ is now a local isometry.

If we have a local isometry $\phi : N \to M$ then any lift $\phi : N \to \widetilde{M}$ is also a local isometry. Deck transformations are lifts of the identify map $M \to M$ and so are diffeomorphisms which are local isometries. Thus deck transformations are in fact, isometries. We conclude that the group of deck transformations of a Riemannian cover is a subgroup of the group of isometries $Isom(\widetilde{M})$.

The simplest example of this construction is $\mathbb{R}^n/\Gamma$ for some lattice Γ and where we use the canonical Riemannian metric on $\mathbb{R}^n$. In case the lattice is isomorphic to $\mathbb{Z}^n$ then $\mathbb{R}^n/\Gamma$ is called a flat torus of dimension n. Now each of these tori are locally isometric but may not be globally so. To be more precise, suppose that $f_1, f_2, ..., f_n$ is a basis for $\mathbb{R}^n$ which is not necessarily orthonormal. Let Γ_f be the lattice consisting of integer linear combinations of $f_1, f_2, ..., f_n$. The question now is what if we have two such lattices Γ_f and $\Gamma_{\bar{f}}$ when is $\mathbb{R}^n/\Gamma_f$ isometric to $\mathbb{R}^n/\Gamma_{\bar{f}}$? Now it may seem that since these are clearly diffeomorphic and since they are locally isometric then they must be (globally) isometric. But this is not the case. We will be able to give a good reason for this shortly but for now we let the reader puzzle over this. (DID I KEEPTHISPROMISS?)

Every smooth manifold that admits partitions of unity also admits at least one (in fact infinitely may) Riemannian metrics. This includes all finite dimensional paracompact manifolds. The reason for this is that the set of all Riemannian metric tensors is, in an appropriate sense, convex. To wit:

Proposition 16.3 *Every smooth manifold that admits a smooth partition of unity admits a Riemannian metric.*

Proof. As in the proof of 16.11 above we can transfer the Euclidean metric onto the domain U_α of any given chart via the chart map ψ_α. The trick is to piece these together in a smooth way. For that we take a smooth partition of unity U_α, ρ_α subordinate to a cover by charts U_α, ψ_α. Let g_α be any metric on U_α and define

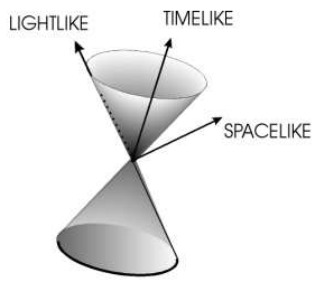
$$\mathrm{g}(p) = \sum \rho_\alpha(p) g_\alpha(p).$$

The sum is finite at each $p \in M$ since the partition of unity is locally finite and the functions $\rho_\alpha \mathrm{g}_\alpha$ are extended to be zero outside of the corresponding U_α. The fact that $\rho_\alpha \geq 0$ and $\rho_\alpha > 0$ at p for at least one α easily gives the result that g positive definite is a Riemannian metric on M. ■

The length of a tangent vector $X_p \in T_pM$ in a *Riemannian* manifold is given by $\sqrt{\mathrm{g}(X_p, X_p)} = \sqrt{\langle X_p, X_p \rangle}$. In the case of an indefinite metric ($\nu > 0$) we will need a classification:

Definition 16.16 *A tangent vector $\nu \in T_pM$ to a semi-Riemannian manifold M is called*

1. *spacelike if* $\langle \nu, \nu \rangle > 0$
2. *lightlike or null if* **??** $\langle \nu, \nu \rangle = 0$



3. *timelike if* $\langle \nu, \nu \rangle < 0$.

4. *nonnull if* ν *is either timelike of spacelike.*

   The terms spacelike,lightlike, timelike indicate the ***causal character*** *of* v.

Definition 16.17 *The set of all timelike vectors* T_pM *in is called the* ***light cone*** *at* p.

Definition 16.18 *Let* $I \subset \mathbb{R}$ *be some interval. A curve* $c : I \to M, \mathrm{g}$ *is called spacelike, lightlike,timelike, or nonnull according as* $\dot{c}(t) \in T_{c(t)}M$ *is spacelike, lightlike, timelike, or nonnull respectively for all* $t \in I$.

For Lorentz spaces, that is for semi-Riemannian manifolds with index equal to 1 and dimension greater than or equal to 2, we may also classify subspaces into three categories:

Definition 16.19 *Let* M, g *be a Lorentz manifold. A subspace* $\mathrm{W} \subset T_pM$ *of the tangents space is called*

1. *spacelike if* $\mathrm{g}|_{\mathrm{W}}$ *is positive definite,*

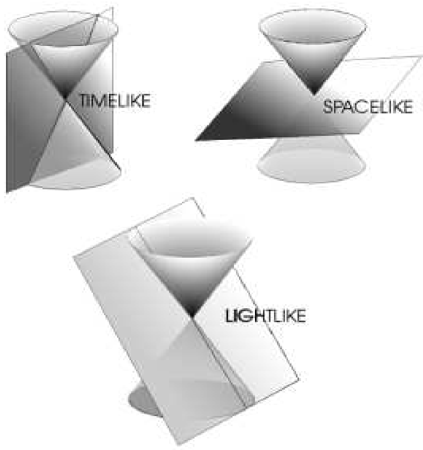
2. *time like if* $\mathrm{g}|_{\mathrm{W}}$ *nondegenerate with index* 1*,*

3. *lightlike if* $\mathrm{g}|_{\mathrm{W}}$ *is degenerate.*

Theorem 16.1 *A smooth manifold admits a an indefinite metric of index* k *if it the tangent bundle has some rank* k *subbundle.*

For the proof of this very plausible theorem see (add here)??

Recall that a continuous curve $c : [a, b] \to M$ into a smooth manifold is called **piecewise smooth** if there exists a partition $a = t_0 < t_1 < \cdots < t_k = b$ such that c restricted to $[t_i, t_{i+1}]$ is smooth for $0 \leq i \leq k - 1$. Also, a curve $c : [a, b] \to M$ is called **regular** if it has a nonzero tangent for all $t \in [a, b]$.



Definition 16.20 *Let M, g be Riemannian. If $c : [a, b] \to M$ is a (piecewise smooth) curve then the length of the curve from $c(a)$ to $c(b)$ is defined by*

$$L(c) := L_{c(a)\to c(b)}(c) := \int_a^t \langle \dot{c}(t), \dot{c}(t)\rangle^{1/2} dt. \tag{16.4}$$

Definition 16.21 *Let M, g be semi-Riemannian. If $c : [a, b] \to M$ is a (piecewise smooth) timelike or spacelike curve then*

$$L_{c(a),c(b)}(c) = \int_a^b |\langle \dot{c}(t), \dot{c}(t)\rangle|^{1/2} \, dt$$

is called the ***length*** *of the curve. For a general (piecewise smooth) curve $c : [a, b] \to M$, where M is a Lorentz manifold, the quantity*

$$\tau_{c(a),c(b)}(c) = \int_a^b \langle \dot{c}(t), \dot{c}(t)\rangle^{1/2} dt$$

will be called the ***proper time*** *of c.*

In general, if we wish to have a positive real number for a length then in the semi-Riemannian case we need to include absolute value signs in the definition so the proper time is just the timelike special case of a generalized arc length defined for any smooth curve by $\int_a^b |\langle \dot{c}(t), \dot{c}(t)\rangle|^{1/2} \, dt$ but unless the curve is either timelike or spacelike this arc length can have some properties that are decidedly not like our ordinary notion of length. In particular, curve may connect two different points and the generalized arc length might still be zero! It becomes clear that we are not going to be able to define a metric distance function as we soon will for the Riemannian case.

Definition 16.22 *A **positive reparametrization** of a piecewise smooth curve $c : I \to M$ is a curve defined by composition $c \circ f^{-1} : J \to M$ where $f : I \to J$ is a piecewise smooth bijection that has $f' > 0$ on each subinterval $[t_{i-1}, t_i] \subset I$ where c is smooth.*

Remark 16.5 (important fact) *The integrals above are well defined since $c'(t)$ is defined except for a finite number of points in $[a, b]$. Also, it is important to notice that by standard change of variable arguments a positive reparametrization $\widetilde{c}(u) = c(f^{-1}(u))$ where $u = f(t)$ does not change the (generalized) length of the curve*

$$\int_a^b |\langle \dot{c}(t), \dot{c}(t)\rangle|^{1/2} dt = \int_{f^{-1}(a)}^{f^{-1}(b)} |\langle \widetilde{c}'(u), \widetilde{c}'(u)\rangle|^{1/2} du.$$

Thus the (generalized) length of a piecewise smooth curve is a geometric property of the curve; i.e. a semi-Riemannian invariant.

16.2.1 Electromagnetism

In this subsection we take short trip into physics. Lorentz manifolds of dimension 4 are the basis of Einstein's General Theory of Relativity. The geometry which is the infinitesimal model of all Lorentz manifolds and provides that background geometry for special relativity is Minkowski space. Minkowski space M^4 is $\mathbb{R}^{1,3}$ acted upon by the Poincaré group. More precisely, Minkowski space is the affine space modeled on $\mathbb{R}^{1,3}$ but we shall just take $\mathbb{R}^{1,3}$ itself and "forget" the preferred origin and coordinates. All coordinates related to the standard coordinate system by an element of the Poincaré group are to be put on equal footing. The reader will recall that these special coordinate systems are referred to as (Lorentz) inertial coordinates. Each tangent space of M^4 is a scalar product space canonically isomorphic $\mathbb{R}^{1,3}$. Of course, M^4 is a very special semi-Riemannian manifold. M^4 is flat (see below definition 16.31 below), simply connected and homogeneous. The Poincaré group is exactly the isometry group of M^4. We shall now gain some insight into Minkowski space by examining some of the physics that led to its discovery. Each component of the electric field associated with an instance of electromagnetic radiation in free space satisfies the wave equation $\Box f = 0$ where $\Box = -\partial_0^2 + \partial_1^2 + \partial_2^2 + \partial_3^2$ is the wave operator). The form of this equation already suggests a connection with the Lorentz metric on M^4. In fact, the groups $\mathrm{O}(1, 3)$ and $P = \mathrm{Euc}(1, 3)$ preserve the operator. On the other hand, things aren't quite so simple since it is the components of a vector that appears in the wave equation and those have their own transformation law. Instead of pursuing this line of reasoning let us take a look at the basic equations of electromagnetism; Maxwell's equations[1]:

[1] Actually, this is the form of Maxwell's equations after a certain convenient choice of units and we are ignoring the somewhat subtle distinction between the two types of electric fields E and D and the two types of magnetic fields B and H and also their relation in terms of dialectic constants.

$$\begin{aligned}
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0 \\
\nabla \cdot \mathbf{E} &= \varrho \\
\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \mathbf{j}
\end{aligned}$$

Here $\mathbf{E}$ and $\mathbf{B}$, the electric and magnetic fields are functions of space and time. We write $\mathbf{E} = \mathbf{E}(t, \mathbf{x})$, $\mathbf{B} = \mathbf{B}(t, \mathbf{x})$. The notation suggests that we have conceptually separated space and time as if we were stuck in the conceptual framework of the Galilean spacetime. Our purpose is to slowly discover how much better the theory becomes when we combine space and time in Minkowski spacetime.

Before continuing we will try to say a few words about the meaning of these equations since the readership may include a few who have not looked at these equations in a while. The electric field $\mathbf{E}$ is produced by the presence of charged particles. Under normal conditions a generic material is composed of a large number of atoms. To simplify matters we will think of the atom as being composed of just three types of particle; electrons, protons and neutrons. Protons carry a positive charge and electrons carry a negative charge and neutrons carry no charge. Normally, each atom will have a zero net charge since it will have an equal number of electrons and protons. It is a testament to the relative strength of electrical force that if a relatively small percent of the atoms in a material a stripped from their atoms and conducted away then there will be a net positive charge. In the vicinity of the material there will be an electric field the evidence of which is that if small "test particle" with a positive charge moves into the vicinity of the positively charged material it will be deflected, that is, accelerated away from what would otherwise presumably be a constant speed straight path. Let us assume for simplicity that the charged body which has the larger, positive charge is stationary at $\mathbf{r}_0$ with respect to a rigid rectangular coordinate system which is also stationary with respect to the laboratory. We must assume that our test particle carries is sufficiently small charge so that the electric field that it creates contribute negligibly to the field we are trying to detect (think of a single electron). Let the test particle be located at $\mathbf{r}$. Careful experiments show that when both charges are positive, the force experience by the test particle is directly away from the charged body located at $\mathbf{r}_0$ and has magnitude proportional to

$$qe/r^2$$

where $r = |\mathbf{r} - \mathbf{r}_0|$ is the distance between the charged body and the test particle, and where q and e are positive numbers which represent the amount of charge carried by the stationary body and the test particle respectively. If the units are chosen in an appropriate way we can say that the force F is given by

$$F = qe/r^2 = qe\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3}.$$

By definition, the electric field at the location $\mathbf{r}$ of the test particle is

$$\mathbf{E} = q\frac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|^3} \tag{16.5}$$

Now if the test particle has charge opposite to that of the source body then one of q or e will be negative an the force is directed toward the source. The direction of the electric field is completely determined by the sign of q and whether q is positive or negative the correct formula for $\mathbf{E}$ is still equation 16.5. The test particle could have been placed anywhere in space and so the electric field is implicitly defined at each point in space and so gives a vector field on $\mathbb{R}^3$. If the charge is modeled as a smoothly distributed charge density q which is nonzero in some region $U \subset \mathbb{R}^3$, then the total charge is given by integration $Q = \int_U q(t, \mathbf{r})dV_{\mathbf{r}}$ and the field at $\mathbf{r}$ is now given by $\mathbf{E}(t, \mathbf{r}) = \int_U q(t, \mathbf{y})\frac{\mathbf{r}-\mathbf{y}}{|\mathbf{r}-\mathbf{y}|^3}dV_{\mathbf{y}}$. Now since the source particle is stationary at $\mathbf{r}_0$ the electric field will be independent of time t. It turns out that a magnetic field is only produced by a changing electric field which would be produced, for example, if the source body was in motion. When charge is put into motion we have an electric current. For example, suppose a region of space is occupied by a mist of charged particles moving in such a way as to be essentially a fluid and so modeled by a smooth vector field $\mathbf{j}$ on $\mathbb{R}^3$ which in general also depend on time. Of course if the fluid of charge have velocity $\mathbf{V}(t, \mathbf{r})$ at $\mathbf{r}$ and the charge density is given by $\rho(t, \mathbf{y})$ then $\mathbf{j} = \mathbf{V}(t, \mathbf{r})\rho(t, \mathbf{y})$. But once again we can imagine a situation where the electric field does not depend of t. This would be the case even when the field is created by a moving fluid of charge as long at the charge density is constant in time. Even then, if the test particle is at rest in the laboratory frame (coordinate system) then there will be no evidence of a magnetic field felt by the test particle. In fact, suppose that there is a magnetic field $\mathbf{B}$ with an accompanying electric field $\mathbf{B}$ produced by a circulating charge density. Then the force felt by the test particle of charge e at $\mathbf{r}$ is $\mathbf{F} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B}$ where $\mathbf{v}$ is the velocity of the test particle. The test particle has to be moving to feel the magnetic part of the field! This is strange already since even within the Galilean framework we would have expected the laws of physics to be the same in any inertial frame. But here we see that the magnetic field would disappear in any frame which is moving with the charge since in that frame the charge would have no velocity. At this point it is worth pointing out that from the point of view of spacetime, we are not staying true to the spirit of differential geometry since a vector field should have a geometric reality that is independent of its expression in an coordinate system. But here it seems that the fields just *are* those local representatives and so are dependent on the choice of spacetime coordinates; they seem to have no reality independent of the choice of inertial coordinate system. This is especially obvious for the magnetic field since a change to a new inertial system in spacetime (Galilean at this point) can effect a disappearance of the magnetic field. Let us ignore this problem for a while by just thinking of time as a parameter and sticking to one coordinate system.

Our next task is to write Maxwell's equations in terms of differential forms. We already have a way to convert (time dependent) vector fields $\mathbf{E}$ and $\mathbf{B}$ on

$\mathbb{R}^3$ into (time dependent) differential forms on $\mathbb{R}^3$. Namely, we use the flatting operation with respect to the standard metric on $\mathbb{R}^3$. For the electric field we have

$$\mathbf{E} = E^x\partial_x + E^y\partial_y + E^z\partial_z \mapsto \mathcal{E} = E_x dx + E_y dy + E_z dz$$

For the magnetic field we do something a bit different. Namely, we flat and then apply the star operator. In rectangular coordinates we have

$$\mathbf{B} = B^x\partial_x + B^y\partial_y + B^z\partial_z \mapsto \mathcal{B} = B_x dy \wedge dz + B_y dy \wedge dx + B_z dx \wedge dz$$

Now if we stick to rectangular coordinates (as we have been) the matrix of the standard metric is just $I = (\delta_{ij})$ and so we see that the above operations do not numerically change the components of the fields. Thus in any rectangular coordinates we have

$$\begin{aligned} E^x &= E_x \\ E^y &= E_y \\ E_z &= E_z \end{aligned}$$

and similarly for the B's. Now it is not hard to check that in the static case where $\mathcal{E}$ and $\mathcal{B}$ are time independent the first pair of (static) Maxwell's equations are equivalent to

$$d\mathcal{E} = 0 \text{ and } d\mathcal{B} = 0$$

This is pretty nice but if we put time dependent back into the picture we need to do a couple more things to get a nice viewpoint. So assume now that $\mathbf{E}$ and $\mathbf{B}$ and hence the forms $\mathcal{E}$ and $\mathcal{B}$ are time dependent and lets view these as differential forms on spacetime M^4 and in fact, ; let us combine $\mathcal{E}$ and $\mathcal{B}$ into a single $2-$form on M^4 by letting

$$\mathcal{F} = \mathcal{B} + \mathcal{E} \wedge dt.$$

Since $\mathcal{F}$ is a $2-$form it can be written in the form $\mathcal{F} = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ where $F_{\mu\nu} = -F_{\nu\mu}$ and where the Greek indices are summed over $\{0, 1, 2, 3\}$. It is traditional in physics to let the Greek indices run over this set and to let Latin indices run over just the space indices $\{1, 2, 3, 4\}$. We will follow this convention for a while. Now if we compare $\mathcal{F} = \mathcal{B} + \mathcal{E} \wedge dt$ with $\frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu$ we see that the $F_{\mu\nu}$'s form a antisymmetric matrix which is none other than

$$\begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

Our goal now is to show that the first pair of Maxwell's equations are equivalent to the single differential form equation

$$d\mathcal{F} = 0$$

Let N be ant manifold and let $M = (a, b)\times N$ for some interval (a, b). Let the coordinate on (a, b) be $t = x^0$ (time). Let $U, (x^1, ..., x^n)$ be a coordinate system on M. With the usual abuse of notation $(x^0, x^1, ..., x^n)$ is a coordinate system on $(a, b)\times N$. One can easily show that the local expression $d\omega = \partial_\mu f_{\mu_1...\mu_k} \wedge dx^\mu \wedge dx^{\mu_1} \wedge\cdots\wedge dx^{\mu_k}$ for the exterior derivative of a form $\omega = f_{\mu_1...\mu_k} dx^{\mu_1} \wedge\cdots\wedge dx^{\mu_k}$ can be written as

$$
\begin{aligned}
d\omega = &\sum_{i=1}^{3} \partial_i \omega_{\mu_1...\mu_k} \wedge dx^i \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k} \\
&+ \partial_0 \omega_{\mu_1...\mu_k} \wedge dx^0 \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}
\end{aligned}
\tag{16.6}
$$

where the μ_i sum over $\{0, 1, 2, ..., n\}$. Thus we may consider spatial d_S part of the exterior derivative operator d on $(a, b)\times S = M$. To wit, we think of a given form ω on $(a, b)\times S$ as a time dependent for on N so that $d_S\omega$ is exactly the first term in the expression 16.6 above. Then we may write $d\omega = d_S\omega + dt \wedge \partial_t\omega$ as a compact version of the expression 16.6. The part $d_S\omega$ contains no dt's. Now we have by definition $\mathcal{F} = \mathcal{B} + \mathcal{E} \wedge dt$ on $\mathbb{R} \times \mathbb{R}^3 = M^4$ and so

$$
\begin{aligned}
d\mathcal{F} &= d\mathcal{B} + d(\mathcal{E} \wedge dt) \\
&= d_S\mathcal{B} + dt \wedge \partial_t\mathcal{B} + (d_S\mathcal{E} + dt \wedge \partial_t\mathcal{E}) \wedge dt \\
&= d_S\mathcal{B} + (\partial_t\mathcal{B} + d_S\mathcal{E}) \wedge dt.
\end{aligned}
$$

Now the $d_S\mathcal{B}$ is the spatial part and contains no dt's. It follows that $d\mathcal{F}$ is zero if and only if both $d_S\mathcal{B}$ and $\partial_t\mathcal{B} + d_S\mathcal{E}$ are zero. But unraveling the definitions shows that the pair of equations $d_S\mathcal{B} = 0$ and $\partial_t\mathcal{B} + d_S\mathcal{E} = 0$ (which we just showed to be equivalent to $d\mathcal{F} = 0$) are Maxwell's first two equations disguised in a new notation so in summary we have

$$
d\mathcal{F} = 0 \iff \begin{array}{c} d_S\mathcal{B} = 0 \\ \partial_t\mathcal{B} + d_S\mathcal{E} = 0 \end{array} \iff \begin{array}{c} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \end{array}
$$

Now we move on to rewrite the last pair of Maxwell's equations where the advantage of combining time and space together is Manifest to an even greater degree. One thing to notice about the about what we have done so far is the following. Suppose that the electric and magnetic fields were really all along most properly thought of a differential forms. Then we see that the equation $d\mathcal{F} = 0$ has nothing to do with the metric on Minkowski space at all. In fact, if $\phi : M^4 \to M^4$ is any diffeomorphism at all we have $d\mathcal{F} = 0$ if and only if $d(\phi^*\mathcal{F}) = 0$ and so the truth of the equation $d\mathcal{F} = 0$ is really a differential topological fact; a certain form is closed. The metric structure of Minkowski space is irrelevant. The same will not be true for the second pair. Even if we start out with the form $\mathcal{F}$ on spacetime it will turn out that the metric will necessarily be implicitly in the differential forms version of the second pair of Maxwell's equations. In fact, what we will show is that if we use the star operator for the Minkowski metric then the second pair can be rewritten as the single equation $*d * \mathcal{F} = *\mathcal{J}$ where $\mathcal{J}$ is formed from $\mathbf{j} = (j^1, j^2, j^3)$ and

ρ as follows: First we form the 4−vector field $J = \rho\partial_t + j^1\partial_x + j^2\partial_y + j^3\partial_z$ (called the 4−current) and then using the flatting operation we produce $\mathcal{J} = -\rho dt + j^1dx + j^2dy + j^3dz = J_0dt + J_1dx + J_2dy + J_3dz$ which is the covariant form of the 4−current. We will only outline the passage from $*d * \mathcal{F} = *\mathcal{J}$ to the pair $\nabla \cdot \mathbf{E} = \varrho$ and $\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}$. Let $*_S$be the operator one gets by viewing differential forms on M^4 as time dependent forms on $\mathbb{R}^3$ and then acting by the star operator with respect to the standard metric on $\mathbb{R}^3$. The first step is to verify that the pair $\nabla \cdot \mathbf{E} = \varrho$ and $\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}$ is equivalent to the pair $*_Sd_S *_S \mathcal{E} = \varrho$ and $-\partial_t\mathcal{E} + *_Sd_S *_S \mathcal{B} = j$ where $j := j^1dx + j^2dy + j^3dz$ and $\mathcal{B}$ and $\mathcal{E}$ are as before. Next we verify that

$$*\mathcal{F} = *_S\mathcal{E} - *_S\mathcal{B} \wedge dt.$$

So the goal has become to get from $*d * \mathcal{F} = *\mathcal{J}$ to the pair $*_Sd_S *_S \mathcal{E} = \varrho$ and $-\partial_t\mathcal{E} + *_Sd_S *_S \mathcal{B} = j$. The following exercise finishes things off.

Exercise 16.3 *Show that* $*d * \mathcal{F} = -\partial_t\mathcal{E} - *_Sd_S *_S \mathcal{E} \wedge dt + *_Sd_S *_S \mathcal{B}$ *and then use this and what we did above to show that* $*d * \mathcal{F} = *\mathcal{J}$ *is equivalent to the pair* $*_Sd_S *_S \mathcal{E} = \varrho$ *and* $-\partial_t\mathcal{E} + *_Sd_S *_S \mathcal{B} = j$.

We have arrived at the pair of equations

$$\begin{aligned} d\mathcal{F} &= 0 \\ *d * \mathcal{F} &= *\mathcal{J} \end{aligned}$$

Now if we just think of this as a pair of equations to be satisfied by a 2−form $\mathcal{F}$ where the 1−form $\mathcal{J}$ is given then this last version of Maxwell's equations make sense on any semi-Riemannian manifold. In fact, on a Lorentz manifold that can be written as $(a, b)\times S = M$ with the twisted product metric $-ds^2 + \overset{3}{g}$ for some Riemannian metric on S then we can write our 2−form $\mathcal{F}$ in the form $\mathcal{F} = \mathcal{B} + \mathcal{E} \wedge dt$ which allows us to identify the electric and magnetic fields in their covariant form.

The central idea in relativity is that the laws of physics should take on a simple and formally identical form when expressed in any Lorentz inertial frame. This is sometimes called covariance. We are now going to show how to interpret two of the four Maxwell's equations in their original form as conservation laws and then show that just these two conservation laws will produce the whole set of Maxwell's equations when combined with a simple assumption involving covariance.

Now it is time to be clear about how we are modeling spacetime. The fact that the Lorentz metric was part of the above procedure that unified and simplified Maxwell's equations suggests that the Minkowski spacetime is indeed the physically correct model.

We now discuss conservation in the context of electromagnetism and then show that we can arrive at the full set of 4 Maxwell's equations in their original form from combining the two simplest of Maxwell's equations with an reasonable assumption concerning Lorentz covariance.

16.3 Levi-Civita Connection

Let M, g be a semi-Riemannian manifold and ∇ a metric connection(deffffine this) for M. Recall that the operator $T_\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ is a tensor called the torsion tensor of ∇. Combining the requirement that a connection be both a metric connection and torsion free pins down the metric completely.

Theorem 16.2 *For a given semi-Riemannian manifold M, g, there is a unique metric connection ∇ such that its torsion is zero; $T_\nabla \equiv 0$. This unique connection is called the* ***Levi-Civita derivative*** *for M, g.*

Proof. We will derive a formula that must be satisfied by ∇ and that can be used to actually define ∇. Let X, Y, Z, W be arbitrary vector fields on $U \subset M$. If ∇ exists as stated then on U we must have

$$
\begin{aligned}
X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\
Y\langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\
Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle.
\end{aligned}
$$

where we have written ∇^U simply as ∇. Now add the first two equations to the third one to get

$$
\begin{aligned}
&X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\
&= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle + \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\
&- \langle \nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle.
\end{aligned}
$$

Now if we assume the torsion zero hypothesis then this reduces to

$$
\begin{aligned}
&X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\
&= \langle Y, [X, Z]\rangle + \langle X, [Y, Z]\rangle \\
&- \langle Z, [X, Y]\rangle + 2\langle \nabla_X Y, Z\rangle.
\end{aligned}
$$

Solving we see that $\nabla_X Y$ must satisfy

$$
\begin{aligned}
2\langle \nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\
&\quad \langle Z, [X, Y]\rangle - \langle Y, [X, Z]\rangle - \langle X, [Y, Z]\rangle.
\end{aligned}
$$

Now since knowing $\langle \nabla_X Y, Z\rangle$ for all Z is tantamount to knowing $\nabla_X Y$ we conclude that if ∇ exists then it is unique. On the other hand, the patient reader can check that if we actually define $\langle \nabla_X Y, Z\rangle$ and hence $\nabla_X Y$ by this equation then all of the defining properties of a connection are satisfied and furthermore T_∇ will be zero. ∎

It is not difficult to check that we may define a system of Christoffel symbols for the Levi-Civita derivative by the formula

$$
\Gamma^\alpha(X_U, Y_U) := (\nabla_X Y)_U - DY_U \cdot X_U
$$

where X_U, Y_U and $(\nabla_X Y)_U$ are the principal representatives of X, Y and $\nabla_X Y$ respectively for a given chart $(U, \mathtt{x})$. Let M be a semi-Riemannian manifold of dimension n and let $U, \mathtt{x} = (x^1, ..., x^n)$ be a chart. Then we have the formula

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

where $g_{jk} g^{ki} = \delta^i_j$.

Let $F : N \to M$ be a smooth map. A **vector field along** F is a map $Z : N \to TM$ such that the following diagram commutes:

$$\begin{array}{ccc} & & TM \\ Z & \nearrow & \downarrow^{\pi} \\ N & \xrightarrow{F} & M \end{array}$$

We denote the set of all smooth vector fields along a map F by $\mathfrak{X}_F$. Let $F : N \to M$ be a smooth map and let the model spaces of M and N be M and N respectively.

We shall say that a pair of charts Let $(U, \mathtt{x})$ be a chart on M and let $(V, \mathtt{u})$ be a chart on N such that $\mathtt{u}(V) \subset U$. We shall say that such a pair of charts is adapted to the map F.

Assume that there exists a system of Christoffel symbols on M. We may define a covariant derivative $\nabla_X Z$ of a vector field Z along F with respect to a field $X \in \mathfrak{X}(N)$ by giving its principal representation with respect to any pair of charts adapted to F. Then $\nabla_X Z$ will itself be a vector fields along F. The map F has a local representation $F_{V,U} : V \to \psi(U)$ defined by $F_{V,U} := \mathtt{x} \circ F \circ \mathtt{u}^{-1}$. Similarly the principal representation $\mathsf{Z} : \mathtt{u}(V) \to \mathsf{M}$ of Z is given by $T\mathtt{x} \circ Z \circ \mathtt{u}^{-1}$ followed by projection onto the second factor of $\mathtt{x}(U) \times \mathsf{M}$. Now given any vector field $X \in \mathfrak{X}(N)$ with principal representation $\mathsf{X} : \psi(U) \to \mathsf{N}$ we define the covariant derivative $\nabla_X Z$ of Z with respect to X as that vector field along F whose principal representation with respect to any arbitrary pair of charts adapted to F is

$$D\mathsf{Z}(u) \cdot \mathsf{X}(u) + \Gamma(F_{V,U}(u))(DF_{V,U}(u) \cdot \mathsf{X}(u), \mathsf{Z}(u)).$$

In traditional notation if $Z = Z^i \frac{\partial}{\partial x^i} \circ F$ and $X = X^r \frac{\partial}{\partial u^r}$

$$(\nabla_X Z)^i = \frac{\partial Z^i}{\partial u^j} X^j + \Gamma^i_{jk} \frac{\partial F^j_{V,U}}{\partial u^r} X^r Z^k$$

The resulting map $\nabla : \mathfrak{X}(N) \times \mathfrak{X}_F \to \mathfrak{X}(N)$ has the following properties:

1. $\nabla : \mathfrak{X}(N) \times \mathfrak{X}_F \to \mathfrak{X}(N)$ is $C^\infty(N)$ linear in the first argument.

2. For the second argument we have

$$\nabla_X (fZ) = f \nabla_X Z + X(f) Z$$

for all $f \in C^\infty(N)$.

3. If Z happens to be of the form $Y \circ F$ for some $Y \in \mathfrak{X}(M)$ then we have

$$\nabla_X (Y \circ F)(p) = (\nabla_{TF \cdot X(p)} Y)(F(p)).$$

4. $(\nabla_X Z)(p)$ depends only on the value of X at $p \in N$ and we write $(\nabla_X Z)(p) = \nabla_{X_p} Z$.

For a curve $c :: \mathbb{R} \to M$ and $Z :: \mathbb{R} \to TM$ we define

$$\frac{\nabla Z}{dt} := \nabla_{d/dt} Z \in \mathfrak{X}_F$$

If Z happens to be of the form $Y \circ c$ then we have the following alternative notations with varying degrees of precision:

$$\nabla_{d/dt}(Y \circ c) = \nabla_{\dot{c}(t)} Y = \nabla_{d/dt} Y = \frac{\nabla Y}{dt}$$

Exercise 16.4 *Show that if* $\alpha : I \to M$ *is a curve and* X, Y *vector fields along* α *then* $\frac{d}{dt}\langle X, Y\rangle = \langle \frac{\nabla}{dt} X, Y\rangle + \langle X, \frac{\nabla}{dt} Y\rangle$.

Exercise 16.5 *Show that if* $h : (a, b) \times (c, d) \to M$ *is smooth then* $\partial h/\partial t$ *and* $\partial h/\partial s$ *are vector fields along* h *we have* $\nabla_{\partial h/\partial t} \partial h/\partial s = \nabla_{\partial h/\partial s} \partial h/\partial t$. *(Hint: Use local coordinates and the fact that* ∇ *is torsion free).*

16.4 Geodesics

In this section I will denote an interval (usually containing 0). The interval may be closed, open or half open. I may be an infinite or "half infinite" such as $(0, \infty)$. Recall that if I contains an endpoint then a curve $\gamma : I \to M$ is said to be smooth if there is a slightly larger open interval containing I to which the curve can be extended to a smooth curve.

Let M, g be a semi-Riemannian manifold. Suppose that $\gamma : I \to M$ is a smooth curve that is self parallel in the sense that

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

along γ. We call γ a **geodesic**. More precisely, $\dot{\gamma}$ is vector field along γ and γ is a geodesic if $\dot{\gamma}$ is parallel: $\frac{\nabla \dot{\gamma}}{dt}(t) = 0$ for all $t \in I$. If M is finite dimensional, $\dim M = n$,and $\gamma(I) \subset U$ for a chart $U, \mathtt{x} = (x^1, ...x^n)$ then the condition for γ to be a geodesic is

$$\frac{d^2 x^i \circ \gamma}{dt^2}(t) + \Gamma^i_{jk}(\gamma(t)) \frac{dx^j \circ \gamma}{dt}(t) \frac{dx^k \circ \gamma}{dt}(t) = 0 \text{ for all } t \in I \text{ and } 1 \le i \le n. \tag{16.7}$$

and this system of equations is (thankfully) abbreviated to $\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$. These are the local geodesic equations. Now if γ is a curve whose image is not necessarily contained in the domain then by continuity we can say that for every

$t_0 \in I$ there is an $\epsilon > 0$ such that $\gamma|_{(t_0-\epsilon,t+\epsilon)}$ is contained in the domain of a chart. Then it is not hard to see that γ is a geodesic if each such restriction satisfies the corresponding local geodesic equations as in 16.7 (for each chart which meets the image of γ). We can convert the local geodesic equations 16.7 , which is a system of n first order equations, into a system of $2n$ first order equations by the usual reduction of order trick. We let v denote a new dependent variable and then we get

$$\frac{dx^i}{dt} = v^i \ , \ 1 \leq i \leq n$$
$$\frac{dv^i}{dt} + \Gamma^i_{jk} v^j v^k = 0 \ , \ 1 \leq i \leq n.$$

We can think of x^i and v^i as coordinates on TM. Once we do this we recognize that the first order system above is the local expression of the equations for the integral curves of vector field on TM.

To be precise, one should distinguish various cases as follows: If $\gamma : I \to M$ is a geodesic we refer to it as a (parameterized) **closed geodesic segment** if $I = [a, b]$ for some finite numbers $a, b \in \mathbb{R}$. In case $I = [a, \infty)$ (resp. $(\infty, a]$) we call γ a positive (resp. negative) **geodesic ray**. If $I = (-\infty, \infty)$ then we call γ a **complete geodesic**.

So far, and for the most part from here on, this all makes sense for infinite dimensional manifolds modeled on a separable Hilbert space. We will write all our local expressions using the standard notation for finite dimensional manifolds. However, we will point out that the local equations for a geodesic can be written in notation appropriate to the infinite dimensional case as follows:

$$\frac{d^2\mathrm{x} \circ \gamma}{dt^2}(t) + \Gamma(\gamma(t), \frac{d\mathrm{x} \circ \gamma}{dt}(t)\frac{d\mathrm{x} \circ \gamma}{dt}(t)) = 0$$

Exercise 16.6 *Show that there is a vector field $G \in \mathfrak{X}(TM)$ such that α is an integral curve of G if and only if $\gamma := \pi_{TM} \circ \alpha$ is a geodesic. Show that the local expression for G is*

$$v^i \frac{\partial}{\partial x^i} + \Gamma^i_{jk} v^j v^k \frac{\partial}{\partial v^i}$$

(The Einstein summation convention is in force here).

The vector field G from this exercise is an example of a spray (see problems 1). The flow if G in the manifold TM is called the **geodesic flow**.

Lemma 16.3 *If $v \in T_pM$ then there is a unique geodesic γ_v such that $\dot{\gamma}_v(0) = v$.*

Proof. This follows from standard existence and uniqueness results from differential equations. One may also deduce this result from the facts about flows since as the exercise above shows, geodesics are projections of integral curves of the vector field G. The reader who did not do the problems on sprays in 1 would do well to look at those problems before going on. ■

Lemma 16.4 *Let γ_1 and γ_2 be geodesics $I \to M$. If $\dot{\gamma}_1(t_0) = \dot{\gamma}_2(t_0)$ for some $t_0 \in I$, then $\gamma_1 = \gamma_2$.*

Proof. If not there must be $t' \in I$ such that $\gamma_1(t') \neq \gamma_2(t')$. Let us assume that $t' > a$ since the proof of the other cases is similar. The set $\{t \in I : t > t_0$ and $\gamma_1(t) \neq \gamma_2(t)\}$has a greatest lower bound $b \geq t_0$. Claim $\dot{\gamma}_1(b) = \dot{\gamma}_2(b)$: Indeed if $b = t_0$ then thus is assumed in the hypothesis and if $b > t_0$ then $\dot{\gamma}_1(t) = \dot{\gamma}_2(t)$ on the interval (t_0, b). By continuity

$$\dot{\gamma}_1(t_0) = \dot{\gamma}_2(t_0)$$

and since $\gamma_1(t_0 + t)$ and $\gamma_2(t_0 + t)$ are clearly geodesics with initial velocity $\dot{\gamma}_1(t_0) = \dot{\gamma}_2(t_0)$ the result follows from lemma 16.3. ■

A geodesic $\gamma : I \to M$ is called maximal if there is no other geodesic with domain J strictly larger $I \subsetneq J$ which agrees with γ in I.

Theorem 16.3 *For any $v \in TM$ there is a unique maximal geodesic γ_v with $\dot{\gamma}_v(0) = v$.*

Proof. Take the class $\mathcal{G}_v$ of all geodesics with initial velocity v. This is not empty by 16.3. If $\alpha, \beta \in \mathcal{G}_v$ and the respective domains I_α and I_β have nonempty intersection then α and β agree on this intersection by 16.4. From this we see that the geodesics in $\mathcal{G}_v$ fit together to form a manifestly maximal geodesic with domain $I = \cup_{\gamma \in \mathcal{G}_v} I_\gamma$.

It is clear this geodesic has initial velocity v. ■

Definition 16.23 *If the domain of every maximal geodesic emanating from a point $p \in T_pM$ is all of R then we say that M is geodesically complete at p. A semi-Riemannian manifold is said to be* ***geodesically complete*** *if and only if it is geodesically complete at each of its points.*

Exercise 16.7 *Let $\mathbb{R}^{n-\nu,\nu}, \eta$ be the semi-Euclidean space of index ν. Show that all geodesics are of the form $t \mapsto x_0 + tv$ for $v \in \mathbb{R}^{n-\nu,\nu}$. (don't confuse v with the index ν (a Greek letter).*

Exercise 16.8 *Show that if γ is a geodesic then a reparametrization $\gamma := \gamma \circ f$ is a geodesic if and only if $f(t) := at + b$ for some $a, b \in \mathbb{R}$ and $a \neq 0$.*

The existence of geodesics emanating from a point $p \in M$ in all possible directions and with all possible speeds allows us to define a very important map called the exponential map $\exp_p$. The domain of $\exp_p$ is the set $\widetilde{\mathcal{D}}_p$ of all $v \in T_pM$ such that the geodesic γ_v is defined at least on the interval $[0, 1]$. The map is define simply by $\exp_p v := \gamma_v(1)$. The map $s \mapsto \gamma_v(ts)$ is a geodesic with initial velocity $t\dot{\gamma}_v(0) = tv$ and the properties of geodesics that we have developed imply that $\exp_p(tv) = \gamma_{tv}(1) = \gamma_v(t)$. Now we have a very convenient situation. The geodesic through p with initial velocity v can always be written in the form $t \mapsto \exp_p tv$. Straight lines through $0_p \in T_pM$ are mapped by $\exp_p$ onto geodesics.

The exponential map is the bridge to comparing manifolds with each other as we shall see and provides a two types of special coordinates around any given p. The basic theorem is the following:

Theorem 16.4 *Let M, g be a Riemannian manifold and $p \in M$. There exists an open neighborhood $\widetilde{U}_p \subset \widetilde{\mathcal{D}}_p$ containing 0_p such that $\exp_p\big|_{\widetilde{U}_p} : \widetilde{U}_p \to \exp_p\big|_{\widetilde{U}_p}(\widetilde{U}_p) := U_p$ is a diffeomorphism.*

Proof. The tangent space T_pM is a vector space which is isomorphic to $\mathbb{R}^n$ (or a Hilbert space) and so has a standard differentiable structure. It is easy to see using the results about smooth dependence on initial conditions for differential equations that $\exp_p$ is well defined and smooth in some neighborhood of $0_p \in T_pM$. The main point is that the tangent map $T\exp_p : T_{0_p}(T_pM) \to T_pM$ is an isomorphism and so the inverse function theorem gives the result. To see that $T\exp_p$ is an isomorphism let $v_{0_p} \in T_{0_p}(T_pM)$ be the velocity of the curve $t \mapsto tv$ in T_pM. Under the canonically identification of $T_{0_p}(T_pM)$ with T_pM this velocity is just v. No unraveling definitions we have $v = v_{0_p} \mapsto T\exp_p \cdot v_{0_p} = \frac{d}{dt}_0 \exp_p \cdot tv = v$ so with this canonically identification $T\exp_p v = v$ so the tangent map is essentially the identity map. ∎

Notation 16.2 *We will let the image set $\exp_p \widetilde{\mathcal{D}}_p$ be denoted by $\mathcal{D}_p$, the image $\exp_p(\widetilde{U}_p)$ is denoted by U_p.*

Definition 16.24 *An subset of a vector space* V *which contains* 0 *is called starshaped about* 0 *if $v \in \mathrm{V}$ implies $tv \in \mathrm{V}$ for all $t \in [0, 1]$.*

Definition 16.25 *If $\widetilde{U} \subset \widetilde{\mathcal{D}}_p$ is a starshaped open set about 0_p in T_pM such that $\exp_p\big|_{\widetilde{U}_p}$ is a diffeomorphism as in the theorem above then the image $\exp_p(\widetilde{U}) = U$ is called a* ***normal neighborhood of*** p. *(By convention such a set U is referred to as a starshaped also.)*

Theorem 16.5 *If $U \subset M$ is a normal neighborhood about p then for every point $q \in U$ there is a* unique *geodesic $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$, $\gamma(1) = q$ and $\exp_p \dot{\gamma}(0) = q$.*

Proof. Let $p \in U$ and $\exp_p \widetilde{U} = U$. By assumption $\widetilde{U}$ is starshaped and so $\rho : t \mapsto tv$, $t \in [0, 1]$ has image in $\widetilde{U}$ and then the geodesic segment $\gamma : t \mapsto \exp_p tv$, $t \in [0, 1]$ has its image inside U. Clearly, $\gamma(0) = p$ and $\gamma(1) = q$. Since $\dot{\rho} = v$ we get

$$\dot{\gamma}(0) = T\exp_p \cdot \dot{\rho}(0) = T\exp_p \cdot v = v$$

under the usual identifications in T_pM.

Now assume that $\gamma_1 : [0, 1] \to M$ is some geodesic with $\gamma_1(0) = p$ and $\gamma_1(1) = q$. If $\dot{\gamma}_1(0) = w$ then $\gamma_1(t) = \exp_p \cdot tv$. We know that $\rho_1 : t \mapsto tw$, ($t \in [0, 1]$) is a ray that stays inside $\widetilde{U}$ and so $w = \rho_1(1) \in \widetilde{U}$. Also, $\exp_p w = \gamma_1(1) = q = \exp_p v$ and since $\exp_p\big|_{\widetilde{U}}$ is a diffeomorphism and hence 1-1 we conclude that $w = v$. Thus by the uniqueness theorems of differential equations the geodesics segments γ and γ_1 are both given by : $t \mapsto \exp_p tv$ and hence they are equal. ■

Definition 16.26 *A continuous curve $\gamma : I \to M$ is called* ***piecewise geodesic, broken geodesic*** *or even "****kinky geodesic****" if it is a piecewise smooth curve whose smooth segments are geodesics. If $t \in I$ (not an endpoint) is a point where γ is not smooth we call t or its image $\gamma(t)$ a* ***kink point****.*

Exercise 16.9 *Prove the following proposition:*

Proposition 16.4 *A semi-Riemannian manifold is connected if and only if every pair of its points can be joined by a broken geodesic $\gamma : [a, b] \to M$.*

We can gather the maps $\exp_p : \widetilde{\mathcal{D}}_p \subset T_pM \to M$ together to get a map $\exp : \widetilde{\mathcal{D}} \to M$ defined by $\exp(v) := \exp_{\pi(v)}(v)$ and where $\widetilde{\mathcal{D}} := \cup_p \widetilde{\mathcal{D}}_p$. To see this let $W \subset \mathbb{R} \times M$ be the domain of the geodesic flow $(s, v) \mapsto \gamma_v'(s)$. This is the flow of a vector field on TM and W is open. W is also the domain of the map $(s, v) \mapsto \pi \circ \gamma_v'(s) = \gamma_v(s)$. Now the map $(1, v) \mapsto \varphi_1^G(v) = \gamma_v(1) = v$ is a diffeomorphism and under this diffeomorphism $\widetilde{\mathcal{D}}$ corresponds to the set $W \cap (\{1\} \times TM)$ so must be open in TM. It also follows that $\widetilde{\mathcal{D}}_p$ is open in T_pM.

Definition 16.27 *A an open subset U of a semi-Riemannian manifold is* ***convex*** *if it is a normal neighborhood of each of its points.*

Notice that U being convex according to the above definition implies that for any two point p and q in U there is a unique geodesic segment $\gamma : [0, 1] \to U$ (image inside U) such that $\gamma(0) = p$ and $\gamma(1) = q$. Thinking about the sphere makes it clear that even if U is convex there may be geodesic segments connecting p and q whose images do not lie in U.

Lemma 16.5 *For each $p \in M$, $\mathcal{D}_p := \exp_p(\widetilde{\mathcal{D}}_p)$ is starshaped*

Proof. By definition $\mathcal{D}_p$ is starshaped if and only if $\widetilde{\mathcal{D}}_p$is starshaped. For $v \in \widetilde{\mathcal{D}}_p$ then γ_v is defined for all $t \in [0, 1]$. On the other hand, $\gamma_{tv}(1) = \gamma_v(t)$ and so $tv \in \widetilde{\mathcal{D}}_p$. ■

Now we take one more step and use the exponential map to get a map EXP from $\widetilde{\mathcal{D}}$ onto a subset containing the diagonal in $M \times M$. The diagonal is the set $\{(p, p) : p \in M\}$. The definition is simply $EXP : v \mapsto (\pi_{TM}(v), \exp_p v)$.

Theorem 16.6 *There is an open set $O \subset \widetilde{\mathcal{D}}$ which is mapped by EXP onto an open neighborhood of the diagonal $\triangle \subset M \times M$ and which is a diffeomorphism when restricted to a sufficiently small neighborhood of any point on $\triangle$.*

Proof. By the inverse mapping theorem and what we have shown about exp we only need to show that if $T_x \exp_p$ is nonsingular for some $x \in \widetilde{\mathcal{D}}_p \subset T_pM$ then T_xEXP is also nonsingular at x. So assume that $T_x \exp_p(v_x) = 0$ and, with an eye towards contradiction, suppose that $T_xEXP(v_x) = 0$. We have $\pi_{TM} = pr_1 \circ EXP$ and so $Tpr_1(v_x) = T\pi_{TM}(T_xEXP(v_x)) = 0$. This means that v_x is vertical (tangent to T_pM). On the other hand, it is easy to see that the following diagram commutes:

$$\begin{array}{ccc} T_pM & \overset{EXP_{T_pM}}{\rightarrow} & \{p\} \times M \\ id \downarrow & & pr_2 \downarrow \\ T_pM & \overset{\exp_p}{\rightarrow} & M \end{array}$$

and hence so does

$$\begin{array}{ccc} T_x\left(T_pM\right) & \overset{T_xEXP_{T_pM}}{\rightarrow} & \{p\} \times M \\ id \downarrow & & pr_2 \downarrow \\ T_x\left(T_pM\right) & \overset{T_x \exp_p}{\rightarrow} & M \end{array}$$

This implies that $T_x \exp_p(v) = 0$ and hence $v = 0$. ■

Theorem 16.7 *Every $p \in M$ has a convex neighborhood.*

Proof. Let $p \in M$ and choose a neighborhood W of 0_p in TM such that $EXP|_W$ is a diffeomorphism onto a neighborhood of $(p,p) \in M \times M$. By a simple continuity argument we may assume that $EXP|_W\,(W)$ is of the form $U(\delta) \times U(\delta)$ for $U(\delta) := \{q : \sum_{i=1}^n (x^i(q))^2 < \delta\}$ and $\mathtt{x} = (x^i)$ some normal coordinate system. Now consider the tensor b on $U(\delta)$ whose components with respect to $\mathtt{x}$ are $b_{ij} = \delta_{ij} - \sum_k \Gamma_{ij}^k x^k$. This is clearly symmetric and positive definite at p and so by choosing δ smaller in necessary we may assume that this tensor is positive definite on $U(\delta)$. Let us show that $U(\delta)$ is a normal neighborhood of each of its points q. Let $W_q := W \cap T_qM$. We know that $EXP|_{W_q}$ is a diffeomorphism onto $\{q\} \times U(\delta)$ and it is easy to see that this means that $\exp_q\big|_{W_q}$ is a diffeomorphism onto $U(\delta)$. We now show that W_q is star shaped about 0_q. Let $q' \in U(\delta)$, $q \neq q'$ and $v = EXP|_{W_q}^{-1}\,(q,q')$. This means that $\gamma_v : [0,1] \to M$ is a geodesic from q to q'. If $\gamma_v([0,1]) \subset U(\delta)$ then $tv \in W_q$ for all $t \in [0,1]$ and so we could conclude that W_q is starshaped. Let us assume that $\gamma_v([0,1])$ is not contained in $U(\delta)$ and work for a contradiction.

If in fact γ_v leaves $U(\delta)$ then the function $f : t \mapsto \sum_{i=1}^n (x^i(\gamma_v(t)))^2$ has a maximum at some $t_0 \in (0,1)$. We have

$$\frac{d^2}{dt^2} f = 2 \sum_{i=1}^n \left(\frac{d\left(x^i \circ \gamma_v\right)}{dt} + x^i \circ \gamma_v \frac{d^2\left(x^i \circ \gamma_v\right)}{dt^2} \right).$$

But γ_v is a geodesic and so using the geodesic equations we get

$$\frac{d^2}{dt^2} f = 2 \sum_{i,j} \left(\delta_{ij} - \sum_k \Gamma_{ij}^k x^k \right) \frac{d\left(x^i \circ \gamma_v\right)}{dt} \frac{d\left(x^i \circ \gamma_v\right)}{dt}.$$

Plugging in t_0 we get

$$\frac{d^2}{dt^2}f(t_0) = 2b(\gamma_v'(t_0), \gamma_v'(t_0)) > 0$$

which contradicts f having a maximum at t_0. ∎

16.4.1 Normal coordinates

Let $M.g$ be finite a semi-Riemannian manifold of dimension n. Let and arbitrary $p \in M$ and pick any orthonormal basis $\{e_1, ..., e_n\}$ for the (semi-)Euclidean scalar product space $T_pM, \langle , \rangle_p$. Now this basis induces an isometry $I : \mathbb{R}^n_v \to T_pM$ by $(x^i) \mapsto x^i e_i$. Now if U is a normal neighborhood containing $p \in M$ then $\mathtt{x}_{Norm} := I \circ \exp_p|_{\tilde{U}}^{-1} : U \to \mathbb{R}^n_v = \mathbb{R}^n$ is a coordinate chart with domain U. These coordinates are referred to as normal coordinates centered at p. Normal coordinates have some very nice properties:

Theorem 16.8 *If $\mathtt{x}_{Norm} = (x^1, ..., x^n)$ are normal coordinates centered at $p \in U \subset M$ then*

$$g_{ij}(p) = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_p = \varepsilon_i \delta_{ij} \ (= \eta_{ij}) \text{ for all } i, j$$
$$\Gamma^i_{jk}(p) = 0 \text{ for all } i, j, k.$$

Proof. Let $v :=\in T_pM$ and let $\{e^i\}$ be the basis of T_p^*M dual to $\{e_i\}$. Then $e^i \circ \exp_p = x^i$. Now $\gamma_v(t) = \exp_p tv$ and so

$$x^i(\gamma_v(t)) = e^i(tv) = te^i(v) = ta^i$$

and so $a^i e_i = v = a^i \left.\frac{\partial}{\partial x^i}\right|_p$. In particular, if $a^i = \delta^i_j$ then $e_i = \left.\frac{\partial}{\partial x^i}\right|_p$ and so the coordinate vectors $\left.\frac{\partial}{\partial x^i}\right|_p \in T_pM$ are orthonormal; $\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle_p = \varepsilon_i \delta_{ij}$.

Now since γ_v is a geodesic and $x^i(\gamma_v(t)) = ta^i$ the coordinate expression for the geodesic equations reduces to $\Gamma^i_{jk}(\gamma_v(t))a^j a^k = 0$ for all i and this hold in particular at $p = \gamma_v(0)$. But (a^i) is arbitrary and so the quadratic form defined on R^n by $Q^k(\vec{a}) = \Gamma^i_{jk}(p)a^j a^k$ is identically zero an by polarization the bilinear form $Q^k : (\vec{a}, \vec{b}) \mapsto \Gamma^i_{jk}(p)a^j b^k$ is identically zero. Of course this means that $\Gamma^i_{jk}(p) = 0$ for all i, j and arbitrary k. ∎

Exercise 16.10 *Sometimes the simplest situation becomes confusing because it is so special. For example, the identification $T_v(T_pM)$ with T_pM can have a confusing psychological effect. For practice with the "unbearably simple" determine $\exp_p$ for $p \in \mathbb{R}^{n-v,v}$ and find normal coordinates about p.*

Theorem 16.9 (Gauss Lemma) *Assume $0 < a < b$. Let $h : [a, b] \times (-\varepsilon, \varepsilon) \to M$ have the following properties:*

(i) $s \mapsto h(s, t)$ is a geodesic for all $t \in (-\varepsilon, \varepsilon)$;

(ii) $\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial s} \rangle$ *is constant along each geodesic curve* $h_t : s \mapsto h(s,t)$.

Then $\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \rangle$ *is constant along each curve* h_t.

Proof. By definition we have

$$\frac{\partial h}{\partial s}(s,t) := T_{(s,t)}h \cdot \frac{\partial}{\partial s}$$
$$\frac{\partial h}{\partial t}(s,t) := T_{(s,t)}h \cdot \frac{\partial}{\partial t}$$

It is enough to show that $\frac{\partial}{\partial s}\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \rangle = 0$. We have

$$\begin{aligned}
\frac{\partial}{\partial s}\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \rangle &= \langle \nabla_{\frac{\partial}{\partial s}} \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \rangle + \langle \frac{\partial h}{\partial s}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial h}{\partial t} \rangle \\
&= \langle \frac{\partial h}{\partial s}, \nabla_{\frac{\partial}{\partial s}} \frac{\partial h}{\partial t} \rangle \text{ (since } s \mapsto h(s,t) \text{ is a geodesic)} \\
&= \langle \frac{\partial h}{\partial s}, \nabla_{\frac{\partial h}{\partial s}} \frac{\partial h}{\partial t} \rangle = \langle \frac{\partial h}{\partial s}, \nabla_{\frac{\partial h}{\partial t}} \frac{\partial h}{\partial s} \rangle \\
&= \frac{1}{2}\frac{\partial}{\partial s}\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial s} \rangle = 0 \text{ by assumption (ii)}
\end{aligned}$$

■

Consider the following set up: $p \in M, x \in T_pM, x \neq 0_p, v_x, w_x \in T_x(T_pM)$ where v_x, w_x correspond to $v, w \in T_pM$. If v_x is radial, i.e. if v is a scalar multiple of x, then the Gauss lemma implies that

$$\langle T_x \exp_p v_x, T_x \exp_p w_x \rangle = \langle v_x, w_x \rangle := \langle v, w \rangle_p.$$

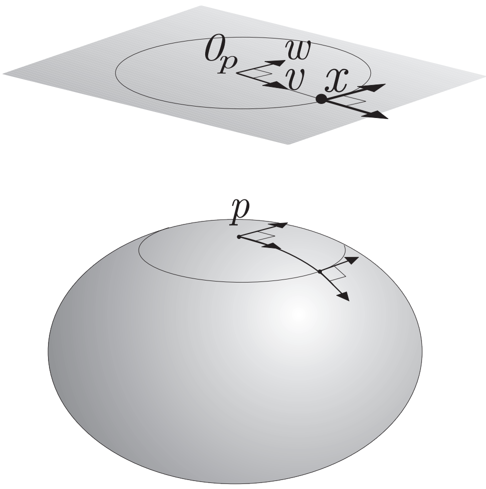
If v_x is not assumed to be radial then the above equality fails in general but we need to have the dimension of the manifold greater than 3 in order to see what can go wrong.

Exercise 16.11 *Show that if a geodesic* $\gamma : [a,b) \to M$ *is extendable to a continuous map* $\overline{\gamma} : [a,b] \to M$ *then there is an* $\varepsilon > 0$ *such that* $\gamma : [a,b) \to M$ *is extendable further to a geodesic* $\widetilde{\gamma} : [a, b+\varepsilon) \to M$ *such that* $\widetilde{\gamma}|_{[a,b]} = \overline{\gamma}$. *This is easy.*

Exercise 16.12 *Show that if* M, g *is a semi-Riemannian manifold then there exists a cover* $\{U_\alpha\}$ *such that each nonempty intersection* $U_{\alpha\beta} = U_\alpha \cap U_\beta$ *also convex.*

Let $v \in T_pM$ and consider the geodesic $\gamma_v : t \mapsto \exp_p tv$. Two rather easy facts that we often use without mention are

1. $\langle \gamma_v(t), \gamma_v(t) \rangle = \langle v, v \rangle$ for all t in the domain of γ_v.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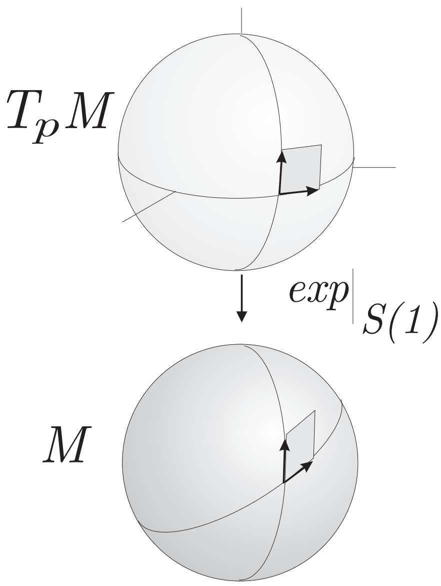

T_pM
$exp|_{S(1)}$
M

2. If $r > 0$ is such that $\exp_p rv$ is defined then $\int_0^r |\langle \gamma_v(t), \gamma_v(t)\rangle|^{1/2}\, dt = r\,|\langle v, v\rangle|^{1/2} = r\,|v|$. In particular, if v is a unit vector then the length of the geodesic $\gamma_v|_{[0,r]}$ is r.

Under certain conditions geodesics can help us draw conclusions about maps. The following result is an example and a main ingredient in the proof of the Hadamard theorem proved later.

Theorem 16.10 *Let $f : M, \mathrm{g} \to N, \mathrm{h}$ be a local isometry of semi-Riemannian manifolds with N connected. Suppose that f has the property that given any geodesic $\gamma : [0,1] \to N$ and a $p \in M$ with $f(p) = \gamma(0)$, there is a curve $\widetilde{\gamma} : [0,1] \to M$ such that $p = \widetilde{\gamma}(0)$ and $\gamma = f \circ \widetilde{\gamma}$. Then ϕ is a semi-Riemannian covering.*

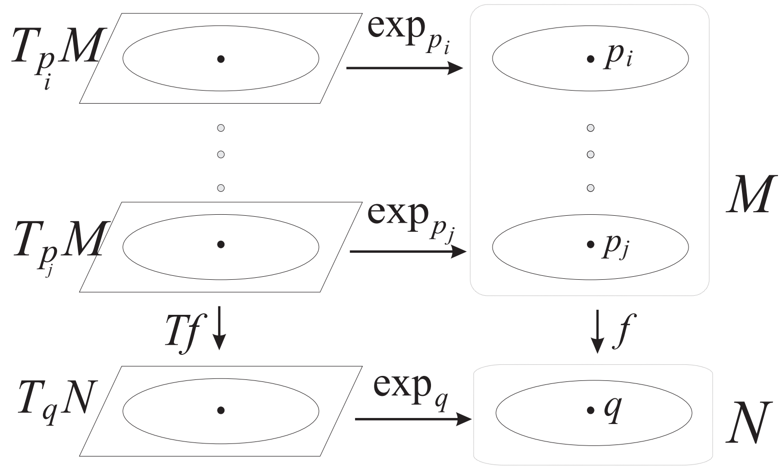
Proof. Since any two points of N can be joined by a broken geodesic it is easy to see that the hypotheses imply that f is onto.

Let U be a normal neighborhood of an arbitrary point $q \in N$ and $\widetilde{U} \subset T_qM$ the open set such that $\exp_q(\widetilde{U}) = U$. We will show that U is evenly covered by f. Choose $p \in f^{-1}\{q\}$. Observe that $T_pf : T_pM \to T_qN$ is a linear isometry (the metrics on the T_pM and T_qN are given by the scalar products $\mathrm{g}(p)$ and $\mathrm{h}(q)$). Thus $\widetilde{V_p} := T_pf^{-1}(\widetilde{U})$ is starshaped about $0_p \in T_pM$. Now if $v \in \widetilde{V}$ then by hypothesis the geodesic $\gamma(t) := \exp_q(t\,(T_pfv))$ has a lift to a curve $\widetilde{\gamma} : [0,1] \to M$ with $\widetilde{\gamma}(0) = p$. But since f is a local isometry this curve must be a geodesic and it is also easy to see that $T_p(\widetilde{\gamma}'(0)) = \gamma'(0) = T_pfv$. It follows that $v = \widetilde{\gamma}'(0)$ and then $\exp_p(v) = \widetilde{\gamma}(1)$. Thus $\exp_p$ is defined on all of $\widetilde{V}$. In fact, it is clear that $f(\exp_p v) = \exp_{f(p)}(Tfv)$ and so we see that f maps $V_p := \exp_p(\widetilde{V}_p)$ onto the set $\exp_q(\widetilde{U}) = U$. We show that V is a normal neighborhood of p: We have $f \circ \exp_p = \exp_{f(p)} \circ Tf$ and it follows that $f \circ \exp_p$ is a diffeomorphism on $\widetilde{V}$. But then $\exp_p : \widetilde{V}_p \to V_p$ is 1-1 and onto and when combined with the fact that $Tf \circ T\exp_p$ is a linear isomorphism at each $v \in \widetilde{V}_p$ and the fact that Tf is a linear isomorphism it follows that $T_v \exp_p$ is a linear isomorphism. It follows that $\widetilde{U}$ is open and $\exp_p : \widetilde{V}_p \longrightarrow V_p$ is a diffeomorphism.

Now if we compose we obtain $f|_{V_p} = \exp_{f(p)}\big|_U \circ Tf \circ \exp_p\big|_{V_p}^{-1}$ which is a diffeomorphism taking V_p onto U.

Now we show that if $p_i, p_j \in f^{-1}\{q\}$ and $p_i \neq p_j$ then the sets V_{p_i} and V_{p_j} (obtained for these points as we did for a generic p above) are disjoint. Suppose to the contrary that $x \in V_{p_i} \cap V_{p_j}$ and let $\gamma_{p_i m}$ and $\gamma_{p_j m}$ be the reverse radial geodesics from m to p_i and p_j respectively. Then $f \circ \gamma_{p_i m}$ and $f \circ \gamma_{p_j m}$ are both reversed radial geodesics from $f(x)$ to q and so must be equal. But then $f \circ \gamma_{p_i m}$ and $f \circ \gamma_{p_j m}$ are equal since they are both lifts of the same curve an start at the same point. It follows that $p_i = p_j$ after all. It remains to prove that $f^{-1}(U) \supset \cup_{p \in f^{-1}(q)} V_p$ since the reverse inclusion is obvious. Let $x \in f^{-1}(U)$ and let $\alpha : [0,1] \to U$ be the reverse radial geodesic from $f(x)$ to the center point q. Now let γ be the lift of α starting at x and let $p = \gamma(1)$.

Then $f(p) = \alpha(1) = q$ which means that $p \in f^{-1}(q)$. One the other hand, the image of γ must lie in $\widetilde{V}_p$ and so $x \in \widetilde{V}_p$. ■



16.5 Riemannian Manifolds and Distance

Once we have a notion of the length of a curve we can then define a distance function (metric in the sense of "metric space") as follow. Let $p, q \in M$. Consider the set path(p, q) of all smooth curves that begin at p and end at q. We define

$$\text{dist}(p, q) = \inf\{l \in \mathbb{R} : l = L(c) \text{ and } c \in \text{path}(p, q)\}. \tag{16.8}$$

or a general manifold just because $\text{dist}(p, q) = r$ does **not necessarily** mean that there must be a curve connecting p to q having length r. To see this just consider the points $(-1, 0)$ and $(1, 0)$ on the punctured plane $\mathbb{R}^2 - 0$.

Definition 16.28 *If $p \in M$ is a point in a Riemannian manifold and $R > 0$ then the set $B_R(p)$ (also denoted $B(p, R)$) defined by $B_R(p) = \{q \in M : dist(p, q) < p\}$ is called a (geodesic)* ***ball*** *centered at p with radius R.*

It is important to notice that unless R is small enough $B_R(p)$ may not be homeomorphic to a ball in a Euclidean space.

Theorem 16.11 (distance topology) *Given a Riemannian manifold, define the distance function* dist *as above. Then M,* dist *is a metric space and the induced topology coincides with the manifold topology on M.*

Proof. That dist is true distance function (metric) we must show that
(1) dist is symmetric,
(2) dist satisfies the triangle inequality,

(3) $\text{dist}(p,q) \geq 0$ and
(4) $\text{dist}(p,q) = 0$ if and only if $p = q$.

Now (1) is obvious and (2) and (3) are clear from the properties of the integral and the metric tensor. To prove (4) we need only show that if $p \neq q$ then $\text{dist}(p,\dot{q}) > 0$. Choose a chart ψ_α, U_α containing p but not q (M is Hausdorff). Now since $\psi_\alpha(U_\alpha) \subset \mathbb{R}^n$ we can transfer the Euclidean distance to U_α and define a small Euclidean ball $B_{Euc}(p,r)$ in this chart. Now any path from p to q must hit the boundary sphere $S(r) = \partial B_{Euc}(p,r)$. Now by compactness of $\bar{B}_{Euc}(p,r)$ we see that there are constants C_0 and C_1 such that $C_1\delta_{ij} \geq \mathrm{g}_{ij}(x) \geq C_0\delta_{ij}$ for all $x \in \bar{B}_{Euc}(p,r)$. Now any piecewise smooth curve $c : [a,b] \to M$ from p to q hits $S(r)$ at some parameter value $b_1 \leq b$ where we may assume this is the first hit (i.e. $c(t) \in B_{Euc}(p,r)$ for $a \leq t < b_0$). Now there is a curve that goes directly from p to q with respect to the Euclidean distance; i.e. a radial curve in the given Euclidean coordinates. This curve is given in coordinates as $\delta_{p,q}(t) = \frac{1}{b-1}(b-t)x(p) + \frac{1}{b-a}(t-a)x(q)$. Thus we have

$$\begin{aligned} L(c) &\geq \int_a^{b_0} \left(\mathrm{g}_{ij} \frac{d(x^i \circ c)}{dt} \frac{d(x^j \circ c)}{dt} \right)^{1/2} dt \geq C_0^{1/2} \int_a^{b_0} \left(\delta_{ij} \frac{d(x^i \circ c)}{dt} \right)^{1/2} dt \\ &= C_0^{1/2} \int_a^{b_0} |c'(t)|\, dt \geq C_0^{1/2} \int_a^{b_0} \left|\delta'_{p,q}(t)\right| dt = C_0^{1/2} r. \end{aligned}$$

Thus we have that $\text{dist}(p,q) = \inf\{L(c) : c \text{ a curve from } p \text{ to } q\} \geq C_0^{1/2} r > 0$. This last argument also shows that if $\text{dist}(p,x) < C_0^{1/2} r$ then $x \in B_{Euc}(p,r)$. This means that if $B(p, C_0^{1/2} r)$ is a ball with respect to dist then $B(p, C_0^{1/2} r) \subset B_{Euc}(p,r)$. Conversely, if $x \in B_{Euc}(p,r)$ then letting $\delta_{p,x}$ a "direct curve" analogous to the one above that connects p to x we have

$$\begin{aligned} \text{dist}(p,x) &\leq L(\delta_{p,x}) \\ &= \int_a^{b_0} \left(\mathrm{g}_{ij} \frac{d(x^i \circ \delta)}{dt} \frac{d(x^j \circ \delta)}{dt} \right)^{1/2} dt \\ &\leq C_1^{1/2} \int_a^{b_0} \left|\delta'_{p,x}(t)\right| dt = C_1^{1/2} r \end{aligned}$$

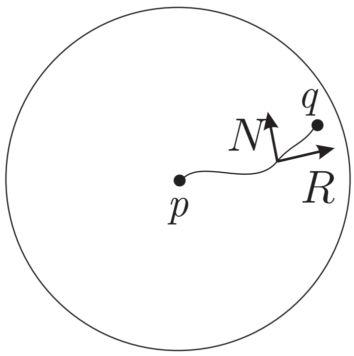
so we conclude that $B_{Euc}(p,r) \subset B(p, C_1^{1/2} r)$. Now we have that inside a chart, every dist-ball contains a Euclidean ball and vice versa. Thus since the manifold topology is generated by open subsets of charts we see that the two topologies coincide as promised. ∎

Lemma 16.6 *Let U be a normal neighborhood of a point p in a Riemannian manifold M, g. If $q \in U$ the radial geodesic $\gamma : [0,1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$, then γ is the unique shortest curve (up to reparameterization) connecting p to q.*

Proof. Let α be a curve connecting p to q. Without loss we may take the domain of α to be $[0,b]$. Let $\frac{\partial}{\partial r}$ be the radial unit vector field in U. Then if we define the vector field R along α by $t \mapsto \frac{\partial}{\partial r}\big|_{\alpha(t)}$ then we may write $\dot{\alpha} = \langle R, \dot{\alpha}\rangle R + N$ for some field N normal to R (but note $N(0) = 0$). We now have

$$\begin{aligned} L(\alpha) &= \int_0^b \langle \dot{\alpha}, \dot{\alpha}\rangle^{1/2} dt = \int_0^b \left[\langle R, \dot{\alpha}\rangle^2 + \langle N, N\rangle\right]^{1/2} dt \\ &\geq \int_0^b |\langle R, \dot{\alpha}\rangle|\, dt \geq \int_0^b \langle R, \dot{\alpha}\rangle dt = \int_0^b \frac{d}{dt}(r \circ \alpha)\, dt \\ &= r(\alpha(b)) = r(q) \end{aligned}$$

On the other hand, if $v = \dot{\gamma}(0)$ then $r(q) = \int_0^1 |v|\, dt = \int_0^1 \langle \dot{\gamma}, \dot{\gamma}\rangle^{1/2} dt$ so $L(\alpha) \geq L(\gamma)$. Now we show that if $L(\alpha) = L(\gamma)$ then α is a reparametrization of γ. Indeed, if $L(\alpha) = L(\gamma)$ then all of the above inequalities must be equalities so that N must be identically zero and $\frac{d}{dt}(r \circ \alpha) = \langle R, \dot{\alpha}\rangle = |\langle R, \dot{\alpha}\rangle|$. It follows that $\dot{\alpha} = \langle R, \dot{\alpha}\rangle R = \left(\frac{d}{dt}(r \circ \alpha)\right) R$ and so α travels radially from p to q and so must be a reparametrization of γ. ∎



16.6 Covariant differentiation of Tensor Fields

Let ∇ be a natural covariant derivative on M. It is a consequence of proposition 6.5 that for each $X \in \mathfrak{X}(U)$ there is a unique tensor derivation ∇_X on $\mathfrak{T}^r_s(U)$ such that ∇_X commutes with contraction and coincides with the given covariant derivative on $\mathfrak{X}(U)$ (also denoted ∇_X) and with $\mathcal{L}_X f$ on $C^\infty(U)$.

To describe the covariant derivative on tensors more explicitly consider $\Upsilon \in \mathfrak{T}^1_1$ with a 1-form Since we have the contraction $Y \otimes \Upsilon \mapsto C(Y \otimes \Upsilon) = \Upsilon(Y)$ we should have

$$\begin{aligned} \nabla_X \Upsilon(Y) &= \nabla_X C(Y \otimes \Upsilon) \\ &= C(\nabla_X (Y \otimes \Upsilon)) \\ &= C(\nabla_X Y \otimes \Upsilon + Y \otimes \nabla_X \Upsilon) \\ &= \Upsilon(\nabla_X Y) + (\nabla_X \Upsilon)(Y) \end{aligned}$$

and so we should define $(\nabla_X \Upsilon)(Y) := \nabla_X(\Upsilon(Y)) - \Upsilon(\nabla_X Y)$. If $\Upsilon \in \mathfrak{T}^1_s$ then

$$(\nabla_X \Upsilon)(Y_1, ..., Y_s) = \nabla_X(\Upsilon(Y_1, ..., Y_s)) - \sum_{i=1}^{s} \Upsilon(..., \nabla_X Y_i, ...)$$

Now if $Z \in \mathfrak{T}^1_0$ we apply this to $\nabla Z \in \mathfrak{T}^1_1$ and get

$$\begin{aligned}(\nabla_X \nabla Z)(Y) &= X(\nabla Z(Y)) - \nabla Z(\nabla_X Y) \\ &= \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z\end{aligned}$$

from which we get the following definition:

Definition 16.29 *The second covariant derivative of a vector field $Z \in \mathfrak{T}^1_0$ is*

$$\nabla^2 Z : (X, Y) \mapsto \nabla^2_{X,Y}(Z) = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z$$

Recall, that associated to any connection we have a curvature operator. In the case of the Levi-Civita Connection on a semi-Riemannian manifold the curvature operator is given by

$$R_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

and the associated $\mathfrak{T}^1_3$ tensor field $R(\alpha, X, Y, Z) := \alpha(R_{X,Y} Z)$ is called the Riemann curvature tensor. In a slightly different form we have the $\mathfrak{T}^0_4$ tensor defined by $R(W, X, Y, Z) := \langle W, R_{X,Y} Z\rangle$

Definition 16.30 *A tensor field Υ is said to be parallel if $\nabla_\xi \Upsilon = 0$ for all ξ. Similarly, if $\sigma : I \to T^r_s(M)$ is a tensor field along a curve $c : I \to M$ satisfies $\nabla_{\partial_t} \sigma = 0$ on I then we say that σ is parallel along c. Just as in the case of a general connection on a vector bundle we then have a parallel transport map $P(c)^t_{t_0} : T^r_s(M)_{c(t_0)} \to T^r_s(M)_{c(t)}$.*

Exercise 16.13 *Prove that*

$$\nabla_{\partial_t} \sigma(t) = \lim_{\epsilon \to 0} \frac{P(c)^t_{t+\epsilon} \sigma(t + \epsilon) - \sigma(t)}{\epsilon}.$$

Also, if $\Upsilon \in \mathfrak{T}^r_s$ then if c^X is the curve $t \mapsto \varphi^X_t(p)$

$$\nabla_X \Upsilon(p) = \lim_{\epsilon \to 0} \frac{P(c^X)^t_{t+\epsilon}(\Upsilon \circ \varphi^X_t(p)) - Y \circ \varphi^X_t(p)}{\epsilon}.$$

The map $\nabla_X : \mathfrak{T}^r_s M \to \mathfrak{T}^r_s M$ just defined commutes with contraction. This means, for instance, that

$$\begin{aligned}\nabla_i(\Upsilon^{jk}{}_k) &= \nabla_i\ \Upsilon^{jk}{}_k \text{ and} \\ \nabla_i(\Upsilon^{ik}{}_j) &= \nabla_i\ \Upsilon^{ik}{}_l\,.\end{aligned}$$

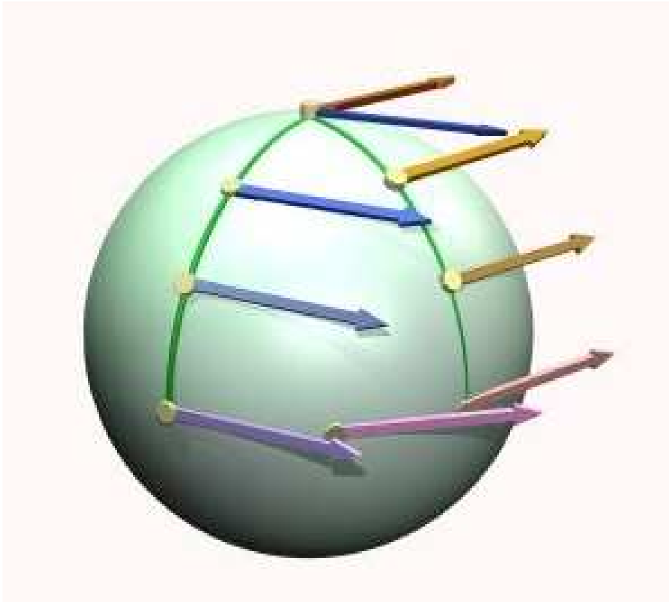


Figure 16.1: Parallel transport around path shows holonomy.

Furthermore, if the connection we are extending is the Levi-Civita connection for semi-Riemannian manifold M, g then

$$\nabla_\xi \mathrm{g} = 0 \text{ for all } \xi$$

To see this recall that

$$\nabla_\xi(\mathrm{g} \otimes Y \otimes W) = \nabla_\xi \mathrm{g} \otimes X \otimes Y + \mathrm{g} \otimes \nabla_\xi X \otimes Y + \mathrm{g} \otimes X \otimes \nabla_\xi Y$$

which upon contraction yields

$$\begin{aligned} \nabla_\xi(\mathrm{g}(X, Y)) &= (\nabla_\xi \mathrm{g})(X, Y) + \mathrm{g}(\nabla_\xi X, Y) + \mathrm{g}(X, \nabla_\xi Y) \\ \xi\langle X, Y\rangle &= (\nabla_\xi \mathrm{g})(X, Y) + \langle \nabla_\xi X, Y\rangle + \langle X, \nabla_\xi Y\rangle. \end{aligned}$$

We see that $\nabla_\xi \mathrm{g} \equiv 0$ for all ξ if and only if $\langle X, Y\rangle = \langle \nabla_\xi X, Y\rangle + \langle X, \nabla_\xi Y\rangle$ for all ξ, X, Y. In other words the statement that the metric tensor is parallel (constant) with respect to ∇ is the same as saying that the connection is a metric connection.

When ∇ is the Levi-Civita connection for the Riemannian manifold M, g we get the interesting formula

$$(\mathcal{L}_X \mathrm{g})(Y, Z) = \mathrm{g}(\nabla_X Y, Z) + \mathrm{g}(Y, \nabla_X Z) \tag{16.9}$$

for vector fields $X, Y, Z \in \mathfrak{X}(M)$.

16.7 Curvature

For M, g a Riemannian manifold with associated Levi-Civita connection ∇ we have the associated curvature which will now be called the Riemann curvature tensor: For $X, Y \in \mathfrak{X}(M)$ by $R_{X,Y} : \mathfrak{X}(M) \to \mathfrak{X}(M)$

$$R_{X,Y}Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.$$

Traditionally two other "forms" of the curvature tensor are defined as

1. A $\mathfrak{T}^1_3$ tensor defined by $R(\alpha, Z, X, Y) = \alpha\,(R_{X,Y}Z)$.
2. A $\mathfrak{T}^0_4$ tensor defined by $R(W, Z, X, Y) = \langle R_{X,Y}Z, W\rangle$
   The seemly odd ordering of the variables is traditional and maybe a bit unfortunate.

Theorem 16.12 $R(W, Z, X, Y) = \langle R_{X,Y}Z, W\rangle$ *is tensorial in all variables and*

(i) $R_{X,Y} = -R_{Y,X}$

(ii) $\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle$
(iii) $R_{X,Y}Z + R_{Y,Z}X + R_{Z,X}Y = 0$
(iv) $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$

Proof. (i) is immediate from the definition of R.
(ii) It is enough to show that $\langle R(X,Y)Z, Z\rangle = 0$. We have

$$\begin{aligned}\langle R(X,Y)Z, Z\rangle &= \langle \nabla_X \nabla_Y Z, Z\rangle - \langle Z, \nabla_X \nabla_Y Z\rangle \\ &= X\langle \nabla_Y Z, Z\rangle - \langle \nabla_X Z, \nabla_Y Z\rangle - Y\langle \nabla_X Z, Z\rangle \\ &+ \langle \nabla_X Z, \nabla_Y Z\rangle \\ &= \frac{1}{2}XY\langle Z, Z\rangle - \frac{1}{2}YX\langle Z, Z\rangle \\ &= \frac{1}{2}[X,Y]\langle Z, Z\rangle\end{aligned}$$

But since R is a tensor we may assume without loss that $[X,Y] = 0$. Thus $\langle R(X,Y)Z, Z\rangle = 0$ and the result follow by polarization.
(iii)

$$\begin{aligned}&R_{X,Y}Z + R_{Y,Z}X + R_{Z,X}Y \\ &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y = 0\end{aligned}$$

(iv) The proof of (iv) is rather unenlightening and is just some combinatorics which we omit: (?) ■

Definition 16.31 *A semi-Riemannian manifold M, g is called **flat** if the curvature is identically zero.*

Theorem 16.13 *For $X, Y, Z \in \mathfrak{X}(M)$ we have*

$$(\nabla_Z R)(X,Y) + \nabla_X R(Y,Z) + \nabla_Y R(Z,X) = 0$$

Proof. *This is the second Bianchi identity for the Levi-Civita connection but we give another proof here. Since this is a tensor equation we only need to prove it under the assumption that all brackets among the X, Y, Z are zero. First we have*

$$\begin{aligned}(\nabla_Z R)(X,Y)W &= \nabla_Z(R(X,Y)W) - R(\nabla_Z X, Y)W \\ &\quad - R(X, \nabla_Z Y)W - R(X,Y)\nabla_Z W \\ &= [\nabla_Z, R_{X,Y}]W - R(\nabla_Z X, Y)W - R(X, \nabla_Z Y)W.\end{aligned}$$

Using this we calculate as follows:

$$\begin{aligned}&(\nabla_Z R)(X,Y)W + (\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W \\ &= [\nabla_Z, R_{X,Y}]W + [\nabla_X, R_{Y,Z}]W + [\nabla_Y, R_{Z,X}]W \\ &- R(\nabla_Z X, Y)W - R(X, \nabla_Z Y)W \\ &- R(\nabla_X Y, Z)W - R(Y, \nabla_X Z)W \\ &- R(\nabla_Y Z, X)W - R(Z, \nabla_Y X)W \\ &= [\nabla_Z, R_{X,Y}]W + [\nabla_X, R_{Y,Z}]W + [\nabla_Y, R_{Z,X}]W \\ &+ R([X,Z],Y)W + R([Z,Y],X)W + R([Y,X],Z)W \\ &= [\nabla_Z, [\nabla_X, \nabla_Y]] + [\nabla_X, [\nabla_Y, \nabla_Z]] + [\nabla_Y, [\nabla_Z, \nabla_X]] = 0\end{aligned}$$

The last identity is the Jacobi identity for commutator (see exercise) ∎

Exercise 16.14 *Show that is $L_i, i = 1,2,3$ are linear operators $V \to V$ and the commutator is defined as usual ($[A,B] = AB - BA$) then we always have the Jacobi identity $[L_1,[L_2,L_3] + [L_2,[L_3,L_1] + [L_3,[L_1,L_2] = 0$.*

These several symmetry properties for the Riemann curvature tensor allow that we have a well defined map

$$\mathfrak{R} : \wedge^2(TM) \to \wedge^2(TM)$$

which is symmetric with respect the natural extension of g to $\wedge^2(TM)$. Recall that the natural extension is defined so that for an orthonormal $\{e_i\}$ the basis $\{e_i \wedge e_j\}$ is also orthonormal. We have

$$g(v_1 \wedge v_2, v_3 \wedge v_4) = \det \begin{pmatrix} g(v_1,v_3) & g(v_1,v_4) \\ g(v_2,v_3) & g(v_2,v_4) \end{pmatrix}$$

$\mathfrak{R}$ is defined implicitly as follows:

$$g(\mathfrak{R}(v_1 \wedge v_2), v_3 \wedge v_4) := \langle R(v_1,v_2)v_4, v_3 \rangle.$$

16.7.1 Tidal Force and Sectional curvature

For each $v \in TM$ the tidal force operator $R_v : T_pM \to T_pM$ is defined by

$$R_v(w) := R_{v,w}v.$$

Another commonly used quantity is the **sectional curvature:**

$$\begin{aligned} K(v \wedge w) &:= -\frac{\langle R_v(w), w\rangle}{\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2} \\ &= \frac{\langle \mathfrak{R}(v \wedge w), v \wedge w\rangle}{\langle v \wedge w, v \wedge w\rangle} \end{aligned}$$

where $v, w \in T_pM$. The value $K(v \wedge w)$ only depends on the oriented plane spanned by the vectors v and w therefore if $P = \text{span}\{v, w\}$ is such a plane we also write $K(P)$ instead of $K(v \wedge w)$. The set of all planes in T_pM is denoted $Gr_p(2)$.

In the following definition V is an $\mathsf{R}-$module. The two cases we have in mind are (1) where V is $\mathfrak{X}$, $\mathsf{R} = C^\infty(M)$ and (2) where V is T_pM, R=$\mathbb{R}$.

Definition 16.32 *A multilinear function* $F : \mathrm{V} \times \mathrm{V} \times \mathrm{V} \times \mathrm{V} \to \mathsf{R}$ *is said to be* ***curvature-like*** $x, y, z, w \in \mathrm{V}$ *it satisfies the symmetries proved for the curvature* R *above; namely,*

(i) $F(x, y, z, w) = -F(y, x, z, w)$
(ii) $F(x, y, z, w) = -F(x, y, w, z)$

(iii) $F(x, y, z, w) + F(y, z, x, w) + F(z, x, y, w) = 0$
(iv) $F(x, y, z, w) = F(w, z, x, y)$

As an example the tensor $C_g(X, Y, Z, W) := g(Y, Z)g(X, W) - g(X, Z)g(Y, W)$ is curvature-like.

Exercise 16.15 *Show that* C_g *is curvature-like.*

Proposition 16.5 *If* F *is curvature-like and* $F(v, w, v, w) = 0$ *for all* $v, w \in \mathrm{V}$ *then* $F \equiv 0$.

Proof. From (iv) it follows that F is symmetric in the second and forth variables so if $F(v, w, v, w) = 0$ for all $v, w \in \mathrm{V}$ then $F(v, w, v, z) = 0$ for all $v, w, z \in \mathrm{V}$. Now is a simple to show that (i) and (ii) imply that $F \equiv 0$. ■

Proposition 16.6 *If* $K(v \wedge w)$ *is know for all* $v, w \in T_pM$ *or if* $\langle R_{v,w}v, w\rangle = g(R_{v,w}v, w)$ *is known for all* $v, w \in T_pM$ *then* R *itself is determined at* p.

Proof. Using an orthonormal basis for T_pM we see that K and ϕ (where $\phi(v, w) := g(R_{v,w}v, w)$) contain the same information so we will just show that

ϕ determines R:

$$\begin{aligned}
&\left.\frac{\partial^2}{\partial s \partial t}\right|_{0,0} (\phi(v+tz, w+su) - \phi(v+tu, w+sz)) \\
&= \left.\frac{\partial^2}{\partial s \partial t}\right|_{0,0} \{g(R(v+tz, w+su)v + tz, w+su) \\
&\quad - g(R(v+tu, w+sz)v + tu, w+sz)\} \\
&= 6R(v, w, z, u)
\end{aligned}$$

■

We are now in a position to prove the following important theorem.

Theorem 16.14 *The following are all equivalent:*

(i) $K(P) = \kappa$ *for all* $P \in Gr_p(2)$

(ii) $g(R_{v_1,v_2}v_3, v_4) = \kappa C_g(v_1, v_2, v_3, v_4)$ *for all* $v_1, v_2, v_3, v_4 \in T_pM$

(iii) $R_v(w) = \kappa(w - g(w,v)v)$ *for all* $w, v \in T_pM$ *with* $|v| = 1$
(iv) $\mathfrak{R}(\omega) = \kappa\omega$ *and* $\omega \in \wedge^2 T_pM$.

Proof. Let $p \in M$. The proof that (ii)$\Longrightarrow$ (iii) and that (iii)$\Longrightarrow$ (i) is left as an easy exercise. We prove that (i) $\Longrightarrow$ (ii)$\Longrightarrow$ (iv)$\Longrightarrow$(i).
(i)$\Longrightarrow$ (ii): Let $T_g := F_\kappa - \kappa C_g$. Then T_g is curvature-like and $T_g(v, w, v, w) = 0$ for all $v, w \in T_pM$ by assumption. It follows from 16.5 that $T_g \equiv 0$.
(ii)$\Longrightarrow$ (iv): Let $\{e_1, ..., e_n\}$ be an orthonormal basis for T_pM. Then $\{e_i \wedge e_j\}_{i<j}$ is an orthonormal basis for $\wedge^2 T_pM$. Using (ii) we see that

$$\begin{aligned}
g(\mathfrak{R}(e_i \wedge e_j), e_k \wedge e_l) &= g(R_{e_i,e_j}, e_k, e_l) \\
&= g(R(e_i, e_j), e_k, e_l) \\
&= \kappa C_g(v_1, v_2, v_3, v_4) \\
&= \kappa g(e_i \wedge e_j, e_k \wedge e_l) \text{ for all } k, l
\end{aligned}$$

Since we are using a basis this implies that $\mathfrak{R}(e_i \wedge e_j) = \kappa e_i \wedge e_j$

(iv)$\Longrightarrow$(i): This follows because if v, w are orthonormal we have $\kappa = g(\mathfrak{R}(v \wedge w), v \wedge w) = K(v \wedge w)$. ■

16.7.2 Ricci Curvature

Definition 16.33 *Let* M, g *be a semi-Riemannian manifold. The Ricci curvature is the* $(1,1)-$*tensor Ric defined by*

$$Ric(v, w) := \sum_{i=1}^{n} \varepsilon_i \langle R_{v,e_i} e_i, w \rangle$$

where $\{e_1, ..., e_n\}$ *is any orthonormal basis of* T_pM *and* $\varepsilon_i := \langle e_i, e_i \rangle$.

We say that the Ricci curvature Ric is **bounded from below by** κ and write $Ric \geq k$ if $Ric(v, w) \geq k\langle v, w\rangle$ for all $v, w \in TM$. Similar and obvious definitions can be given for $Ric \leq k$ and the strict bounds $Ric > k$ and $Ric < k$. Actually, it is usually the case that the bound on Ricci curvature is given in the form $Ric \geq \kappa(n-1)$ where $n = \dim(M)$.

There is a very important an interesting class of manifolds Einstein manifolds. A semi-Riemannian manifold M, g is called an Einstein manifold with Einstein constant k if and only if $Ric(v, w) = k\langle v, w\rangle$ for all $v, w \in TM$. For example, if M, g has constant sectional curvature κ then M, g called an **Einstein manifold** with Einstein constant $k = \kappa(n-1)$. The effect of this condition depends on the signature of the metric. Particularly interesting is the case where the index is 0 (Riemannian) and also the case where the index is 1 (Lorentz). Perhaps the first question one should ask is whether there exists any Einstein manifolds that do not have constant curvature. It turns out that there are many interesting Einstein manifolds that do not have constant curvature.

Exercise 16.16 *Show that if M is connected and $\dim(M) > 2$ then $Ric = f\mathrm{g}$ where $f \in C^\infty(M)$ then $Ric = k\mathrm{g}$ for some $k \in \mathbb{R}$ (M, g is Einstein).*

16.8 Jacobi Fields

One again we consider a semi-Riemannian manifold M, g of arbitrary index. We shall be dealing with two parameter maps $h : [\epsilon_1, \epsilon_2] \times [a, b] \to M$. The partial maps $t \mapsto h_s(t) = h(s, t)$ are called the longitudinal curves and the curves $s \mapsto h(s, t)$ are called the transverse curves. Let α be the center longitudinal curve $t \mapsto h_0(t)$. The vector field along α defined by $V(t) = \frac{d}{ds}\big|_{s=0} h_s(t)$ is called the **variation vector field** along α. We will use the following important result more than once:

Lemma 16.7 *Let Y be a vector field along the map $h : [\epsilon_1, \epsilon_2] \times [a, b] \to M$. Then*

$$\nabla_{\partial_s} \nabla_{\partial_t} Y - \nabla_{\partial_s} \nabla_{\partial_t} Y = R(\partial_s h, \partial_t h) Y$$

Proof. This rather plausible formula takes a little case since h may not even be an immersion. Nevertheless, if one computes in a local chart the result falls out after a mildly tedious computation which we leave to the curious reader. ∎

Suppose we have the special situation that, for each s, the partial maps $t \mapsto h_s(t)$ are geodesics. In this case let us denote the center geodesic $t \mapsto h_0(t)$ by γ. We call h a variation of γ through geodesics. Let h be such a special variation and V the variation vector field. Using the previous lemma 16.7 the result of exercise 16.5 we compute

$$\begin{aligned}
\nabla_{\partial_t} \nabla_{\partial_t} V &= \nabla_{\partial_t} \nabla_{\partial_t} \partial_s h = \nabla_{\partial_t} \nabla_{\partial_s} \partial_t h \\
&= \nabla_{\partial_s} \nabla_{\partial_t} \partial_t h + R(\partial_t h, \partial_s h) \partial_t h \\
&= R(\partial_t h, \partial_s h) \partial_t h
\end{aligned}$$

and evaluating at $s = 0$ we get $\nabla_{\partial_t}\nabla_{\partial_t}V(t) = R(\dot{\gamma}(t), V(t))\dot{\gamma}(t)$. This equation is important and shows that V is a Jacobi field:

Definition 16.34 *Let $\gamma : [a,b] \to M$ be a geodesic and $J \in \mathfrak{X}_\gamma(M)$ be a vector field along γ. The field J is called a **Jacobi field** if*

$$\nabla_{\partial_t} J = R(\dot{\gamma}(t), J(t))\dot{\gamma}(t)$$

for all $t \in [a,b]$.

In local coordinates we recognize the above as a second order (system of) linear differential equations and we easily arrive at the following

Theorem 16.15 *Let M, g and $\gamma : [a,b] \to M$ be a geodesic as above. Given $w_1, w_2 \in T_{\gamma(a)}M$, there is a unique Jacobi field $J^{w_1,w_2} \in \mathfrak{X}_\gamma(M)$ such that $J(a) = w_1$ and $\frac{\nabla J}{dt}(a) = w_2$. The set $Jac(\gamma)$ of all Jacobi fields along γ is a vector space isomorphic to $T_{\gamma(a)}M \times T_{\gamma(a)}M$.*

If $\gamma_v(t) = \exp_p(tv)$ defined on $[0,b]$ then $J_\gamma^{0,w}$ denotes the unique Jacobi field along $\gamma_v(t)$ with initial conditions $J(0) = 0$ and $\nabla_{\partial_t} J(0) = w$.

Proposition 16.7 *If $w = rv$ for some $r \in \mathbb{R}$ then $J_\gamma^{0,rv}(t) = tr\gamma_v(t)$. If $w \perp v$ then $\langle J^{0,w}(t), \dot{\gamma}_v(t)\rangle = 0$ for all $t \in [0,b]$.*

Proof. First let $w = rv$. Let $J := rt\dot{\gamma}_v(t)$. Then clearly $J(0) = 0 \cdot \dot{\gamma}_v(0) = 0$ and $\nabla_{\partial_t} J(t) = rt\nabla_{\partial_t}\gamma_v(t) + r\gamma_v(t)$ so $\nabla_{\partial_t} J(0) = r\gamma_v(0) = rv$. Thus J satisfied the correct initial conditions. Now we have $\nabla^2_{\partial_t} J(t) = r\nabla^2_{\partial_t}\gamma_v(t) = r\nabla_{\partial_t}\dot{\gamma}_v(0) = 0$ since γ_v is a geodesic. One the other hand,

$$\begin{aligned}
&R(\dot{\gamma}_v(t), J(t))\dot{\gamma}_v(t) \\
&= R(\dot{\gamma}_v(t), r\dot{\gamma}_v(t))\dot{\gamma}_v(t) \\
&= rR(\dot{\gamma}_v(t), \dot{\gamma}_v(t))\dot{\gamma}_v(t) = 0
\end{aligned}$$

Thus $J = J^{0,rv}$.

Now for the case $w \perp v$ we have $\langle J^{0,w}(0), \dot{\gamma}_v(0)\rangle = 0$ and

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_0 \langle J^{0,w}(t), \dot{\gamma}_v(t)\rangle \\
&= \langle \nabla_{\partial_t} J^{0,w}(0), \dot{\gamma}_v(0)\rangle + \langle J^{0,w}(t0), \nabla_{\partial_t}\dot{\gamma}_v(0)\rangle \\
&= \langle w, v\rangle = 0
\end{aligned}$$

Thus the function $f(t) = \langle J^{0,w}(t), \dot{\gamma}_v(t)\rangle$ satisfies $f'(0) = 0$ and $f(0) = 0$ and so $f \equiv 0$ on $[0,b]$. ■

Corollary 16.1 *Every Jacobi field $J^{0,w}$ along $\exp_v tv$ with $J^{0,w}(0) = 0$ has the form $J^{0,w} := rt\dot{\gamma}_v + J^{0,w_1}$ where $w = \nabla_{\partial_t} J(0) = rv + w_1$ and $w_1 \perp v$. Also, $J^{0,w}(t) \perp \dot{\gamma}_v(t)$ for all $t \in [0,b]$.*

We now examine the more general case of a Jacobi field J^{w_1,w_2} along a geodesic $\gamma : [a,b] \to M$. First notice that for any curve $\alpha : [a,b] \to M$ with $|\langle \dot{\alpha}(t), \dot{\alpha}(t)\rangle| > 0$ for all $t \in [a,b]$, any vector field Y along α decomposes into an orthogonal sum $Y^\top + Y^\perp$. This means that $Y^\top$ is a multiple of $\dot{\alpha}$ and $Y^\perp$ is normal to $\dot{\alpha}$. If $\gamma : [a,b] \to M$ is a geodesic then $\nabla_{\partial_t} Y^\perp$ is also normal to $\dot{\gamma}$ since $0 = \frac{d}{dt}\langle Y^\perp, \dot{\gamma}\rangle = \langle \nabla_{\partial_t} Y^\perp, \dot{\gamma}\rangle + \langle Y^\perp, \nabla_{\partial_t}\dot{\gamma}\rangle = \langle \nabla_{\partial_t} Y^\perp, \dot{\gamma}\rangle$. Similarly, $\nabla_{\partial_t} Y^\top$ is parallel to $\dot{\gamma}$ all along γ.

Proposition 16.8 *Let $\gamma : [a,b] \to M$ be a geodesic.*

(i) If $Y \in \mathfrak{X}_\gamma(M)$ is tangent to γ then Y is a Jacobi field if and only if $\nabla^2_{\partial_t} Y = 0$ along γ. In this case $Y(t) = (at+b)\dot{\gamma}(t)$.

(ii) If J is a Jacobi field along γ and there is some distinct $t_1, t_2 \in [a,b]$ with $J(t_1) \perp \dot{\gamma}(t_1)$ and $J(t_2) \perp \dot{\gamma}(t_2)$ then $J(t) \perp \dot{\gamma}(t)$ for all $t \in [a,b]$.

(iii) If J is a Jacobi field along γ and there is some $t_0 \in [a,b]$ with $J(t_0) \perp \dot{\gamma}(t_0)$ and $\nabla_{\partial_t} J(t_0) \perp \dot{\gamma}(t_0)$ then $J(t) \perp \dot{\gamma}(t)$ for all $t \in [a,b]$.

(iv) If γ is not a null geodesic then Y is a Jacobi field if and only if both $Y^\top$ and $Y^\perp$ are Jacobi fields.

Proof. (i) Let $Y = f\dot{\gamma}$. Then the Jacobi equation reads

$$
\begin{gathered}
\nabla^2_{\partial_t} f\dot{\gamma}(t) = R(\dot{\gamma}(t), f\dot{\gamma}(t))\dot{\gamma}(t) = 0 \\
\text{or} \\
\nabla^2_{\partial_t} f\dot{\gamma}(t) \ = 0 \text{ or } f'' \ = 0
\end{gathered}
$$

(ii) and (iii) $\frac{d^2}{dt^2}\langle Y, \dot{\gamma}\rangle = \langle R(\dot{\gamma}(t), Y(t))\dot{\gamma}(t), \dot{\gamma}(t)\rangle = 0$ (from the symmetries of the curvature tensor). Thus $\langle Y(t), \dot{\gamma}(t)\rangle = at + b$ for some $a, b \in \mathbb{R}$. The reader can now easily deduce both (ii) and (iii).

(iv) The operator $\nabla^2_{\partial_t}$ preserves the normal and tangential parts of Y. We now show that the same is true of the map $Y \mapsto R(\dot{\gamma}(t), Y)\dot{\gamma}(t)$. Since we assume that γ is not null we have $Y^\top = f\dot{\gamma}$ for some $\dot{\gamma}$. Thus $R(\dot{\gamma}(t), Y^\top)\dot{\gamma}(t) = R(\dot{\gamma}(t), f\dot{\gamma}(t))\dot{\gamma}(t) = 0$ which is trivially tangent to $\dot{\gamma}(t)$. On the other hand, $\langle R(\dot{\gamma}(t), Y^\perp(t))\dot{\gamma}(t), \dot{\gamma}(t)\rangle = 0$.

$$
\begin{aligned}
\left(\nabla^2_{\partial_t} Y\right)^\top + \left(\nabla^2_{\partial_t} Y\right)^\perp &= \nabla^2_{\partial_t} Y = R(\dot{\gamma}(t), Y(t))\dot{\gamma}(t) \\
&= R(\dot{\gamma}(t), Y^\top(t))\dot{\gamma}(t) + R(\dot{\gamma}(t), Y^\perp(t))\dot{\gamma}(t) \\
&= 0 + R(\dot{\gamma}(t), Y^\perp(t))\dot{\gamma}(t)
\end{aligned}
$$

So the Jacobi equation $\nabla^2_{\partial_t} J(t) = R(\dot{\gamma}(t), J(t))\dot{\gamma}(t)$ splits into two equations

$$
\begin{aligned}
\nabla^2_{\partial_t} J^\top(t) &= 0 \\
\nabla^2_{\partial_t} J^\perp(t) &= R_{\dot{\gamma}(t), J^\perp(t)}\dot{\gamma}(t)
\end{aligned}
$$

and the result follows from this. ■

The proof of the last result shows that a Jacobi field decomposes into a parallel vector field along γ which is just a multiple of the velocity $\dot{\gamma}$ and a "normal Jacobi field" $J^{\perp}$ which is normal to γ. Of course, the important part is the normal part and so we now restrict to that case. Thus we consider the Jacobi equation $\nabla^2_{\partial_t} J(t) = R(\dot{\gamma}(t), J(t))\dot{\gamma}(t)$ with initial conditions $J(a) = w_1$ with $\nabla_{\partial_t} J(a) = w_2$ and $w_1, w_2 \in (\dot{\gamma}(a))^{\perp}$. Notice that in terms of the tidal force operator the Jacobi equation is

$$\nabla^2_{\partial_t} J(t) = R_{\dot{\gamma}(t)}(J(t))$$

Exercise 16.17 *Prove that for $v \in T_{\dot{\gamma}(t)}M$ the operator R_v maps $(\dot{\gamma}(t))^{\perp}$ to itself.*

Definition 16.35 *Let $\gamma : [a, b] \to M$ be a geodesic. Let $\mathcal{J}_0(\gamma, a, b)$ denote the set of all Jacobi fields J such that $J(a) = J(b) = 0$.*

Definition 16.36 *If there exists a geodesic $\gamma : [a, b] \to M$ and a nonzero Jacobi field $J \in \mathcal{J}_0(\gamma, a, b)$ then we say that $\gamma(a)$ is conjugate to $\gamma(b)$ along γ.*

From standard considerations from the theory of linear differential equations the set $\mathcal{J}_0(\gamma, a, b)$ is a vector space. The dimension of the vector space $\mathcal{J}_0(\gamma, a, b)$ the order of the conjugacy. Since the Jacobi fields in $\mathcal{J}_0(\gamma, a, b)$ vanish twice and we have seen that this means that such fields are normal to $\dot{\gamma}$ all along γ it follows that the dimension of . $\mathcal{J}_0(\gamma, a, b)$ is at most $n-1$ where $n = \dim M$. We have seen that a variation through geodesics is a Jacobi field so if we can find a nontrivial variation h of a geodesic γ such that all of the longitudinal curves $t \mapsto h_s(t)$ begin and end at the same points $\gamma(a)$ and $\gamma(b)$ then the variation vector field will be a nonzero element of $\mathcal{J}_0(\gamma, a, b)$. Thus we conclude that $\gamma(a)$ is conjugate to $\gamma(b)$.

Let us get the exponential map into play. Let $\gamma : [0, b] \to M$ be a geodesic as above. Let $v = \dot{\gamma}(0) \in T_pM$. Then $\gamma : t \mapsto \exp_p tv$ is exactly our geodesic γ which begins at p and ends at q at time b. Now we create a variation of γ by

$$h(s, t) = \exp_p t(v + sw).$$

where $w \in T_pM$ and s ranges in $(-\epsilon, \epsilon)$ for some sufficiently small ϵ. Now we know that $J(t) = \frac{\partial}{\partial s}\big|_{s=0} h(s, t)$ is a Jacobi field. It is clear that $J(0) := \frac{\partial}{\partial s}\big|_{s=0} h(s, 0) = 0$. If w_{bv} is the vector tangent in $T_{bv}(T_pM)$ which canonically corresponds to w, in other words, if w_{bv} is the velocity vector at $s = 0$ for the curve $s \mapsto bv + sw$ in T_pM, then

$$\begin{aligned} J(b) &= \left.\frac{\partial}{\partial s}\right|_{s=0} h(s, b) \\ &= \left.\frac{\partial}{\partial s}\right|_{s=0} \exp_p t(bv + sw) = T_{bv} \exp_p(w_{bv}) \end{aligned}$$

Also,

$$\begin{aligned}\nabla_{\partial_t} J(0) &= \nabla_{\partial_t} \nabla_{\partial_s} \exp_p t(bv + sw)\big|_{s=0,t=0} \\ &= \nabla_{\partial_s}\big|_{s=0} \nabla_{\partial_t}\big|_{t=0} \exp_p t(bv + sw)\end{aligned}$$

But $X(s) := \frac{\partial}{\partial t}\big|_{t=0} \exp_p t(bv+sw) = bv+sw$ is a vector field along the constant curve $t \mapsto p$ and so by exercise 15.5 we have $\nabla_{\partial_s}\big|_{s=0} X(s) = X'(0) = w$. The equality $J(b) = T_{bv} \exp_p(v_{bv})$ is important because it shows that if $T_{bv} \exp_p : T_{bv}(T_pM) \to T_{\gamma(b)}M$ is not an isomorphism then we can find a vector $w_{bv} \in T_{bv}(T_pM)$ such that $T_{bv} \exp_p(w_{bv}) = 0$. But then if w is the vector in T_pM which corresponds to w_{bv} as above then for this choice of w the Jacobi field constructed above is such that $J(0) = J(b) = 0$ and so $\gamma(0)$ is conjugate to $\gamma(b)$ along γ. Also, if J is a Jacobi field with $J(0) = 0$ and $\nabla_{\partial_t} J(0) = w$ then this uniquely determines J and it must have the form $\frac{\partial}{\partial s}\big|_{s=0} \exp_p t(bv + sw)$ as above.

Theorem 16.16 *Let $\gamma : [0, b] \to M$ be a geodesic. Then the following are equivalent:*

(i) $\gamma(0)$ is conjugate to $\gamma(b)$ along γ.

(ii) There is a nontrivial variation h of γ through geodesics which all start at $p = \gamma(0)$ such that $J(b) := \frac{\partial h}{\partial s}(0, b) = 0$.

(iii) If $v = \gamma'(0)$ then $T_{bv} \exp_p$ is singular.

Proof. (ii)$\Longrightarrow$(i): We have already seen that a variation through geodesics is a Jacobi field J and if (ii) then by assumption $J(0) = J(b) = 0$ and so we have (i).

(i)$\Longrightarrow$(iii): If (i) is true then there is a nonzero Jacobi field J with $J(0) = J(b) = 0$. Now let $w = \nabla_{\partial_t} J(0)$ and $h(s, t) = \exp_p t(bv + sw)$. Then $h(s, t)$ is a variation through geodesics and $0 = J(b) = \frac{\partial}{\partial s}\big|_{s=0} \exp_p t(bv + sw) = T_{bv} \exp_p(w_{bv})$ so that $T_{bv} \exp_p$ is singular.

(iii)$\Longrightarrow$(ii): Let $v = \gamma'(0)$. If $T_{bv} \exp_p$ is singular then there is a w with $T_{bv} \exp_p \cdot w_{bv} = 0$. Thus the variation $h(s, t) = \exp_p t(bv + sw)$ does the job. ∎

16.9 First and Second Variation of Arc Length

Let us restrict attention to the case where α is either spacelike of timelike and let $\varepsilon = +1$ if α is spacelike and $\varepsilon = -1$ if α is timelike. This is just the condition that $\left|\langle \dot{h}_0(t), \dot{h}_0(t)\rangle\right| > 0$. By a simple continuity argument we may choose $\epsilon > 0$ small enough that $\left|\langle \dot{h}_s(t), \dot{h}_s(t)\rangle\right| > 0$ for all $s \in (-\epsilon, \epsilon)$. Consider the arc length functional defined by

$$L(\alpha) = \int_a^b \varepsilon\langle\dot{\alpha}(t), \dot{\alpha}(t)\rangle^{1/2} dt.$$

Now if $h : (-\epsilon, \epsilon) \times [a, b] \to M$ is a variation of α as above with variation vector field V then formally V is a tangent vector at α in the space of curves $[a, b] \to M$. Then we have the variation of the arc length functional defined by

$$\delta L|_\alpha(V) := \frac{d}{ds}_{s=0} L(h_s) := \frac{d}{ds}_{s=0} \int_a^b \varepsilon\langle\dot{h}_s(t), \dot{h}_s(t)\rangle^{1/2} dt.$$

So we are interesting in studying the critical points of $L(s) := L(h_s)$ and so we need to find $L'(0)$ and $L''(0)$. For the proof of the following proposition we use the result of exercise 16.5 to the effect that $\nabla_{\partial_s}\partial_t h = \nabla_{\partial_t}\partial_s h$.

Proposition 16.9 *Let $h : (-\epsilon, \epsilon) \times [a, b] \to M$ be a variation of a curve $\alpha := h_0$ such that $\left|\langle\dot{h}_s(t), \dot{h}_s(t)\rangle\right| > 0$ for all $s \in (-\epsilon, \epsilon)$. Then*

$$L'(s) = \int_a^b \varepsilon\langle\nabla_{\partial_s}\partial_t h(s,t), \partial_t h(s,t)\rangle \left(\varepsilon\langle\partial_t h(s,t), \partial_t h(s,t)\rangle\right)^{-1/2} dt$$

Proof.

$$\begin{aligned}
L'(s) &= \frac{d}{ds}\int_a^b \varepsilon\langle\dot{h}_s(t), \dot{h}_s(t)\rangle dt \\
&= \int_a^b \frac{d}{ds}\varepsilon\langle\dot{h}_s(t), \dot{h}_s(t)\rangle dt \\
&= \int_a^b 2\varepsilon\langle\nabla_{\partial_s}\dot{h}_s(t), \dot{h}_s(t)\rangle \frac{1}{2}\left(\varepsilon\langle\dot{h}_s(t), \dot{h}_s(t)\rangle\right)^{-1/2} dt \\
&= \int_a^b \varepsilon\langle\nabla_{\partial_s}\partial_t h(s,t), \partial_t h(s,t)\rangle \left(\varepsilon\langle\partial_t h(s,t), \partial_t h(s,t)\rangle\right)^{-1/2} dt \\
&= \varepsilon\int_a^b \langle\nabla_{\partial_t}\partial_s h(s,t), \partial_t h(s,t)\rangle \left(\varepsilon\langle\partial_t h(s,t), \partial_t h(s,t)\rangle\right)^{-1/2} dt
\end{aligned}$$

■

Corollary 16.2

$$\delta L|_\alpha(V) = L'(0) = \varepsilon\int_a^b \langle\nabla_{\partial_t}V(t), \dot{\alpha}(t)\rangle \left(\varepsilon\langle\dot{\alpha}(t), \dot{\alpha}(t)\rangle\right)^{-1/2} dt.$$

Let us now consider a more general situation where $\alpha : [a, b] \to M$ is only piecewise smooth (but still continuous). Let us be specific by saying that there is a partition $a < t_1 < ... < t_k < b$ so that α is smooth on each $[t_i, t_{i+1}]$. A variation appropriate to this situation is a continuous map $h : (-\epsilon, \epsilon) \times [a, b] \to M$ with $h(0, t) = \alpha(t)$ such that h is smooth on each set of the form $(-\epsilon, \epsilon) \times [t_i, t_{i+1}]$.

This is what we shall mean by a piecewise smooth variation of a piecewise smooth curve. The velocity $\dot{\alpha}$ and the variation vector field $V(t) := \frac{\partial h(0,t)}{\partial s}$ are only piecewise smooth. At each "kink" point t_i we have the jump vector $\triangle\dot{\alpha}(t_i) := V(t_i+) - V(t_i-)$ which measure this discontinuity of $\dot{\alpha}$ at t_i. Using this notation we have the following theorem which gives the **first variation formula**:

Theorem 16.17 *Let $h : (-\epsilon, \epsilon) \times [a,b] \to M$ be a piecewise smooth variation of a piecewise smooth curve $\alpha : [a,b]$. If α has constant speed $c = (\varepsilon\langle\dot{\alpha},\dot{\alpha}\rangle)^{1/2}$ and variation vector field V then*

$$\delta L|_{\alpha}(V) = L'(0) = -\frac{\varepsilon}{c}\int_a^b \langle \nabla_{\partial_t}\dot{\alpha}, V\rangle dt - \frac{\varepsilon}{c}\sum_{i=1}^{k}\langle \triangle V(t_i), V(t_i)\rangle + \frac{\varepsilon}{c}\left.\langle\dot{\alpha}, V\rangle\right|_a^b$$

Proof. Since $c = (\varepsilon\langle\dot{\alpha},\dot{\alpha}\rangle)^{1/2}$ the proposition 16.2 gives $L'(0) = \frac{\varepsilon}{c}\int_a^b \langle\nabla_{\partial_t}V(t), \dot{\alpha}(t)\rangle dt$. Now since we have $\langle\dot{\alpha}, \nabla_{\partial_t}V\rangle = \frac{d}{dt}\langle\dot{\alpha}, V\rangle - \langle\nabla_{\partial_t}\dot{\alpha}, V\rangle$ we can employ integration by parts: On each interval $[t_i, t_{i+1}]$ we have

$$\frac{\varepsilon}{c}\int_{t_i}^{t_{i+1}} \langle\nabla_{\partial_t}V, \dot{\alpha}\rangle dt = \frac{\varepsilon}{c}\left.\langle\dot{\alpha}, V\rangle\right|_{t_i}^{t_{i+1}} - \frac{\varepsilon}{c}\int_{t_i}^{t_{i+1}}\langle\nabla_{\partial_t}\dot{\alpha}, V\rangle dt.$$

Taking the convention that $t_0 = a$ and $t_{k+1} = b$ we sum from $i = 0$ to $i = k$ to get

$$\frac{\varepsilon}{c}\int_a^b \langle\dot{\alpha}, \nabla_{\partial_t}V\rangle dt = \frac{\varepsilon}{c}\langle\dot{\alpha}, V\rangle - \frac{\varepsilon}{c}\sum_{i=1}^{k}\langle\triangle\dot{\alpha}(t_i), V(t_i)\rangle$$

which equivalent to the result. ∎

A variation $h : (-\epsilon, \epsilon) \times [a,b] \to M$ of α is called a **fixed endpoint variation** if $h(s,a) = \alpha(a)$ and $h(s,b) = \alpha(b)$ for all $s \in (-\epsilon, \epsilon)$. In this situation the variation vector field V is zero at a and b.

Corollary 16.3 *A piecewise smooth curve $\alpha : [a,b] \to M$ with constant speed $c > 0$ on each subinterval where α is smooth is a (nonnull) geodesic if and only if $\delta L|_{\alpha}(V) = 0$ for all fixed end point variations of α. In particular, if M is a Riemannian manifold and $\alpha : [a,b] \to M$ minimizes length among nearby curves then α is an (unbroken) geodesic.*

Proof. If α is a geodesic then it is smooth and so $\triangle\dot{\alpha}(t_i) = 0$ for all t_i (even though α is smooth the variation still only need to be piecewise smooth). It follows that $L'(0) = 0$.

Now if we suppose that α is a piecewise smooth curve and that $L'(0) = 0$ for any variation then we can conclude that α is a geodesic by picking some clever variations. As a first step we show that $\alpha|_{[t_i, t_{i+1}]}$ is a geodesic for each segment $[t_i, t_{i+1}]$. Let $t \in (t_i, t_{i+1})$ be arbitrary and let v be any nonzero vector in $T_{\alpha(t)}M$. Let b be a cut-off function on $[a,b]$ with support in $(t-\delta, t+\delta)$ and δ chosen small and then let $V(t) := b(t)Y(t)$ where Y is the parallel translation of

y along α. We can now easily produce a fixed end point variation with variation vector field V by the formula

$$h(s,t) := \exp_{\alpha(t)} sV(t).$$

With this variation the last theorem gives

$$L'(0) = -\frac{\varepsilon}{c}\int_a^b \langle \nabla_{\partial_t}\dot{\alpha}, V\rangle dt = -\frac{\varepsilon}{c}\int_{t-\delta}^{t+\delta} \langle \nabla_{\partial_t}\dot{\alpha}, b(t)Y(t)\rangle dt$$

which must hold no matter what our choice of y and for any $\delta > 0$. From this it is straightforward to show that $\nabla_{\partial_t}\dot{\alpha}(t) = 0$ and since t was an arbitrary element of (t_i, t_{i+1}) we conclude that $\alpha|_{[t_i,t_{i+1}]}$ is a geodesic. All that is left is to show that there can be no discontinuities of $\dot{\alpha}$ (recall exercise 16.12). One again we choose a vector y but this time $y \in T_{\alpha(t_i)}M$ where t_i is a potential kink point. take another cut-off function b with $\operatorname{supp} b \subset [t_{i-1}, t_{i+1}] = [t_{i-1}, t_i] \cup [t_i, t_{i+1}]$, $b(t_i) = 1$, and i a fixed but arbitrary element of $\{1, 2, ..., k\}$. Extend y to a field Y as before and let $V = bY$. Since we now have that α is a geodesic on each segment and we are assume the variation is zero, the first variation formula for any variation with variation vector field V reduces to

$$0 = L'(0) = -\frac{\varepsilon}{c}\langle \triangle\dot{\alpha}(t_i), y\rangle$$

for all y. This means that $\triangle\dot{\alpha}(t_i) = 0$ and since i was arbitrary we are done. ∎

We now see that for fixed endpoint variations $L'(0) = 0$ implies that α is a geodesic. The geodesics are the critical "points" (or curves) of the arc length functional restricted to all curves with fixed endpoints. In order to classify the critical curves we look at the second variation but we only need the formula for variations of geodesics. For a variation h of a geodesic γ we have the variation vector field V as before but we also now consider the transverse acceleration vector field $A(t) := \nabla_{\partial_s}\partial_s h(0,t)$. Recall that for a curve γ with $|\langle\dot{\gamma},\dot{\gamma}\rangle| > 0$ a vector field Y along γ has an orthogonal decomposition $Y = Y^\top + Y^\perp$ (tangent and normal to γ). Also we have $(\nabla_{\partial_t}Y)^\perp = (\nabla_{\partial_t}Y)^\perp$ and so we can use $\nabla_{\partial_t}Y^\perp$ to denote either of these without ambiguity.

We now have the **second variation formula of Synge**:

Theorem 16.18 *Let $\gamma : [a,b] \to M$ be a (nonnull) geodesic of speed $c > 0$. Let $\varepsilon = sgn\langle\dot{\gamma},\dot{\gamma}\rangle$ as before. If $h : (-\epsilon,\epsilon)\times[a,b]$ is a variation of γ with variation vector field V and acceleration vector field A then the second variation of $L(s) := L(h_s(t))$ at $s = 0$ is*

$$\begin{aligned} L''(0) &= \frac{\varepsilon}{c}\int_a^b \left(\langle\nabla_{\partial_t}Y^\perp, \nabla_{\partial_t}Y^\perp\rangle + \langle R_{\dot{\gamma},V}\dot{\gamma}, V\rangle\right)dt + \frac{\varepsilon}{c}\left.\langle\dot{\gamma}, A\rangle\right|_a^b \\ &= \frac{\varepsilon}{c}\int_a^b \left(\langle\nabla_{\partial_t}Y^\perp, \nabla_{\partial_t}Y^\perp\rangle + \langle R_{\dot{\gamma}}(V), V\rangle\right)dt + \frac{\varepsilon}{c}\left.\langle\dot{\gamma}, A\rangle\right|_a^b \end{aligned}$$

Proof. Let $H(s,t) := \left|\langle \frac{\partial h}{\partial s}(s,t), \frac{\partial h}{\partial s}(s,t)\rangle\right|^{1/2} = \left(\varepsilon\langle \frac{\partial h}{\partial s}(s,t), \frac{\partial h}{\partial s}(s,t)\rangle\right)^{1/2}$. We have $L'(s) = \int_a^b \frac{\partial}{\partial s^2} H(s,t)dt$. Now computing as before we see that $\frac{\partial H(s,t)}{\partial s} = \frac{\varepsilon}{H}\langle \frac{\partial \gamma}{\partial s}(s,t), \nabla_{\partial_s}\frac{\partial \gamma}{\partial t}(s,t)\rangle$. Taking another derivative we have

$$\begin{aligned}
\frac{\partial^2 H(s,t)}{\partial s^2} &= \frac{\varepsilon}{H^2}\left(H\frac{\partial}{\partial s}\langle \frac{\partial h}{\partial t}, \nabla_{\partial_s}\frac{\partial h}{\partial t}\rangle - \langle \frac{\partial h}{\partial t}, \nabla_{\partial_s}\frac{\partial h}{\partial t}\rangle \frac{\partial H}{\partial s}\right) \\
&= \frac{\varepsilon}{H}\left(\langle \nabla_{\partial_s}\frac{\partial h}{\partial t}, \nabla_{\partial_s}\frac{\partial h}{\partial t}\rangle + \langle \frac{\partial h}{\partial t}, \nabla^2_{\partial_s}\frac{\partial h}{\partial t}\rangle - \frac{1}{H}\langle \frac{\partial h}{\partial t}, \nabla_{\partial_s}\frac{\partial h}{\partial t}\rangle \frac{\partial H}{\partial s}\right) \\
&= \frac{\varepsilon}{H}\left(\langle \nabla_{\partial_s}\frac{\partial h}{\partial t}, \nabla_{\partial_s}\frac{\partial h}{\partial t}\rangle + \langle \frac{\partial h}{\partial t}, \nabla^2_{\partial_s}\frac{\partial h}{\partial t}\rangle - \frac{\varepsilon}{H}\langle \frac{\partial h}{\partial t}, \nabla_{\partial_s}\frac{\partial h}{\partial t}\rangle^2\right).
\end{aligned}$$

Now using $\nabla_{\partial_t}\partial_s h = \nabla_{\partial_s}\partial_t h$ and lemma 16.7 we obtain

$$\nabla_{\partial_s}\nabla_{\partial_s}\frac{\partial h}{\partial t} = \nabla_{\partial_s}\nabla_{\partial_t}\frac{\partial h}{\partial s} = R(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t})\frac{\partial h}{\partial s} + \nabla_{\partial_t}\nabla_{\partial_s}\frac{\partial h}{\partial s}$$

and then

$$\begin{aligned}
\frac{\partial^2 H}{\partial s^2} = \frac{\varepsilon}{H}\Big\{&\langle \nabla_{\partial_t}\frac{\partial \gamma}{\partial s}, \nabla_{\partial_t}\frac{\partial \gamma}{\partial s}\rangle + \langle \frac{\partial h}{\partial t}, R(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t})\frac{\partial h}{\partial s}\rangle \\
&+\langle \frac{\partial h}{\partial t}, \nabla_{\partial_t}\nabla_{\partial_s}\frac{\partial h}{\partial s}\rangle - \frac{\varepsilon}{H^2}\langle \frac{\partial h}{\partial t}, \nabla_{\partial_t}\frac{\partial h}{\partial s}\rangle^2\Big\}
\end{aligned}$$

Now we let $s = 0$ and get

$$\begin{aligned}
\frac{\partial^2 H}{\partial s^2}(0,t) = \frac{\varepsilon}{c}\{&\langle \nabla_{\partial_t}V, \nabla_{\partial_t}V\rangle + \langle \dot{\gamma}, R(V,\dot{\gamma})V\rangle \\
&+\langle \dot{\gamma}, \nabla_{\partial_t}A\rangle - \frac{\varepsilon}{c^2}\langle \dot{\gamma}, \nabla_{\partial_t}V\rangle^2\Big\}
\end{aligned}$$

Now before we integrate the above expression we use the fact that $\langle \dot{\gamma}, \nabla_{\partial_t}A\rangle = \frac{d}{dt}\langle \dot{\gamma}, A\rangle$ (γ is a geodesic) and the fact that the orthogonal decomposition of $\frac{\nabla V}{dt}$ is

$$\nabla_{\partial_t}V = \frac{\varepsilon}{c^2}\langle \dot{\gamma}, \nabla_{\partial_t}V\rangle\dot{\gamma} + \nabla_{\partial_t}V^{\perp}$$

so that $\langle \nabla_{\partial_t}V, \nabla_{\partial_t}V\rangle = \frac{\varepsilon}{c^2}\langle \dot{\gamma}, \nabla_{\partial_t}V\rangle^2 + \langle \nabla_{\partial_t}V^{\perp}, \nabla_{\partial_t}V^{\perp}\rangle$. Plugging these identities in, observing the cancellation, and integrating we get

$$\begin{aligned}
L''(0) &= \int_a^b \frac{\partial^2 H}{\partial s^2}(0,t)dt \\
&= \frac{\varepsilon}{c}\int_a^b \left(\langle \nabla_{\partial_t}V^{\perp}, \nabla_{\partial_t}V^{\perp}\rangle + \langle \dot{\gamma}, R(V,\dot{\gamma})V\right) \\
&+ \frac{\varepsilon}{c}\langle \dot{\gamma}, A\rangle\Big|_a^b
\end{aligned}$$

∎

It is important to notice that the term $\frac{\varepsilon}{c}\langle \dot{\gamma}, A\rangle\big|_a^b$ depends not just on A but also on the curve itself. Thus the right hand side of the main equation of the second variation formula just proved depends only on V and A except for the last term. But if the variation is a fixed endpoint variation then this dependence drops out.

For our present purposes we will not loose anything by assume that $a = 0$. On the other hand it will pay to refrain from assuming $b = 1$.It is traditional to think of the set $\Omega_{0,b}(p,q)$ of all piecewise smooth curves $\alpha : [0,b] \to M$ from p to q as an infinite dimensional manifold. Then a variation vector field V along a curve $\alpha \in \Omega(p,q)$ which is zero at the endpoints is the tangent at α to the curve in $\Omega_{0,b}(p,q)$ given by the corresponding fixed endpoint variation h. Thus the "tangent space" $T(\Omega) = T_\alpha\left(\Omega_{0,b}(p,q)\right)$ at α is the set of all piecewise smooth vector fields V along α such that $V(0) = V(b) = 0$. We then think of L as being a function on $\Omega_{0,b}(p,q)$ whose (nonnull[2]) critical points we have discovered to be nonnull geodesics beginning at p and ending at q at times 0 and b respectively. Further thinking along these lines leads to the idea of the index form.

Definition 16.37 *For a given nonnull geodesic $\gamma : [0,b] \to M$, the index form $I_\gamma : T_\gamma\Omega_{0,b} \times T_\gamma\Omega_{0,b} \to \mathbb{R}$ is defined by $I_\gamma(V,V) = L''_\gamma(0)$ where $L_\gamma(s) = \int_0^b \left|\langle \dot{h}_s(t), \dot{h}_s(t)\rangle\right| dt$ and $\frac{\nabla h}{\partial s}(0,t) = V$.*

Of course this definition makes sense because $L''_\gamma(0)$ only depends on V and not on h itself. Also, we have defined the quadratic form $I_\gamma(V,V)$ but not directly $I_\gamma(V,W)$. On the other hand, polarizations gives a but if $V,W \in T_\gamma\Omega_{0,b}$ then it is not hard to see from the second variation formula that

$$I_\gamma(V,W) = \frac{\varepsilon}{c}\int_0^b \left\{\langle \nabla_{\partial_t} V^\perp, \nabla_{\partial_t} W^\perp\rangle + \langle R(V,\dot{\gamma})W\rangle, \dot{\gamma}\right\} dt \tag{16.10}$$

It is important to notice that the right hand side of the above equation is in fact symmetric in V and W. It is also not hard to prove that if even one of V or W is tangent to $\dot{\gamma}$ then $I_\gamma(V,W) = 0$ and so $I_\gamma(V,W) = I_\gamma(V^\perp, W^\perp)$ and so we may as well restrict I_γ to

$$T_\gamma^\perp\Omega_{0,b} = \{V \in T_\gamma\Omega_{0,b} : V \perp \dot{\gamma}\}.$$

This restriction will be denoted by $I_\gamma^\perp$.

It is important to remember that the variations and variation vector fields we are dealing with are allowed to by only piecewise smooth even if the center curve is smooth. So let $0 = t_0 < t_1 < ... < t_k < t_{k+1} = b$ as before and let V and W be vector fields along a geodesic γ. We now derive another formula for $I_\gamma(V,W)$. Rewrite formula16.10 as

$$I_\gamma(V,W) = \frac{\varepsilon}{c}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} \left\{\langle \nabla_{\partial_t} V^\perp, \nabla_{\partial_t} W^\perp\rangle + \langle R(V,\dot{\gamma})W, \dot{\gamma}\rangle\right\} dt$$

On each interval $[t_i, t_{i+1}]$ we have

$$\langle \nabla_{\partial_t} V^\perp, \nabla_{\partial_t} W^\perp\rangle = \nabla_{\partial_t}\langle \nabla_{\partial_t} V^\perp, W^\perp\rangle - \langle \nabla_{\partial_t}^2 V^\perp, W^\perp\rangle$$

[2]By nonnull we just mean that the geodesic is nonnull.

and substituting this into the above formula we have

$$I_\gamma(V,W) = \frac{\varepsilon}{c}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} \left\{\nabla_{\partial_t}\langle\nabla_{\partial_t}V^\perp, W^\perp\rangle - \langle\nabla^2_{\partial_t}V^\perp, W^\perp\rangle + \langle R(V,\dot\gamma)W,\dot\gamma\rangle\right\} dt.$$

As for the curvature term we use

$$\begin{aligned}\langle R(V,\dot\gamma)W,\dot\gamma\rangle &= \langle R(\dot\gamma,V)\dot\gamma, W\rangle \\ &= \langle R(\dot\gamma,V^\perp)\dot\gamma, W^\perp\rangle.\end{aligned}$$

Substituting we get

$$I_\gamma(V,W) = \frac{\varepsilon}{c}\sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} \left\{\nabla_{\partial_t}\langle\nabla_{\partial_t}V^\perp, W^\perp\rangle - \langle\nabla^2_{\partial_t}V^\perp, W^\perp\rangle + \langle R(\dot\gamma,V^\perp)\dot\gamma, W^\perp\rangle\right\} dt.$$

Using the fundamental theorem of calculus on each interval $[t_i, t_{i+1}]$ and the fact that W vanishes at a and b we obtain alternate formula:

Proposition 16.10 (Formula for Index Form) *Let $\gamma : [0,b] \to M$ be a nonnull geodesic. Then*

$$I_\gamma(V,W) = -\frac{\varepsilon}{c}\int_0^b \langle\nabla^2_{\partial_t}V^\perp + R(\dot\gamma,V^\perp)\dot\gamma, W^\perp\rangle dt - \frac{\varepsilon}{c}\sum_{i=1}^{k}\langle\triangle\nabla_{\partial_t}V^\perp(t_i), W^\perp(t_i)\rangle$$

where $\triangle\nabla_{\partial_t}V^\perp(t_i) = \nabla_{\partial_t}V^\perp(t_i+) - \nabla_{\partial_t}V^\perp(t_i-)$.

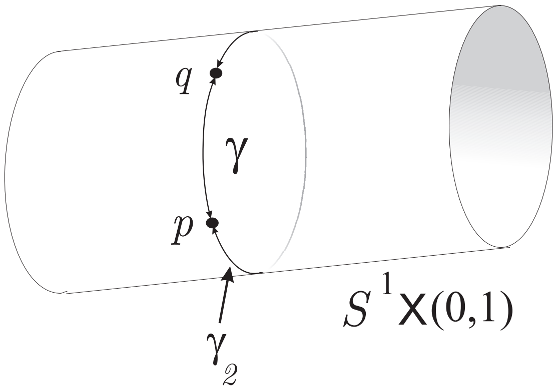
Letting $V = W$ we have

$$I_\gamma(V,V) = -\frac{\varepsilon}{c}\int_0^b \langle\nabla^2_{\partial_t}V^\perp + R(\dot\gamma,V^\perp)\dot\gamma, V^\perp\rangle dt - \frac{\varepsilon}{c}\sum_{i=1}^{k}\langle\triangle\nabla_{\partial_t}V^\perp(t_i), V^\perp(t_i)\rangle$$

and the presence of the term $R(\dot\gamma,V^\perp)\dot\gamma, V^\perp\rangle$ indicates a connection with Jacobi fields.

Definition 16.38 *A geodesic segment $\gamma : [a,b] \to M$ is said to be* ***relatively length minimizing*** *if for all piecewise smooth fixed endpoint variations h of γ the function $L(s) := \int_a^b \left|\langle\dot h_s(t), \dot h_s(t)\rangle\right| dt$ has a local minimum at $s = 0$ (where $\gamma = h_0(t) := h(0,t)$).*

If $\gamma : [a,b] \to M$ is a relatively length minimizing nonnull geodesic then L is at least twice differentiable at $s = 0$ and clearly $L''(0) = 0$ which means that $I_\gamma(V,V) = 0$ for any $V \in T_\gamma\Omega_{a,b}$. The adjective "relatively" is included in the terminology because of the possibility that there may be curve in $\Omega_{a,b}$ which are "far away from γ" which has smaller length than γ. A simple example of this is depicted in the figure below where γ_2 has greater length than γ even though γ_2 is relatively length minimizing. We are assume that the metric on $(0,1)\times S^1$ is the



usual definite metric dx^2+dy^2 induced from that on $\mathbb{R}\times(0,1)$ where we identify $S^1\times(0,1)$ with the quotient $\mathbb{R}\times(0,1)/((x,y)\sim(x+2\pi,y))$. On the other hand, one sometimes hears the statement that geodesics in a Riemannian manifold are **locally length minimizing**. This means that for any geodesic $\gamma:[a,b]\to M$, the restrictions to small intervals $[a,\epsilon]$ is always relatively length minimizing but this is only true for Riemannian manifolds. For a semi-Riemannian manifold with indefinite metric a small geodesic segment can have nearby curves that increase the length. To see an example of this consider the metric $-dx^2+dy^2$ on $\mathbb{R}\times(0,1)$ and the induced metric on the quotient $S^1\times(0,1)=\mathbb{R}\times(0,1)/\sim$. In this case, the geodesic γ has length less than all nearby geodesics; the index form I_γ is now negative semidefinite.

Exercise 16.18 *Prove the above statement concerning I_γ for $S^1\times(0,1)$ with the index 1 metric $-dx^2+dy^2$.*

Exercise 16.19 *Let M be a Riemannian manifold. Show that if $I_\gamma(V,V)=0$ for all $V\in T_\gamma^\perp\Omega_{a,b}$ then a nonnull geodesic $\gamma$is relatively length minimizing.*

Theorem 16.19 *Let $\gamma:[0,b]\to M$ be a nonnull geodesic. The nullspace $\mathcal{N}(I_\gamma^\perp)$(deefffine!!) of $I_\gamma^\perp : T_\gamma^\perp\Omega_{0,b}\to R$ is exactly the space $\mathcal{J}_0(\gamma,0,b)$ of Jacobi fields vanishing at $\gamma(0)$ and $\gamma(b)$.*

Proof. It follow from the formula of proposition 16.10 makes it clear that $\mathcal{J}_0(\gamma,0,b)\subset\mathcal{N}(I_\gamma^\perp)$.

Suppose that $V\in\mathcal{N}(I_\gamma^\perp)$. Let $t\in(t_i,t_{i+1})$ where the t_i is the partition of V. Pick an arbitrary nonzero element $y\in T_{\gamma(t)}M$ and let Y be the unique parallel field along $\gamma|_{[t_i,t_{i+1}]}$ such that $Y(t)=y$. Picking a cut-off function β with support in $[t+\delta,t_i-\delta]\subset(t_i,t_{i+1})$ as before we extend βY to a field W along γ with $W(t)=y$. Now V is normal to the geodesic and so $I_\gamma(V,W)=I_\gamma^\perp(V,W)$ and

$$I_\gamma(V,W)=-\frac{\varepsilon}{c}\int_{t-\delta}^{t+\delta}\langle\nabla_{\partial_t}^2V+R(\dot\gamma,V)\dot\gamma,\beta Y^\perp\rangle dt$$

for small δ, $\beta Y^{\perp}$ is approximately the arbitrary nonzero y and it follows that $\nabla^2_{\partial_t} V + R(\dot{\gamma}, V)\dot{\gamma}$ is zero at t since t was arbitrary it is identically zero on (t_i, t_{i+1}). Thus V is a Jacobi field on each interval (t_i, t_{i+1}) and since $V$is continuous on $[0, b]$ it follows from standard theory of differential equations that V is a smooth Jacobi field along all of γ and since $V \in T_{\gamma(t)}M$ we already have $V(0) = V(b) = 0$ so $V \in \mathcal{J}_0(\gamma, 0, b)$. ■

Proposition 16.11 *Let M, g be a semi-Riemannian manifold of index $\nu = \mathrm{ind}(M)$ and $\gamma : [a, b] \to M$ a nonnull geodesic. If the index I_γ is positive semidefinite then $\nu = 0$ or n (thus the metric is definite and so, up to sign of g, the manifold is Riemannian). On the other hand, if I_γ is negative semidefinite then $\nu = 1$ or $n - 1$ (so that up to sign convention M is a Lorentz manifold).*

Proof. Let I_γ be positive semi-definite and assume that $0 < \nu < n$. In this case there must be a unit vector u in $T_{\gamma(a)}M$ which is normal to $\dot{\gamma}(0)$ and has the opposite causal character of $\dot{\gamma}(a)$. This means that if $\varepsilon = \langle \dot{\gamma}(a), \dot{\gamma}(a) \rangle$ then $\varepsilon \langle u, u \rangle = -1$. Let U be the field along γ which is the parallel translation of u. By choosing $\delta > 0$ appropriately we can arrange that δ is as small as we like and simultaneously that $\sin(t/\delta)$ is zero at $t = a$ and $t = b$. Let $V := \delta \sin(t/\delta) U$ and make the harmless assumption that $|\dot{\gamma}| = 1$. Notice that by construction $V \perp \dot{\gamma}$. We compute:

$$
\begin{aligned}
I_\gamma(V, V) &= \varepsilon \int_a^b \{ \langle \nabla_{\partial_t} V, \nabla_{\partial_t} V \rangle + \langle R(\dot{\gamma}, V)\dot{\gamma}, V \rangle \} \, dt \\
&= \varepsilon \int_a^b \{ \langle \nabla_{\partial_t} V, \nabla_{\partial_t} V \rangle + \langle R(\dot{\gamma}, V)V, \dot{\gamma} \rangle \} \, dt \\
&\varepsilon \int_a^b \{ \langle \nabla_{\partial_t} V, \nabla_{\partial_t} V \rangle + K(V \wedge \dot{\gamma}) \langle V \wedge \dot{\gamma}, V \wedge \dot{\gamma} \rangle \} \, dt \\
&= \varepsilon \int_a^b \{ \langle \nabla_{\partial_t} V, \nabla_{\partial_t} V \rangle + K(V \wedge \dot{\gamma}) \langle V, V \rangle \varepsilon \} \, dt
\end{aligned}
$$

where $K(V \wedge \dot{\gamma}) := \frac{\langle \mathfrak{R}(V \wedge \dot{\gamma}), V \wedge \dot{\gamma} \rangle}{\langle V \wedge \dot{\gamma}, V \wedge \dot{\gamma} \rangle} = \frac{\langle \mathfrak{R}(V \wedge \dot{\gamma}), V \wedge \dot{\gamma} \rangle}{\langle V, V \rangle^2}$ defined earlier. Continuing the computation we have

$$
\begin{aligned}
I_\gamma(V, V) &= \varepsilon \int_a^b \left\{ \langle u, u \rangle \cos^2(t/\delta) + K(V \wedge \dot{\gamma}) \delta^2 \sin^2(t/\delta) \right\} dt \\
&= \int_a^b \left\{ -\cos^2(t/\delta) + \varepsilon K(V \wedge \dot{\gamma}) \delta^2 \sin^2(t/\delta) \right\}
\end{aligned}
$$

Not as we said, we can choose δ as small as we like and since $K(V(t) \wedge \dot{\gamma}(t))$ is bounded on the (compact) interval this clearly means that $I_\gamma(V, V) < 0$ which contradicts the fact that I_γ is positive semidefinite. Thus our assumption that $0 < \nu < n$ is impossible.

Now let I_γ be negative semi-definite. Suppose that we assume that contrary to what we wish to show, ν is not 1 or $n - 1$. In this case one can find a unit vector $u \in T_{\gamma(a)}M$ normal to $\dot{\gamma}(a)$ such that $\varepsilon\langle u, u\rangle = +1$. The the same sort of calculation as we just did shows that I_γ cannot be semi-definite; again a contradiction. ∎

By changing the sign of the metric the cases handled by this last theorem boil down to the two important cases 1) where M, g is Riemannian, γ is arbitrary and 2) where M, g is Lorentz and γ is timelike. We consolidate these two cases by a definition:

Definition 16.39 *A geodesic $\gamma : [a, b] \to M$ is cospacelike if the subspace $\dot{\gamma}(s)^\perp \subset T_{\gamma(s)}M$ is spacelike for some (and consequently all) $s \in [a, b]$.*

Exercise 16.20 *Show that if $\gamma : [a, b] \to M$ is cospacelike then γ is nonnull, $\dot{\gamma}(s)^\perp \subset T_{\gamma(s)}M$ is spacelike for all $s \in [a, b]$ and also show that M, g is either Riemannian of Lorentz.*

A useful observation about Jacobi fields along a geodesic is the following:

Lemma 16.8 *If we have two Jacobi fields, say J_1 and J_2 along a geodesic γ then $\langle \nabla_{\partial_t} J_1, J_2\rangle - \langle J_1, \nabla_{\partial_t} J_2\rangle$ is constant along γ.*

To see this we note that

$$\begin{aligned}
\nabla_{\partial_t}\langle \nabla_{\partial_t} J_1, J_2\rangle &= \langle \nabla^2_{\partial_t} J_1, J_2\rangle + \langle \nabla_{\partial_t} J_1, \nabla_{\partial_t} J_2\rangle \\
&= \langle R(J_{1,}\dot{\gamma})J_2, \dot{\gamma}\rangle + \langle \nabla_{\partial_t} J_1, \nabla_{\partial_t} J_2\rangle \\
&= \langle R(J_2\dot{\gamma})J_{1,}, \dot{\gamma}\rangle + \langle \nabla_{\partial_t} J_2, \nabla_{\partial_t} J_1\rangle \\
&= \nabla_{\partial_t}\langle \nabla_{\partial_t} J_2, J_1\rangle.
\end{aligned}$$

In particular, if $\langle \nabla_{\partial_t} J_1, J_2\rangle = \langle J_1, \nabla_{\partial_t} J_2\rangle$ at $t = 0$ then $\langle \nabla_{\partial_t} J_1, J_2\rangle - \langle J_1, \nabla_{\partial_t} J_2\rangle = 0$ for all t.

We now need another simple but rather technical lemma.

Lemma 16.9 *If $J_1, ..., J_k$ are Jacobi fields along a geodesic γ such that $\langle \nabla_{\partial_t} J_i, J_j\rangle = \langle J_i, \nabla_{\partial_t} J_j\rangle$ for all $i, j \in \{1, ..., k\}$ then and field Y which can be written $Y = \varphi^i J_i$ has the property that*

$$\langle \nabla_{\partial_t} Y, \nabla_{\partial_t} Y\rangle + \langle R(Y, \dot{\gamma})Y, \dot{\gamma}\rangle = \langle \left(\partial_t \varphi^i\right) J_i, \left(\partial_t \varphi^i\right) J_i\rangle + \partial_t\langle Y, \varphi^r \left(\nabla_{\partial_t} J_r\right)\rangle$$

Proof. $\nabla_{\partial_t} Y = \left(\partial_t \varphi^i\right) J_i + \varphi^r \left(\nabla_{\partial_t} J_r\right)$ and so

$$\begin{aligned}
\partial_t\langle Y, \varphi^r \left(\nabla_{\partial_t} J_r\right)\rangle &= \langle \left(\nabla_{\partial_t} Y\right), \varphi^r \left(\nabla_{\partial_t} J_r\right)\rangle + \langle \left(\partial_t \varphi^i\right) J_i, \nabla_{\partial_t} \left[\varphi^r \left(\nabla_{\partial_t} J_r\right)\right]\rangle \\
&= \langle \left(\partial_t \varphi^i\right) J_i, \varphi^r \left(\nabla_{\partial_t} J_r\right)\rangle + \langle \varphi^r \left(\nabla_{\partial_t} J_r\right), \varphi^r \left(\nabla_{\partial_t} J_r\right)\rangle \\
&+ \langle Y, \partial_t \varphi^r \nabla_{\partial_t} J_r\rangle + \langle Y, \varphi^r \nabla^2_{\partial_t} J_r\rangle
\end{aligned}$$

The last term $\langle Y, \varphi^r \nabla^2_{\partial_t} J_r\rangle$ equals $\langle R(Y, \dot{\gamma})Y, \dot{\gamma}\rangle$ by the Jacobi equation. Using this and the fact that $\langle Y, \partial_t \varphi^r \nabla_{\partial_t} J_r\rangle = \langle \left(\partial_t \varphi^i\right) J_i, \varphi^r \left(\nabla_{\partial_t} J_r\right)\rangle$ which follows from a short calculation using the hypotheses on the J_i we arrive at

$$\begin{aligned}
\partial_t\langle Y, \varphi^r \left(\nabla_{\partial_t} J_r\right)\rangle &= 2\langle \left(\partial_t \varphi^i\right) J_i, \varphi^r \left(\nabla_{\partial_t} J_r\right)\rangle + \langle \varphi^r \left(\nabla_{\partial_t} J_r\right), \varphi^r \left(\nabla_{\partial_t} J_r\right)\rangle \\
&+ \langle R(Y, \dot{\gamma})Y, \dot{\gamma}\rangle.
\end{aligned}$$

Using the last equation together with $\nabla_{\partial_t} Y = \left(\partial_t \varphi^i\right) J_i + \varphi^r \left(\nabla_{\partial_t} J_r\right)$ gives the result (check it!). ∎

Exercise 16.21 *Work through the details of the proof of the lemma above.*

Through out the following discussion $\gamma : [0, b] \to M$ will be a cospacelike geodesic with sign $\varepsilon := \langle \dot{\gamma}, \dot{\gamma} \rangle$.

Suppose that, for whatever reason, there are no conjugate points of $p = \gamma(0)$ along γ. There exist Jacobi fields $J_1, ..., J_{n-1}$ along γ which vanish at $t = 0$ and such that the vectors $\nabla_{\partial_t} J_1(0), ..., \nabla_{\partial_t} J_{n-1}(0) \in T_p M$ are a basis for the space $\dot{\gamma}(0)^\perp \subset T_{\gamma(0)} M$. We know that these J_i are all normal to the geodesic and since we are assuming that there are no conjugate points to p along γ it follows that the fields remain linearly independent and at each t with $0 < t \leq b$ form a basis of $\dot{\gamma}(t)^\perp \subset T_{\gamma(t)} M$. Now let $Y \in T_\gamma(\Omega)$ be a piecewise smooth variation vector field along γ and write $Y = \varphi^i J_i$ for some piecewise smooth functions φ^i on $(0, b]$ which can be show to extend continuously to $[0, b]$. Since $\langle \nabla_{\partial_t} J_i, J_j \rangle = \langle J_i, \nabla_{\partial_t} J_j \rangle = 0$ at $t = 0$ we have $\langle \nabla_{\partial_t} J_i, J_j \rangle - \langle J_i, \nabla_{\partial_t} J_j \rangle = 0$ for all t by lemma 16.8. This allows the use of the lemma 16.9 to arrive at

$$\langle \nabla_{\partial_t} Y, \nabla_{\partial_t} Y \rangle + \langle R(Y, \dot{\gamma}) Y, \dot{\gamma} \rangle = \langle \left(\partial_t \varphi^i\right) J_i, \left(\partial_t \varphi^i\right) J_i \rangle + \partial_t \langle Y, \varphi^r \left(\nabla_{\partial_t} J_r\right) \rangle$$

and then

$$\varepsilon I_\gamma(Y, Y) = \frac{1}{c} \int_0^b \langle \left(\partial_t \varphi^i\right) J_i, \left(\partial_t \varphi^i\right) J_i \rangle dt + \frac{1}{c} \langle Y, \varphi^r \left(\nabla_{\partial_t} J_r\right) \rangle \Big|_0^b . \tag{16.11}$$

On the other hand, Y is zero at a and b and so the last term above vanishes. Now we notice that since γ is cospacelike and the acceleration field $\nabla_{\partial_t} \dot{\gamma}$ is normal to the geodesic we must have $\langle \nabla_{\partial_t} \dot{\gamma}, \nabla_{\partial_t} \dot{\gamma} \rangle \geq 0$ (Exercise: prove this last statement). Now by equation 16.11 above we conclude that $\varepsilon I_\gamma(Y, Y) \geq 0$. On the other hand, if $I_\gamma(Y, Y) = 0$ identically then $\int_0^b \langle \left(\partial_t \varphi^i\right) J_i, \left(\partial_t \varphi^i\right) J_i \rangle dt = 0$ and $\langle \left(\partial_t \varphi^i\right) J_i, \left(\partial_t \varphi^i\right) J_i \rangle = 0$. In turn this implies that $A = 0$ and that each φ^i is zero and finally that Y itself is identically zero along γ. A moments thought shows that all we have assumed about Y is that it is in the domain of the restricted index $I_\gamma^\perp$ and so we have proved the following:

Proposition 16.12 *If $\gamma \in \Omega_{0,b}(\gamma)$ is cospacelike and there is no conjugate points to $p = \gamma(0)$ along γ then $\varepsilon I_\gamma^\perp(Y, Y) \geq 0$ and $Y = 0$ along γ if and only if $I_\gamma^\perp(Y, Y) = 0$.*

We may paraphrase the above result as follows: For a cospacelike geodesic γ without conjugate points, the restricted index for $I_\gamma^\perp$ is definite; positive definite if $\varepsilon = +1$ and negative definite if $\varepsilon = -1$. The first case ($\varepsilon = +1$) is exactly the case where M, g is Riemannian (exercise 16.20).

Next we consider the situation where the cospacelike geodesic $\gamma : [0, b] \to M$ is such that $\gamma(b)$ is the only point conjugate to $p = \gamma(0)$ along γ. In this case, theorem 16.19 tells use that $I_\gamma^\perp$ has a nontrivial nullspace and so I_γ cannot be definite. Claim: I_γ is semidefinite. To see this let $Y \in T_\gamma\Omega_{0,b}(\gamma)$ and write Y is the form $(b-t)Z(t)$ for some (continuous) piecewise smooth Z. Let $b_i \to b$ and define Y_i to be $(b_i - t)Z(t)$ on $[0, b_i]$. Our last proposition applied to $\gamma_i := \gamma|_{[0,b_i]}$ shows that $\varepsilon I_{\gamma_i}(Y_i, Y_i) \geq 0$. Now $\varepsilon I_{\gamma_i}(Y_i, Y_i) \to \varepsilon I_\gamma(Y, Y)$ (some uninteresting details are omitted) and so the claim is true.

Now we consider the case where there is a conjugate point to p before $\gamma(b)$. Suppose that J is a nonzero Jacobi field along $\gamma|_{[0,r]}$ with $0 < r < b$ such that $J(0) = J(r) = 0$. We can extend J to a field J_{ext} on $[0, b]$ by defining it to be 0 on $[r, b]$. Notice that $\nabla_{\partial_t} J_{ext}(r-)$ is equal to $\nabla_{\partial_t} J(r)$ which is not 0 since otherwise J would be identically zero (over determination). On the other hand, $\nabla_{\partial_t} J_{ext}(r+) = 0$ and so the "kink" $\triangle J'_{ext}(r) := \nabla_{\partial_t} J_{ext}(r+) - \nabla_{\partial_t} J_{ext}(r-)$ is not zero. We will now show that if $W \in T_\gamma(\Omega)$ such that $W(r) = \triangle J'_{ext}(r)$ (and there are plenty of such fields), then $\varepsilon I_\gamma(J_{ext} + \delta W, J_{ext} + \delta W) < 0$ for small enough $\delta > 0$. This will allow us to conclude that I_γ *cannot be definite* since by 16.11 we can always find a Z with $\varepsilon I_\gamma(Z, Z) > 0$. We have

$$\varepsilon I_\gamma(J_{ext} + \delta W, J_{ext} + \delta W) = \varepsilon I_\gamma(J_{ext}, J_{ext}) + 2\delta\varepsilon I_\gamma(J_{ext}, W) + \varepsilon\delta^2 I_\gamma(W, W)$$

Now it is not hard to see from the formula of theorem 16.10 that $I_\gamma(J_{ext}, J_{ext})$ is zero since it is Piecewise Jacobi and is zero at the single kink point r. But suing the formula again, $\varepsilon I_\gamma(J_{ext}(r), W(r))$ reduces to

$$-\frac{1}{c}\langle \triangle J'_{ext}(r), W(r)\rangle = -\frac{1}{c}\left|\triangle J'_{ext}(r)\right|^2 < 0$$

and so taking δ small enough gives the desired conclusion.

Summarizing the conclusion the above discussion (together with the result of proposition 16.12) we obtain the following nice theorem:

Theorem 16.20 *If $\gamma : [0, b] \to M$ is a cospacelike geodesic of sign $\varepsilon := \langle \dot\gamma, \dot\gamma\rangle$ then M, g is either Riemannian of Lorentz and we have the following three cases:*

(i) If there are no conjugate to $\gamma(0)$ along γ then $\varepsilon I_\gamma^\perp$ is definite (positive if $\varepsilon = 1$ and negative if $\varepsilon = -1$)

(ii) If $\gamma(b)$ is the only conjugate point to $\gamma(0)$ along γ then I_γ is not definite but must be semidefinite.

(iii) If there is a conjugate $\gamma(r)$ to $\gamma(0)$ with $0 < r < b$ then I_γ is not semidefinite (or definite).

Corollary 16.4 *If $\gamma : [0, b] \to M$ is a cospacelike geodesic of sign $\varepsilon := \langle \dot\gamma, \dot\gamma\rangle$ and suppose that Y is a vector field along γ and that J is a Jacobi field along γ such that*

$$\begin{aligned} Y(0) &= J(0) \\ Y(b) &= J(b) \\ (Y - J) &\perp \dot\gamma \end{aligned}$$

Then $\varepsilon I_\gamma(J,J) \le \varepsilon I_\gamma(Y,Y)$.

Proof of Corollary. From the previous theorem we have $0 \le \varepsilon I_\gamma^\perp(Y-J, Y-J) = \varepsilon I_\gamma(Y-J, Y-J)$ and so

$$0 \le \varepsilon I_\gamma(Y,Y) - 2\varepsilon I_\gamma(J,Y) + \varepsilon I_\gamma(J,J)$$

On the other hand,

$$\begin{aligned} \varepsilon I_\gamma(J,Y) &= \varepsilon \left\langle \nabla_{\partial_t} J, Y \right\rangle\big|_0^b - \varepsilon \int_0^b \langle \nabla_{\partial_t}^2 J, Y\rangle - \langle R_{\dot\gamma,J}\dot\gamma, J\rangle \\ &= \varepsilon \left\langle \nabla_{\partial_t} J, Y \right\rangle\big|_0^b = \varepsilon \left\langle \nabla_{\partial_t} J, J \right\rangle\big|_0^b \\ &= \varepsilon I_\gamma(J,J) \text{ (since } J \text{ is a Jacobi field)} \end{aligned}$$

Thus $0 \le \varepsilon I \varepsilon_\gamma(Y,Y) - 2\varepsilon I_\gamma(J,Y) + \varepsilon I_\gamma(J,J) = \varepsilon I_\gamma(Y,Y) - \varepsilon I_\gamma(J,J)$. ■

As we mentioned the Jacobi equation can be written in terms of the tidal force operator: $R_v : T_pM \to T_pM$ as

$$\nabla_{\partial_t}^2 J(t) = R_{\dot\gamma(t)}(J(t))$$

The meaning of the term force here is that $R_{\dot\gamma(t)}$ controls the way nearby families of geodesics attract or repel each other. Attraction tends to create conjugate points while repulsion tends to prevent conjugate points. If γ is cospacelike then if we take any unit vector u normal to $\dot\gamma(t)$ then we can look at the component of $R_{\dot\gamma(t)}(u)$ in the u direction; $\langle R_{\dot\gamma(t)}(u), u\rangle u = \langle R_{\dot\gamma(t),u}(\dot\gamma(t)), u\rangle u = -\langle \mathfrak{R}(\dot\gamma(t) \wedge u), \dot\gamma(t) \wedge u\rangle u$. In terms of sectional curvature

$$\langle R_{\dot\gamma(t)}(u), u\rangle u = K(\dot\gamma(t) \wedge u)\langle \dot\gamma(t), \langle \dot\gamma(t)\rangle.$$

It follows from the Jacobi equation that if $\langle R_{\dot\gamma(t)}(u), u\rangle \ge 0$, i.e. if $K(\dot\gamma(t) \wedge u)\langle \dot\gamma(t), \dot\gamma(t)\rangle \le 0$ then we have repulsion and if this always happens anywhere along γ we expect that $\gamma(0)$ has no conjugate point along γ. This intuition is indeed correct:

Proposition 16.13 *Let $\gamma : [0,b] \to M$ be a cospacelike geodesic. If for every t and every vector $v \in \gamma(t)^\perp$ we have $\langle R_{\dot\gamma(t)}(v), v\rangle \ge 0$ (i.e. if $K(\dot\gamma(t) \wedge v)\langle \dot\gamma(t), \dot\gamma(t)\rangle \le 0$) then $\gamma(0)$ has no conjugate point along γ.*

In particular, a Riemannian manifold with sectional curvature $K \le 0$ has no conjugate pairs of points. Similarly, a Lorentz manifold with sectional curvature $K \ge 0$ has no conjugate pairs along any timelike geodesics.

Proof. Take J to be a Jacobi field along γ such that $J(0)$ and $J \perp \dot\gamma$. We have $\frac{d}{dt}\langle J, J\rangle = 2\langle \nabla_{\partial_t} J, J\rangle$ and

$$\begin{aligned} \frac{d^2}{dt^2}\langle J, J\rangle &= 2\langle \nabla_{\partial_t} J, \nabla_{\partial_t} J\rangle + 2\langle \nabla_{\partial_t}^2 J, J\rangle \\ &= 2\langle \nabla_{\partial_t} J, \nabla_{\partial_t} J\rangle + 2\langle R_{\dot\gamma(t),J}(\dot\gamma(t)), J\rangle \\ &= 2\langle \nabla_{\partial_t} J, \nabla_{\partial_t} J\rangle + 2\langle R_{\dot\gamma(t)}(J), J\rangle \end{aligned}$$

and by the hypotheses $\frac{d^2}{dt^2}\langle J, J\rangle \ge 0$. On the other hand, we have $\langle J(0), J(0)\rangle = 0$ and $\left.\frac{d}{dt}\right|_0 \langle J, J\rangle = 0$. It follows that since $\langle J, J\rangle$ is not identically zero we must have $\langle J, J\rangle > 0$ for all $t \in (0, b]$. ■

16.10 More Riemannian Geometry

Recall that a manifold is geodesically complete at p if and only if $\exp_p$ is defined on all of T_pM. The following lemma is the essential ingredient in the proof of the Hopf-Rinow theorem stated and proved below. In fact, this lemma itself is sometimes referred to as the Hopf-Rinow theorem. Note that this is a theorem about Riemannian manifolds.

Lemma 16.10 *Let M, g be a (finite dimensional) connected Riemannian manifold. Suppose that $\exp_p$ is defined on the ball $B_\rho(p)$ for $\rho > 0$. Then each point $q \in B_\rho(p)$ can be connected to p by an absolutely minimizing geodesic. In particular, if M is geodesically complete at $p \in M$ then each point $q \in M$ can be connected to p by an absolutely minimizing geodesic.*

Proof. Let $q \in B_\rho(p)$ with $p \neq q$ and let $R = dist(p, q)$. Choose $\epsilon > 0$ small enough that $B_{2\epsilon}(p)$ is the domain of a normal coordinate system. By lemma 16.6 we already know the theorem is true if $B_\rho(p) \subset B_\epsilon(p)$ so we will assume that $\epsilon < R < \rho$. Because $\partial B_\epsilon(p)$ is diffeomorphic to $S^{n-1} \subset \mathbb{R}^n$ it is compact and so there is a point $p_\epsilon \in \partial B_\epsilon(p)$ such that $x \mapsto dist(x, q)$ achieves its minimum at p_ϵ. This means that

$$\begin{aligned} dist(p,q) &= dist(p,p_\epsilon) + dist(p_\epsilon, q) \\ &= \epsilon + dist(p_\epsilon, q). \end{aligned}$$

Let $\gamma : [0, \rho] \to M$ be the constant speed geodesic with $|\dot{\gamma}| = 1$, $\gamma(0) = p$, and $\gamma(\epsilon) = p_\epsilon$. It is not difficult to see that the set

$$T = \{t \in [0, R] : dist(p, \gamma(t)) + dist(\gamma(t), q)$$

is closed in $[0, R]$ and is nonempty since $\epsilon \in T$. Let $t_{\sup} = \sup T > 0$. We will show that $t_{\max} = R$ from which it will follow that $\gamma|_{[0,R]}$ is a minimizing geodesic from p to q. With an eye toward a contradiction, assume that $t_{\sup} < R$. Let $x := \gamma(t_{\sup})$ and choose ϵ_1 with $0 < \epsilon_1 < R - t_{\sup}$ and small enough that $B_{2\epsilon_1}(x)$ is the domain of normal coordinates about x. Arguing as before we see that there must be a point $x_{\epsilon_1} \in \partial B_{\epsilon_1}(x)$ such that

$$dist(x, q) = dist(x, x_{\epsilon_1}) + dist(x_{\epsilon_1}, q) = \epsilon_1 + dist(x_{\epsilon_1}, q).$$

Now let γ_1 be the unit speed geodesic such that $\gamma_1(0) = x$ and $\gamma_1(\epsilon_1) = x_{\epsilon_1}$. Combining, we now have

$$dist(p, q) = dist(p, x) + dist(x, x_{\epsilon_1}) + dist(x_{\epsilon_1}, q).$$

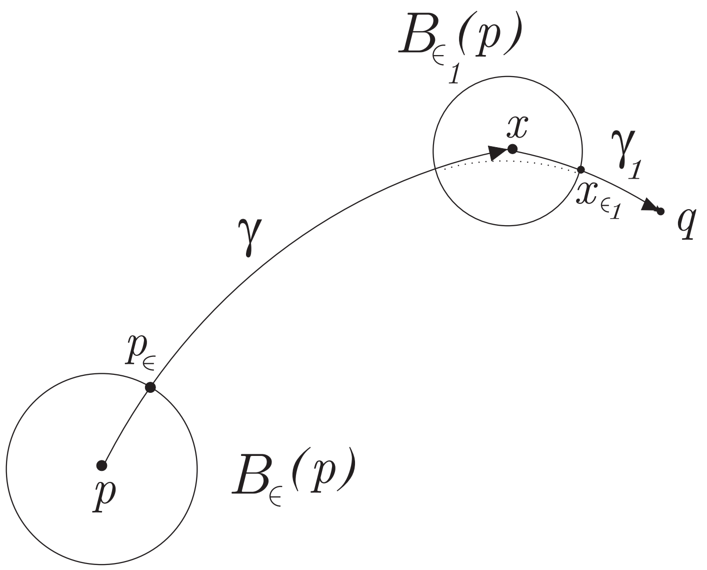
By the triangle inequality $dist(p, q) \leq dist(p, x_{\epsilon_1}) + dist(x_{\epsilon_1}, q)$ and so

$$dist(p, x) + dist(x, x_{\epsilon_1}) \leq dist(p, x_{\epsilon_1}).$$

But also $dist(p, x_{\epsilon_1}) \leq dist(p, x) + dist(x, x_{\epsilon_1})$ and so

$$dist(p, x_{\epsilon_1}) = dist(p, x) + dist(x, x_{\epsilon_1})$$

Now examining the implications of this last equality we see that the concatenation of $\gamma|_{[0,t_{\text{sup}}]}$ with γ_1 forms a curve from p to x_{ϵ_1} of length $dist(p, x_{\epsilon_1})$ which must therefore be a minimizing curve. By exercise 16.22 below, this potentially broken geodesic must in fact be unbroken and so must actually be the geodesic $\gamma|_{[0,t_{\text{sup}}+\epsilon_1]}$ and so $t_{\text{sup}} + \epsilon_1 \in T$ which contradicts the definition of t_{sup}. This contradiction forces us to conclude that $t_{\max} = R$ and we are done. ∎



Exercise 16.22 *Show that a piecewise smooth curve connecting two points p_0 and p_1 in a Riemannian manifold must be smooth if it is length minimizing. Hint: Suppose the curve has a corner and look in a small normal neighborhood of the corner. Show that the curves can be shortened by rounding off the corner.*

Theorem 16.21 (Hopf-Rinow) *If M, g is a connected Riemannian manifold then the following statements are equivalent:*

(i) The metric space M, dist is complete. That is every Cauchy sequence is convergent.

(ii) There is a point $p \in M$ such that M is geodesically complete at p.

(iii) M is geodesically complete.

(iv) Every closed and bounded subset of M is compact.

Proof. (i)⇔(iv) is the famous Heine-Borel theorem and we shall not reproduce the proof here.

(i)⇒(iii): Let p be arbitrary and let $\gamma_v(t)$ be the geodesic with $\dot{\gamma}_v(0) = v$ and J its maximal domain of definition. We can assume without loss that $\langle v, v\rangle = 1$ so that $L(\gamma_v|_{[t_1,t_2]}) = t_2 - t_1$ for all relevant t_1, t_2. We want to show that there can be no upper bound for the set J. We argue by contradiction: Assume that $t_+ = \sup J$ is finite. Let $\{t_n\} \subset J$ be a Cauchy sequence such that $t_n \to t_+ < \infty$. Since $dist(\gamma_v(t), \gamma_v(s)) \leq |t - s|$ it follows that $\gamma_v(t_n)$ is a Cauchy sequence in M which by assumption must converge. Let $q := \lim_{n\to\infty} \gamma_v(t_n)$ and choose a small ball $B_\epsilon(q)$ which is small enough to be a normal neighborhood. Take t_1 with $0 < t_+ - t_1 < \epsilon/2$ and let γ_1 be the (maximal) geodesic with initial velocity $\dot{\gamma}_v(t_1)$. Then in fact $\gamma_1(t) = \gamma_v(t_1 + t)$ and so γ_1 is defined $t_1 + \epsilon/2 > t_+$ and this is a contradiction.

(iii)⇒(ii) is a trivial implication.

(ii)⇒(i): Suppose M is geodesically complete at p. Now let $\{x_n\}$ be any Cauchy sequence in M. For each x_n there is (by assumption) a minimizing geodesic from p to x_n which we denote by γ_{px_n}. We may assume that each γ_{px_n} is unit speed. It is easy to see that the sequence $\{l_n\}$, where $l_n := L(\gamma_{px_n}) = dist(p, x_n)$, is a Cauchy sequence in $\mathbb{R}$ with some limit, say l. The key fact is that the vectors $\dot{\gamma}_{px_n}$ are all unit vectors in T_pM and so form a sequence in the (compact) unit sphere in T_pM. Replacing$\{\dot{\gamma}_{px_n}\}$ by a subsequence if necessary we have $\dot{\gamma}_{px_n} \to u \in T_pM$ for some unit vector u. Continuous dependence on initial velocities implies that $\{x_n\} = \{\gamma_{px_n}(l_n)\}$ has the limit $\gamma_u(l)$. ■

If M, g is a complete connected Riemannian manifold with sectional curvature $K \leq 0$ then for each point $p \in M$ the geodesics emanating from p have no conjugate points and so $T_{v_p} \exp_p : T_{v_p}T_pM \to M$ is nonsingular for each $v_p \in \mathrm{dom}(\exp_p) \subset T_pM$. This means that $\exp_p$ is a local diffeomorphism. If we give T_pM the metric $\exp_p^*(\mathrm{g})$ then $\exp_p$ is a local isometry. Also, since the rays $t \mapsto tv$ in T_pM map to geodesics we see that M is complete at p and then by the Hopf-Rinow theorem M is complete. It now follows from theorem 16.10 that $\exp_p : T_pM \to M$ is a Riemannian covering. Thus we arrive at the Hadamard theorem

Theorem 16.22 (Hadamard) *If M, g is a complete* simply connected *Riemannian manifold with sectional curvature $K \leq 0$ then $\exp_p : T_pM \to M$ is a diffeomorphism and each two points of M can be connected by unique geodesic segment.*

Definition 16.40 *If M, g is a Riemannian manifold then the diameter of M is defined to be*

$$\mathrm{diam}(M) := \sup\{\mathrm{dist}(p, q) : p, q \in M\}$$

The injectivity radius at $p \in M$, denoted $inj(p)$, is the supremum over all $\epsilon > 0$ such that $\exp_p : \widetilde{B}(0_p, \epsilon) \to B(p, \epsilon)$ is a diffeomorphism. The injectivity radius of M is $\mathrm{inj}(M) := \inf_{p\in M}\{\mathrm{inj}(p)\}$.

The Hadamard theorem above has as a hypothesis that the sectional curvature is nonpositive. A bound on the sectional curvature is stronger that a bound on Ricci curvature since the latter is a sort of average sectional curvature. In the sequel, statement like $Ric \geq C$ should be interpreted to mean $Ric(v,v) \geq C\langle v,v\rangle$ for all $v \in TM$.

Lemma 16.11 *Let M, g be an n-dimensional Riemannian manifold and Let $\gamma : [0,L] \to M$ be a unit speed geodesic. Suppose that $Ric \geq (n-1)\,\kappa > 0$ for some constant $\kappa > 0$ (at least along γ). If the length L of γ is greater than or equal to $\pi/\sqrt{\kappa}$ then there is a point conjugate to $\gamma\,(0)$ along γ.*

Proof. Suppose $b = \pi/\sqrt{\kappa}$ for $0 < b \leq L$. Letting $\varepsilon = \langle\dot{\gamma},\dot{\gamma}\rangle$, if we can show that $\varepsilon I_{\perp}$ is not positive definite then 15.6 implies the result. To show that $I_{\perp}$ is not positive definite we find an appropriate vector field $V \neq 0$ along γ such that $I(V,V) \leq 0$. Choose orthonormal fields $E_2,...,E_n$ so that $\dot{\gamma}, E_2,...,E_n$ is an orthonormal frame along γ. Now for a function $f : [0,\pi/\sqrt{\kappa}] \to \mathbb{R}$ which vanishes we form the fields fE_i using 16.10 we have

$$I(fE_j, fE_j) = \int_0^{\pi/\sqrt{\kappa}} \left\{ f'(s)^2 + f(s)^2 \langle R_{E_j,\dot{\gamma}}(E_j(s)), \dot{\gamma}(s)\rangle \right\} ds$$

and then

$$\begin{aligned}\sum_{j=2}^{n} I(fE_j, fE_j) &= \int_0^{\pi/\sqrt{\kappa}} \left\{ (n-1)f'^2 - f^2 Ric(\dot{\gamma},\dot{\gamma}) \right\} ds \\ &\leq (n-1)\int_0^{\pi/\sqrt{\kappa}} \left(f'^2 - \kappa f^2 \right) ds\end{aligned}$$

Letting $f(s) = \sin(\sqrt{\kappa}s)$ we get

$$\sum_{j=2}^{n} I(fE_j, fE_j) \leq (n-1)\int_0^{\pi/\sqrt{\kappa}} \kappa \left(\cos^2(\sqrt{\kappa}s) - \sin(\sqrt{\kappa}s) \right) ds = 0$$

and so $I(fE_j, fE_j) \leq 0$ for some j. ■

The next theorem also assumes only a bound on the Ricci curvature and is one of the most celebrated theorems of Riemannian geometry.

Theorem 16.23 (Myers) *Let M, g be a complete Riemannian manifold of dimension n. If $Ric \geq (n-1)\,\kappa > 0$ then*

(i) $\mathrm{diam}(M) \leq \pi/\sqrt{\kappa}$, *$M$ is compact and*

(ii) $\pi_1(M)$ is finite.

Proof. Since M complete there is always a shortest geodesic γ_{pq} between any two given points p and q. We can assume that γ_{pq} is parameterized by arc length:

$$\gamma_{pq} : [0, dist(p,q)] \to M$$

It follows that $\gamma_{pq}|_{[0,a]}$ is arc length minimizing for all $a \in [0, dist(p,q)]$. From 16.11we see that the only possible conjugate to p along γ_{pq} is q. From preceding lemma we see that $\pi/\sqrt{\kappa} > 0$ is impossible.

Since the point p and q were arbitrary we must have $\operatorname{diam}(M) \leq \pi/\sqrt{\kappa}$. It follows from the Hopf-Rinow theorem that M is compact.

For (ii) we consider the simply connected covering $\wp : \widetilde{M} \to M$ (which is a local isometry). Since $\widetilde{M}$ is also complete, and has the same Ricci curvature bound as M so $\widetilde{M}$ is also compact. It follows easily that $\wp^{-1}(p)$ is finite for any $p \in M$ from which (ii) follows. ■

16.11 Cut Locus

Related to the notion of conjugate point is the notion of a cut point. For a point $p \in M$ and a geodesic γ emanating from $p = \gamma(0)$, a **cut point of** p along γ is the first point $q = \gamma(t')$ along γ such that for any point $r = \gamma(t'')$ beyond p, (i.e. $t'' > t'$) there is a geodesic shorter that $\gamma|_{[0,t']}$ which connects p with r. To see the difference between this notion and that of a point conjugate to p it suffices to consider the example of a cylinder $S^1 \times \mathbb{R}$ with the obvious flat metric. If $p = (e^{i\theta}, 0) \in S^1 \times \mathbb{R}$ then for any$x \in \mathbb{R}$, the point $(e^{i(\theta+\pi)}, x)$ is a cut point of p along the geodesic $\gamma(t) := ((e^{i(\theta+t\pi)}, x))$. We know that beyond a conjugate point, a geodesic is not (locally) minimizing but the last example shows that a cut point need not be a conjugate point. In fact, $S^1 \times \mathbb{R}$ has no conjugate points along any geodesic. Let us agree that all geodesics referred to in this section are parameterized by arc length unless otherwise indicated.

Definition 16.41 *Let M, g be a complete Riemannian manifold and let $p \in M$. The set $C(p)$ of all cut points to p along geodesics emanating from p is called the **cut locus** of p.*

For a point $p \in M$, the situations is summarized by the fact that if $q = \gamma(t')$ is a cut point of p along a geodesic γ then for any $t'' > t'$ there is a geodesic connecting p with q which is shorter than $\gamma|_{[0,t']}$ while if $t'' < t'$ then not only is there geodesic connecting p and $\gamma(t'')$ with shorter length but there is no geodesic connecting p and $\gamma(t'')$ whose length is even equal to that of $\gamma|_{[0,t'']}$.

Consider the following two conditions:

a $\gamma(t_0)$ is the first conjugate point of $p = \gamma(0)$ along γ.

b There is a geodesic α different from $\gamma|_{[0,t_0]}$ such that $L(\alpha) = L(\gamma|_{[0,t_0]})$.

Proposition 16.14 *Let M be a complete Riemannian manifold.*
(i) If for a given unit speed geodesic γ, either condition (a) or (b) holds, then there is a $t_1 \in (0, t_0]$ such that $\gamma(t_1)$ is the cut point of p along γ.

(ii) If $\gamma(t_0)$ is the cut point of $p = \gamma(0)$ along the unit speed geodesic ray γ. Then either condition (a) or (b) holds.

Proof. (i) This is already clear from our discussion: for suppose (a) holds, then $\gamma|_{[0,t']}$ cannot minimize for $t' > t_0$ and so the cut point must be $\gamma(t_1)$ for some $t_1 \in (0, t_0]$. Now if (b) hold then choose $\epsilon > 0$ small enough that $\alpha(t_0 - \epsilon)$ and $\gamma(t_0 + \epsilon)$ are both contained in a convex neighborhood of $\gamma(t_0)$. The concatenation of $\alpha|_{[0,t_0]}$ and $\gamma|_{[t_0,t_0+\epsilon]}$ is a curve c that has a kink at $\gamma(t_0)$. But there is a unique minimizing geodesic τ joining $\alpha(t_0 - \epsilon)$ to $\gamma(t_0 + \epsilon)$ and we can concatenate the geodesic $\alpha|_{[0,t_0-\epsilon]}$ with τ to get a curve with arc length strictly less than $L(c) = t_0 + \epsilon$. It follows that the cut point to p along γ must occur at $\gamma(t')$ for some $t' \leq t_0 + \epsilon$. But ϵ can be taken arbitrarily small and so the result (i) follows.

Now suppose that $\gamma(t_0)$ is the cut point of $p = \gamma(0)$ along a unit speed geodesic ray γ. We let $\epsilon_i \to 0$ and consider a sequence $\{\alpha_i\}$ of minimizing geodesics with α_i connecting p to $\gamma(t_0 + \epsilon_i)$. We have a corresponding sequence of initial velocities $u_i := \dot{\alpha}_i(0) \in S^1 \subset T_pM$. The unit sphere in T_pM is compact so replacing u_i by a subsequence we may assume that $u_i \to u \in S^1 \subset T_pM$. Let α be the unit speed segment joining p to $\gamma(t_0 + \epsilon_i)$ with initial velocity u. Arguing from continuity, we see that α is also a minimizing and $L(\alpha) = L(\gamma|_{[0,t_0+\epsilon]})$. If $\alpha \neq \gamma|_{[0,t_0+\epsilon]}$ then we are done. If $\alpha = \gamma|_{[0,t_0]}$ then since $\gamma|_{[0,t_0+\epsilon]}$ is minimizing it will suffice to show that $T_{t_0\dot{\gamma}(0)} \exp_p$ is singular since that would imply that condition (a) holds. The proof of this last statement is by contradiction: Suppose that $\alpha = \gamma|_{[0,t_0]}$ (so that $\dot{\gamma}(0) = u$) and that $T_{t_0\dot{\gamma}(0)} \exp_p$ is not singular. Take U to be an open neighborhood of $t_0\dot{\gamma}(0)$ in T_pM such that $\exp_p\big|_U$ is a diffeomorphism. Now $\alpha_i(t_0 + \epsilon_i') = \gamma(t_0 + \epsilon_i)$ for $0 < \epsilon_i' \leq \epsilon_i$ since the α_i are minimizing. We now restrict attention to i such that ϵ_i is small enough that $(t_0 + \epsilon_i')u_i$ and $(t_0 + \epsilon_i)u$ are in U. Then we have

$$\begin{aligned}\exp_p(t_0 + \epsilon_i)u &= \gamma(t_0 + \epsilon_i) = \\ &\alpha_i(t_0 + \epsilon_i') = \exp_p(t_0 + \epsilon_i')u_i\end{aligned}$$

and so $(t_0 + \epsilon_i)u = (t_0 + \epsilon_i')u_i$ and then since $\epsilon_i \to 0$ we have $\dot{\gamma}(0) = u = u_i$ for sufficiently large i. But then $\alpha_i = \gamma$ on $[0, t_0]$ which contradicts the fact that $\gamma|_{[0,t_0]}$ is not minimizing. ∎

Exercise 16.23 *Show that if q is the cut point of p along γ then p is the cut point of q along $\gamma^{\leftarrow}$ (where $\gamma^{\leftarrow}(t) := \gamma(L - t)$ and $L = L(\gamma)$).*

It follows from the development so far that if $q \in M \backslash C(p)$ then there is a unique minimizing geodesic joining p to q and that if $B(p, R)$ is the ball of radius R centered at p then $\exp_p$ is a diffeomorphism on $B(p, R)$ if $R \leq d(p, C(p))$. In fact, an alternative definition of the injectivity radius at p is $d(p, C(p))$ and the injectivity radius of M is

$$inj(M) = \inf_{p \in M} \{d(p, C(p))\}$$

Intuitively, the complexities of the topology of M begin at the cut locus of a given point.

Let T_1M denote the unit tangent bundle of the Riemannian manifold:

$$T_1M = \{u \in TM : \|u\| = 1\}.$$

Define a function $c_M : T_1M \to (0, \infty]$ by

$$c_M(u) := \begin{cases} t_0 & \text{if } \gamma_u(t_0) \text{ is the cut point of } \pi_{TM}(u) \text{ along } \gamma_u \\ \infty & \text{if there is no cut point in the direction } u \end{cases}$$

Recall that the topology on $(0, \infty]$ is such that a sequence t_k converges to the point ∞ if $\lim_{k\to\infty} t_k = \infty$ in the usual sense. We now have

Theorem 16.24 *If M, g is complete Riemannian manifold then the function $c_M : T_1M \to (0, \infty]$ is continuous.*

Proof. If u_i is a sequence in T_1M converging to $u \in T_1M$ then $\pi_{TM}(u_i) = p_i$ converges to $p = \pi_{TM}(u)$. Let u_i be such a sequence and let γ_i be the unit speed geodesic connecting p_i to the corresponding cut point $\gamma_i(t_{0i})$ in the direction u_i and where $t_{0i} = \infty$ if there is no such cut point (in this case just let $\gamma_i(t_{0i})$ be arbitrary). Note that $u_i = \dot{\gamma}_i(0)$. Also let γ be the geodesic with initial velocity u. Let $t_0 \in (0, \infty]$ be the distance to the cut point in the direction u. Our task is to show that t_{0i} converges to t_0.

Claim 1: $\limsup t_{0i} \leq t_0$. If $t_0 = \infty$ then this claim is trivially true so assume that $t_0 < \infty$. Given any $\epsilon > 0$ there is only a finite number of i such that $t_0 + \epsilon < t_{0i}$ for otherwise we would have a sequence i_k such that $d(p_{i_k}, \gamma_{i_k}(t_0 + \epsilon)) = t_0 + \epsilon$ which would give $d(p, \gamma(t_0 + \epsilon)) = t_0 + \epsilon$. But this last equality contradicts the fact that $\gamma(t_0)$ is the cut point of p along γ. Thus we must have $\limsup t_{0i} \leq t_0 + \epsilon$ and since ϵ was arbitrarily small we deduce the truth of the claim.

Claim 2: $\liminf t_{0i} \geq t$. The theorem is proved once we prove this claim. For the proof of this claim we suppose that $\liminf t_{0i} < \infty$ since otherwise there is nothing to prove. Once again let t_{0i} be a sequence which (after a reduction to a subsequence we may assume) converges to $t_{\inf} := \liminf t_{0i}$. The reader may easily prove that if (after another reduction to a subsequence) the points $\gamma_i(t_{0i})$ are conjugate to p_i along γ_i then $\gamma(t_{\inf})$ is conjugate to p along γ. If this is the case then $t_{\inf} := \liminf t_{0i} \geq t_0$ and the claim is true. Suppose therefore that there is a sequence of indices i_k so that $t_{0i_k} \to t_{\inf}$ and such that $\gamma_{i_k}(t_{0i_k})$ is *not* conjugate to p_i along γ_{i_k}. In this case there must be a (sub)sequence of geodesics α_i different from γ_i such that $\alpha_i(0) = \gamma_i(0) = p_i$, $\alpha_{i_k}(t_{0i_k}) = \gamma_{i_k}(t_{0i_k})$ and $L(\alpha_{i_k}) = L(\gamma_{i_k})$. After another reduction to a subsequence we may assume that $\dot{\alpha}_i(0) \to v \in T_1M$ and that a geodesic α in with $\dot{\alpha}(0) = v$ connects p to $\gamma(t_{\inf})$. If α is different than γ then by 16.14 $t_0 \leq t_{\inf}$ and we are done. If, on the other hand, $\alpha = \gamma$ then an argument along the lines of proposition 16.14(see exercise 16.24 below) $\gamma(t_{\inf})$ is conjugate to p along γ which again implies $t_0 \leq t_{\inf}$. ∎

Exercise 16.24 *Fill in the final details of the proof of theorem 16.24.*

Chapter 17

Geometry of Submanifolds

17.1 Definitions

Let M be a d dimensional submanifold of a semi-Riemannian manifold $\overline{M}$ of dimension n where $d < n$. The metric $\mathrm{g}(.,.) = \langle .,. \rangle$ on $\overline{M}$ restricts to tensor on M which we denote by h. Since h is a restriction of g we shall also use the notation $\langle .,. \rangle$ for h. If the restriction h of is nondegenerate on each space T_pM then h is a metric tensor on M and we say that M is a semi-Riemannian submanifold of $\bar{M}$. If $\bar{M}$ is Riemannian then this nondegeneracy condition is automatic and the metric h is automatically Riemannian. More generally, if $\phi : M \to \bar{M}, \mathrm{g}$ is an immersion we can consider the pull-back tensor $\phi^*\mathrm{g}$ defined by

$$\phi^*\mathrm{g}(X, Y) = \mathrm{g}(T\phi \cdot X, T\phi \cdot Y).$$

If $\phi^*\mathrm{g}$ is nondegenerate on each tangent space then it is a metric on M called the pull-back metric and we call ϕ a semi-Riemannian immersion. If M is already endowed with a metric g_M then if $\phi^*\mathrm{g} = \mathrm{g}_M$ then we say that $\phi : M, \mathrm{g}_M \to \bar{M}, g$ is an isometric immersion. Of course, if $\phi^*\mathrm{g}$ is a metric at all, as it always is if $\bar{M}, g$ is Riemannian, then the map $\phi : M, \phi^*\mathrm{g} \to \bar{M}, \mathrm{g}$ is an isometric immersion. Since every immersion restricts locally to an embedding we may, for many purposes, assume that M is a submanifold and that ϕ is just the inclusion map.

Definition 17.1 *Let $\overline{M}, \mathrm{g}$ be a Lorentz manifold. A submanifold M is said to be spacelike (resp. timelike, lightlike) if $T_pM \subset T_p\overline{M}$ is spacelike (resp. timelike, lightlike).*

There is an obvious bundle on M which is the restriction of $T\overline{M}$ to M. This is the bundle $T\overline{M}\big|_M = \bigsqcup_{p \in M} T_p\overline{M}$. Each tangent space $T_p\overline{M}$ decomposes as

$$T_p\overline{M} = T_pM \oplus (T_pM)^\perp$$

where $(T_pM)^\perp = \{v \in T_p\overline{M} : \langle v, w \rangle = 0 \text{ for all } w \in T_pM\}$. Then $TM^\perp = \bigsqcup_p (T_pM)^\perp$ its natural structure as a smooth vector bundle called the **normal**

bundle to M in $\overline{M}$. The smooth sections of the normal bundle will be denoted by $\Gamma\left(TM^{\perp}\right)$ or $\mathfrak{X}(M)^{\perp}$. Now the orthogonal decomposition above is globalizes as

$$T\overline{M}\big|_{M} = TM \oplus TM^{\perp}$$

A vector field on M is always the restriction of some (not unique) vector field on a neighborhood of $\overline{M}$. The same is true of any not necessarily tangent vector field along M. The set of all vector fields along M will be denoted by $\mathfrak{X}(\overline{M})\big|_{M}$. Since any function on M is also the restriction of some function on $\overline{M}$ we may consider $\mathfrak{X}(M)$ as a submodule of $\mathfrak{X}(\overline{M})\big|_{M}$. If $\overline{X} \in \mathfrak{X}(\overline{M})$ then we denote its restriction to M by $\overline{X}\big|_{M}$ or sometimes just X. Notice that $\mathfrak{X}(M)^{\perp}$ is a submodule of $\mathfrak{X}(\overline{M})\big|_{M}$. We have two projection maps : $T_p\overline{M} \to N_pM$ and tan : $T_p\overline{M} \to T_pM$ which in turn give module projections nor : $\mathfrak{X}(\overline{M})\big|_{M} \to \mathfrak{X}(M)^{\perp}$ and : $\mathfrak{X}(\overline{M})\big|_{M} \to \mathfrak{X}(M)$. The reader should contemplate the following diagrams:

$$\begin{array}{ccccc} C^{\infty}(\overline{M}) & \overset{restr}{\to} & C^{\infty}(M) & = & C^{\infty}(M) \\ \times\downarrow & & \times\downarrow & & \times\downarrow \\ \mathfrak{X}(\overline{M}) & \overset{restr}{\to} & \mathfrak{X}(\overline{M})\big|_{M} & \overset{\tan}{\to} & \mathfrak{X}(M) \end{array}$$

$$\text{and} \quad \begin{array}{ccccc} C^{\infty}(\overline{M}) & \overset{restr}{\to} & C^{\infty}(M) & = & C^{\infty}(M) \\ \times\downarrow & & \times\downarrow & & \times\downarrow \\ \mathfrak{X}(\overline{M}) & \overset{restr}{\to} & \mathfrak{X}(\overline{M})\big|_{M} & \overset{\text{nor}}{\to} & \mathfrak{X}(M)^{\perp} \end{array}.$$

Now we also have an exact sequence of modules

$$0 \to \mathfrak{X}(M)^{\perp} \to \mathfrak{X}(\overline{M})\big|_{M} \overset{\tan}{\to} \mathfrak{X}(M) \to 0$$

which is in fact a split exact sequence since we also have

$$0 \longleftarrow \mathfrak{X}(M)^{\perp} \overset{\text{nor}}{\longleftarrow} \mathfrak{X}(\overline{M})\big|_{M} \longleftarrow \mathfrak{X}(M) \longleftarrow 0$$

The extension map $\mathfrak{X}(\overline{M})\big|_{M} \longleftarrow \mathfrak{X}(M)$ is not canonical but in the presence of a connection it is almost so: If $U_\epsilon(M)$ is the open tubular neighborhood of M given, for sufficiently small ϵ by

$$U_\epsilon(M) = \{p \in \overline{M} : \text{dist}(p, M) < \epsilon\}$$

then we can use the following trick to extend any $X \in \mathfrak{X}(M)$ to $\mathfrak{X}(U_\epsilon(M))|_M$. First choose a smooth frame field $E_1, ..., E_n$ defined along M so that $E_i \in \mathfrak{X}(\overline{M})\big|_{M}$. We may arrange if need to have the first d of these tangent to M. Now parallel translate each frame radially outward a distance ϵ to obtain a smooth frame field $\overline{E}_1, ..., \overline{E}_n$ on $U_\epsilon(M)$.

Now we shall obtain a sort of splitting of the Levi-Civita connection of $\overline{M}$ along the submanifold M. The reader should recognize a familiar theme here especially if elementary surface theory is fresh in his or her mind. First we notice that the Levi-Civita connection $\bar{\nabla}$ on $\overline{M}$ restrict nicely to a connection on the bundle $T\overline{M}_M \to M$. The reader should be sure to realize that the space of sections of this bundle is exactly $\mathfrak{X}(\overline{M})\big|_M$ and so the restricted connection is a map $\bar{\nabla}\big|_M : \mathfrak{X}(M) \times \mathfrak{X}(\overline{M})\big|_M \to \mathfrak{X}(\overline{M})\big|_M$. The point is that if $X \in \mathfrak{X}(M)$ and $W \in \mathfrak{X}(\overline{M})\big|_M$ then $\bar{\nabla}_X W$ doesn't seem to be defined since X and W are not elements of $\mathfrak{X}(\overline{M})$. But we may extend X and W to elements of $\mathfrak{X}(\overline{M})$ and then restrict again to get an element of $\mathfrak{X}(\overline{M})\big|_M$.Then recalling the local properties of a connection we see that the result does not depend on the extension.

Exercise 17.1 *Use local coordinates to prove the claimed independence on the extension.*

We shall write simply $\bar{\nabla}$ in place of $\bar{\nabla}\big|_M$ since the context make it clear when the later is meant. Thus $\bar{\nabla}_X W := \bar{\nabla}_{\bar{X}} \bar{W}$ where $\bar{X}$ and $\bar{W}$ are any extensions of X and W respectively.

Clearly we have $\bar{\nabla}_X \langle Y_1, Y_2 \rangle = \langle \bar{\nabla}_X Y_1, Y_2 \rangle + \langle Y_1, \bar{\nabla}_X Y_2 \rangle$ and so $\bar{\nabla}$ is a metric connection on $T\overline{M}\big|_M$. For a fixed $X, Y \in \mathfrak{X}(M)$ we have the decomposition of $\bar{\nabla}_X Y$ into tangent and normal parts. Similarly, for $V \in \mathfrak{X}(M)^\perp$ we can consider the decomposition of $\bar{\nabla}_X V$ into tangent and normal parts. Thus we have

$$
\begin{aligned}
\bar{\nabla}_X Y &= (\bar{\nabla}_X Y)^{\tan} + (\bar{\nabla}_X Y)^\perp \\
\bar{\nabla}_X V &= (\bar{\nabla}_X V)^{\tan} + (\bar{\nabla}_X V)^\perp
\end{aligned}
$$

We make the following definitions:

$$
\begin{aligned}
\nabla_X Y &:= (\bar{\nabla}_X Y)^{\tan} \text{ for all } X, Y \in \mathfrak{X}(M) \\
b_{12}(X, Y) &:= (\bar{\nabla}_X Y)^\perp \text{ for all } X, Y \in \mathfrak{X}(M) \\
b_{21}(X, V) &:= (\bar{\nabla}_X V)^{\tan} \text{ for all } X \in \mathfrak{X}(M),\ V \in \mathfrak{X}(M)^\perp \\
\nabla_X^\perp V &:= (\bar{\nabla}_X V)^\perp \text{ for all } X \in \mathfrak{X}(M),\ V \in \mathfrak{X}(M)^\perp
\end{aligned}
$$

Now if $X, Y \in \mathfrak{X}(M)$, $V \in \mathfrak{X}(M)^\perp$ then $0 = \langle Y, V \rangle$ and so

$$
\begin{aligned}
0 &= \bar{\nabla}_X \langle Y, V \rangle \\
&= \langle \bar{\nabla}_X Y, V \rangle + \langle Y, \bar{\nabla}_X V \rangle \\
&= \langle \left(\bar{\nabla}_X Y\right)^\perp, V \rangle + \langle Y, \left(\bar{\nabla}_X V\right)^{\tan} \rangle \\
&= \langle b_{12}(X, Y), V \rangle + \langle Y, b_{21}(X, V) \rangle.
\end{aligned}
$$

It follows that $\langle b_{12}(X, Y), V \rangle = -\langle Y, b_{21}(X, V) \rangle$. Now we from this that $b_{12}(X, Y)$ is not only $C^\infty(M)$ linear in X but also in Y. Thus b_{12} is tensorial and so for each fixed $p \in M$, $b_{12}(X_p, Y_p)$ is a well defined element of $T_p M^\perp$ for each fixed $X_p, Y_p \in T_p M$. Also, for any $X_1, X_2 \in \mathfrak{X}(M)$ we have

$$
\begin{aligned}
& b_{12}(X_1, X_2) - b_{12}(X_2, X_1) \\
&= \left(\bar{\nabla}_{X_1} X_2 - \bar{\nabla}_{X_2} X_1\right)^\perp \\
&= ([X_1, X_2])^\perp = 0.
\end{aligned}
$$

So b_{12} is symmetric. The classical notation for b_{12} is II and the form is called the **second fundamental tensor** or the **second fundamental form**. For $\xi \in T_pM$ we define the linear map $B_\xi(\cdot) := b_{12}(\xi, \cdot)$. With this in mind we can easily deduce the following facts which we list as a theorem:

1. $\nabla_X Y := (\bar{\nabla}_X Y)^{\tan}$ defines a connection on M which is identical with the Levi-Civita connection for the induced metric on M.
2. $(\bar{\nabla}_X V)^{\perp} := \nabla_X^{\perp} V$ defines a metric connection on the vector bundle $TM^{\perp}$.
3. $(\bar{\nabla}_X Y)^{\perp} := b_{12}(X, Y)$ defines a symmetric $C^\infty(M)$-bilinear form with values in $\mathfrak{X}(M)^{\perp}$.
4. $(\bar{\nabla}_X V)^{\tan} := b_{21}(X, V)$ defines a symmetric $C^\infty(M)$-bilinear form with values in $\mathfrak{X}(M)$.

Corollary 17.1 *b_{21} is tensorial and so for each fixed $p \in M$, $b_{21}(X_p, Y_p)$ is a well defined element of $T_pM^{\perp}$ for each fixed $X_p, Y_p \in T_pM$ and we have a bilinear form $b_{21} : T_pM \times T_pM^{\perp} \to T_pM$.*

Corollary 17.2 *The map $b_{21}(\xi, \cdot) : T_pM^{\perp} \to T_pM$ is equal to $-B_\xi^t : T_pM \to T_pM^{\perp}$.*

Writing any $Y \in \mathfrak{X}(\overline{M})\big|_M$ as $Y = (Y^{\tan}, Y^{\perp})$ we can write the map $\bar{\nabla}_X : \mathfrak{X}(\overline{M})\big|_M \to \mathfrak{X}(\overline{M})\big|_M$ as a matrix of operators:

$$\begin{bmatrix} \nabla_X & B_X \\ -B_X^t & \nabla_X^{\perp} \end{bmatrix}$$

Next we define the shape operator which is also called the Weingarten map. There is not a perfect agreement on sign conventions; some author's shape operator is the negative of the shape operator as defined by other others. We shall define S^+ and S^- which only differ in sign. This way we can handle both conventions simultaneously and the reader can see where the sign convention does or does not make a difference in any given formula.

Definition 17.2 *Let $p \in M$. For each unit vector u normal to M at p we have a map called the ($\pm$)* ***shape operator*** *$S_u^{\pm}$ associated to u. defined by $S_u^{\pm}(v) := \left(\pm\bar{\nabla}_v U\right)^{\tan}$ where U is any unit normal field defined near p such that $U(p) = u$.*

The shape operators $\{S_u^{\pm}\}_{u \text{ a unit normal}}$ contain essentially the same information as the second fundamental tensor $II = b_{12}$. This is because for any $X, Y \in \mathfrak{X}(M)$ and $U \in \mathfrak{X}(M)^{\perp}$ we have

$$\begin{aligned} \langle S_U^{\pm} X, Y\rangle &= \langle \left(\pm\bar{\nabla}_X U\right)^{\tan}, Y\rangle = \langle U, \pm\bar{\nabla}_X Y\rangle \\ &= \langle U, \left(\pm\bar{\nabla}_X Y\right)^{\perp}\rangle = \langle U, \pm b_{12}(X, Y)\rangle \\ &= \langle U, \pm II(X, Y)\rangle. \end{aligned}$$

Note: $\langle U, b_{12}(X, Y)\rangle$ is tensorial in U, X and Y. Of course , $S_{-U}^{\pm} X = -S_U^{\pm} X$.

Theorem 17.1 *Let M be a semi-Riemannian submanifold of $\bar{M}$. We have the* ***Gauss equation***

$$\begin{aligned}\langle R_{VW}X,Y\rangle &= \langle \bar{R}_{VW}X,Y\rangle \\ &\quad - \langle II(V,X), II(W,Y)\rangle + \langle II(V,Y), II(W,X)\rangle\end{aligned}$$

Proof. Since this is clearly a tensor equation we may assume that $[V,W]=0$. With this assumption we have we have $\langle \bar{R}_{VW}X,Y\rangle = (VW)-(WV)$ where $(VW)=\langle \bar{\nabla}_V\bar{\nabla}_W X,Y\rangle$

$$\begin{aligned}\langle \bar{\nabla}_V\bar{\nabla}_W X,Y\rangle &= \langle \bar{\nabla}_V\nabla_W X,Y\rangle + \langle \bar{\nabla}_V(II(W,X)),Y\rangle \\ &= \langle \nabla_V\nabla_W X,Y\rangle + \langle \bar{\nabla}_V(II(W,X)),Y\rangle \\ &= \langle \nabla_V\nabla_W X,Y\rangle + V\langle II(W,X),Y\rangle - \langle II(W,X),\bar{\nabla}_V Y\rangle \\ &= \langle \nabla_V\nabla_W X,Y\rangle + V\langle II(W,X),Y\rangle - \langle II(W,X),\bar{\nabla}_V Y\rangle\end{aligned}$$

Since

$$\begin{aligned}&\langle II(W,X),\bar{\nabla}_V Y\rangle \\ &= \langle II(W,X),\left(\bar{\nabla}_V Y\right)^{\perp}\rangle = \langle II(W,X), II(V,Y)\rangle\end{aligned}$$

we have $(VW)=\langle \nabla_V\nabla_W X,Y\rangle - \langle II(W,X),II(V,Y)\rangle$. Interchanging the roles of V and W and subtracting we get the desired conclusion. ■

Another formula that follows easily from the Gauss equation is the following formula (also called the Gauss formula):

$$K(v\wedge w) = \bar{K}(v\wedge w) + \frac{\langle II(v,v), II(w,w)\rangle - \langle II(v,w), II(v,w)\rangle}{\langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2}$$

Exercise 17.2 *Prove the last formula (second version of the Gauss equation).*

From this last version of the Gauss equation we can show that a sphere $S^n(r)$ of radius r in $\mathbb{R}^{n+1}$ has constant sectional curvature $1/r^2$ for $n>1$. If (u_i) is the standard coordinates on $\mathbb{R}^{n+1}$ then the position vector field in $\mathbb{R}^{n+1}$ is $\mathbf{r}=\sum_{i=1}^{n+1} u_i\partial_i$. Recalling that the standard (flat connection) D on $\mathbb{R}^{n+1}$ is just the Lie derivative we see that $D_X\mathbf{r}=\sum_{i=1}^{n+1} Xu_i\partial_i = X$. Now using the usual identifications, the unit vector field $U=\mathbf{r}/r$ is the outward unit vector field on $S^n(r)$. We have

$$\begin{aligned}\langle II(X,Y),U\rangle &= \langle D_X Y,U\rangle \\ &= \frac{1}{r}\langle D_X Y,\mathbf{r}\rangle = -\frac{1}{r}\langle Y,D_X\mathbf{r}\rangle \\ &= -\frac{1}{r}\langle Y,X\rangle = -\frac{1}{r}\langle X,Y\rangle.\end{aligned}$$

Now letting $\bar{M}$ be $\mathbb{R}^{n+1}$ and M be $S^n(r)$ and using the fact that the Gauss curvature of $\mathbb{R}^{n+1}$ is identically zero, the Gauss equation gives $K=1/r$.

The second fundamental form contains information about how the semi-Riemannian submanifold M bends about in $\bar{M}$. First we need a definition:

Definition 17.3 *Let M be semi-Riemannian submanifold of $\bar{M}$ and N a semi-Riemannian submanifold of $\bar{N}$. A pair isometry $\Phi : (\bar{M}, M) \to (\bar{N}, N)$ consists of an isometry $\Phi : \bar{M} \to \bar{N}$ such that $\Phi(M) = N$ and such that $\Phi|_M : M \to N$ is an isometry.*

Proposition 17.1 *A pair isometry $\Phi : (\bar{M}, M) \to (\bar{N}, N)$ preserves the second fundamental tensor:*

$$T_p\Phi \cdot II(v, w) = II(T_p\Phi \cdot v, T_p\Phi \cdot w)$$

for all $v, w \in T_pM$ and all $p \in M$.

Proof. Let $p \in M$ and extend $v, w \in T_pM$ to smooth vector fields V and W. Since an isometry respects the Levi-Civita connections we have $\Phi_*\bar{\nabla}_V W = \bar{\nabla}_{\Phi_* V}\Phi_* W$. Now since Φ is a pair isometry we have $T_p\Phi(T_pM) \subset T_{\Phi(p)}N$ and $T_p\Phi(T_pM^\perp) \subset \left(T_{\Phi(p)}N\right)^\perp$. This means that $\Phi_* : \mathfrak{X}(\bar{M})\big|_M \to \mathfrak{X}(\bar{N})\big|_N$ preserves normal and tangential components $\Phi_*(\mathfrak{X}(M)) \subset \mathfrak{X}(N)$ and $\Phi_*(\mathfrak{X}(M)^\perp) \subset \mathfrak{X}(N)^\perp$. We have

$$\begin{aligned}
T_p\Phi \cdot II(v, w) &= \Phi_* II(V, W)(\Phi(p)) \\
&= \Phi_* \left(\bar{\nabla}_V W\right)^\perp (\Phi(p)) \\
&= \left(\Phi_* \bar{\nabla}_V W\right)^\perp (\Phi(p)) \\
&= \left(\bar{\nabla}_{\Phi_* V}\Phi_* W\right)^\perp (\Phi(p)) \\
&= II(\Phi_* V, \Phi_* W)(\Phi(p)) \\
&= II(\Phi_* V, \Phi_* W)(\Phi(p)) \\
&= II(T_p\Phi \cdot v, T_p\Phi \cdot w)
\end{aligned}$$

■

The following example is simple but conceptually very important.

Example 17.1 *Let M be the strip 2 dimensional strip $\{(x, y, 0) : -\pi < x < \pi\}$ considered as submanifold of $\mathbb{R}^3$ (with the canonical Riemannian metric). Let N be the subset of $\mathbb{R}^3$ given by $\{(x, y, \sqrt{1 - x^2}) : -1 < x < 1\}$.* ***Exercise:*** *Show that M is isometric to M. Show that there is no pair isometry $(\mathbb{R}^3, M) \to (\mathbb{R}^3, N)$.*

17.2 Curves in Submanifolds

If $\gamma : I \to M$ is a curve in M and M is a semi-Riemannian submanifold of $\bar{M}$ then we have $\bar{\nabla}_{\partial_t} Y = \nabla_{\partial_t} Y + II(\dot{\gamma}, Y)$ for any vector field Y **along** γ. If Y is a vector field in $\mathfrak{X}(\bar{M})\big|_M$ or in $\mathfrak{X}(\bar{M})$ then $Y \circ \gamma$ is a vector field along γ. In this case we shall still write $\bar{\nabla}_{\partial_t} Y = \nabla_{\partial_t} Y + II(\dot{\gamma}, Y)$ rather than $\bar{\nabla}_{\partial_t}(Y \circ \gamma) = \nabla_{\partial_t}(Y \circ \gamma) + II(\dot{\gamma}, Y \circ \gamma)$.

Recall that $\dot{\gamma}$ is a vector field along γ. We also have $\ddot{\gamma} := \bar{\nabla}_{\partial_t}\dot{\gamma}$ which in this context will be called the **extrinsic acceleration** (or acceleration in $\bar{M}$. By definition we have $\nabla_{\partial_t} Y = \left(\bar{\nabla}_{\partial_t} Y\right)^{\perp}$. The **intrinsic acceleration** (acceleration in M) is $\nabla_{\partial_t}\dot{\gamma}$. Thus we have

$$\ddot{\gamma} = \nabla_{\partial_t}\dot{\gamma} + II(\dot{\gamma}, \dot{\gamma}).$$

From this definitions we can immediately see the truth of the following

Proposition 17.2 *If $\gamma : I \to M$ is a curve in M and M is a semi-Riemannian submanifold of $\bar{M}$ then γ is a geodesic in M if and only if $\ddot{\gamma}(t)$ is normal to M for every $t \in I$.*

Exercise 17.3 *A constant speed parameterization of a great circle in $S^n(r)$ is a geodesic. Every geodesic in $S^n(r)$ is of this form.*

Definition 17.4 *A semi-Riemannian manifold $M \subset \bar{M}$ is called totally geodesic if every geodesic in M is a geodesic in $\bar{M}$.*

Theorem 17.2 *For a semi-Riemannian manifold $M \subset \bar{M}$ the following conditions are equivalent*

i) M is totally geodesic

ii) $II \equiv 0$

iii) For all $v \in TM$ the $\bar{M}$ geodesic γ_v with initial velocity v is such that $\gamma_v[0, \epsilon] \subset M$ for $\epsilon > 0$ sufficiently small.

iv) For any curve $\alpha : I \to M$, parallel translation along α induced by $\bar{\nabla}$ in $\bar{M}$ is equal to parallel translation along α induced by ∇ in M.

Proof. (i)$\Longrightarrow$(iii) follows from the uniqueness of geodesics with a given initial velocity.

iii)$\Longrightarrow$(ii); Let $v \in TM$. Applying 17.2 to γ_v we see that $II(v, v) = 0$. Since v was arbitrary we conclude that $II \equiv 0$.

(ii)$\Longrightarrow$(iv); Suppose $v \in T_pM$. If V is a parallel vector field with respect to ∇ that is defined near p such that $V(p) = v$. Then $\bar{\nabla}_{\partial_t} V = \nabla_{\partial_t} V + II(\dot{\gamma}, V) = 0+0$ for any γ with $\gamma(0) = p$ so that V is a parallel vector field with respect to $\bar{\nabla}$.

(iv)$\Longrightarrow$(i); Assume (iv). If γ is a geodesic in M then γ' is parallel along γ with respect to ∇. Then by assumption γ' is parallel along γ with respect to $\bar{\nabla}$. Thus γ is also a $\bar{M}$ geodesic. ∎

17.3 Hypersurfaces

If the codimension of M in $\bar{M}$ is equal to 1 then we say that M is a **hypersurface**. If M is a semi-Riemannian hypersurface in $\bar{M}$ and $\langle u,u\rangle > 0$ for every $u \in (T_pM)^{\perp}$ we call M a positive hypersurface. If $\langle u,u\rangle < 0$ for every $u \in (T_pM)^{\perp}$ we call M a negative hypersurface. Of course, if $\bar{M}$ is Riemannian then every hypersurface in $\bar{M}$ is positive. The sign of M, in $\bar{M}$ denoted $\operatorname{sgn} M$ is $\operatorname{sgn}\langle u,u\rangle$.

Exercise 17.4 *Suppose that c is a regular value of $f \in C^{\infty}(\bar{M})$ then $M = f^{-1}(c)$ is a semi-Riemannian hypersurface if $\langle df, df\rangle > 0$ on all of M or if $\langle df, df\rangle < 0$. $\operatorname{sgn}\langle df, df\rangle = \operatorname{sgn}\langle u,u\rangle$.*

From the preceding exercise it follows if $M = f^{-1}(c)$ is a semi-Riemannian hypersurface then $U = \nabla f / \|\nabla f\|$ is a unit normal for M and $\langle U,U\rangle = \operatorname{sgn} M$. Notice that this implies that $M = f^{-1}(c)$ is orientable if $\bar{M}$ is orientable. Thus not every semi-Riemannian hypersurface is of the form $f^{-1}(c)$. On the other hand every hypersurface is locally of this form.

In the case of a hypersurface we have (locally) only two choices of unit normal. Once we have chosen a unit normal u the shape operator is denoted simply by S rather than S_u.

We are already familiar with the sphere $S^n(r)$ which is $f^{-1}(r^2)$ where $f(x) = \langle x,x\rangle = \sum_{i=1}^n x^i x^i$. A similar example arises when we consider the semi-Euclidean space $\mathbb{R}^{n+1-\nu,\nu}$ where $\nu \neq 0$. In this case, the metric is $\langle x,y\rangle_\nu = -\sum_{i=1}^{\nu} x^i y^i + \sum_{i=\nu+1}^{n} x^i y^i$. We let $f(x) := -\sum_{i=1}^{\nu} x^i x^i + \sum_{i=\nu+1}^{n} x^i x^i$ and then for $r > 0$ we have that $f^{-1}(\epsilon r^2)$ is a semi-Riemannian hypersurface in $\mathbb{R}^{n+1-\nu,\nu}$ with sign ε and unit normal $U = \mathbf{r}/r$. We shall divide these hypersurfaces into two classes according to sign.

Definition 17.5 *For $n > 1$ and $0 \le v \le n$, we define*

$$S_\nu^n(r) = \{x \in \mathbb{R}^{n+1-\nu,\nu} : \langle x,x\rangle_\nu = r^2.$$

$S_\nu^n(r)$ is called the pseudo-sphere of index ν.

Definition 17.6 *For $n > 1$ and $0 \le \nu \le n$, we define*

$$H_\nu^n(r) = \{x \in \mathbb{R}^{n+1-(\nu+1),\nu+1} : \langle x,x\rangle_\nu = -r^2.$$

$H_\nu^n(r)$ is called the pseudo-hyperbolic space of radius r and index ν.

Chapter 18

Killing Fields and Symmetric Spaces

Following the excellent treatment given by Lang [L1], we will treat Killing fields in a more general setting than is traditional. We focus attention on pairs (M, ∇) where M is a smooth manifold and ∇ is a (not necessarily metric) torsion free connection. Now let (M, ∇^M) and (N, ∇^N) be two such pairs. For a diffeomorphism $\varphi : M \to N$ the pull-back connection $\varphi^*\nabla$ is defined so that

$$(\varphi^*\nabla^N)_{\varphi^* X}\varphi^* Y = \varphi^*(\nabla^M_X Y)$$

for $X, Y \in \mathfrak{X}(N)$ and $\varphi^* X$ is defined as before by $\varphi^* X = T\varphi^{-1} \circ X \circ \varphi$. An isomorphism of pairs $(M, \nabla^M) \to (N, \nabla^N)$ is a diffeomorphism $\varphi : M \to M$ such that $\varphi^*\nabla = \nabla$. Automorphism of a pair (M, ∇) is defined in the obvious way. Because we will be dealing with the flows of vector fields which are not necessarily complete we will have to deal with maps $\varphi :: M \to M$ with domain a proper open subset $O \subset M$. In this case we say that φ is a local isomorphism if $\varphi^* \left.\nabla\right|_{\varphi^{-1}(O)} = \left.\nabla\right|_O$.

Definition 18.1 *A vector field $X \in \mathfrak{X}(M)$ is called a $\nabla-$**Killing field** if φ_t^X is a local isomorphism of the pair (M, ∇) for all sufficiently small t.*

Definition 18.2 *Let M, g be a semi-Riemannian manifold. A vector field $X \in \mathfrak{X}(M)$ is called a $\mathrm{g}-$**Killing field** if X is a local isometry.*

It is clear that if ∇^{g} is the Levi-Civita connection for M, g then any isometry of M, g is an automorphism of (M, ∇^{g}) and any $\mathrm{g}-$Killing field is a $\nabla^{\mathrm{g}}-$Killing field. The reverse statements are not always true. Letting $Kill_g(M)$ denote the $\mathrm{g}-$Killing fields and $Kill_\nabla(M)$ the $\nabla-$Killing fields we have

$$Kill_{\mathrm{g}}(M) \subset Kill_{\nabla^{\mathrm{g}}}(M)$$

Lemma 18.1 *If (M, ∇) is as above then for any $X, Y, Z \in \mathfrak{X}(M)$ we have*

$$[X, \nabla_Z Y] = \nabla_{Y,Z} - R_{Y,Z}X + \nabla_{[X,Z]}Y + \nabla_Z[X, Y]$$

Proof. The proof is a straightforward calculation left to the reader. ∎

Theorem 18.1 *Let M be a smooth manifold and ∇ a torsion free connection. For $X \in \mathfrak{X}(M)$, the following three conditions are equivalent:*

(i) X is a $\nabla-$Killing field

(ii) $[X, \nabla_Z Y] = \nabla_{[X,Z]}Y + \nabla_Z[X, Y]$ for all $Y, Z \in \mathfrak{X}(M)$

(iii) $\nabla_{Y,Z}X = R_{Y,Z}X$ for all $Y, Z \in \mathfrak{X}(M)$.

Proof. The equivalence of (ii) and (iii) follows from the previous lemma.

Let $\phi_t := \varphi_t^X$. If X is Killing (so (i) is true) then locally we have $\frac{d}{dt}\varphi_t^* Y = \varphi_t^* \mathcal{L}_X Y = \varphi_t^*[X, Y]$ for all $Y \in \mathfrak{X}(M)$. We also have $\phi_t^* X = X$. One calculates that

$$\begin{aligned} \frac{d}{dt}\phi_t^* (\nabla_Z Y) &= \phi_t^*[X, \nabla_Z Y] = [\phi_t^* X, \phi_t^* \nabla_Z Y] \\ &= [X, \phi_t^* \nabla_Z Y] \end{aligned}$$

and on the other hand

$$\frac{d}{dt}\nabla_{\phi_t^* Z}\phi_t^* Y = \nabla_{\phi_t^*[X,Z]}\phi_t^* Y + \nabla_{\phi_t^* Z}(\phi_t^*[X, Y]).$$

Setting $t = 0$ and comparing we get (ii).

Now assume (ii). We would like to show that $\phi_{-t}^* \nabla_{\phi_t^* Z}\phi_t^* Y = \nabla_Z Y$. We show that $\frac{d}{dt}\phi_{-t}^* \nabla_{\phi_t^* Z}\phi_t^* Y = 0$ for all sufficiently small t. The thing to notice here is that since the difference of connections is tensorial $\tau(Y, Z) = \frac{d}{dt}\phi_{-t}^* \nabla_{\phi_t^* Z}\phi_t^* Y$ is tensorial being the limit of the a difference quotient. Thus we can assume that $[X, Y] = [X, Z] = 0$. Thus $\phi_t^* Z = Z$ and $\phi_t^* Y = Y$. We now have

$$\begin{aligned} &\frac{d}{dt}\phi_{-t}^* \nabla_{\phi_t^* Z}\phi_t^* Y \\ &= \frac{d}{dt}\phi_{-t}^* (\nabla_Z Y) = \phi_{-t}^*[X, \nabla_Z Y] \\ &= \phi_{-t}^*(\nabla_{[X,Z]}Y + \nabla_Z[X, Y]) = 0 \end{aligned}$$

But since $\phi_{-t}^* \nabla_{\phi_t^* Z}\phi_t^* Y$ is equal to $\nabla_Z Y$ when $t = 0$, we have that $\phi_{-t}^* \nabla_{\phi_t^* Z}\phi_t^* Y = \nabla_Z Y$ for all t which is (i). ∎

Clearly the notion of a Jacobi field makes sense in the context of a general (torsion free) connection on M. Notice also that an automorphism ϕ of a pair (M, ∇) has the property that $\gamma \circ \phi$ is a geodesic if and only if ϕ is a geodesic.

Proposition 18.1 *X is a $\nabla-$Killing field if and only if $X \circ \gamma$ is a Jacobi field along γ for every geodesic γ.*

Proof. If X is Killing then $(s,t) \mapsto \varphi_s^X(\gamma(t))$ is a variation of γ through geodesics and so $t \mapsto X \circ \gamma(t) = \frac{\partial}{\partial s}\big|_{s=0} \varphi_s^X(\gamma(t))$ is a Jacobi field. The proof of the converse (Karcher) is as follows: Suppose that X is such that its restriction to any geodesic is a Jacobi field. Then for γ a geodesic, we have

$$\begin{aligned}\nabla_{\dot\gamma}^2 (X \circ \gamma) &= R(\dot\gamma, X \circ \gamma)\dot\gamma \\ &= \nabla_{\dot\gamma}\nabla_X \dot\gamma - \nabla_X \nabla_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\nabla_X \dot\gamma\end{aligned}$$

where we have used $\dot\gamma$ to denote not only a field along γ but also an extension to a neighborhood with $[\dot\gamma, X] = 0$. But

$$\begin{aligned}\nabla_{\dot\gamma}\nabla_X \dot\gamma &= \nabla_{\dot\gamma}\nabla_{\dot\gamma} X - \nabla_{\dot\gamma}[\dot\gamma, X] = \nabla_{\dot\gamma}\nabla_{\dot\gamma} X \\ &= \nabla_{\dot\gamma}\nabla_{\dot\gamma} X - \nabla_{\nabla_{\dot\gamma}\dot\gamma} X = \nabla_{\dot\gamma,\dot\gamma} X\end{aligned}$$

and so $\nabla_{\dot\gamma}^2 (X \circ \gamma) = \nabla_{\dot\gamma,\dot\gamma} X$. Now let $v, w \in T_pM$ for $p \in M$. Then there is a geodesic γ with $\dot\gamma(0) = v + w$ and so

$$\begin{aligned}&R(v,X)v + R(w,X)w + R(v,X)w + R(w,X)v \\ &= R(v+w,X)(v+w) = \nabla_{\dot\gamma,\dot\gamma} X = \nabla_{v+w,v+w} X \\ &= \nabla_{v,v} X + \nabla_{w,w} X + \nabla_{v,w} X + \nabla_{w,v} X\end{aligned}$$

Now replace w with $-w$ and subtract (polarization) to get

$$\nabla_{v,w} X + \nabla_{w,v} X = R(v,X)w + R(w,X)v.$$

On the other hand, $\nabla_{v,w} X - \nabla_{w,v} X = R(v,w)X$ and adding this equation to the previous one we get $2\nabla_{v,w} X = R(v,X)w - R(X,w)v - R(w,v)X$ and then by the Bianchi identity $2\nabla_{v,w} X = 2R(v,w)X$. The result now follows from theorem 18.1. ∎

We now give equivalent conditions for $X \in \mathfrak{X}(M)$ to be a $g-$Killing field for a semi-Riemannian (M,g). First we need a lemma.

Lemma 18.2 *For any vector fields $X, Y, Z \in \mathfrak{X}(M)$ we have*

$$\begin{aligned}\mathcal{L}_X \langle Y, Z\rangle &= \langle \mathcal{L}_X Y, Z\rangle + \langle Y, \mathcal{L}_X Z\rangle + (\mathcal{L}_X g)(Y,Z) \\ &= \langle \mathcal{L}_X Y, Z\rangle + \langle Y, \mathcal{L}_X Z\rangle + \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle\end{aligned}$$

Proof. This is again a straightforward calculation using for the second equality

$$\begin{aligned}(\mathcal{L}_X g)(Y,Z) &= \mathcal{L}_X \langle Y, Z\rangle - \langle \mathcal{L}_X Y, Z\rangle - \langle Y, \mathcal{L}_X Z\rangle \\ &= \mathcal{L}_X \langle Y, Z\rangle + \langle \nabla_X Y - \nabla_Y X, Z\rangle + \langle Y, \nabla_X Z - \nabla_Z X\rangle \\ &= \mathcal{L}_X \langle Y, Z\rangle + \langle \nabla_X Y - \nabla_Y X, Z\rangle + \langle Y, \nabla_X Z - \nabla_Z X\rangle \\ &= \mathcal{L}_X \langle Y, Z\rangle - \langle \nabla_Y X, Z\rangle - \langle Y, \nabla_Z X\rangle + \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ & 0 + \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle\end{aligned}$$

∎

Theorem 18.2 *Let (M, g) be semi-Riemannian. $X \in \mathfrak{X}(M)$ is a $g-$Killing field if and only if any one of the following conditions hold:*

(i) $\mathcal{L}_X g = 0$.

(ii) $\mathcal{L}_X \langle Y, Z\rangle = \langle \mathcal{L}_X Y, Z\rangle + \langle Y, \mathcal{L}_X Z\rangle$ for all $Y, Z \in \mathfrak{X}(M)$.

(iii) For each $p \in M$, the map $T_pM \to T_pM$ given by $v \longmapsto \nabla_v X$ is skewsymmetric for the inner product $\langle ., .\rangle_p$.

Proof. Since if X is Killing then $\varphi_t^{X*} g = g$ and in general for a vector field X we have

$$\frac{d}{dt}\varphi_t^{X*} g = \varphi_t^{X*} \mathcal{L}_X g$$

the equivalence of (i) with the statement that X is $g-$Killing is immediate.

If (ii) holds then by the previous lemma, (i) holds and conversely. The equivalence of (ii) and (i) also follows from the lemma and is left to the reader.
■

Chapter 19

Comparison Theorems

19.0.1 Rauch's Comparison Theorem

In this section we deal strictly with Riemannian manifolds. Recall that for a (semi-) Riemannian manifold M, the sectional curvature $K^M(P)$ of a $2-$plane $P \subset T_pM$ is

$$\langle \mathfrak{R}\left(e_1 \wedge e_2\right), e_1 \wedge e_2 \rangle$$

for any orthonormal pair e_1, e_2 that span P.

Lemma 19.1 *If Y is a vector field along a curve $\alpha : [a,b] \to M$ then if $Y^{(k)}$ is the parallel transport of $\nabla^k_{\partial_t} Y(a)$ along α we have the following Taylor expansion:*

$$Y(t) = \sum_{k=0}^{m} \frac{Y^{(k)}(t)}{k!}(t-a)^k + O(|t-a|^{m+1})$$

Proof. Exercise. ∎

Definition 19.1 *If M, g and N, h are Riemannian manifolds and $\gamma^M : [a,b] \to M$ and $\gamma^N : [a,b] \to N$ are unit speed geodesics defined on the same interval $[a,b]$ then we say that $K^M \geq K^N$ along the pair (γ^M, γ^N) if $K^M(Q_{\gamma^M(t)}) \geq K^N(P_{\gamma^N(t)})$ for every all $t \in [a,b]$ and every pair of 2-planes $Q_{\gamma^M(t)} \in T_{\gamma^M(t)}M$, $P_{\gamma^N(t)} \in T_{\gamma^N(t)}N$.*

We develop some notation to be used in the proof of Rauch's theorem. Let M be a given Riemannian manifold. If Y is a vector field along a unit speed geodesic γ^M such that $Y(a) = 0$ then let

$$I^M_s(Y,Y) := I_{\gamma^M|[a,s]}(Y,Y) = \int_a^s \langle \nabla_{\partial_t} Y(t), \nabla_{\partial_t} Y(t) \rangle + \langle R_{\dot\gamma^M, Y}\dot\gamma^M, Y\rangle(t)dt.$$

If Y is an orthogonal Jacobi field then $I^M_s(Y,Y) = \langle \nabla_{\partial_t} Y, Y\rangle(s)$ by theorem 16.10.

Theorem 19.1 (Rauch) *Let M, g and N, h be Riemannian manifolds of the same dimension and let $\gamma^M : [a, b] \to M$ and $\gamma^N : [a, b] \to N$ unit speed geodesics defined on the same interval $[a, b]$. Let J^M and J^N be Jacobi fields along γ^M and γ^N respectively and orthogonal to their respective curves. Suppose that the following four conditions hold:*

(i) $J^M(a) = J^N(a)$ *and neither of* $J^M(t)$ *or* $J^N(t)$ *is zero for* $t \in (a, b]$
(ii) $\left\|\nabla_{\partial_t} J^M(a)\right\| = \left\|\nabla_{\partial_t} J^N(a)\right\|$
(iii) $L(\gamma^M) = dist(\gamma^M(a), \gamma^M(b))$
(iv) $K^M \geq K^N$ *along the pair* (γ^M, γ^N)

Then $\left\|J^M(t)\right\| \leq \left\|J^N(t)\right\|$ *for all* $t \in [a, b]$.

Proof. Let f_M be defined by $f_M(s) := \left\|J^M(s)\right\|$ and h_M by $h_M(s) := I_s^M(J^M, J^M)/\left\|J^M(s)\right\|^2$ for $s \in (a, b]$. Define f_N and h_N analogously. We have

$$f_M'(s) = 2I_s^M(J^M, J^M) \text{ and } f_M'/f_M = 2h_M$$

and the analogous equalities for f_N and h_N. If $c \in (a, b)$ then

$$\ln(\left\|J^M(s)\right\|^2) = \ln(\left\|J^M(c)\right\|^2) + 2\int_c^s h_M(s')ds'$$

with the analogous equation for N. Thus

$$\ln\left(\frac{\left\|J^M(s)\right\|^2}{\left\|J^N(s)\right\|^2}\right) = \ln\left(\frac{\left\|J^M(c)\right\|^2}{\left\|J^N(c)\right\|^2}\right) + 2\int_c^s [h_M(s') - h_N(s')]ds'$$

From the assumptions (i) and (ii) and the Taylor expansions for J^M and J^N we have

$$\lim_{c \to a} \frac{\left\|J^M(c)\right\|^2}{\left\|J^N(c)\right\|^2} = 0$$

and so

$$\ln\left(\frac{\left\|J^M(s)\right\|^2}{\left\|J^N(s)\right\|^2}\right) = 2\lim_{c \to a}\int_c^s [h_M(s') - h_N(s')]ds'$$

If we can show that $h_M(s) - h_N(s) \leq 0$ for $s \in (a, b]$ then the result will follow. So fix $s_0 \in (a, b]$ let $Z^M(s) := J^M(s)/\left\|J^M(r)\right\|$ and $Z^N(s) := J^N(s)/\left\|J^N(r)\right\|$. We now define a parameterized families of sub-tangent spaces along γ^M by $W_M(s) := \dot{\gamma}^M(s)^{\perp} \subset T_{\gamma^M(s)}M$ and similarly for $W_N(s)$. We can choose a linear isometry $L_r : W_N(r) \to W_M(r)$ such that $L_r(J^N(r)) = J^M(r)$. We now want to extend L_r to a family of linear isometries $L_s : W_N(s) \to W_M(s)$. We do this using parallel transport by

$$L_s := P(\gamma^M)_r^s \circ L_r \circ P(\gamma^N)_s^r.$$

Define a vector field Y along γ^M by $Y(s) := L_s(J^M(s))$. Check that

$$\begin{aligned}
Y(a) &= J^M(a)\\
Y(r) &= J^M(r)\\
\|Y\|^2 &= \left\|J^N\right\|^2\\
\|\nabla_{\partial_t} Y\|^2 &= \left\|\nabla_{\partial_t} J^N\right\|^2
\end{aligned}$$

The last equality is a result of exercise 15.10 where in the notation of that exercise $\beta(t) := P(\gamma^M)_t^r \circ Y(t)$. Since (iii) holds there can be no conjugates along γ^M up to r and so I_r^M is positive definite. Now $Y - J^M$ is orthogonal to the geodesic γ^M and so by corollary 16.4we have $I_r^M(J^M, J^M) \leq I_r^M(Y, Y)$ and in fact

$$\begin{aligned}
I_r^M(J^M, J^M) \leq I_r^M(Y, Y) &= \int_a^r \|\nabla_{\partial_t} Y\|^2 + R^M(\dot{\gamma}^M, Y, \dot{\gamma}^M, Y)\\
&\leq \int_a^r \|\nabla_{\partial_t} Y\|^2 + R^N(\dot{\gamma}^N, J^N, \dot{\gamma}^N, J^N) \text{ (by (iv))}\\
&= I_r^N(J^N, J^N)
\end{aligned}$$

Recalling the definition of Y we obtain

$$I_r^M(J^M, J^M) / \left\|J^M(r)\right\|^2 \leq I_r^N(J^N, J^N) / \left\|J^N(r)\right\|^2$$

and so $h_M(r) - h_N(r) \leq 0$ but r was arbitrary and so we are done. ■

19.0.2 Bishop's Volume Comparison Theorem

`under construction`

19.0.3 Comparison Theorems in semi-Riemannian manifolds

`under construction`

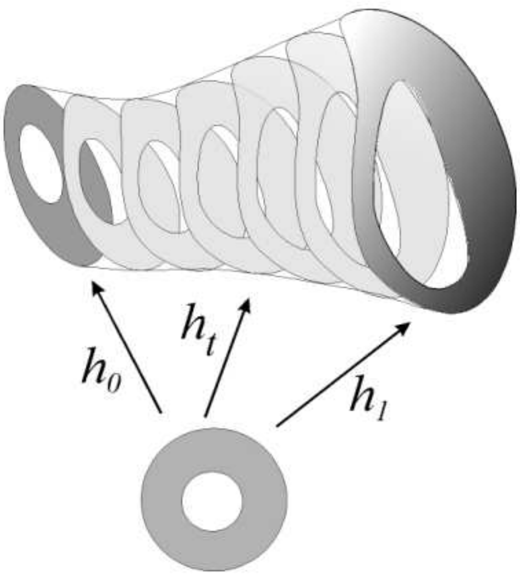
Chapter 20

Algebraic Topology

UNDER CONSTRUCTION

20.1 Topological Sum

20.2 Homotopy



Homotopy as a family of maps.

Definition 20.1 *Let $f_0, f_1 : X \to Y$ be maps. A homotopy from f_0 to f_1 is a one parameter family of maps $\{h_t : X \to Y : 0 \le t \le 1\}$ such that $h_0 = f_0$, $h_1 = f_1$ and such that $(x, t) \mapsto h_t(x)$ defines a (jointly continuous) map*

$X \times [0,1] \to Y$. If there exists such a homotopy we write $f_0 \simeq f_1$ and say that f_0 is homotopic to f_1. If there is a subspace $A \subset X$ such that $h_t|\,A = f_0|\,A$ for all $t \in [0,1]$ then we say that f_0 is homotopic to f_1 relative to A and we write $f_0 \simeq f_1(\mathrm{rel}A)$.

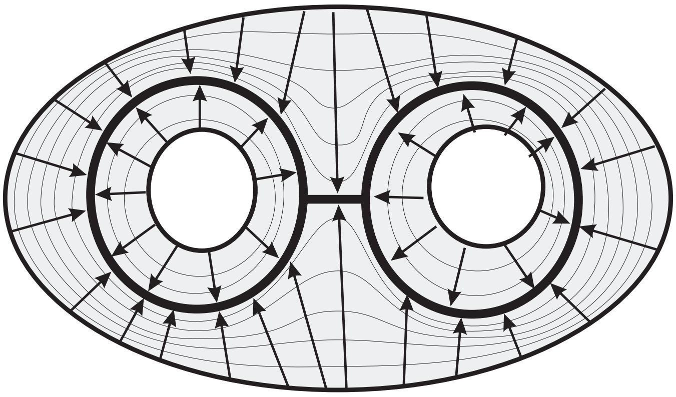
It is easy to see that homotopy equivalence is in fact an equivalence relation. The set of homotopy equivalence classes of maps $X \to Y$ is denoted $[X, Y]$ or $\pi(X, Y)$.

Definition 20.2 *Let $f_0, f_1 : (X, A) \to (Y\ , B)$ be maps of topological pairs. A homotopy from f_0 to f_1 is a homotopy h of the underlying maps $f_0, f_1 : X \to Y$ such that $h_t(A) \subset B$ for all $t \in [0,1]$. If $S \subset X$ then we say that f_0 is homotopic to f_1 relative to S if $h_t|\,S = f_0|\,S$ for all $t \in [0,1]$.*

The set of homotopy equivalence classes of maps $(X, A) \to (Y\ , B)$ is denoted $[(X, A), (Y, B)]$ or $\pi((X, A), (Y, B))$. As a special case we have the notion of a homotopy of **pointed maps** $f_0, f_1 : (X, x_0) \to (Y, y_0)$. The points x_0 and y_0 are called the **base points** and are commonly denoted by the generic symbol $*$. The set of all homotopy classes of pointed maps between pointed topological spaced is denoted $[(X, x_0), (Y, y_0)]$ or $\pi((X, x_0), (Y, y_0))$ but if the base points are fixed and understood then we denote the space of pointed homotopy classes as $[X, Y]_0$ or $\pi(X, Y)_0$. We may also wish to consider morphisms of **pointed pairs** such as $f : (X, A, a_0) \to (Y, B, b_0)$ which is given by a map $f : (X, A) \to (Y\ , B)$ such that $f(a_0) = b_0$. Here usually have $a_0 \in A$ and $b_0 \in B$. A homotopy between two such morphisms, say f_0 and $f_1 : (X, A, a_0) \to (Y, B, b_0)$ is a homotopy h of the underlying maps $(X, A) \to (Y\ , B)$ such that $h_t(a_0) = b_0$ for all $t \in [0,1]$. Clearly there are many variations on this theme of restricted homotopy.

Remark 20.1 *Notice that if $f_0, f_1 : (X, A) \to (Y\ , y_0)$ are homotopic as maps of topological pairs then we automatically have $f_0 \simeq f_1$(rel A). However, this is not necessarily the case if $\{y_0\}$ is replaced by a set $B \subset Y$ with more than one element.*

Definition 20.3 *A (strong)* ***deformation retraction*** *of X onto subspace $A \subset X$ is a homotopy f_t from $f_0 = \mathrm{id}_X$ to f_1 such that $f_1(X) \subset A$ and $f_t|\,A = \mathrm{id}_A$ for all $t \in [0,1]$. If such a retraction exists then we say that A is a (strong)* ***deformation retract*** *of X.*

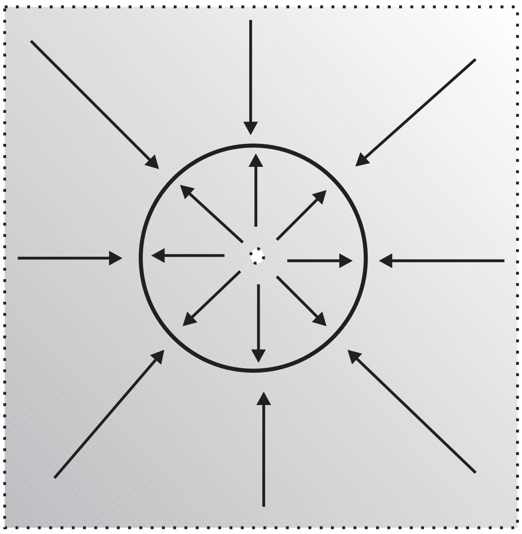


Retraction onto "eyeglasses"

Example 20.1 *Let $f_t : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ be defined by*

$$f_t(x) := t\frac{x}{|x|} + (1-t)x$$

for $0 \le t \le 1$. Then f_t gives a deformation retraction of $\mathbb{R}^n \backslash \{0\}$ onto $S^{n-1} \subset \mathbb{R}^n$.



Retraction of Punctured Plane

Definition 20.4 *A map $f : X \to Y$ is called a* ***homotopy equivalence*** *if there is a map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. The maps are then said to be homotopy inverses of each other. In this case we say that X and Y are* ***homotopy equivalent*** *and are said to be of the same* ***homotopy type****. We denote this relationship by $X \simeq Y$*

Definition 20.5 *A space X is called* ***contractible*** *if it is homotopy equivalent to a one point space.*

Definition 20.6 *A map $f : X \to Y$ is called **null-homotopic** if it is homotopic to a constant map.*

Equivalently, one can show that X is contractible if and only if every map $f : X \to Y$ is null-homotopic.

20.3 Cell Complexes

Let I denote the closed unit interval and let $I^n := I \times \cdots \times I$ be the n-fold Cartesian product of I with itself. The boundary of I is $\partial I = \{0, 1\}$ and the boundary of I^2 is $\partial I^2 = (I \times \{0, 1\}) \cup (\{0, 1\} \times I)$. More generally, the boundary of I^n is the union of the sets of the form $I \times \cdots \times \partial I \cdots \times I$. Also, recall that the closed unit n-disk D^n is the subset of $\mathbb{R}^n$ given by $\{|x|^2 \leq 1\}$ and has as boundary the sphere S^{n-1}. From the topological point of view the pair $(I^n, \partial I^n)$ is indistinguishable from the pair (D^n, S^{n-1}). In other words , $(I^n, \partial I^n)$ is homeomorphic to (D^n, S^{n-1}).

There is a generic notation for any homeomorphic copy of $I^n \cong D^n$ which is simply $\overline{e}^n$. Any such homeomorph of D^n is referred to as a closed n-cell. If we wish to distinguish several copies of such a space we might add an index to the notation as in $\overline{e}^n_1, \overline{e}^n_2$...etc. The interior of $\overline{e}^n$ is called an open n-cell and is generically denoted by e^n. The boundary is denoted by $\partial \overline{e}^n$ (or just ∂e^n). Thus we always have $(\overline{e}^n, \partial \overline{e}^n) \cong (D^n, S^{n-1})$.

An important use of the attaching idea is the construction of so called cell complexes . The open unit ball in $\mathbb{R}^n$ or any space homeomorphic to it is referred to as an open n-cell and is denoted by e^n. The closed ball is called a closed n-cell and has as boundary the $n - 1$ sphere. A 0-cell is just a point and a 1-cell is a (homeomorph of) the unit interval the boundary of which is a pair of points. We now describe a process by which one can construct a large and interesting class of topological spaces called cell complexes. The steps are as follows:

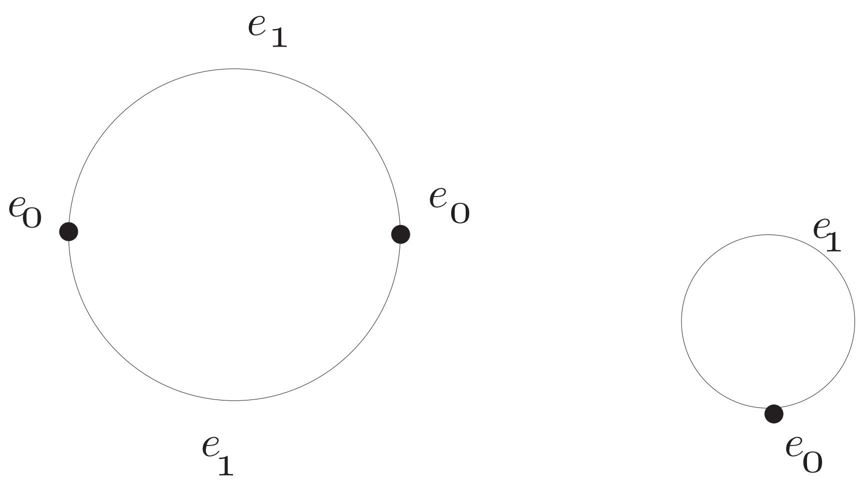
1. Start with any discrete set of points and regard these as 0-cells.

2. Assume that one has completed the $n - 1$ step in the construction with a resulting space X^{n-1}, construct X^n by attaching some number of copies of n-cells $\{e^n_\alpha\}_{\alpha \in A}$ (indexed by some set A) by attaching maps $f_\alpha : \partial e^n_\alpha = S^{n-1} \to X^{n-1}$.

3. Stop the process with a resulting space X^n called a **finite cell complex** or continue indefinitely according to some recipe and let $X = \bigcup_{n \geq 0} X^n$ and define a topology on X as follows: A set $U \subset X$ is defined to be open if and only if $U \cap X^n$ is open in X^n (with the relative topology). The space X is called a **CW complex** or just a **cell complex** .

Definition 20.7 *Given a cell complex constructed as above the set X^n constructed at the n-th step is called the n-**skeleton**. If the cell complex is finite* *then the highest step n reached in the construction is the whole space and the cell complex is said to have dimension n. In other words, a finite cell complex has* ***dimension*** *n if it is equal to its own n-skeleton.*

It is important to realize that the stratification of the resulting topological space by the via the skeletons and also the open cells that are homeomorphically embedded are part of the definition of a cell complex and so two different cell complexes may in fact be homeomorphic without being the same cell complex. For example, one may realize the circle S^1 by attaching a 1-cell to a 0-cell or by attaching two 1-cells to two different 0-cells as in figure **??**.



Two cell structures for circle

Another important example is the projective space $\mathbb{R}P^n$ which can be thought of as a the hemisphere $\overline{S^n_+} = \{x \in \mathbb{R}^{n+1} : x^{n+1} \geq 0\}$ modulo the identification of antipodal points of the boundary $\partial\overline{S^n_+} = S^{n-1}$. But S^{n-1} with antipodal point identified becomes $\mathbb{R}P^{n-1}$ and so we can obtain $\mathbb{R}P^n$ by attaching an n-cell e^n to $\mathbb{R}P^{n-1}$ with the attaching map $\partial e^n = S^{n-1} \to \mathbb{R}P^{n-1}$ given as the quotient map of S^{n-1} onto $\mathbb{R}P^{n-1}$. By repeating this analysis inductively we conclude that $\mathbb{R}P^n$ can be obtained from a point by attaching one cell from each dimension up to n :

$$\mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$$

and so $\mathbb{R}P^n$ is a finite cell complex of dimension n.

20.4 Axioms for a Homology Theory

Consider the category $\mathcal{TP}$ of all topological pairs (X, A) where X is a topological space, A is a subspace of X and where a morphism $f : (X, A) \to (X', A')$ is given by a map $f : X \to X'$ such that $f(A) \subset A'$. We may consider the category of topological spaces and maps as a subcategory of $\mathcal{TP}$ by identifying $(X, \emptyset)$ with X. We will be interested in functors from some subcategory $\mathcal{NTP}$ to the category $\mathbb{Z}{-}\mathcal{GAG}$ of $\mathbb{Z}$-graded abelian groups. The subcategory $\mathcal{NTP}$ (tentatively called "nice topological pairs") will vary depending of the situation but one example for which things work out nicely is the category of finite cell complex pairs. Let $\sum A_k$ and $\sum B_k$ be graded abelian groups. A morphism of $\mathbb{Z}$-graded abelian groups is a sequence $\{h_k\}$ of group homomorphisms $h_k : A_k \to B_k$. Such a morphism may also be thought of as combined to give a degree preserving map on the graded group; $h : \sum A_k \to \sum B_k$.

In the following we write $H_p(X)$ for $H_p(X, \emptyset)$. A homology theory H with coefficient group G is a covariant functor h_G from a category of nice topological pairs $\mathcal{NTP}$ to the category $\mathbb{Z}{-}\mathcal{GAG}$ of $\mathbb{Z}$-graded abelian groups;

$$h_G : \begin{cases} (X, A) \mapsto H(X, A, G) = \sum_{p \in \mathbb{Z}} H_p(X, A, G) \\ \qquad f \mapsto f_* \end{cases}$$

and that satisfies the following axioms:

1. $H_p(X, A) = 0$ for $p < 0$.

2. (**Dimension axiom**) $H_p(pt) = 0$ for all $p \geq 1$ and $H_0(pt) = G$.

3. If $f : (X, A) \to (X', A')$ is homotopic to $g : (X, A) \to (X', A')$ then $f_* = g_*$

4. (**Boundary map axiom**) To each pair (X, A) and each $p \in \mathbb{Z}$ there is a boundary homomorphism $\partial_p : H_p(X, A; G) \to H_{p-1}(A; G)$ such that for all maps $f : (X, A) \to (X', A')$ the following diagram commutes:

$$\begin{array}{ccc} H_p(X, A; G) & \overset{f_*}{\to} & H_p(X', A'; G) \\ \partial_p \downarrow & & \partial_p \downarrow \\ H_{p-1}(A; G) & \underset{(f|A)_*}{\to} & H_{p-1}(A'; G) \end{array}$$

5. (**Excision axiom**) For each inclusion $\iota : (B, B \cap A) \to (A \cup B, A)$ the induced map $\iota_* : H(B, B \cap A; G) \to H(A \cup B, A; G)$ is an isomorphism.

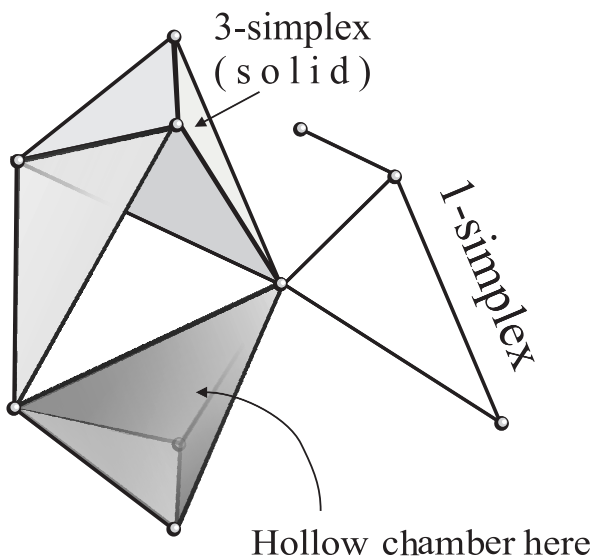
6. For each pair (X, A) and inclusions $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$ there is a long exact sequence

$$\begin{array}{ccccc} \cdots \to H_{p+1}(A) \overset{i_*}{\to} H_{p+1}(X) & \overset{j_*}{\to} & H_{p+1}(X, A) & & \\ & & \partial_{p+1} \swarrow & & \\ & & H_{p+1}(A) \overset{i_*}{\to} & H_{p+1}(X) \overset{j_*}{\to} \cdots & \end{array}$$

where we have suppressed the reference to G for brevity.

20.5 Simplicial Homology

Simplicial homology is a perhaps the easiest to understand in principle.
And we have



Simplicial Complex

20.6 Singular Homology

These days an algebraic topology text is like to emphasize singular homology.

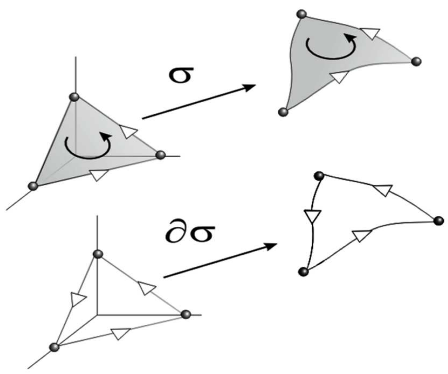


Figure 20.1: Singular 2-simplex and boundary

20.7 Cellular Homology

UNDER CONSTRUCTION

20.8 Universal Coefficient theorem

UNDER CONSTRUCTION

20.9 Axioms for a Cohomology Theory

UNDER CONSTRUCTION

20.10 Topology of Vector Bundles

In this section we study vector bundles with finite rank. Thus, the typical fiber may be taken to be $\mathbb{R}^n$ (or $\mathbb{C}^n$ for a complex vector bundle) for some positive integer n. We would also like to study vectors bundles over spaces that are not necessarily differentiable manifolds; although this will be our main interest. All the spaces in this section will be assumed to be paracompact Hausdorff spaces. We shall refer to continuous maps simply as maps. In many cases the theorems will makes sense in the differentiable category and in this case one reads map as "smooth map".

Recall that a (rank n) real vector bundle is a triple (π_E, E, M) where E and M are paracompact spaces and $\pi_E : E \to M$ is a surjective map such that there is a cover of M by open sets U_α together with corresponding trivializing maps (VB-charts) $\phi_\alpha : \pi_E^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ of the form $\phi_\alpha = (\pi_E, \Phi_\alpha)$. Here $\Phi_\alpha : \pi_E^{-1}(U_\alpha) \to \mathbb{R}^n$ has the property that $\Phi_\alpha|_{E_x} : E_x \to \mathbb{R}^n$ is a diffeomorphism for each fiber $E_x := \pi_E^{-1}(x)$. Furthermore, in order that we may consistently transfer the linear structure of $\mathbb{R}^n$ over to E_x we must require that when $U_\alpha \cap U_\beta \neq \emptyset$ and $x \in U_\alpha \cap U_\beta$ then function

$$\Phi_{\beta\alpha;x} = \Phi_\beta|_{E_x} \circ \Phi_\alpha|_{E_x}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$$

is a linear isomorphism. Thus the fibers are vector spaces isomorphic to $\mathbb{R}^n$. For each nonempty overlap $U_\alpha \cap U_\beta$ we have a map $U_\alpha \cap U_\beta \to Gl(n)$

$$x \mapsto \Phi_{\beta\alpha;x}.$$

We have already seen several examples of vector bundles but let us add one more to the list:

Example 20.2 *The normal bundle to $S^n \subset \mathbb{R}^{n+1}$ is the subset $N(S^n)$ of $S^n \times \mathbb{R}^{n+1}$ given by*

$$N(S^n) := \{(x, v) : x \cdot v = 0\}.$$

The bundle projection $\pi_{N(S^n)}$ *is given by* $(x, v) \mapsto x$. *We may define bundle charts by taking opens sets* $U_\alpha \subset S^n$ *that cover* S^n *and then since any* $(x, v) \in \pi^{-1}_{N(S^n)}(U_\alpha)$ *is of the form* (x, tx) *for some* $t \in \mathbb{R}$ *we may define*

$$\phi_\alpha : (x, v) = (x, tx) \mapsto (x, t).$$

Now there is a very important point to be made from the last example. Namely, it seems we could have taken *any* cover $\{U_\alpha\}$ with which to build the VB-charts. But can we just take the cover consisting of the single open set $U_1 := S^n$ and thus get a VB-chart $N(S^n) \to S^n \times \mathbb{R}$? The answer is that in this case we can. This is because $N(S^n)$ is itself a trivial bundle; that is, it is isomorphic to the product bundle $S^n \times \mathbb{R}$. This is not the case for vector bundles in general. In particular, we will later be able to show that the tangent bundle of an even dimensional sphere is always nontrivial. Of course, we have already seen that the Möbius line bundle is nontrivial.

20.11 De Rham's Theorem

UNDER CONSTRUCTION

20.12 Sheaf Cohomology

UNDER CONSTRUCTION

20.13 Characteristic Classes

UNDER CONSTRUCTION

Chapter 21

Ehresmann Connections and Cartan Connections

Halmos, Paul R.

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?

Halmos, Paul R.

I Want to be a Mathematician, Washington: MAA Spectrum, 1985.

21.1 Principal and Associated $G-$Bundles

In this chapter we take a (C^r) **fiber bundle** over M with typical fiber F to be a quadruple (π, E, M, F) where $\pi : E \to M$ is a smooth C^r-submersion such that for every $p \in M$ there is an open set U containing p with a C^r-isomorphism $\phi = (\pi, \Phi) : \pi^{-1}(U) \to U \times F$. To review; we denote the fiber at p by $E_p = \pi^{-1}(p)$ and for each $p \in U$ the map $\Phi|_{E_p} : E_p \to F$ is a C^r-diffeomorphism. Given two such trivializations $(\pi, \Phi_\alpha) : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ and $(\pi, \Phi_\beta) : \pi^{-1}(U_\beta) \to U_\beta \times F$ then for each $p \in U_\alpha \cap U_\beta$ there is a diffeomorphism $\Phi_{\alpha\beta}|_p : E_p \to E_p$ $U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$ defined by $p \mapsto \Phi_{\alpha\beta}(p) = \Phi_{\alpha\beta}|_p$. These are called **transition maps** or **transition functions**.

Remark 21.1 *Recall that a group action $\rho : G \times F \to F$ is equivalently thought of as a representation $\overline{\rho} : G \to \mathrm{Diff}(F)$ given by $\overline{\rho}(g)(f) = \rho(g, f)$. We will forgo the separate notation $\overline{\rho}$ and simple write ρ for the action and the corresponding representation.*

Returning to our discussion of fiber bundles, suppose that there is a Lie group action $\rho : G \times F \to F$ and cover of E by trivializations $\{U_\alpha, \phi_\alpha\}$ such that for each α, β we have

$$\Phi_{\alpha\beta}(p)(f) = \rho(g_{\alpha\beta}(p), f) := g_{\alpha\beta}(p) \cdot f$$

for some smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ then we say that we have presented (π, E, M, F) as G **bundle under the representation** ρ. We also say that the transition functions **live in** G (via ρ). In many but not all cases the representation ρ will be faithful, i.e. the action will be **effective** and so G can be considered as a subgroup of $\mathrm{Diff}(F)$. A notable exception is the case of spin bundles. We call (π, E, M, F, G) a (G, ρ) **bundle** or just a $G-$bundle if the representation is understood or standard in some way. It is common to call G the **structure group** but since the action in question may not be effective we should really refer to the **structure group representation (or action)** ρ.

A fiber bundle is determined if we are given an open cover $\mathcal{U} = \{U_\alpha\}$ and maps $\Phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{diff}(F)$ such that for all α, β, γ

$$\begin{aligned} \Phi_{\alpha\alpha}(p) &= \mathrm{id}\,\text{for } p \in U_\alpha \\ \Phi_{\alpha\beta}(p) &= \Phi_{\beta\alpha}^{-1}(p) \text{ for } p \in U_\alpha \cap U_\beta \\ \Phi_{\alpha\beta}(p) \circ \Phi_{\beta\gamma}(p) \circ \Phi_{\gamma\alpha}(p) &= \text{ id for } p \in U_\alpha \cap U_\beta \cap U_\gamma \end{aligned}$$

If we want a G bundle under a representation ρ then we further require that $\Phi_{\alpha\beta}(p)(f) = \rho(g_{\alpha\beta}(p))(f)$ as above and that the maps $g_{\alpha\beta}$ themselves satisfy the cocycle condition:

$$\begin{aligned} g_{\alpha\alpha}(p) &= \mathrm{id} \text{ for } p \in U_\alpha \\ g_{\alpha\beta}(p) &= g_{\beta\alpha}^{-1}(p) \text{ for } p \in U_\alpha \cap U_\beta \\ g_{\alpha\beta}(p) \circ g_{\beta\gamma}(p) \circ g_{\gamma\alpha}(p) &= \mathrm{id} \text{ for } p \in U_\alpha \cap U_\beta \cap U_\gamma \end{aligned} \tag{21.1}$$

We shall also call the maps $g_{\alpha\beta}$ **transition functions** or **transition maps**. Notice that if ρ is effective the last condition follows from the first. The family $\{U_\alpha\}$ together with the maps $\Phi_{\alpha\beta}$ form a **cocycle** and we can construct a bundle by taking the disjoint union $\bigsqcup(U_\alpha \times F) = \bigcup U_\alpha \times F \times \{\alpha\}$ and then taking the equivalence classes under the relation $(p, f, \beta) \backsim (p, \Phi_{\alpha\beta}(p)(f), \alpha)$ so that

$$E = \left(\bigcup U_\alpha \times F \times \{\alpha\}\right) / \backsim$$

and $\pi([p, f, \beta]) = p$.

Let $H \subset G$ be a closed subgroup. Suppose that we can, by throwing out some of the elements of $\{ U_\alpha, \Phi_\alpha\}$ arrange that all of the transition functions live in H. That is, suppose we have that $\Phi_{\alpha\beta}(p)(f) = \rho(g_{\alpha\beta}(p), f)$ where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to H$. Then we have a **reduction the structure group** (or **reduction of the structure representation** in case the action needs to be specified).

Next, suppose that we have an surjective Lie group homomorphism $h : \overline{G} \to G$ **.** We then have the lifted representation $\overline{\rho} : \overline{G} \times F \to F$ given by $\overline{\rho}(\overline{g}, f) = \rho(h(\overline{g}), f)$. Under suitable topological conditions we may be able to lift the maps $g_{\alpha\beta}$ to maps $\overline{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \overline{G}$ and by choosing a subfamily we can even arrange that the $\overline{g}_{\alpha\beta}$ satisfy the cocycle condition. Note that $\Phi_{\alpha\beta}(p)(f) = \rho(g_{\alpha\beta}, f) = \rho(h(\overline{g}_{\alpha\beta}), f) = \overline{\rho}(\overline{g}_{\alpha\beta}(p), f)$. In this case we say that we have **lifted** the structure representation to $\overline{\rho}$.

Example 21.1 *The simplest class of examples of fiber bundles over a manifold M are the product bundles. These are just Cartesian products $M \times F$ together with the projection map $pr_1 : M \times F \to M$. Here, the structure group can be reduced to the trivial group $\{e\}$ acting as the identity map on F. On the other hand, this bundle can also be prolonged to any Lie group acting on F.*

Example 21.2 *A covering manifold $\pi : \widetilde{M} \to M$ is a G-bundle where G is the group of deck transformations. In this example the group G is a discrete (0-dimensional) Lie group.*

Example 21.3 (The Hopf Bundle) *Identify S^1 as the group of complex numbers of unit modulus. Also, we consider the sphere S^3 as it sits in $\mathbb{C}^2$:*

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

The group S^1 acts on S^2 by $u \cdot (z_1, z_2) = (uz_1, uz_2)$. Next we get S^2 into the act. We want to realize S^2 as the sphere of radius 1/2 in $\mathbb{R}^3$ and having two coordinate maps coming from stereographic projection from the north and south poles onto copies of $\mathbb{C}$ embedded as planes tangent to the sphere at the two poles. The chart transitions then have the form $w = 1/z$. Thus we may view S^2 as two copies of $\mathbb{C}$, say the z plane $\mathbb{C}_1$ and the w plane $\mathbb{C}_2$ glued together under the identification $\phi : z \mapsto 1/z = w$

$$S^2 = \mathbb{C}_1 \cup_\phi \mathbb{C}_2.$$

With this in mind define a map $\pi : S^3 \subset \mathbb{C}^2 \to S^2$ by

$$\pi(z_1, z_2) = \begin{cases} z_2/z_1 \in \mathbb{C}_2 \subset \mathbb{C}_1 \cup_\phi \mathbb{C}_2 & \textit{if} \quad z_1 \neq 0 \\ z_1/z_2 \in \mathbb{C}_1 \subset \mathbb{C}_1 \cup_\phi \mathbb{C}_2 & \textit{if} \quad z_2 \neq 0 \end{cases}.$$

Note that this gives a well defined map onto S^2.

Claim 21.1 $u \cdot (z_1, z_2) = u \cdot (w_1, w_2)$ *if and only if* $\pi(z_1, z_2) = \pi(w_1, w_2)$.

Proof. If $u \cdot (z_1, z_2) = u \cdot (w_1, w_2)$ and $z_1 \neq 0$ then $w_1 \neq 0$ and $\pi(w_1, w_2) = w_2/w_1 = uw_2/uw_1 = \pi(w_1, w_2) = \pi(z_1, z_2) = uz_2/uz_1 = z_2/z_1 = \pi(z_1, z_2)$. A similar calculation show applies when $z_2 \neq 0$. On the other hand, if $\pi(w_1, w_2) = \pi(z_1, z_2)$ then by a similar chain of equalities we also easily get that $u \cdot (w_1, w_2) = ... = \pi(w_1, w_2) = \pi(z_1, z_2) = ... = u \cdot (z_1, z_2)$. ■

Using these facts we see that there is a fiber bundle atlas on $\pi_{Hopf} = \pi : S^3 \to S^2$ given by the following trivializations:

$$\varphi_1 : \pi^{-1}(C_1) \to C_1 \times S^1$$
$$\varphi_1 : (z_1, z_2) = (z_2/z_1, z_1/|z_1|)$$

and

$$\varphi_2 : \pi^{-1}(C_2) \to C_2 \times S^1$$
$$\varphi_2 : (z_1, z_2) = (z_1/z_2, z_2/|z_2|).$$

The transition map is

$$(z, u) \mapsto (1/z, \frac{z}{|z|}u)$$

which is of the correct form since $u \mapsto \frac{z}{|z|} \cdot u$ is a circle action. Thus the Hopf bundle is an S^1-bundle with typical fiber S^1 itself. It can be shown that the inverse image of a circle on S^2 by the Hopf projection π_{Hopf} is a torus. Since the sphere S^2 is foliated by circles degenerating at the poles we have a foliation of S^3-{two circles} by tori degenerating to circles at the fiber over the two poles. Since $S^3 \backslash \{\text{pole}\}$ is diffeomorphic to $\mathbb{R}^3$ we expect to be able to get a picture of this foliation by tori. In fact, the following picture depicts this foliation.

21.2 Principal and Associated Bundles

An important case for a bundle with structure group G is where the typical fiber is the group itself. In fact we may obtain such a bundle by taking the transition functions $g_{\alpha\beta}$ from any effective G bundle $E \to M$ or just any smooth maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ that form a cocycle with respect to some cover $\{U_\alpha\}$ of M. We let G act on itself by left multiplication and then use the bundle construction method above. Thus if $\{U_\alpha\}$ is the cover of M corresponding to the cocycle $\{g_{\alpha\beta}\}_{\alpha,\beta}$ then we let

$$P = \left(\bigcup U_\alpha \times G \times \{\alpha\}\right) / \backsim$$

where $(p, g, \alpha) \backsim (p, g_{\alpha\beta}(p)g, \beta)$ gives the equivalence relation. In this way we construct what is called the a principal bundle. Notice that for $g \in G$ we have $(p, g_1, \beta) \backsim (p, g_2, \alpha)$ if and only if $(p, g_1 g, \beta) \backsim (p, g_2 g, \alpha)$ and so there is a well defined right action on any bundle principal bundle. On the other hand there is a more direct way chart free way to define the notion of principal bundle. The advantage of defining a principal bundle without explicit reference to transitions functions is that we may then use the principal bundle to give another definition of a G-bundle that doesn't appeal directly to the notion of transition functions. We will see that every G bundle is given by a choice of a principal G-bundle and an action of G on some manifold F (the typical fiber).

First we define the **trivial principal G bundle over** U to be the trivial bundle $pr_1 : U \times G \to M$ together with the right G action $(U \times G) \times G$ given by

$$(x, g_1)g := (x, g_1 g).$$

An automorphism of the G-space $U \times G$ is a bundle map $\delta : U \times G \to U \times G$ such that $\delta(x, g_1 g) = \delta(x, g_1)g$ for all $g_1, g \in G$ and all $x \in U$. Now δ must have the form given by $\delta(x, g) = (x, \Delta(x, g))$ and so

$$\Delta(x, g) = \Delta(x, e)g.$$

If we then let the function $x \mapsto \Delta(x, e)$ be denoted by $g_\delta()$ then we have $\delta(x, g) = (x, g_\delta(x)g)$. Thus we obtain the following

Lemma 21.1 *Every automorphism of a trivial principal G bundle over and open set U has the form $\delta : (x, g) \mapsto (x, g_\delta(x)g)$ for some smooth map $g_\delta : U \to G$.*

Definition 21.1 *A **principal G-bundle** is a fiber bundle $\pi_P : P \to M$ together with a right G action $P \times G \to P$ that is locally equivalent as a right G space to the trivial principal G bundle over M. This means that for each point $x \in M$ there is an open neighborhood U_x and a trivialization ϕ*

$$\begin{array}{ccc} \pi_P^{-1}(U_x) & \xrightarrow{\phi} & U_x \times G \\ & \searrow \quad \swarrow \ pr_1 & \\ & U_x & \end{array}$$

*that is G equivariant. Thus we require that $\phi(pg) = \phi(p)g$. We shall call such a trivialization an **equivariant trivialization**.*

Note that $\phi(pg) = (\pi_P(pg), \Phi(pg))$ while on the other hand $\phi(p)g = (\pi_P(pg), \Phi(p)g)$ so it is necessary and sufficient that $\Phi(p)g = \Phi(pg)$. Now we want to show that this means that the structure representation of $\pi_P : P \to M$ is left multiplication by elements of G. Let ϕ_1, U_1 and ϕ_2, U_2 be two equivariant trivializations such that $U_1 \cap U_2 \neq \emptyset$. On the overlap we have the diagram

$$\begin{array}{ccccc} U_1 \cap U_2 \times G & \overset{\phi_2}{\leftarrow} & \pi_P^{-1}(U_1 \cap U_2) & \overset{\phi_1}{\rightarrow} & U_1 \cap U_2 \times G \\ & \searrow & \downarrow & \swarrow & \\ & & U_1 \cap U_2 & & \end{array}$$

The map $\phi_2 \circ \phi_1^{-1}\big|_{U_1 \cap U_2}$ clearly must an G-bundle automorphism of $U_1 \cap U_2 \times G$ and so by 21.1 must have the form $\phi_2 \circ \phi_1^{-1}\big|_{U_1 \cap U_2}(x, g) = (x, \Phi_{12}(x)g)$. We conclude that a principal G-bundle is a G-bundle with typical fiber G as defined in section 21.1. The maps on overlaps such as Φ_{12} are the transition maps. Notice that $\Phi_{12}(x)$ acts on G by left multiplication and so the structure representation is left multiplication.

Proposition 21.1 *If $\pi_P : P \to M$ is a principal G-bundle then the right action $P \times G \to P$ is free and the action restricted to any fiber is transitive.*

Proof. Suppose $p \in P$ and $pg = p$ for some $g \in G$. Let $\pi_P^{-1}(U_x) \xrightarrow{\phi} U_x \times G$ be an (equivariant) trivialization over U_x where U_x contains $\pi_P(p) = x$. Then we have

$$\begin{aligned} \phi(pg) = \phi(p) \quad &\Rightarrow \\ (x, g_0 g) = (x, g_0) \quad &\Rightarrow \\ g_0 g = g_0 \end{aligned}$$

and so $g = e$.

Now let $P_x = \pi_P^{-1}(x)$ and let $p_1, p_2 \in P_x$. Again choosing an (equivariant) trivialization over U_x as above we have that $\phi(p_i) = (x, g_i)$ and so letting $g := g_1^{-1} g_2$ we have $\phi(p_1 g) = (x, g_1 g) = (x, g_2) = \phi(p_2)$ and since ϕ is injective $p_1 g = p_2$. ■

The reader should realize that this result is in some sense "obvious" since the upshot is just that the result is true for the trivial principal bundle and then it follows for the general case since a general principal bundle is locally G-bundle isomorphic to a trivial principal bundle.

Remark 21.2 *Some authors define a principal bundle to be fiber bundle with typical fiber G and with a free right action that is transitive on each fiber. This approach turns out to be equivalent to the present one.*

Our first and possibly most important example of a principal bundle is the frame bundle associated to a rank k vector bundle $\pi : E \to M$. To achieve an appropriate generality let us assume that the structure group of the bundle can be reduced to some matrix group G (e.g. $SO(k)$). In this case we may single out a special class of frames in each fiber which are related to each other by elements of the group G. Let us call these frames "$G-$frames". A moving frame $\sigma_\alpha = (F_1, ..., F_k)$ which is such that $\sigma_\alpha(x)$ is always a $G-$frame will be called a moving G-frame. Let $P_{G,x}(\pi)$ be the set of all $G-$frames at x and define

$$P_G(\pi) = \bigcup_{x \in M x} P_{G,x}(\pi).$$

Also, let $\wp : GF(\pi) \to M$ be the natural projection map that takes any frame $\mathcal{F}_x \in GF(\pi)$ with $\mathcal{F}_x \in GF_x(\pi)$ to its base x. A moving $G-$frame is clearly the same thing as a local section of $GF(\pi)$. $GF(\pi)$ is a smooth manifold and in fact a principal bundle. We describe a smooth atlas for $GF(\pi)$. Let us adopt the convention that $(\mathcal{F}_x)_i := f_i$ is the i-th vector in the frame $\mathcal{F}_x = (f_1, ..., f_n)$. Let $\{(U_\alpha, \mathtt{x}_\alpha)\}_{\alpha \in A}$ be an atlas for M. We can assume without loss that each chart domains U_α is also the domain of a moving $G-$frame $\sigma_\alpha : U_\alpha \to GF(\pi)$ and use the notation $\sigma_\alpha = (F_1, ..., F_k)$. For each chart α on we define a chart $\widetilde{U}_\alpha, \widetilde{\mathtt{x}}_\alpha$ by letting

$$\widetilde{U}_\alpha := \wp^{-1}(U_\alpha) = \bigcup_{x \in U_\alpha} GF_x(\pi)$$

and

$$\widetilde{\mathtt{x}}_\alpha(\mathcal{F}_x) := (x^i(x), f_i^j) \in \mathbb{R}^n \times \mathbb{R}^{k \times k}$$

where (f_j^i) is matrix such that $(\mathcal{F}_x)_i = \sum f_i^j F_j(x)$. We leave it to the reader to find the change of coordinate maps and see that they are smooth. The right G action on $GF(\pi)$ is given by matrix multiplication $(\mathcal{F}_x, g) \mapsto \mathcal{F}_x g$ where we think of $\mathcal{F}_x$ as row of basis vectors.

Example 21.4 (Full frame bundle of a manifold) *The frame bundle of the tangent bundle of a manifold M is the set of all frames without restriction and* *so the structure group is* $Gl(n)$ *where* $n = \dim(M)$. *This frame bundle (usually called* the *frame bundle of* M*) is also traditionally denoted* $L(M)$.

Example 21.5 (Orthonormal Frame Bundle) *If* $E \to M$ *is a real vector bundle with a bundle metric then the structure group is reduced to* $\mathrm{O}(k)$ *and the corresponding frame bundle is denoted* $P_{\mathrm{O}(k)}(\pi)$ *or* $P_{\mathrm{O}(k)}(E)$ *and is called the orthonormal frame bundle of* E. *In the case of the tangent bundle of a Riemannian manifold* M, g *we denote the orthonormal frame of* TM *by* $P_{\mathrm{O}(n)}(TM)$ *although the notation* $F(M)$ *is also popular in the literature.*

21.3 X Connections

`UNDER CONSTRUCTION`

As we have pointed out, a moving $G-$frame on a vector bundle may clearly be thought of as a local section of the frame bundle $\sigma : U \to F(E)$. If

21.4 Gauge Fields

`UNDER CONSTRUCTION`

21.5 Cartan Connections

`UNDER CONSTRUCTION`

Part II

Part II

Chapter 22

Analysis on Manifolds

The best way to escape from a problem is to solve it.
-Alan Saporta

22.1 Basics

Now E is a Hermitian or a Riemannian vector bundle. First, if E is a Riemannian vector bundle then so is $E \otimes T^*M^{\otimes k}$ since T^*M also has a metric (locally given by g^{ij}). If $\xi = \xi^r_{i_1 \dots i_k} \epsilon_r \otimes \theta^{i_1} \otimes \dots \otimes \theta^{i_k}$ and $\mu = \mu^r_{i_1 \dots i_k} \epsilon_r \otimes \theta^{i_1} \otimes \dots \otimes \theta^{i_k}$ then

$$\langle \xi, \mu \rangle = g^{i_1 j_1} \cdots g^{i_1 j_1} h_{sr} \xi^r_{i_1 \dots i_k} \mu^s_{j_1 \dots j_k}$$

We could denote the metric on $E \otimes T^*M^{\otimes k}$ by $h \otimes (g^{-1})^{\otimes k}$ but we shall not often have occasion to use this notation will stick with the notation $\langle ., . \rangle$ whenever it is clear which metric is meant.

Let M, g be an oriented Riemannian manifold. The volume form vol_g allows us to integrate smooth functions of compact support. This gives a functional $C^\infty_c(M) \longrightarrow \mathbb{C}$ which extends to a positive linear functional on $C^0(M)$. A standard argument from real analysis applies here and gives a measure μ_g on the Borel sigma algebra $\mathcal{B}(M)$ generated by open set on M which is characterized by the fact that for every open set $\mu_g(O) = \sup\{\int f \, vol_g : f \prec O\}$ where $f \prec O$ means that $\operatorname{supp} f$ is a compact subset of O and $0 \leq f \leq 1$. We will denote integration with respect to this measure by $f \mapsto \int_M f(x) \mu_g(dx)$ or by a slight abuse of notation $\int f \, vol_g$. In fact, if f is smooth then $\int_M f(x) \mu_g(dx) = \int_M f \, vol_g$. We would like to show how many of the spaces and result from analysis on $\mathbb{R}^n$ still make sense in a more global geometric setting.

22.1.1 L^2, L^p, L^∞

Let $\pi : E \to M$ be a Riemannian vector bundle or a Hermitian vector bundle. Thus there is a real or complex inner product $h_p() = \langle ., . \rangle_p$ on every fiber E_p which varies smoothly with $p \in M$. Now if $v_x \in E_x$ we let $|v_x| := \langle v_x, v_x \rangle^{1/2} \in \mathbb{R}$

and for each section $\sigma \in \Gamma(M)$ let $|\sigma| = \langle \sigma, \sigma \rangle^{1/2} \in C^0(M)$. We need a measure on the base space M and so for convenience we assume that M is oriented and has a Riemannian metric. Then the associated volume element vol_g induces a Radon measure which is equivalent to Lebesgue measure in every coordinate chart. We now define the norms

$$\|\sigma\|_p := \left(\int_M |\sigma|^p \, vol_g \right)^{1/p}$$
$$\|\sigma\|_\infty := \sup_{x \in M} |\sigma(x)|$$

First of all we must allow sections of the vector bundle which are not necessarily C^∞. It is quite easy to see what it means for a section to be continuous and a little more difficult but still rather easy to see what it means for a section to be measurable.

Definition 22.1 $L_g^p(M, E)$ *is the space of measurable sections of* $\pi : E \to M$ *such that* $\left(\int_M |\sigma|^p \, vol_g \right)^{1/p} < \infty$.

With the norm $\|\sigma\|_p$ this space is a Banach space. For the case $p = 2$ we have the obvious Hilbert space inner product on $L_g^2(M, E)$ defined by

$$(\sigma, \eta) := \int_M \langle \sigma, \eta \rangle vol_g$$

Many, in fact, most of the standard facts about L^p spaces of functions on a measure space still hold in this context. For example, if $\sigma \in L^p(M, E), \eta \in L^q(M, E)$, $\frac{1}{p} + \frac{1}{q} = 1, p, q \geq 1$ then $|\sigma| \, |\eta| \in L^1(M)$ and Hölder's inequality holds:

$$\int_M |\sigma| \, |\eta| \, vol_g \leq \left(\int_M |\sigma|^p \, vol_g \right)^{1/p} \left(\int_M |\eta|^q \, vol_g \right)^{1/q}$$

To what extent do the spaces $L_g^p(M, E)$ depend on the metric? If M is compact, it is easy to see that for any two metrics g_1 and g_2 there is a constant $C > 0$ such that

$$\frac{1}{C} \left(\int_M |\sigma|^p \, vol_{g_2} \right)^{1/p} \leq \left(\int_M |\sigma|^p \, vol_{g_1} \right)^{1/p} \leq C \left(\int_M |\sigma|^p \, vol_{g_2} \right)^{1/p}$$

uniformly for all $\sigma \in L_{g_1}^p(M, E)$. Thus $L_{g_1}^p(M, E) = L_{g_2}^p(M, E)$ and the norms are equivalent. For this reason we shall forego the subscript which references the metric.

Now we add one more piece of structure into the mix.

22.1.2 Distributions

For every integer $m \geq 0$ and compact subset $K \subset M$ we define (semi) norm $p_{K,m}$ on the space $\mathfrak{X}_l^k(M)$ by

$$p_{K,m}(\tau) := \sum_{j \leq m} \sup_{x \in K} \{ \left| \nabla^{(j)} \tau \right| (x) \}$$

Let $\mathfrak{X}^k_l(K)$ denote the set of restrictions of elements of $\mathfrak{X}^k_l(M)$ to K. The set $\mathfrak{X}^k_l(K)$ is clearly a vector space and $\{p_{K,m}\}_{1\leq m<\infty}$ is a family of norms that turns $\mathfrak{X}^k_l(K)$ into a Frechet space. Now let $\mathcal{D}(M,\mathfrak{X}^k_l)$ denote the space of (k,l)-tensor fields with compact support $\mathfrak{X}^k_l(M)_c \subset \mathfrak{X}^k_l(M)$ but equipped with the inductive limit topology of the family of Frechet spaces $\{\mathfrak{X}^k_l(K) : K \subset M \text{ compact}\}$. What we need to know is what this means in practical terms. Namely, we need a criterion for the convergence of a sequence (or net) of elements from $\mathfrak{X}^k_l(M)_c$.

Criterion 22.1 *Let $\{\tau_\alpha\} \subset \mathcal{D}(M,\mathfrak{X}^k_l) = \mathfrak{X}^k_l(M)_c$. Then $\tau_\alpha \to \tau$ if and only if given any $\epsilon > 0$ there is a compact set K_ϵ and $N > 0$ such that $\operatorname{supp}\tau_\alpha \subset K$ and such that $p_{K,m}(\tau_\alpha) < \epsilon$ whenever $\alpha > N$.*

Now the space of generalized tensors or tensor distributions $\mathcal{D}'(M,\mathfrak{X}^k_l)$ of type (k,l) is the set of all linear functionals on the space $\mathcal{D}(M,\mathfrak{X}^k_l)$ which are continuous in the following sense:

Criterion 22.2 *A linear functional $F : \mathcal{D}(M,\mathfrak{X}^k_l) \to \mathbb{C}$ is continuous if and only if for every compact set $K \subset M$ there are constants $C(K)$ and $m(K)$ such that*

$$|\langle F,\tau\rangle| \leq C(K)p_{K,m}(\tau) \text{ for all } \tau \in \mathfrak{X}^k_l(K)$$

Definition 22.2 *For each integer $p \geq 1$ let $\|\tau\|_p := \left(\int_M |\tau|^p \, dV\right)^{1/p}$. Let $L^p\mathfrak{X}^k_l(M)$ denote the completion of $\mathfrak{X}^k_l(M)$ with respect to this norm. Similarly, for every compact subset $K \subset M$ define $\|\tau\|_{p,K} := \left(\int_K |\tau|^p \, dV\right)^{1/p}$.*

The family of norms $\{\|.\|_{p,K}\}$, where K runs over all compact K, provides $\mathfrak{X}^k_l(M)$ with a Frechet space structure which we denote by $\mathfrak{X}^k_l(M)$

Definition 22.3 *For each integer $p \geq 1$ and integer $r \geq 0$ let $\|T\|_{m,p} := \sum_{|r|\leq m} \|\nabla^r T\|_p$. This defines a norm on $\mathfrak{X}^k_l(M)$ called the Sobolev norm. Let $W_{r,p}\mathfrak{X}^k_l(M)$ denote the completion of $\mathfrak{X}^k_l(M)$ with respect to this norm. For $k,l = 0,0$ so that we are dealing with functions we write $W_{r,p}(M)$ instead of $W_{r,p}\mathfrak{X}^0_0(M)$.*

22.1.3 Elliptic Regularity

22.1.4 Star Operator II

The definitions and basic algebraic results concerning the star operator on a scalar product space globalize to the tangent bundle of a Riemannian manifold in a straightforward way.

Definition 22.4 *Let M, g be a semi-Riemannian manifold. Each tangent space is a scalar product space and so on each tangent space T_pM we have a metric volume element vol_p and then the map $p \mapsto \mathrm{vol}_p$ gives a section of $\bigwedge^n T^*M$*

called the metric volume element of M, g. *Also on each fiber* $\bigwedge T_p^* M$ *of* $\bigwedge T^* M$ *we have a star operator* $*_p : \bigwedge^k T_p^* M \to \bigwedge^{n-k} T_p^* M$. *These induce a bundle map* $* : \bigwedge^k T^* M \to \bigwedge^{n-k} T^* M$ *and thus a map on sections (i.e. smooth forms)* $* : \Omega^k(M) \to \Omega^{n-k}(M)$.

Definition 22.5 *The star operator is sometimes referred to as the* ***Hodge star operator****.*

Definition 22.6 *Globalizing the scalar product on the Grassmann algebra we get a* ***scalar product bundle*** $\Omega(M), \langle ., . \rangle$ *where for every* $\eta, \omega \in \Omega^k(M)$ *we have a smooth function* $\langle \eta, \omega \rangle$ *defined by*

$$p \mapsto \langle \eta(p), \omega(p) \rangle$$

and thus a $C^\infty(M)$*-bilinear map* $\langle ., . \rangle : \Omega^k(M) \times \Omega^k(M) \to C^\infty(M)$. *Declaring forms of differing degree to be orthogonal as before we extend to a* C^∞ *bilinear map* $\langle ., . \rangle : \Omega(M) \times \Omega(M) \to C^\infty(M)$.

Theorem 22.1 *For any forms* $\eta, \omega \in \Omega(M)$ *we have* $\langle \eta, \omega \rangle \operatorname{vol} = \eta \wedge *\omega$

Now let M, g be a Riemannian manifold so that $\mathrm{g} = \langle ., . \rangle$ is positive definite. We can then define a Hilbert space of square integrable differential forms:

Definition 22.7 *Let an inner product be defined on* $\Omega_c(M)$, *the elements of* $\Omega(M)$ *with compact support, by*

$$(\eta, \omega) := \int_M \eta \wedge *\omega = \int_M \langle \eta, \omega \rangle \operatorname{vol}$$

and let $L^2(\Omega(M))$ *denote the* L^2 *completion of* $\Omega_c(M)$ *with respect to this inner product.*

22.2 The Laplace Operator

The exterior derivative operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ has a formal adjoint $\delta : \Omega^{k+1}(M) \to \Omega^k(M)$ defined by the requirement that for all $\alpha, \beta \in \Omega_c^k(M)$ with compact support we have

$$(d\alpha, \beta) = (\alpha, \delta\beta).$$

On a Riemannian manifold M the Laplacian of a function $f \in C(M)$ is given in coordinates by

$$\Delta f = -\frac{1}{\sqrt{g}} \sum_{j,k} \partial_j (g^{jk} \sqrt{g} \partial_k f)$$

where g^{ij} is the inverse of g_{ij} the metric tensor and g is the determinant of the matrix $G = (g_{ij})$. We can obtain a coordinate free definition as follows. First we recall that the divergence of a vector field $X \in \mathfrak{X}(M)$ is given at $p \in M$ by the trace of the map $\nabla X|_{T_pM}$. Here $\nabla X|_{T_pM}$ is the map

$$v \mapsto \nabla_v X.$$

Thus

$$\operatorname{div}(X)(p) := tr(\nabla X|_{T_pM}).$$

Then we have

$$\Delta f := \operatorname{div}(\operatorname{grad}(f))$$

Eigenvalue problem: For a given compact Riemannian manifold M one is interested in finding all $\lambda \in R$ such that there exists a function $f \neq 0$ in specified subspace $S \subset L^2(M)$ satisfying $\Delta f = \lambda f$ together with certain boundary conditions in case $\partial M \neq 0$.

The reader may be a little annoyed that we have not specified S more clearly. The reason for this is twofold. First, the theory will work even for relatively compact open submanifolds with rather unruly topological boundary and so regularity at the boundary becomes and issue. In general, our choice of S will be influenced by boundary conditions. Second, even though it may appear that S must consist of C^2 functions, we may also seek "weak solutions" by extending Δ in some way. In fact, Δ is essentially self adjoint in the sense that it has a unique extension to a self adjoint unbounded operator in $L^2(M)$ and so eigenvalue problems could be formulated in this functional analytic setting. It turns out that under very general conditions on the form of the boundary conditions, the solutions in this more general setting turn out to be smooth functions. This is the result of the general theory of elliptic regularity.

Definition 22.8 *A boundary operator is a linear map $b : S \to C^0(\partial M)$.*

Using this notion of a boundary operator we can specify boundary conditions as the requirement that the solutions lie in the kernel of the boundary map. In fact, the whole eigenvalue problem can be formulated as the search for λ such that the linear map

$$(\triangle - \lambda) \oplus b : S \to L^2(M) \oplus C^0(\partial M)$$

has a nontrivial kernel. If we find such a λ then this kernel is denoted $E_\lambda \subset L^2(M)$ and by definition $\Delta f = \lambda f$ and $bf = 0$ for all $f \in E_\lambda$. Such a function is called an **eigenfunction** corresponding to the eigenvalue λ. We shall see below that in each case of interest (for compact M) the eigenspaces E_λ will be finite dimensional and the eigenvalues form a sequence of nonnegative numbers increasing without bound. The dimension $\dim(E_\lambda)$ is called the **multiplicity** of λ. We shall present the sequence of eigenvalues in two ways:

1. If we write the sequence so as to include repetitions according to multiplicity then the eigenvalues are written as $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \uparrow \infty$. Thus it is possible, for example, that we might have $\lambda_2 = \lambda_3 = \lambda_4$ if $\dim(E_{\lambda_2}) = 3$.

2. If we wish to list the eigenvalues without repetition then we use an overbar:

$$0 \leq \bar{\lambda}_1 < \bar{\lambda}_2 < \ldots \uparrow \infty$$

The sequence of eigenvalues is sometimes called the **spectrum** of M.

To make thing more precise we divide things up into four cases:

The closed eigenvalue problem: In this case M is a compact Riemannian manifold without boundary the specified subspace of $L^2(M)$ can be taken to be $C^2(M)$. The kernel of the map $\Delta - \lambda : C^2(M) \to C^0(M)$ is the λ eigenspace and denoted by E_λ It consists of eigenfunctions for the eigenvalue λ.

The Dirichlet eigenvalue problem: In this case M is a compact Riemannian manifold without nonempty boundary ∂M. Let $\overset{\circ}{M}$ denote the interior of M. The specified subspace of $L^2(M)$ can be taken to be $C^2(\overset{\circ}{M}) \cap C^0(M)$ and the boundary conditions are $f|\,\partial M \equiv 0$ (**Dirichlet boundary conditions)** so the appropriate boundary operator is the restriction map $b_D : f \longmapsto f|\,\partial M$. The solutions are called **Dirichlet eigenfunctions** and the corresponding sequence of numbers λ for which a nontrivial solution exists is called the **Dirichlet spectrum** of M**.**

The Neumann eigenvalue problem: In this case M is a compact Riemannian manifold without nonempty boundary ∂M but . The specified subspace of $L^2(M)$ can be taken to be $C^2(\overset{\circ}{M}) \cap C^1(M)$. The problem is to find nontrivial solutions of $\Delta f = \lambda f$ with $f \in C^2(\overset{\circ}{M}) \cap C^0(\partial M)$ that satisfy $\nu f|\,\partial M \equiv 0$ (**Neumann boundary conditions)**.Thus the boundary map here is $b_N : C^1(M) \to C^0(\partial M)$ given by $f \mapsto \nu f|\,\partial M$ where ν is a smooth unit normal vector field defined on ∂M and so the νf is the normal derivative of f . The solutions are called **Neumann eigenfunctions** and the corresponding sequence of numbers λ for which a nontrivial solution exists is called the **Neumann spectrum** of M**.**

Recall that the completion of $C^k(M)$ (for any $k \geq 0$) with respect to the inner product

$$(f, g) = \int_M fg dV$$

is the Hilbert space $L^2(M)$. The Laplace operator has a natural extension to a self adjoint operator on $L^2(M)$ and a careful reformulation of the above eigenvalue problems in this Hilbert space setting together with the theory of elliptic regularity lead to the following

Theorem 22.2 *1) For each of the above eigenvalue problems the set of eigenvalues (the spectrum) is a sequence of nonnegative numbers which increases without bound:* $0 \leq \bar{\lambda}_1 < \bar{\lambda}_2 < \cdots \uparrow \infty$.

2) Each eigenfunction is a C^∞ *function on* $M = \overset{\circ}{M} \cup \partial M$.

3) Each eigenspace $E_{\bar{\lambda}_i}$ *(or* $E^D_{\bar{\lambda}_i}$ *or* $E^N_{\bar{\lambda}_i}$*) is finite dimensional, that is, each eigenvalue has finite multiplicity.*

4) If $\varphi_{\bar{\lambda}_i,1},...,\varphi_{\bar{\lambda}_i,m_i}$ is an orthonormal basis for the eigenspace $E_{\bar{\lambda}_i}$ (or $E^D_{\bar{\lambda}_i}$ or $E^N_{\bar{\lambda}_i}$) then the set $B = \cup_i\{\varphi_{\bar{\lambda}_i,1},...,\varphi_{\bar{\lambda}_i,m_i}\}$ is a complete orthonormal set for $L^2(M)$. In particular, if we write the spectrum with repetitions by multiplicity, $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \uparrow \infty$, then we can reindex this set of functions B as $\{\varphi_1, \varphi_2, \varphi_3, ...\}$ to obtain an ordered orthonormal basis for $L^2(M)$ such that φ_i is an eigenfunction for the eigenvalue λ_i.

The above can be given the following physical interpretation. If we think of M as a vibrating homogeneous membrane then the transverse motion of the membrane is described by a function $f : M \times (0, \infty) \to R$ satisfying

$$\Delta f + \frac{\partial^2 f}{\partial t^2} = 0$$

and if $\partial M \neq \emptyset$ then we could require $f|\partial M \times (0, \infty) = 0$ which means that we are holding the boundary fixed. A similar discussion for the Neumann boundary conditions is also possible and in this case the membrane is free at the boundary. If we look for the solutions of the form $f(x, t) = \phi(x)T(t)$ then we are led to conclude that ϕ must satisfy $\Delta\phi = \lambda\phi$ for some real number λ with $\phi = 0$ on ∂M. This is the Dirichlet eigenvalue problem discussed above.

Theorem 22.3 *For each of the eigenvalue problems defined above*

Now explicit solutions of the above eigenvalue problems are very difficult to obtain except in the simplest of cases. It is interesting therefore, to see if one can tell something about the eigenvalues from the geometry of the manifold. For instance we may be interested in finding upper and/or lower bounds on the eigenvalues of the manifold in terms of various geometric attributes of the manifold. A famous example of this is the Faber–Krahn inequality which states that if Ω is a regular domain in say $\mathbb{R}^n$ and D is a ball or disk of the same volume then

$$\lambda(\Omega) \geq \lambda(D)$$

where $\lambda(\Omega)$ and $\lambda(D)$ are the lowest nonzero Dirichlet eigenvalues of Ω and D respectively. Now it is of interest to ask whether one can obtain geometric information about the manifold given a degree of knowledge about the eigenvalues. There is a 1966 paper by M. Kac entitled “Can One Hear the Shape of a Drum?” which addresses this question. Kac points out that Weyl’s asymptotic formula shows that the sequence of eigenvalues does in fact determine the volume of the manifold. Weyl’s formula is

$$(\lambda_k)^{n/2} \sim \left(\frac{(2\pi)^n}{\omega_n}\right)\frac{k}{vol(M)} \quad \text{as} \quad k \longrightarrow \infty$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and M is the given compact manifold. In particular,

$$\left(\frac{(2\pi)^n}{\omega_n}\right)\lim_{k\to\infty}\frac{k}{(\lambda_k)^{n/2}} = vol(M).$$

So the volume is indeed determined by the spectrum[1].

22.3 Spectral Geometry

Legend has it that Pythagoras was near a black-smith's shop one day and hearing the various tones of a hammer hitting an anvil was lead to ponder the connection between the geometry (and material composition) of vibrating objects and the pitches of the emitted tones. This lead him to experiment with vibrating strings and a good deal of mathematics ensued. Now given a string of uniform composition it is essentially the length of the string that determines the possible pitches. Of course, there isn't much Riemannian geometry in a string because the dimension is 1. Now we have seen that a natural mathematical setting for vibration in higher dimensions is the Riemannian manifold and the wave equation associated with the Laplacian. The spectrum of the Laplacian corresponds the possible frequencies of vibration and it is clearly only the metric together with the total topological structure of the manifold that determines the spectrum. If the manifold is a Lie group or is a homogeneous space acted on by a Lie group, then the topic becomes highly algebraic but simultaneously involves fairly heavy analysis. This is the topic of harmonic analysis and is closely connected with the study of group representations. One the other hand, the Laplacian and its eigenvalue spectrum are defined for arbitrary (compact) Riemannian manifolds and, generically, a Riemannian manifold is far from being a Lie group or homogeneous space. The isometry group may well be trivial. Still the geometry must determine the spectrum. But what is the exact relationship between the geometry of the manifold and the spectrum? Does the spectrum determine the geometry. Is it possible that two manifolds can have the same spectrum without being isometric to each other? That the answer is yes has been known for quite a while now it wasn't until (??whenn) that the question was answered for planar domains? This was Mark Kac's original question: "Can one hear the shape of a drum?" It was shown by Carolyn Gordon and (??WhoW?) that the following two domains have the same Dirichlet spectrum but are not isometric:

finsh>

[1]Notice however, one may ask still how far out into the spectrum must one "listen" in order to gain an estimate of $vol(M)$ to a given accuracy.

22.4 Hodge Theory

22.5 Dirac Operator

It is often convenient to consider the differential operator $D = i\frac{\partial}{\partial x}$ instead of $\frac{\partial}{\partial x}$ even when one is interested mainly in real valued functions. For one thing $D^2 = -\frac{\partial^2}{\partial x^2}$ and so D provides a sort of square root of the positive Euclidean Laplacian $\triangle = -\frac{\partial^2}{\partial x^2}$ in dimension 1. Dirac wanted a similar square root for the wave operator $\square = \partial_0^2 - \sum_{i=1}^3 \partial_i^2$ (the Laplacian in $\mathbb{R}^4$ for the Minkowski inner metric) and found that an operator of the form $D = \partial_0 - \sum_{i=1}^3 \gamma_i \partial_i$ would do the job if it could be arranged that $\gamma_i\gamma_j + \gamma_i\gamma_j = 2\eta_{ij}$ where

$$(\eta_{ij}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

One way to do this is to allow the γ_i to be matrices.

Now lets consider finding a square root for $\triangle = -\sum_{i=1}^n \partial_i^2$. We accomplish this by an $\mathbb{R}$-linear embedding of $\mathbb{R}^n$ into an $N \times N$ real or complex matrix algebra A by using n linearly independent matrices $\{\gamma_i : i = 1, 2, ..., n\}$ (so called "gamma matrices") and mapping

$$(x^1, ..., x^n) \mapsto x^i \gamma_i \text{ (sum)}.$$

and where $\gamma_1, ..., \gamma_n$ are matrices satisfying the basic condition

$$\gamma_i\gamma_j + \gamma_i\gamma_j = -2\delta_{ij}.$$

We will be able to arrange[2] that $\{1, \gamma_1, ..., \gamma_n\}$ generates an algebra of dimension 2^n spanned as vector space by the identity matrix 1 and all products of the form $\gamma_{i_1} \cdots \gamma_{i_k}$ with $i_1 < i_2 < \cdots < i_k$. Thus we aim to identify $\mathbb{R}^n$ with the linear span of these gamma matrices. Now if we can find matrices with the property that $\gamma_i\gamma_j + \gamma_i\gamma_j = -2\delta_{ij}$ then our "Dirac operator" will be

$$D = \sum_{i=1}^n \gamma_i \partial_i$$

which is now acting on N-tuples of smooth functions.

Now the question arises: What are the differential operators $\partial_i = \frac{\partial}{\partial x^i}$ acting on exactly. The answer is that they act on whatever we take the algebra spanned by the gamma matrices to be acting on. In other words we should have some vector space S that is a module over the algebra spanned by the gamma matrices.

[2]It is possible that gamma matrices might span a space of half the dimension we are interested in. This fact has gone unnoticed in some of the literature. The dimension condition is to assure that we get a universal Clifford algebra.

Then we take as our "fields" smooth maps $f : \mathbb{R}^n \to S$. Of course since the $\gamma_i \in \mathbb{M}_{N\times N}$ we may always take $S = \mathbb{R}^N$ with the usual action of $\mathbb{M}_{N\times N}$ on $\mathbb{R}^N$. The next example shows that there are other possibilities.

Example 22.1 *Notice that with* $\frac{\partial}{\partial z} := \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ *and* $\frac{\partial}{\partial \bar{z}} := \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$ *we have*

$$
\begin{aligned}
&\begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \bar{z}} & 0 \end{bmatrix}\begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \bar{z}} & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} \\ -\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} & 0 \end{bmatrix} = \begin{bmatrix} \triangle & 0 \\ 0 & \triangle \end{bmatrix} \\
&= \triangle 1
\end{aligned}
$$

where $\triangle = -\sum \partial_i^2$. *On the other hand*

$$
\begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \bar{z}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{\partial}{\partial x} + \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\frac{\partial}{\partial y}.
$$

From this we can see that appropriate gamma matrices for this case are $\gamma_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ *and* $\gamma_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

Now let E^0 *be the span of* $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\gamma_2\gamma_1 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$. *Let* E^1 *be the span of* γ_1 *and* γ_2. *Refer to* E^0 *and* E^1 *the even and odd parts of* $Span\{1, \gamma_1, \gamma_2, \gamma_2\gamma_1\}$. *Then we have that* $D = \begin{bmatrix} 0 & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \bar{z}} & 0 \end{bmatrix}$ *maps* E^0 *to* E^1 *and writing a typical element of* E^0 *as* $f(x, y) = u(x, y) + \gamma_2\gamma_1 v(x, y)$ *is easy to show that* $Df = 0$ *is equivalent to the Cauchy-Riemann equations.*

The reader should keep this last example in mind as this kind of decomposition into even and odd part will be a general phenomenon below.

22.5.1 Clifford Algebras

A Clifford algebra is the type of algebraic object that allows us to find differential operators that square to give Laplace type operators. The matrix approach described above is in fact quite general but there are other approaches that are more abstract and encourage one to think about a Clifford algebra as something that contains the scalar and the vector space we start with. The idea is similar to that of the complex numbers. We seldom think about complex numbers as "pairs of real numbers" while we are calculating unless push comes to shove. After all, there are other good ways to represent complex numbers; as matrices for example. And yet there is one underlying abstract object called the complex numbers which is quite concrete once one get used to using them. Similarly we encourage the reader to learn to think about abstract Clifford algebras in the same way. Just compute!

Clifford algebras are usually introduced in connection with a quadratic form q on some vector space but in fact we are just as interested in the associated symmetric bilinear form and so in this section we will generically use the same symbol for a quadratic form and the bilinear form obtained by polarization and write both $q(v)$ and $q(v, w)$.

Definition 22.9 *Let* V *be an* n *dimensional vector space over a field* $\mathbb{K}$ *with characteristic not equal to* 2. *Suppose that* q *is a quadratic form on* V *and let* q *be the associated symmetric bilinear form obtained by polarization. A* ***Clifford algebra based on*** V,q *is an algebra with unity* 1 $Cl(\mathrm{V},q,\mathbb{K})$ *containing* V *(or an isomorphic image of* V*) such that the following relations hold:*

$$vw + wv = -2q(v, w)1$$

and such that $Cl(\mathrm{V},q,\mathbb{K})$ *is universal in the following sense: Given any linear map* $L : \mathrm{V} \to A$ *into an associative* $\mathbb{K}-$*algebra with unity* **1** *such that*

$$L(v)L(w) + L(w)L(v) = -2q(v, w)\mathbf{1}$$

then there is a unique extension of L *to an algebra homomorphism* $\bar{L} : Cl(\mathrm{V},q,\mathbb{K}) \to A$.

If $e_1, ..., e_n$ is an orthonormal basis for V, q then we must have

$$e_i e_j + e_j e_i = 0 \text{ for } i \neq j$$
$$e_i^2 = -q(e_i) = \pm 1 \text{ or } 0$$

A common choice is the case when q is a nondegenerate inner product on a real vector space. In this case we have a particular realization of the Clifford algebra obtained by introducing a new product into the Grassmann vector space $\wedge\mathrm{V}$. The said product is the unique linear extension of the following rule for $v \in \wedge^1\mathrm{V}$ and $w \in \wedge^k\mathrm{V}$:

$$v \cdot w := v \wedge w - v^\flat \lrcorner w$$
$$w \cdot v := (-1)^k (v \wedge w + v^\flat \lrcorner w)$$

We will refer to this as a **geometric algebra** on $\wedge\mathrm{V}$ and this version of the Clifford algebra will be called the **form presentation of** $Cl(\mathrm{V},q)$. Now once we have a definite inner product on V we have an inner product on V^* and $\mathrm{V} \cong \mathrm{V}^*$. The Clifford algebra on V^* is generated by the following more natural looking formulas

$$\alpha \cdot \beta := \alpha \wedge \beta - (\sharp\alpha) \lrcorner \beta$$
$$\beta \cdot \alpha := (-1)^k (\alpha \wedge \beta + (\sharp\alpha) \lrcorner \beta)$$

for $\alpha \in \wedge^1\mathrm{V}$ and $\beta \in \wedge\mathrm{V}$.

Now we have seen that one can turn $\wedge\mathrm{V}$ (or $\wedge\mathrm{V}^*$) into a Clifford algebra and we have also seen that one can obtain a Clifford algebra whenever appropriate gamma matrices can be found. A slightly more abstract construction is also common: Denote by $I(q)$ the ideal of the full tensor algebra $T(\mathrm{V})$ generated by elements of the form $x \otimes x - q(x) \cdot 1$. The **Clifford algebra** is (up to isomorphism) given by

$$Cl(\mathrm{V}, q, \mathbb{K}) = T(\mathrm{V})/I(q).$$

We can use the canonical injection

$$i : \mathrm{V} \longrightarrow C_K$$

to identify V with its image in $Cl(\mathrm{V}, q, \mathbb{K})$. (The map turns out that i is 1–1 onto $i(\mathrm{V})$ and we will just accept this without proof.)

Exercise 22.1 *Use the universal property of* $Cl(\mathrm{V}, q, \mathbb{K})$ *to show that it is unique up to isomorphism.*

Remark 22.1 *Because of the form realization of a Clifford algebra we see that* $\wedge \mathrm{V}$ *is a* $Cl(\mathrm{V}, q, \mathbb{R})-$*module. But even if we just have some abstract* $Cl(\mathrm{V}, q, \mathbb{R})$ *we can use the universal property to extend the action of* V *on* $\wedge\mathrm{V}$ *given by*

$$v \mapsto v \cdot w := v \wedge w - v^\flat \lrcorner w$$

to an action of $Cl(\mathrm{V}, q, \mathbb{K})$ *on* $\wedge\mathrm{V}$ *thus making* $\wedge\mathrm{V}$ *a* $Cl(\mathrm{V}, q, \mathbb{R})-$*module.*

Definition 22.10 *Let* $\mathbb{R}^n_{(r,s)}$ *be the real vector space* $\mathbb{R}^n$ *with the inner product of signature* (r, s) *given by*

$$\langle x, y \rangle := \sum_{i=1}^{r} x_i y_i - \sum_{i=r+1}^{r+s=n} x_i y_i.$$

The Clifford algebra formed from this inner product space is denoted $Cl_{r,s}$. *In the special case of* $(p, q) = (n, 0)$ *we write* Cl_n.

Definition 22.11 *Let* $\mathbb{C}^n$ *be the complex vector space of* n*-tuples of complex numbers together with the standard symmetric* $\mathbb{C}-$*bilinear form*

$$b(z, w) := \sum_{i=1}^{n} z_i w_i.$$

The (complex) Clifford algebra obtained is denoted $\mathbb{C}l_n$.

Remark 22.2 *The complex Clifford algebra* $\mathbb{C}l_n$ *is based on a complex symmetric form and* not *on a Hermitian form.*

Exercise 22.2 *Show that for any nonnegative integers* p, q *with* $p + q = n$ *we have* $Cl_{p,q} \otimes \mathbb{C} \cong \mathbb{C}l_n$.

Example 22.2 *The Clifford algebra based on* $\mathbb{R}^1$ *itself with the relation* $x^2 = -1$ *is just the complex number system.*

The Clifford algebra construction can be globalized in the obvious way. In particular, we have the option of using the form presentation so that the above formulas $\alpha \cdot \beta := \alpha \wedge \beta - (\sharp\alpha) \lrcorner \beta$ and $\beta \cdot \alpha := (-1)^k(\alpha \wedge \beta + (\sharp\alpha) \lrcorner \beta)$ are interpreted as equations for differential forms $\alpha \in \wedge^1 T^*M$ and $\beta \in \wedge^k T^*M$ on a semi-Riemannian manifold M, g. In any case we have the following

Definition 22.12 *Given a Riemannian manifold M, g, the Clifford algebra bundle is $Cl(T^*M, \mathrm{g}) = Cl(T^*M) := \cup_x Cl(T_x^*M)$.*

Since we take each tangent space to be embedded $T_x^*M \subset Cl(T_x^*M)$, the elements θ^i of a local orthonormal frame $\theta^1, ..., \theta^n \in \Omega^1$ are also local sections of $Cl(T^*M, \mathrm{g})$ and satisfy

$$\theta^i\theta^j + \theta^j\theta^i = -\langle\theta^i, \theta^j\rangle = -\epsilon^i\delta^{ij}$$

Recall that $\varepsilon^1, ..., \varepsilon^n$ is a list of numbers equal to ± 1 (or even 0 if we allow degeneracy) and giving the index of the metric $\mathrm{g}(.,.) = \langle .,. \rangle$.

Obviously, we could also work with the bundle $Cl(TM) := \cup_x Cl(T_xM)$ which is naturally isomorphic to $Cl(T^*M)$ in which case we would have

$$e^ie^j + e^je^i = -\langle e^i, e^j\rangle = -\varepsilon^i\delta^{ij}$$

for orthonormal frames. Of course it shouldn't make any difference to our development since one can just identify TM with T^*M by using the metric. On the other hand, we could define $Cl(T^*M, \mathrm{b})$ even if b is a *degenerate* bilinear tensor and then we recover the Grassmann algebra bundle $\wedge T^*M$ in case $\mathrm{b} \equiv 0$. These comments should make it clear that $Cl(T^*M, \mathrm{g})$ is in general a sort of deformation of the Grassmann algebra bundle.

There are a couple of things to notice about $Cl(T^*M)$ when we realize it as $\wedge T^*M$ with a new product. First of all if $\alpha, \beta \in \wedge T^*M$ and $\langle\alpha, \beta\rangle = 0$ then $\alpha \cdot \beta = \alpha \wedge \beta$ where as if $\langle\alpha, \beta\rangle \neq 0$ then in general $\alpha\beta$ is not a homogeneous element. Second, $Cl(T^*M)$ is locally generated by $\{1\} \cup \{\theta^i\} \cup \{\theta^i\theta^j : i < j\} \cup \cdots \cup \{\theta^1\theta^2 \cdots \theta^n\}$ where $\theta^1, \theta^2, ..., \theta^n$ is a local orthonormal frame. Now we can immediately define our current objects of interest:

Definition 22.13 *A **bundle of modules over** $Cl(T^*M)$ is a vector bundle $\Sigma = (E, \pi, M)$ such that each fiber E_x is a module over the algebra $Cl(T_x^*M)$ and such that for each $\theta \in \Gamma(Cl(T^*M))$ and each $\sigma \in \Gamma(\Sigma)$ the map $x \mapsto \theta(x)\sigma(x)$ is smooth. Thus we have an induced map on smooth sections: $\Gamma(Cl(T^*M)) \times \Gamma(\Sigma) \to \Gamma(\Sigma)$.*

Proposition 22.1 *The bundle $Cl(T^*M)$ is a Clifford module over itself and the Levi-Civita connection ∇ on M induces a connection on $Cl(T^*M)$ (this connection is also denoted ∇) such that*

$$\nabla(\sigma_1\sigma_2) = (\nabla\sigma_1)\sigma_2 + \sigma_1\nabla\sigma_2$$

for all $\sigma_1, \sigma_2 \in \Gamma(Cl(T^*M))$. In particular, if $X, Y \in \mathfrak{X}(M)$ and $\sigma \in \Gamma(Cl(T^*M))$ then

$$\nabla_X(Y\sigma) = (\nabla_XY)\sigma + Y\nabla_X\sigma.$$

Proof. Realize $Cl(T^*M)$ as $\wedge T^*M$ with Clifford multiplication and let ∇ be usual induced connection on $\wedge T^*M \subset \otimes T^*M$. We have for an local orthonormal frame $e_1, ..., e_n$ with dual frame $\theta^1, \theta^2, ..., \theta^n$. Then $\nabla_\xi \theta^i = -\Gamma^i_j(\xi)\theta^j$

$$\begin{aligned}
\nabla_\xi(\theta^i\theta^j) &= \nabla_\xi(\theta^i \wedge \theta^j) \\
&= \nabla_\xi\theta^i \wedge \theta^j + \theta^i \wedge \nabla_\xi\theta^j \\
&= -\Gamma^i_k(\xi)\theta^k \wedge \theta^j - \Gamma^j_k(\xi)\theta^k \wedge \theta^i \\
&= -\Gamma^i_k(\xi)\theta^k\theta^j - \Gamma^j_k(\xi)\theta^k\theta^i = (\nabla_\xi\theta^i)\theta^j + \theta^i\nabla_\xi\theta^j
\end{aligned}$$

The result follows by linearity and a simple induction since a general section σ can be written locally as $\sigma = \sum a_{i_1 i_2 \ldots i_k}\theta^{i_1}\theta^{i_2}\cdots\theta^{i_k}$. ∎

Definition 22.14 *Let M, g be a (semi-) Riemannian manifold. A compatible connection for a* ***bundle of modules*** *Σ* ***over*** *$Cl(T^*M)$ is a connection ∇^Σ on Σ such that*

$$\nabla^\Sigma(\sigma \cdot s) = (\nabla\sigma) \cdot s + \sigma \cdot \nabla^\Sigma s$$

for all $s \in \Gamma(\Sigma)$ and all $\sigma \in \Gamma(Cl(T^*M))$

Definition 22.15 *Let $\Sigma = (E, \pi, M)$ be a bundle of modules over $Cl(T^*M)$ with a compatible connection $\nabla = \nabla^\Sigma$. The associated* ***Dirac operator*** *is defined as a differential operator Σ on by*

$$Ds := \sum \theta^i \cdot \nabla^\Sigma_{e_i} s$$

for $s \in \Gamma(\Sigma)$.

Notice that Clifford multiplication of $Cl(T^*M)$ on $\Sigma = (E, \pi, M)$ is a zeroth order operator and so is well defined as a fiberwise operation $Cl(T^*_xM) \times E_x \to E_x$.

There are still a couple of convenient properties that we would like to have. These are captured in the next definition.

Definition 22.16 *Let $\Sigma = (E, \pi, M)$ be a bundle of modules over $Cl(T^*M)$ such that Σ carries a Riemannian metric and compatible connection $\nabla = \nabla^\Sigma$. We call $\Sigma = (E, \pi, M)$ a* ***Dirac bundle*** *if the following equivalent conditions hold:*

1) $\langle es_1, es_2\rangle = \langle s_1, s_2\rangle$ for all $s_1, s_2 \in E_x$ and all $e \in T^*_xM \subset Cl(T^*_xM))$ with $|e| = 1$. In other words, Clifford multiplication by a unit (co)vector is required to be an isometry of the Riemannian metric on each fiber of Σ. Since , $e^2 = -1$ it follows that this is equivalent to requiring .

2) $\langle es_1, s_2\rangle = -\langle s_1, es_2\rangle$ for all $s_1, s_2 \in E_x$ and all $e \in T^*_xM \subset Cl(T^*_xM))$ with $|e| = 1$.

Assume in the sequel that q is nondegenerate. Let denote the subalgebra generated by all elements of the form $x_1 \cdots x_k$ with k even. And similarly, $Cl_1(\mathrm{V},q)$, with k odd. Thus $Cl(\mathrm{V},q)$ has the structure of a Z_2-graded algebra :

$$Cl(\mathrm{V},q) = Cl_0(\mathrm{V},q) \oplus Cl_1(\mathrm{V},q)$$

$$Cl_0(\mathrm{V},q) \cdot Cl_0(\mathrm{V},q) \subset Cl_0(\mathrm{V},q)$$
$$Cl_0(\mathrm{V},q) \cdot Cl_1(\mathrm{V},q) \subset Cl_1(\mathrm{V},q)$$
$$Cl_1(\mathrm{V},q) \cdot Cl_1(\mathrm{V},q) \subset Cl_0(\mathrm{V},q)$$

$Cl_0(\mathrm{V},q)$ and $Cl_1(\mathrm{V},q)$ are referred to as the **even** and **odd** part respectively. A Z_2-graded algebra is also called a **superalgebra**. There exists a fundamental automorphism α of $Cl(\mathrm{V},q)$ such that $\alpha(x) = -x$ for all $x \in \mathrm{V}$. Note that $\alpha^2 = \mathrm{id}$. It is easy to see that $Cl_0(\mathrm{V},q)$ and $Cl_1(\mathrm{V},q)$ are the $+1$ and -1 eigenspaces of $\alpha : Cl(\mathrm{V},q) \to Cl(\mathrm{V},q)$.

22.5.2 The Clifford group and spinor group

Let G be the group of all invertible elements $s \in C_K$ such that $s\mathrm{V}s^{-1} = \mathrm{V}$. This is called the **Clifford group** associated to q. The special Clifford group is $G^+ = G \cap C_0$. Now for every $s \in G$ we have a map $\phi_s : v \longrightarrow svs^{-1}$ for $v \in \mathrm{V}$. It can be shown that ϕ is a map from G into $\mathrm{O}(q)$, the orthogonal group of q. The kernel is the invertible elements in the center of C_K.

It is a useful and important fact that if $x \in G \cap \mathrm{V}$ then $q(x) \neq 0$ and $-\phi_x$ is reflection through the hyperplane orthogonal to x. Also, if s is in G^+ then ϕ_s is in $S\mathrm{O}(q)$. In fact, $\phi(G^+) = S\mathrm{O}(q)$.

Besides the fundamental automorphism α mentioned above, there is also a fundament anti-automorphism or **reversion** $\beta : Cl(\mathrm{V},q) \to Cl(\mathrm{V},q)$ which is determined by the requirement that $\beta(v_1 v_2 \cdots v_k) = v_k v_{k-1} \cdots v_1$ for $v_1, v_2, ..., v_k \in \mathrm{V} \subset Cl(\mathrm{V},q)$. We can use this anti-automorphism β to put a kind of "norm" on G^+;

$$N; G^+ \longrightarrow \mathbb{K}^*$$

where $\mathbb{K}^*$ is the multiplicative group of nonzero elements of $\mathbb{K}$ and $N(s) = \beta(s)s$. This is a homomorphism and if we "mod out" the kernel of N we get the so called reduced Clifford group G_0^+.

We now specialize to the real case $\mathbb{K} = \mathbb{R}$. The identity component of G_0^+ is called the **spin group** and is denoted by $\mathrm{Spin}(\mathrm{V}, q)$.

22.6 The Structure of Clifford Algebras

Now if $\mathbb{K} = \mathbb{R}$ and

$$q(x) = \sum_{i=1}^{r} (x_i)^2 - \sum_{i=r+1}^{r+s} (x_i)^2$$

we write $C(\mathbb{R}^{r+s}, q, \mathbb{R}) = Cl(r, s)$. Then one can prove the following isomorphisms.

$$Cl(r+1, s+1) \cong Cl(1,1) \otimes C(r,s)$$
$$Cl(s+2, r) \cong Cl(2,0) \otimes Cl(r,s)$$
$$Cl(s, r+2) \cong Cl(0,2) \otimes Cl(r,s)$$

and

$$Cl(p,p) \cong \bigotimes^{p} Cl(1,1)$$

$$Cl(p+k,p) \cong \bigotimes^{p} Cl(1,1) \bigotimes Cl(k,0)$$

$$Cl(k,0) \cong Cl(2,0) \otimes Cl(0,2) \otimes Cl(k-4,0) \qquad k>4$$

Using the above type of periodicity relations together with

$$Cl(2,0) \cong Cl(1,1) \cong M_2(\mathbb{R})$$

$$Cl(1,0) \cong \mathbb{R} \oplus \mathbb{R}$$

and

$$Cl(0,1) \cong \mathbb{C}$$

we can piece together the structure of $Cl(r,s)$ in terms of familiar matrix algebras. We leave out the resulting table since for one thing we are more interested in the simpler complex case. Also, we will explore a different softer approach below.

The complex case . In the complex case we have a much simpler set of relations;

$$Cl(2r) \cong Cl(r,r) \otimes \mathbb{C} \cong M_{2^r}(\mathbb{C})$$

$$Cl(2r+1) \cong Cl(1) \otimes Cl(2r)$$

$$\cong Cl(2r) \oplus Cl(2r) \cong M_{2^r}(\mathbb{C}) \oplus M_{2^r}(\mathbb{C})$$

These relations remind us that we may use matrices to represent our Clifford algebras. Lets return to this approach and explore a bit.

22.6.1 Gamma Matrices

Definition 22.17 *A set of real or complex matrices $\gamma_1, \gamma_2, ..., \gamma_n$ are called gamma matrices for $Cl(r,s)$ if*

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2g_{ij}$$

where $(g_{ij}) = diag(1,...,1,-1,...,)$ is the diagonalized matrix of signature (r,s).

Example 22.3 *Recall the Pauli matrices:*

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

It is easy to see that σ_1, σ_2 serve as gamma matrices for $Cl(0,2)$ while $-i\sigma_1, -i\sigma_2$ serve as gamma matrices for $Cl(2,0)$.

$Cl(2,0)$ is spanned as a vector space of matrices by $\sigma_0, -i\sigma_1, -i\sigma_2, -i\sigma_3$ and is (algebra) isomorphic to the quaternion algebra $\mathbb{H}$ under the identification

$$\begin{aligned} \sigma_0 &\mapsto 1 \\ -i\sigma_1 &\mapsto I \\ -i\sigma_2 &\mapsto J \\ -i\sigma_3 &\mapsto K \end{aligned}$$

22.7 Clifford Algebra Structure and Representation

22.7.1 Bilinear Forms

We will need some basic facts about bilinear forms. We review this here.

(1) Let E be a module over a commutative ring R. Typically E is a vector space over a field $\mathbb{K}$. A bilinear map $g : \mathrm{E} \times \mathrm{E} \longrightarrow R$ is called symmetric if $g(x,y) = g(y,x)$ and antisymmetric if $g(x,y) = -g(y,x)$ for $(x,y) \in \mathrm{E} \times \mathrm{E}$. If R has an automorphism of order two, $a \mapsto \bar{a}$ we say that g is Hermitian if $g(ax,y) = ag(x,y)$ and $g(x,ay) = \bar{a}g(x,y)$ for all $a \in R$ and $(x,y) \in \mathrm{E} \times \mathrm{E}$. If g is any of symmetric,antisymmetric,or Hermitian then the " left kernel" of g is equal to the "right kernel". That is

$$\begin{aligned} \ker g &= \{x \in \mathrm{E} : g(x,y) = 0 \quad \forall y \in \mathrm{E}\} \\ &= \{y \in \mathrm{E} : g(x,y) = 0 \quad \forall x \in \mathrm{E}\} \end{aligned}$$

If $\ker g = 0$ we say that g is nondegenerate. In case E is a vector space of finite dimension g is nondegenerate if and only if $x \mapsto g(x,\cdot) \in \mathrm{E}^*$ is an isomorphism. An orthogonal basis for g is a basis $\{v_i\}$ for E such that $g(v_i, v_i) = 0$ for $i \neq j$.

Definition 22.18 *Let E be a vector space over a three types above. If $\mathrm{E} = \mathrm{E} \oplus \mathrm{E}_2$ for subspaces $\mathrm{E}_i \subset \mathrm{E}$ and $g(x_1, x_2) = 0 \qquad \forall x_1 \in \mathrm{E}, x_2 \in \mathrm{E}_2$ then we write*

$$\mathrm{E} = \mathrm{E}_1 \perp \mathrm{E}_2$$

and say that E is the orthogonal direct sum of E_1 and E_2.

Proposition 22.2 *Suppose E, g is as above with*

$$\mathrm{E} = \mathrm{E}_1 \perp \mathrm{E}_2 \perp \cdots \perp \mathrm{E}_k$$

Then g is non-degenerate if and only if its restrictions $g|_{\mathrm{E}_i}$ are and

$$\ker \mathrm{E} = \mathrm{E}_1^o \perp \mathrm{E}_2^o \perp \cdots \perp \mathrm{E}_k^o$$

Proof. Nearly obvious. ■

Terminology: If g is one of symmetric, antisymmetric or Hermitian we say that g is geometric.

Proposition 22.3 *Let g be a geometric bilinear form on a vector space* E *(over* $\mathbb{K}$*). Suppose g is nondegenerate. Then g is nondegenerate on a subspace F if and only if* $\mathrm{E} = F \perp F^{\perp}$ *where*

$$F^{\perp} = \{x \in \mathrm{E} : g(x, f) = 0 \qquad \forall f \in F\}$$

Definition 22.19 *A map q is called quadratic if there is a symmetric g such that $q(x) = g(x, x)$. Note that g can be recovered from q:*

$$2g(x, y) = q(x + y) - q(x) - q(y)$$

22.7.2 Hyperbolic Spaces And Witt Decomposition

E, g is a vector space with symmetric form g. If E has dimension 2 we call E a hyperbolic plane. If $\dim \mathrm{E} \geq 2$ and $\mathrm{E} = \mathrm{E}_1 \perp \mathrm{E}_2 \perp \cdots \perp \mathrm{E}_k$ where each E_i is a hyperbolic plane for $g|_{\mathrm{E}_i}$ then we call E a hyperbolic space. For a hyperbolic plane one can easily construct a basis f_1, f_2 such that $g(f_1, f) = g(f_2, f_2) = 0$ and $g(f_1, f_2) = 1$. So that with respect to this basis g is given by the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This pair $\{f_1, f_2\}$ is called a hyperbolic pair for E, g. Now we return to $\dim\mathrm{E} \geq 2$. Let $\operatorname{rad} F \equiv F^{\perp} \cap F = \ker g|_F$

Lemma 22.1 *There exists a subspace $U \subset \mathrm{E}$ such that* $\mathrm{E} = \operatorname{rad} \mathrm{E} \perp U$ *and U is nondegenerate.*

Proof. It is not to hard to see that $\operatorname{rad} U = \operatorname{rad} U^{\perp}$. If $\operatorname{rad} U = 0$ then $\operatorname{rad} U^{\perp} = 0$ and vice versa. Now $U + U^{\perp}$ is clearly direct since $0 = \operatorname{rad} U = U \cap U^{\perp}$. Thus $\mathrm{E} = U \perp U^{\perp}$. ∎

Lemma 22.2 *Let g be nondegenerate and $U \subset \mathrm{E}$ some subspace. Suppose that $U = \operatorname{rad} U \perp W$ where $\operatorname{rad} W = 0$. Then given a basis $\{u_1, \cdots, u_s\}$ for $\operatorname{rad} U$ there exists $v_1, \cdots, v_s \in W^{\perp}$ such that each $\{u_i, v_i\}$ is a hyperbolic pair. Let $P_i = span\{u_i, v_i\}$. Then*

$$\mathrm{E} = W \perp P_1 \perp \cdots \perp P_s.$$

Proof. Let $W_1 = \operatorname{span}\{u_2, u_3, \cdots, u_s\} \oplus W$. Then $W_1 \subsetneq \operatorname{rad} U \oplus W$ so $(\operatorname{rad} U \oplus W)^{\perp} \subsetneq W_1^{\perp}$. Let $w_1 \in W_1^{\perp}$ but assume $w_1 \notin (\operatorname{rad} U \oplus W)^{\perp}$. Then we have $g(u_1, w_1) \neq 0$ so that $P_1 = span\{u_1, w_1\}$ is a hyperbolic plane. Thus we can find v_1 such that u_1, v_1 is a hyperbolic pair for P_1. We also have

$$U_1 = (u_2, u_3 \cdots u_s) \perp P_1 \perp W$$

so we can proceed inductively since $u_2, U_3, \ldots u_s \in \operatorname{rad} U_1$. ∎

Definition 22.20 *A subspace $U \subset \mathrm{E}$ is called totally isotropic if $g|_U \equiv 0$.*

Proposition 22.4 *(Witt decomposition) Suppose that $U \subset \mathrm{E}$ is a maximal totally isotropic subspace and $e_1, e_2, \ldots e_r$ a basis for U. Then there exist (null) vectors $f_1, f_2, \ldots, f_r$ such that each $\{e_i, f_i\}$ is a hyperbolic pair and $U' = span\{f_i\}$ is totally isotropic. Further*

$$\mathrm{E} = U \oplus U' \perp G$$

where $G = (U \oplus U')^{\perp}$.

Proof. Using the proof of the previous theorem we have $\operatorname{rad} U = U$ and $W = 0$. The present theorem now follows. ■

Proposition 22.5 *If g is symmetric then $g|_G$ is definite.*

Example 22.4 *Let $\mathrm{E}, g = \mathbb{C}^{2k}, g_0$ where*

$$g_0(z, w) = \sum_{i=1}^{2k} z_i w_i$$

Let $\{e_1, ..., e_k, e_{k+1}, ..., e_{2k}\}$ be the standard basis of $\mathbb{C}^{2k}$. Define

$$\epsilon_j = \frac{1}{\sqrt{2}}(e_j + ie_{k+j}) \qquad j = 1, ..., k$$

and

$$\eta_j = \frac{1}{\sqrt{2}}(e_i - ie_{k+j}).$$

Then letting $F = span\{\epsilon_i\}$, $F' = span\{\eta_j\}$ we have $\mathbb{C}^{2k} = F \oplus F'$ and F is a maximally isotropic subspace. Also, each $\{\epsilon_j, \eta_j\}$ is a hyperbolic pair.

This is the most important example of a neutral space:

Proposition 22.6 *A vector space E with quadratic form is called neutral if the rank, that is, the dimension of a totally isotropic subspace, is $r = dim\mathrm{E}/2$. The resulting decomposition $F \oplus F'$ is called a (weak) polarization.*

22.7.3 Witt's Decomposition and Clifford Algebras

Even Dimension Suppose that V, Q is quadratic space over $\mathbb{K}$. Let $\dim \mathrm{V} = r$ and suppose that V, Q is neutral. Then we have that $C_{\mathbb{K}}$ is isomorphic to $\mathrm{End}(S)$ for an r dimensional space S (spinor space). In particular, $C_{\mathbb{K}}$ is a simple algebra.

Proof. Let $F \oplus F'$ be a polarization of V. Here, F and F' are maximal totally isotropic subspaces of V. Now let $\{x_1, ..., x_r, y_1, ..., y_r\}$ be a basis for V such that $\{x_i\}$ is a basis for F and $\{y_i\}$ a basis for F'. Set $f = y_1 y_2 \cdots y_h$. Now let S be the span of elements of the form $x_{i_1} x_{i_2} \cdots x_{i_r} f$ where $1 \leq i_1 < ... < i_h \leq r$. S is an ideal of $C_{\mathbb{K}}$ of dimension 2^r. We define a representation ρ of $C_{\mathbb{K}}$ in S by

$$\rho(u)s = us$$

This can be shown to be irreducible so that we have the desired result. ■

Now since we are interested in Spin which sits inside and in fact generates C_0 we need the following

Proposition 22.7 *C_0 is isomorphic to* $\mathrm{End}(S^+) \times \mathrm{End}(S^-)$ *where* $S^+ = C_0 \cap S$ *and* $S^- = C_1 \cap S$.

This follows from the obvious fact that each of $C_0 f$ and $C_1 f$ are invariant under multiplication by C_0.

Now consider a real quadratic space V, Q where Q is positive definite. We have $\mathrm{Spin}(n) \subset Cl^0(0) \subset C_0$ and $\mathrm{Spin}(n)$ generates C_0. Thus the complex spin group representation of is just given by restriction and is semisimple factoring as $S^+ \oplus S^-$.

Odd Dimension In the odd dimensional case we can not expect to find a polarization but this cloud turns out to have a silver lining. Let x_0 be a non-isotropic vector from V and set $\mathrm{V}_1 = (x_0)^\perp$. On V_1 we define a quadratic form Q_1 by

$$Q_1(y) = -Q(x_0)Q(y)$$

for $y \in \mathrm{V}_1$. It can be shown that Q_1 is non-degenerate. Now notice that for $y \in \mathrm{V}_1$ then $x_0 y = -y x_0$ and further

$$(x_0 y)^2 = -x_0^2 y^2 = -Q(x_0)Q(y) = Q_1(y)$$

so that by the universal mapping property the map

$$y \longrightarrow x_0 y$$

can be extended to an algebra morphism h from $Cl(Q_1, \mathrm{V}_1)$ to $C_{\mathbb{K}}$. Now these two algebras have the same dimension and since C_o is simple it must be an isomorphism. Now if Q has rank r then Q_1, V_1 is neutral and we obtain the following

Theorem 22.4 *If the dimension of* V *is odd and Q has rank r then C_0 is represented irreducibly in a space S^+ of dimension 2^r. In particular $C_0 \cong$ End(S^+).*

22.7.4 The Chirality operator

Let V be a Euclidean vector space with associated positive definite quadratic form Q. Let $\{e_1, ..., e_n\}$ be an oriented orthonormal. frame for V. We define the Chirality operator τ to be multiplication in the associated (complexified) Clifford algebra by the element

$$\tau = (\sqrt{-1})^{n/2} e_1 \cdots e_n$$

if n is even and by

$$\tau = (\sqrt{-1})^{(n+1)/2} e_1 \cdots e_n$$

if n is odd. Here $\tau \in Cl(n)$ and does not depend on the choice of orthonormal. oriented frame. We also have that $\tau v = -v\tau$ for $v \in$ V and $\tau^2 = 1$.

Let us consider the case of n even. Now we have seen that we can write V $\otimes C = F \oplus \bar{F}$ where F is totally isotropic and of dimension n. In fact we may assume that F has a basis $\{e_{2j-1} - ie_{2j} : 1 \leq j \leq n/2\}$, where the e_i come from an oriented orthonormal basis. Lets use this polarization to once again construct the spin representation.

First note that Q (or its associated bilinear form) places F and $\bar{F}$ in duality so that we can identify $\bar{F}$ with the dual space F'. Now set $S = \wedge F$. First we show how V act on S. Given $v \in$ Vconsider $v \in$ V$\otimes C$ and decompose $v = w + \bar{w}$ according to our decomposition above. Define $\phi_w s = \sqrt{2} w \wedge s$ and

$$\phi_{\bar{w}} s = -\iota(\bar{w}) s.$$

where ι is interior multiplication. Now extend ϕ linearly to V. Exercise Show that ϕ extends to a representation of $C \otimes Cl(n)$. Show that $S^+ = \wedge^+ F$ is invariant under C_0. It turns out that ϕ_τ is $(-1)^k$ on $\wedge^k F$

22.7.5 Spin Bundles and Spin-c Bundles

22.7.6 Harmonic Spinors

Chapter 23

Classical Mechanics

Every body continues in its state of rest or uniform motion in a straight line, except insofar as it doesn't.

Arthur Eddington, Sir

23.1 Particle motion and Lagrangian Systems

If we consider a *single* particle of mass m then Newton's law is

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}(\mathbf{x}(t), t)$$

The force $\mathbf{F}$ is **conservative** if it doesn't depend on time and there is a potential function $V : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F}(\mathbf{x}) = -\operatorname{grad} V(\mathbf{x})$. Assume this is the case. Then Newton's law becomes

$$m\frac{d^2}{dt^2}\mathbf{x}(t) + \operatorname{grad} V(\mathbf{x}(t)) = 0.$$

Consider the Affine space $C(I, \mathbf{x}_1, \mathbf{x}_2)$ of all C^2 paths from $\mathbf{x}_1$ to $\mathbf{x}_2$ in $\mathbb{R}^3$ defined on the interval $I = [t_1, t_2]$. This is an Affine space modeled on the Banach space $C_0^r(I)$ of all C^r functions $\varepsilon : I \to \mathbb{R}^3$ with $\varepsilon(t_1) = \varepsilon(t_1) = 0$ and with the norm

$$\|\varepsilon\| = \sup_{t\in I}\{|\varepsilon(t)| + |\varepsilon'(t)| + |\varepsilon''(t)|\}.$$

If we define the fixed affine linear path $\mathbf{a} : I \to \mathbb{R}^3$ by $\mathbf{a}(t) = \mathbf{x}_1 + \frac{t-t_1}{t_2-t_1}(\mathbf{x}_2 - \mathbf{x}_1)$ then all we have a coordinatization of $C(I, \mathbf{x}_1, \mathbf{x}_2)$ by $C_0^r(I)$ given by the single chart $\psi : \mathbf{c} \mapsto \mathbf{c} - \mathbf{a} \in C_0^2(I)$. Then the tangent space to $C(I, \mathbf{x}_1, \mathbf{x}_2)$ at a fixed path c_0 is just $C_0^2(I)$. Now we have the function S defined on $C(I, \mathbf{x}_1, \mathbf{x}_2)$ by

$$S(\mathbf{c}) = \int_{t_1}^{t_2} \left(\frac{1}{2}m\,\|\mathbf{c}'(t)\|^2 - V(\mathbf{c}(t))\right) dt$$

The variation of S is just the 1-form $\delta S : C_0^2(I) \to \mathbb{R}$ defined by

$$\delta S \cdot \varepsilon = \left.\frac{d}{d\tau}\right|_{\tau=0} S(\mathbf{c}_0 + \tau\varepsilon)$$

Let us suppose that $\delta S = 0$ at c_0. Then we have

$$\begin{aligned}
0 &= \left.\frac{d}{d\tau}\right|_{\tau=0} S(\mathbf{c}_0'(t) + \tau\varepsilon) \\
&= \left.\frac{d}{d\tau}\right|_{\tau=0} \int_{t_1}^{t_2} \left(\frac{1}{2}m\left\|\mathbf{c}_0'(t) + \tau\varepsilon'(t)\right\|^2 - V(\mathbf{c}_0(t) + \tau\varepsilon(t))\right) dt \\
&= \int_{t_1}^{t_2} \left(m\left\langle \mathbf{c}_0'(t), \varepsilon'(t)\right\rangle - \frac{\partial V}{\partial x^i}(\mathbf{c}_0)\frac{d\varepsilon^i}{dt}(0)\right) dt \\
&= \int_{t_1}^{t_2} \left(m\mathbf{c}_0'(t) \cdot \frac{d}{dt}\varepsilon(t) - \operatorname{grad} V(\mathbf{c}_0(t)) \cdot \varepsilon(t)\right) dt \\
&\int_{t_1}^{t_2} (m\mathbf{c}_0''(t) - \operatorname{grad} V(\mathbf{c}_0(t))) \cdot \varepsilon(t) dt
\end{aligned}$$

Now since this is true for every choice of $\varepsilon \in C_0^2(I)$ we see that

$$m\mathbf{c}_0''(t) - \operatorname{grad} V(\mathbf{c}_0(t)) = 0$$

thus we see that $\mathbf{c}_0(t) = \mathbf{x}(t)$ is a critical point in $C(I, \mathbf{x}_1, \mathbf{x}_2)$, that is, a stationary path, if and only if **??** is satisfied.

23.1.1 Basic Variational Formalism for a Lagrangian

In general we consider a differentiable manifold Q as our state space and then a Lagrangian density function L is given on TQ. For example we can take a potential function $V : Q \to \mathbb{R}$, a Riemannian metric g on Q and define the action functional S on the space of smooth paths $I \to Q$ beginning and ending at a fixed points p_1 and p_2 given by

$$\begin{aligned}
S(c) &= \int_{t_1}^{t_2} \mathrm{L}(c'(t))dt = \\
&\int_{t_1}^{t_2} \frac{1}{2}m\left\langle c'(t), c'(t)\right\rangle - V(c(t))dt
\end{aligned}$$

The tangent space at a fixed c_0 is the Banach space $\Gamma_0^2(c_0^*TQ)$ of C^2 vector fields ε: $I \to TQ$ along c_0 that vanish at t_1 and t_2. A curve with tangent ε at c_0 is just a variation $v : (-\epsilon, \epsilon) \times I \to Q$ such that $\varepsilon(t) = \left.\frac{\partial}{\partial s}\right|_{s=0} v(s,t)$ is the variation vector field. Then we have

$$\begin{aligned}
\delta S \cdot \varepsilon &= \left.\frac{\partial}{\partial s}\right|_{s=0} \int_{t_1}^{t_2} \mathrm{L}\left(\frac{\partial v}{\partial t}(s,t)\right) dt \\
&= \int_{t_1}^{t_2} \left.\frac{\partial}{\partial s}\right|_{s=0} \mathrm{L}\left(\frac{\partial v}{\partial t}(s,t)\right) dt \\
&= etc.
\end{aligned}$$

Let us examine this in the case of $Q = U \subset \mathbb{R}^n$. With $\mathbf{q} = (q^1, ...q^n)$ being (general curvilinear) coordinates on U we have natural (tangent bundle chart) coordinates $\mathbf{q}, \dot{\mathbf{q}}$ on $TU = U \times \mathbb{R}^n$. Assume that the variation has the form $\mathbf{q}(s,t) = \mathbf{q}(t) + s\varepsilon(t)$. Then we have

$$\begin{aligned}
\delta S \cdot \varepsilon &= \left.\frac{\partial}{\partial s}\right|_{s=0} \int_{t_1}^{t_2} \mathrm{L}(\mathbf{q}(s,t), \dot{\mathbf{q}}(s,t)) dt \\
&\int_{t_1}^{t_2} \left.\frac{\partial}{\partial s}\right|_{s=0} \mathrm{L}(\mathbf{q} + s\varepsilon, \dot{\mathbf{q}} + s\varepsilon) dt \\
&= \int_{t_1}^{t_2} (\frac{\partial \mathrm{L}}{\partial \mathbf{q}}(\mathbf{q}, \dot{\mathbf{q}}) \cdot \varepsilon + \frac{\partial \mathrm{L}}{\partial \dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}) \cdot \varepsilon) dt \\
&= \int_{t_1}^{t_2} (\frac{\partial \mathrm{L}}{\partial \mathbf{q}}(\mathbf{q}, \dot{\mathbf{q}}) \cdot \varepsilon - \frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}) \cdot \varepsilon) dt \\
&= \int_{t_1}^{t_2} (\frac{\partial \mathrm{L}}{\partial \mathbf{q}}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}})) \cdot \varepsilon dt
\end{aligned}$$

and since ε was arbitrary we get the Euler-Lagrange equations for the motion

$$\frac{\partial \mathrm{L}}{\partial \mathbf{q}}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}) = 0$$

In general, a time-independent Lagrangian on a manifold Q is a smooth function on the tangent bundle:

$$\mathrm{L} : TQ \to Q$$

and the associated action functional is a map from the space of smooth curves $C^\infty([a,b], Q)$ defined for $c : [a,b] \to Q$ by

$$\mathcal{S}_L(c) = \int_a^b \mathrm{L}(\dot{c}(t)) dt$$

where $\dot{c} : [a,b] \to TQ$ is the canonical lift (velocity). A time dependent Lagrangian is a smooth map

$$L : \mathbb{R} \times TQ \to Q$$

where the first factor $\mathbb{R}$ is the time t, and once again we have the associated action functional $\mathcal{S}_L(c) = \int_a^b L(t, \dot{c}(t)) dt$.

Let us limit ourselves initially to the time independent case.

Definition 23.1 *A smooth variation of a curve $c : [a, b] \to Q$ is a smooth map $\nu : [a, b] \times (-\epsilon, \epsilon) \to Q$ for small ϵ such that $\nu(t, 0) = c(t)$. We call the variation a* ***variation with fixed endpoints*** *if $\nu(a, s) = c(a)$ and $\nu(b, s) = c(b)$ for all $s \in (-\epsilon, \epsilon)$. Now we have a family of curves $\nu_s = \nu(., s)$. The infinitesimal variation at ν_0 is the vector field along c defined by $V(t) = \frac{d\nu}{ds}(t, 0)$. This V is called the* ***variation vector field*** *for the variation. The differential of the functional δS_L (classically called the first variation) is defined as*

$$\begin{aligned}\delta S_{\mathrm{L}}(c) \cdot V &= \left.\frac{d}{ds}\right|_{s=0} S_{\mathrm{L}}(\nu_s) \\ &= \left.\frac{d}{ds}\right|_{s=0} \int_a^b \mathrm{L}(\nu_s(t))dt\end{aligned}$$

Remark 23.1 *Every smooth vector field along c is the variational vector field coming from some variation of c and for any other variation ν' with $V(t) = \frac{d\nu'}{ds}(t, 0)$ the about computed quantity $\delta S_L(c) \cdot V$ will be the same.*

At any rate, if $\delta S_L(c) \cdot V = 0$ for all variations vector fields V along c and vanishing at the endpoints then we write $\delta S_L(c) = 0$ and call c **critical** (or **stationary**) for L.

Now consider the case where the image of c lies in some coordinate chart $U, \psi = q^1, q^2, ...q^n$ and denote by $TU, T\psi = (q^1, q^2, ..., q^n, \dot{q}^1, \dot{q}^2, ..., \dot{q}^n)$ the natural chart on $TU \subset TQ$. In other words, $T\psi(\xi) = (q^1 \circ \tau(\xi), q^2 \circ \tau(\xi), ..., q^n \circ \tau(\xi), dq^1(\xi), dq^2(\xi), ..., dq^n(\xi))$. Thus the curve has coordinates

$$(c, \dot{c}) = (q^1(t), q^2(t), ..., q^n(t), \dot{q}^1(t), \dot{q}^2(t), ..., \dot{q}^n(t))$$

where now the $\dot{q}^i(t)$ *really are* time derivatives. In this local situation we can choose our variation to have the form $q^i(t) + s\delta q^i(t)$ for some functions $\delta q^i(t)$ vanishing at a and b and some parameter s with respect to which we will differentiate. The lifted variation is $(\mathbf{q}(t) + s\delta(t), \dot{\mathbf{q}}(t) + s\delta\dot{\mathbf{q}}(t))$ which is the obvious abbreviation for a path in $T\psi(TU) \subset \mathbb{R}^n \times \mathbb{R}^n$. Now we have seen above that the path c will be critical if

$$\left.\frac{d}{ds}\right|_{s=0} \int \mathrm{L}(\mathbf{q}(t) + s\delta(t), \dot{\mathbf{q}}(t) + s\delta\dot{\mathbf{q}}(t))dt = 0$$

for all such variations and the above calculations lead to the result that

$$\frac{\partial}{\partial \mathbf{q}}\mathrm{L}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) - \frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{q}}}\mathrm{L}(\mathbf{q}(t), \dot{\mathbf{q}}(\mathbf{t})) = \mathbf{0} \quad \text{Euler-Lagrange}$$

for any L-critical path (with image in this chart). Here $n = \dim(Q)$.

It can be show that even in the case that the image of c does not lie in the domain of a chart that c is L-critical path if it can be subdivided into sub-paths lying in charts and L-critical in each such chart.

23.1.2 Two examples of a Lagrangian

Example 23.1 *Suppose that we have a 1-form $\theta \in \mathfrak{X}^*(Q)$. A 1-form is just a map $\theta : TQ \to \mathbb{R}$ that happens to be linear on each fiber T_pQ. Thus we may examine the special case of $\mathrm{L} = \theta$. In canonical coordinates $(q, \dot{q})$ again,*

$$\mathrm{L} = \theta = \sum a_i(q) dq^i$$

for some functions $a_i(q)$. An easy calculation shows that the Euler-Lagrange equations become

$$\left(\frac{\partial a_i}{\partial q^k} - \frac{\partial a_k}{\partial q^i} \right) \dot{q}^i = 0$$

but on the other hand

$$d\theta = \frac{1}{2} \sum \left(\frac{\partial a_j}{\partial q^i} - \frac{\partial a_i}{\partial q^j} \right) \partial q^i \wedge \partial q^j$$

and one can conclude that if $c = (q^i(t))$ is critical for $L = \theta$ then for any vector field X defined on the image of c we have

$$\frac{1}{2} \sum \left(\frac{\partial a_j}{\partial q^i}(q^i(t)) - \frac{\partial a_i}{\partial q^j}(q^i(t)) \right) \dot{q}^i(t) X^j$$

or $d\theta(\dot{c}(t), X) = 0$. This can be written succinctly as

$$\iota_{\dot{c}(t)} d\theta = 0.$$

Example 23.2 *Now let us take the case of a Riemannian manifold M, g and let $\mathrm{L}(v) = \frac{1}{2}\mathrm{g}(v, v)$. Thus the action functional is the "energy"*

$$S_g(c) = \int g(\dot{c}(t), \dot{c}(t)) dt$$

In this case the critical paths are just geodesics.

23.2 Symmetry, Conservation and Noether's Theorem

Let G be a Lie group acting on a smooth manifold M.

$$\lambda : G \times M \to M$$

As usual we write $g \cdot x$ for $\lambda(g, x)$. We have a fundamental vector field $\xi^\natural$ associated to every $\xi \in \mathfrak{g}$ defined by the rule

$$\xi^\natural(p) = T_{(e,p)}\lambda \cdot (., 0)$$

or equivalently by the rule

$$\xi^{\natural}(p) = \left.\frac{d}{dt}\right|_0 \exp(t\xi) \cdot p$$

The map $\natural : \xi \mapsto \xi^{\natural}$ is a Lie algebra anti-homomorphism. Of course, here we are using the flow associated to ξ

$$\varphi^{\xi}(t,p) := \varphi^{\xi^{\natural}}(t,p) = \exp(t\xi) \cdot p$$

and it should be noted that $t \mapsto \exp(t\xi)$ is the one parameter subgroup associated to ξ and to get the corresponding left invariant vector field $X^{\xi} \in \mathfrak{X}^L(G)$ we act on the right:

$$X^{\xi}(g) = \left.\frac{d}{dt}\right|_0 g \cdot \exp(t\xi)$$

Now a diffeomorphism acts on a covariant k-tensor field contravariantly according to

$$(\phi^* K)(p)(v_1, ...v_k) = K(\phi(p))(T\phi v_1, ...T\phi v_k)$$

Suppose that we are given a covariant tensor field $\Upsilon \in \mathfrak{T}(M)$ on M. We think of Υ as defining some kind of extra structure on M. The two main examples for our purposes are

1. $\Upsilon = \langle .,. \rangle$ a nondegenerate covariant symmetric 2-tensor. Then $M, \langle .,. \rangle$ is a (semi-) Riemannian manifold.

2. $\Upsilon = \omega \in \Omega^2(M)$ a non-degenerate 2-form. Then M, ω is a **symplectic manifold**.

Then G acts on Υ since G acts on M as diffeomorphisms. We denote this natural (left) action by $g \cdot \Upsilon$. If $g \cdot \Upsilon = \Upsilon$ for all $g \in G$ we say that G acts by symmetries of the pair M, Υ.

Definition 23.2 *In general, a vector field X on M, Υ is called an* ***infinitesimal symmetry*** *of the pair M, Υ if $\mathcal{L}_X \Upsilon = 0$. Other terminology is that X is a $\Upsilon-$**Killing field** . The usual notion of a Killing field in (pseudo-) Riemannian geometry is the case when $\Upsilon = \langle , \rangle$ is the metric tensor.*

Example 23.3 *A group G is called a symmetry group of a symplectic manifold M, ω if G acts by symplectomorphisms so that $g \cdot \omega = \omega$ for all $g \in G$. In this case, each $\xi \in \mathfrak{g}$ is an infinitesimal symmetry of M, ω meaning that*

$$\mathcal{L}_{\xi} \omega = 0$$

where $\mathcal{L}_{\xi}$ is by definition the same as $\mathcal{L}_{\xi^{\natural}}$. This follows because if we let $g_t = \exp(t\xi)$ then each g_t is a symmetry so $g_t^* \omega = 0$ and

$$\mathcal{L}_{\xi} \omega = \left.\frac{d}{dt}\right|_0 g_t^* \omega = 0$$

23.2.1 Lagrangians with symmetries.

We need two definitions

Definition 23.3 *If $\phi : M \to M$ is a diffeomorphism then the induced tangent map $T\phi : TM \to TM$ is called the* ***canonical lift***.

Definition 23.4 *Given a vector field $X \in \mathfrak{X}(M)$ there is a lifting of X to $\widetilde{X} \in \mathfrak{X}(TM) = \Gamma(TM, TTM)$*

$$\begin{array}{lll} \widetilde{X}: & TM \to & TTM \\ & \downarrow & \downarrow \\ X: & M \to & TM \end{array}$$

such that the flow $\varphi^{\widetilde{X}}$ is the canonical lift of φ^X

$$\begin{array}{lll} \varphi_t^{\widetilde{X}}: & TM \to & TM \\ & \downarrow & \downarrow \\ \varphi_t^X: & M \to & M \end{array}.$$

In other words, $\varphi^{\widetilde{X}} = T\varphi_t^X$. We simply define $\widetilde{X}(v) = \frac{d}{dt}(T\varphi_t^X \cdot v)$.

Definition 23.5 *Let ω_{L} denote the unique 1-form on Q that in canonical coordinates is $\omega_{\mathrm{L}} = \sum_{i=1}^n \frac{\partial \mathrm{L}}{\partial \dot{q}^i} dq^i$.*

Theorem 23.1 (E. Noether) *If X is an infinitesimal symmetry of the Lagrangian then the function $\omega_{\mathrm{L}}(\widetilde{X})$ is constant along any path $c : I \subset \mathbb{R}$ that is stationary for the action associated to* L.

Let's prove this using local coordinates $(q^i, \dot{q}^i)$ for $TU_\alpha \subset TQ$. It turn out that locally,

$$\widetilde{X} = \sum_i \left(a^i \frac{\partial}{\partial q^i} + \sum_j \frac{\partial a^i}{\partial q^j} \frac{\partial}{\partial \dot{q}^i} \right)$$

where a^i is defined by $X = \sum a^i(q) \frac{\partial}{\partial q^i}$. Also, $\omega_L(\widetilde{X}) = \sum a^i \frac{\partial L}{\partial \dot{q}^i}$. Now suppose that $q^i(t), \dot{q}^i(t) = \frac{d}{dt} q^i(t)$ satisfies the Euler-Lagrange equations. Then

$$\begin{aligned} & \frac{d}{dt}\omega_{\mathrm{L}}(\widetilde{X})(q^i(t), \dot{q}^i(t)) \\ & = \frac{d}{dt} \sum a^i(q^i(t)) \frac{\partial \mathrm{L}}{\partial \dot{q}^i}(q^i(t), \dot{q}^i(t)) \\ & = \sum \frac{da^i}{dt}(q^i(t)) \frac{\partial \mathrm{L}}{\partial \dot{q}^i}(q^i(t), \dot{q}^i(t)) + a^i(q^i(t)) \frac{d}{dt} \frac{\partial \mathrm{L}}{\partial \dot{q}^i}(q^i(t), \dot{q}^i(t)) \\ & = \sum_i \left[\sum_j \frac{da^i}{dq^j} \dot{q}^j(t) \frac{\partial \mathrm{L}}{\partial \dot{q}^i}(q^i(t), \dot{q}^i(t)) + a^i(q^i(t)) \frac{\partial \mathrm{L}}{\partial q^i}(q^i(t), \dot{q}^i(t)) \right] \\ & = d\mathrm{L}(X) = \mathcal{L}_X \mathrm{L} = 0 \end{aligned}$$

This theorem tells us one case when we get a conservation law. A conservation law is a function C on TQ (or T^*Q for the Hamiltonian flow) such that C is constant along solution paths. (i.e. stationary for the action or satisfying the Euler-Lagrange equations.)

$$\mathrm{L} : TQ \to Q$$

let $X \in T(TQ)$.

23.2.2 Lie Groups and Left Invariants Lagrangians

Recall that G act on itself by left translation $l_g : G \to G$. The action lifts to the tangent bundle $Tl_g : TG \to TG$. Suppose that $\mathrm{L} : TG \to \mathbb{R}$ is invariant under this left action so that $\mathrm{L}(Tl_g X_h) = \mathrm{L}(X_p)$ for all $g, h \in G$. In particular, $\mathrm{L}(Tl_g X_e) = L(X_e)$ so L is completely determined by its restriction to $T_eG = \mathfrak{g}$. Define the restricted Lagrangian function by $\Lambda = \mathrm{L}|_{T_eG}$. We view the differential $d\Lambda$ as a map $d\Lambda : \mathfrak{g} \to \mathbb{R}$ and so in fact $d\lambda \in \mathfrak{g}^*$. Next, recall that for any $\xi \in \mathfrak{g}$ the map $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{ad}_\xi v = [\xi, v]$ and we have the adjoint map $\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$. Now let $t \mapsto g(t)$ be a motion of the system and define the "body velocity" by $\nu_c(t) = Tl_{c(t)^{-1}} \cdot c'(t) = \omega_G(c'(t))$. Then we have

Theorem 23.2 *Assume L is invariant as above. The curve $c(.)$ satisfies the Euler-Lagrange equations for* L *if and only if*

$$\frac{d}{dt} d\Lambda(\nu_c(t)) = \mathrm{ad}^*_{\nu_c(t)} d\Lambda$$

23.3 The Hamiltonian Formalism

Let us now examine the change to a description in cotangent chart $\mathbf{q}, \mathbf{p}$ so that for a covector at $\mathbf{q}$ given by $\mathbf{a}(\mathbf{q}) \cdot d\mathbf{q}$ has coordinates $\mathbf{q}, \mathbf{a}$. Our method of transfer to the cotangent side is via the Legendre transformation induced by L. In fact, this is just the fiber derivative defined above. We must assume that the map $F : (\mathbf{q}, \dot{\mathbf{q}}) \mapsto (\mathbf{q}, \mathbf{p}) = (\mathbf{q}, \frac{\partial L}{\partial \dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}))$ is a diffeomorphism (this is written with respect to the two natural charts on TU and T^*U). Again this just means that the Lagrangian is nondegenerate. Now if $v(t) = (\mathbf{q}(t), \dot{\mathbf{q}}(t))$ is (a lift of) a solution curve then defining the **Hamiltonian** function

$$\widetilde{H}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial \mathrm{L}}{\partial \dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}) \cdot \dot{\mathbf{q}} - \mathrm{L}(\mathbf{q}, \dot{\mathbf{q}})$$

we compute with $\dot{\mathbf{q}} = \frac{d}{dt}\mathbf{q}$

$$\begin{aligned}
\frac{d}{dt}\widetilde{H}(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{d}{dt}(\frac{\partial \mathrm{L}}{\partial \dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}) \cdot \dot{\mathbf{q}}) - \frac{d}{dt}\mathrm{L}(\mathbf{q}, \dot{\mathbf{q}}) \\
&= \frac{\partial \mathrm{L}}{\partial \dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}) \frac{d}{dt}\dot{\mathbf{q}} + \frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \dot{\mathbf{q}}}(\mathbf{q}, \dot{\mathbf{q}}) \cdot \dot{\mathbf{q}} - \frac{d}{dt}\mathrm{L}(\mathbf{q}, \dot{\mathbf{q}}) \\
&= 0
\end{aligned}$$

we have used that the Euler-Lagrange equations $\frac{\partial \mathrm{L}}{\partial \mathbf{q}}(\mathbf{q},\dot{\mathbf{q}})-\frac{d}{dt}\frac{\partial \mathrm{L}}{\partial \dot{\mathbf{q}}}(\mathbf{q},\dot{\mathbf{q}})=0$. Thus differential form $d\widetilde{H}=\frac{\partial \widetilde{H}}{\partial \mathbf{q}}d\mathbf{q}+\frac{\partial \widetilde{H}}{\partial \dot{\mathbf{q}}}d\dot{\mathbf{q}}$ is zero on the velocity $v'(t)=\frac{d}{dt}(\mathbf{q},\dot{\mathbf{q}})$

$$\begin{aligned} d\widetilde{H}\cdot v'(t) &= d\widetilde{H}\cdot\frac{d}{dt}(\mathbf{q},\dot{\mathbf{q}}) \\ &= \frac{\partial \widetilde{H}}{\partial \mathbf{q}}\frac{d\mathbf{q}}{dt}+\frac{\partial \widetilde{H}}{\partial \dot{\mathbf{q}}}\frac{d\dot{\mathbf{q}}}{dt}=0 \end{aligned}$$

We then use the inverse of this diffeomorphism to transfer the Hamiltonian function to a function $H(\mathbf{q},\mathbf{p})=F^{-1*}\widetilde{H}(\mathbf{q},\dot{\mathbf{q}})=\mathbf{p}\cdot\dot{\mathbf{q}}(\mathbf{q},\mathbf{p})-\mathrm{L}(\mathbf{q},\dot{\mathbf{q}}(\mathbf{q},\mathbf{p}))$.. Now if $\mathbf{q}(t),\dot{\mathbf{q}}(t)$ is a solution curve then its image $b(t)=F\circ v(t)=(\mathbf{q}(t),\mathbf{p}(t))$ satisfies

$$\begin{aligned} dH(b'(t)) &= (dH\cdot TF.v'(t)) \\ &= (F^{*}dH)\cdot v'(t) \\ &= d(F^{*}H)\cdot v'(t) \\ &= d\widetilde{H}\cdot v'(t)=0 \end{aligned}$$

so we have that

$$0=dH(b'(t))=\frac{\partial H}{\partial \mathbf{q}}\cdot\frac{d\mathbf{q}}{dt}+\frac{\partial H}{\partial \mathbf{p}}\cdot\frac{d\mathbf{p}}{dt}$$

but also

$$\frac{\partial}{\partial \mathbf{p}}H(\mathbf{q},\mathbf{p})=\dot{\mathbf{q}}+\mathbf{p}\cdot\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{p}}-\frac{\partial L}{\partial \dot{\mathbf{q}}}\cdot\frac{\partial \dot{\mathbf{q}}}{\partial \mathbf{p}}=\dot{\mathbf{q}}=\frac{d\mathbf{q}}{dt}$$

solving these last two equations simultaneously we arrive at Hamilton's equations of motion:

$$\begin{aligned} \frac{d}{dt}\mathbf{q}(t) &= \frac{\partial H}{\partial \mathbf{p}}(\mathbf{q}(t),\mathbf{p}(t)) \\ \frac{d}{dt}\mathbf{p}(t) &= -\frac{\partial H}{\partial \mathbf{q}}(\mathbf{q}(t),\mathbf{p}(t)) \end{aligned}$$

or

$$\frac{d}{dt}\begin{pmatrix}\mathbf{q}\\ \mathbf{p}\end{pmatrix}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{pmatrix}\frac{\partial H}{\partial \mathbf{q}}\\ \frac{\partial H}{\partial \mathbf{p}}\end{pmatrix}$$

Remark 23.2 *One can calculate directly that* $\frac{dH}{dt}(\mathbf{q}(t),\mathbf{p}(t))=0$ *for solutions these equations. If the Lagrangian was originally given by* $\mathrm{L}=\frac{1}{2}K-V$ *for some kinetic energy function and a potential energy function then this amounts to conservation of energy. We will see that this follows from a general principle below.*

Chapter 24

Symplectic Geometry

Equations are more important to me, because politics is for the present, but an equation is something for eternity
-Einstein

24.1 Symplectic Linear Algebra

A (real) **symplectic vector space** is a pair V, α where V is a (real) vector space and α is a nondegenerate alternating (skew-symmetric) bilinear form $\alpha : \mathrm{V} \times \mathrm{V} \to \mathbb{R}$. The basic example is $\mathbb{R}^{2n}$ with

$$\alpha_0(x, y) = x^t J_n y$$

where

$$J_n = \begin{pmatrix} 0 & I_{n\times n} \\ -I_{n\times n} & 0 \end{pmatrix}.$$

The standard symplectic form on α_0 is typical. It is a standard fact from linear algebra that for any N dimensional symplectic vector space V, α there is a basis $e_1, ..., e_n, f^1, ..., f^n$ called a **symplectic basis** such that the matrix that represents α with respect to this basis is the matrix J_n. Thus we may write

$$\alpha = e^1 \wedge f_1 + ... + e^n \wedge f_n$$

where $e^1, ..., e^n, f_1, ..., f_n$ is the dual basis to $e_1, ..., e_n, f^1, ..., f^n$. If V, η is a vector space with a not necessarily nondegenerate alternating form η then we can define the null space

$$N_\eta = \{v \in \mathrm{V} : \eta(v, w) = 0 \text{ for all } w \in \mathrm{V}\}.$$

On the quotient space $\overline{\mathrm{V}} = \mathrm{V}/N_\eta$ we may define $\overline{\eta}(\overline{v}, \overline{w}) = \eta(v, w)$ where v and w represent the elements $\overline{v}, \overline{w} \in \overline{\mathrm{V}}$. Then $\overline{\mathrm{V}}, \overline{\eta}$ is a symplectic vector space called the symplectic reduction of V, η.

Proposition 24.1 *For any $\eta \in \bigwedge V^*$ (regarded as a bilinear form) there is linearly independent set of elements $e^1, ..., e^k, f_1, ..., f_k$ from V^* such that*

$$\eta = e^1 \wedge f_1 + ... + e^k \wedge f_k$$

where $\dim(V) - 2k \geq 0$ *is the dimension of* N_η*.*

Definition 24.1 *Note: The number k is called the* ***rank*** *of η. The matrix that represents η actually has rank $2k$ and so some might call k the* ***half rank*** *of η.*

Proof. Consider the symplectic reduction $\overline{V}, \overline{\eta}$ of V, η and choose set of elements $e^1, ..., e^k, f_1, ..., f_k$ such that $\bar{e}^1, ..., \bar{e}^k, \bar{f}_1, ..., \bar{f}_k$ form a symplectic basis of $\overline{V}, \overline{\eta}$. Add to this set a basis $b_1, ..., b_l$ a basis for N_η and verify that $e^1, ..., e^k, f_1, ..., f_k, b_1, ..., b_l$ must be a basis for V. Taking the dual basis one can check that

$$\eta = e^1 \wedge f_1 + ... + e^k \wedge f_k$$

by testing on the basis $e^1, ..., e^k, f_1, ..., f_k, b_1, ..., b_l$. ∎

Now if W is a subspace of a symplectic vector space then we may define

$$W^\perp = \{v \in V : \eta(v, w) = 0 \text{ for all } w \in W\}$$

and it is true that $\dim(W) + \dim(W^\perp) = \dim(V)$ but it is **not** necessarily the case that $W \cap W^\perp = 0$. In fact,we classify subspaces W by two numbers: $d = \dim(W)$ and $\nu = \dim(W \cap W^\perp)$. If $\nu = 0$ then $\eta|_W, W$ is a symplectic space and so we call W a **symplectic subspace** . At the opposite extreme, if $\nu = d$ then W is called a **Lagrangian subspace** . If $W \subset W^\perp$ we say that W is an **isotropic subspace**.

A linear transformation between symplectic vector spaces $\ell : V_1, \eta_1 \to V_2, \eta_2$ is called a **symplectic linear map** if $\eta_2(\ell(v), \ell(w)) = \eta_1(v, w)$ for all $v, w \in V_1$; In other words, if $\ell^*\eta_2 = \eta_1$. The set of all symplectic linear isomorphisms from V, η to itself is called the **symplectic group** and denoted $Sp(V,\eta)$. With respect to a symplectic basis $\mathcal{B}$ a symplectic linear isomorphism ℓ is represented by a matrix $A = [\ell]_\mathcal{B}$ that satisfies

$$A^t J\, A = J$$

where $J = J_n$ is the matrix defined above and where $2n = \dim(V)$. Such a matrix is called a symplectic matrix and the group of all such is called the **symplectic matrix group** and denoted $Sp(n, \mathbb{R})$. Of course if $\dim(V) = 2n$ then $Sp(V,\eta) \cong Sp(n, \mathbb{R})$ the isomorphism depending a choice of basis. If η is a symplectic from on V with $\dim(V) = 2n$ then $\eta^n \in \wedge^{2n}V$ is nonzero and so orients the vector space V.

Lemma 24.1 *If $A \in Sp(n, \mathbb{R})$ then* $\det(A) = 1$.

Proof. If we use A as a linear transformation $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ then $A^*\alpha_0 = \alpha_0$ and $A^*\alpha_0^n = \alpha_0^n$ where α_0 is the standard symplectic form on $\mathbb{R}^{2n}$ and $\alpha_0^n \in \wedge^{2n}\mathbb{R}^{2n}$ is top form. Thus $\det A = 1$. ∎

Theorem 24.1 (Symplectic eigenvalue theorem) *If λ is a (complex) eigenvalue of a symplectic matrix A then so is $1/\lambda$, $\bar{\lambda}$ and $1/\bar{\lambda}$.*

Proof. Let $p(\lambda) = \det(A - \lambda I)$ be the characteristic polynomial. It is easy to see that $J^t = -J$ and $JAJ^{-1} = (A^{-1})^t$. Using these facts we have

$$\begin{aligned} p(\lambda) &= \det(J(A - \lambda I)J^{-1}) = \det(A^{-1} - \lambda I) \\ &= \det(A^{-1}(I - \lambda A)) = \det(I - \lambda A) \\ &= \lambda^{2n} \det(\frac{1}{\lambda} I - A)) = \lambda^{2n} p(1/\lambda). \end{aligned}$$

So we have $p(\lambda) = \lambda^{2n} p(1/\lambda)$. Using this and remembering that 0 is not an eigenvalue one concludes that $1/\lambda$ and $\bar{\lambda}$ are eigenvalues of A. ■

Exercise 24.1 *With respect to the last theorem, show that λ and $1/\lambda$ have the same multiplicity.*

24.2 Canonical Form (Linear case)

Suppose one has a vector space W with dual W*. We denote the pairing between W and W* by $\langle ., . \rangle$. There is a simple way to produce a symplectic form on the space Z = W × W* which we will call the **canonical symplectic form**. This is defined by

$$\Omega((v_1, \alpha_1), (v_2, \alpha_2)) := \langle \alpha_2, v_1 \rangle - \langle \alpha_1, v_2 \rangle.$$

If W is an inner product space with inner product $\langle ., . \rangle$ then we may form the canonical symplectic from on Z = W × W by the same formula. As a special case we get the standard symplectic form on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ given by

$$\Omega((x, y), (\widetilde{x}, \widetilde{y})) = \widetilde{y} \cdot x - y \cdot \widetilde{x}.$$

24.3 Symplectic manifolds

Definition 24.2 *A **symplectic form** on a manifold M is a* nondegenerate closed *2-form $\omega \in \Omega^2(M) = \Gamma(M, T^*M)$. A **symplectic manifold** is a pair (M, ω) where ω is a symplectic form on M. If there exists a symplectic form on M we say that M has a symplectic structure or admits a symplectic structure.*

A map of symplectic manifolds, say $f : (M, \omega) \to (N, \varpi)$ is called a **symplectic map** if and only if $f^*\varpi = \omega$. We will reserve the term **symplectomorphism** to refer to diffeomorphisms that are symplectic maps. Notice that since a symplectic form such as ω is nondegenerate, the $2n$ form $\omega^n = \omega \wedge \cdots \wedge \omega$ is nonzero and global. Hence a symplectic manifold is orientable (more precisely, it is oriented).

Definition 24.3 *The form $\Omega_\omega = \frac{(-1)^n}{(2n)!} \omega^n$ is called the **canonical volume form** or **Liouville volume**.*

We immediately have that if $f : (M, \omega) \to (M, \omega)$ is a symplectic diffeomorphism then $f^*\Omega_\omega = \Omega_\omega$.

Not every manifold admits a symplectic structure. Of course if M does admit a symplectic structure then it must have even dimension but there are other more subtle obstructions. For example, the fact that $H^2(S^4) = 0$ can be used to show that S^4 does not admit ant symplectic structure. To see this, suppose to the contrary that ω is a closed nondegenerate 2-form on S^4. Then since $H^2(S^4) = 0$ there would be a 1-form θ with $d\theta = \omega$. But then since $d(\omega \wedge \theta) = \omega \wedge \omega$ the 4-form $\omega \wedge \omega$ would be exact also and Stokes' theorem would give $\int_{S^4} \omega \wedge \omega = \int_{S^4} d(\omega \wedge \theta) = \int_{\partial S^4 = \emptyset} \omega \wedge \theta = 0$. But as we have seen $\omega^2 = \omega \wedge \omega$ is a nonzero top form so we must really have $\int_{S^4} \omega \wedge \omega \neq 0$. So in fact, S^4 does not admit a symplectic structure. We will give a more careful examination to the question of obstructions to symplectic structures but let us now list some positive examples.

Example 24.1 (surfaces) *Any orientable surface with volume form (area form) qualifies since in this case the volume ω itself is a closed nondegenerate two form.*

Example 24.2 (standard) *The form $\omega_{can} = \sum_{i=1}^n dx^i \wedge dx^{i+n}$ on $\mathbb{R}^{2n}$ is the prototypical symplectic form for the theory and makes $\mathbb{R}^n$ a symplectic manifold. (See Darboux's theorem 24.2 below)*

Example 24.3 (cotangent bundle) *We will see in detail below that the cotangent bundle of any smooth manifold has a natural symplectic structure. The symplectic form in a natural bundle chart (q, p) has the form $\omega = \sum_{i=1}^n dq^i \wedge dp_i$. (warning: some authors use $-\sum_{i=1}^n dq^i \wedge dp_i = \sum_{i=1}^n dp_i \wedge dq^i$ instead).*

Example 24.4 (complex submanifolds) *The symplectic $\mathbb{R}^{2n}$ may be considered the realification of $\mathbb{C}^n$ and then multiplication by i is thought of as a map $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. We have that $\omega_{can}(v, Jv) = -|v|^2$ so that ω_{can} is nondegenerate on any complex submanifold M of $\mathbb{R}^{2n}$ and so $M, \omega_{can}|_M$ is a symplectic manifold.*

Example 24.5 (coadjoint orbit) *Let G be a Lie group. Define the coadjoint map $\mathrm{Ad}^\dagger : G \to GL(\mathfrak{g}^*)$, which takes g to $\mathrm{Ad}_g^\dagger$, by*

$$\mathrm{Ad}_g^\dagger(\xi)(x) = \xi(\mathrm{Ad}_{g^{-1}}(x)).$$

The action defined by $\mathrm{Ad}^\dagger$,

$$g \to g \cdot \xi = \mathrm{Ad}_g^\dagger(\xi),$$

is called the ***coadjoint action****. Then we have an induced map $\mathrm{ad}^\dagger : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ at the Lie algebra level;*

$$\mathrm{ad}^\dagger(x)(\xi)(y) = -\xi([x, y]).$$

The orbits of the action given by Ad^* are called coadjoint orbits and we will show in theorem below that each orbit is a symplectic manifold in a natural way.

24.4 Complex Structure and Kähler Manifolds

Recall that a complex manifold is a manifold modeled on $\mathbb{C}^n$ and such that the chart overlap functions are all biholomorphic. Every (real) tangent space T_pM of a complex manifold M has a complex structure $J_p : T_pM \to T_pM$ given in biholomorphic coordinates $z = x + iy$ by

$$J_p(\left.\frac{\partial}{\partial x^i}\right|_p) = \left.\frac{\partial}{\partial y^i}\right|_p$$
$$J_p(\left.\frac{\partial}{\partial y^i}\right|_p) = -\left.\frac{\partial}{\partial x^i}\right|_p$$

and for any (biholomorphic) overlap function $\Delta = \varphi \circ \psi^{-1}$ we have $T\Delta \circ J = J \circ T\Delta$.

Definition 24.4 *An **almost complex structure** on a smooth manifold M is a bundle map $J : TM \to TM$ covering the identity map such that $J^2 = -\operatorname{id}$. If one can choose an atlas for M such that all the coordinate change functions (overlap functions) Δ satisfy $T\Delta \circ J = J \circ T\Delta$ then J is called a **complex structure** on M.*

Definition 24.5 *An **almost symplectic structure** on a manifold M is a nondegenerate smooth 2-form ω that is not necessarily closed.*

Theorem 24.2 *A smooth manifold M admits an almost complex structure if and only if it admits an almost symplectic structure.*

Proof. First suppose that M has an almost complex structure J and let g be any Riemannian metric on M. Define a quadratic form q_p on each tangent space by

$$q_p(v) = \mathrm{g}_p(v, v) + \mathrm{g}_p(Jv, Jv).$$

Then we have $q_p(Jv) = q_p(v)$. Now let h be the metric obtained from the quadratic form q by polarization. It follows that $\mathrm{h}(v, w) = \mathrm{h}(Jv, Jw)$ for all $v, w \in TM$. Now define a two form ω by

$$\omega(v, w) = \mathrm{h}(v, Jw).$$

This really is skew-symmetric since $\omega(v, w) = \mathrm{h}(v, Jw) = \mathrm{h}(Jv, J^2w) = -\mathrm{h}(Jv, w) = \omega(w, v)$. Also, ω is nondegenerate since if $v \neq 0$ then $\omega(v, Jv) = \mathrm{h}(v, v) > 0$.

Conversely, let ω be a nondegenerate two form on a manifold M. Once again choose a Riemannian metric g for M. There must be a vector bundle map $\Omega : TM \to TM$ such that

$$\omega(v, w) = \mathrm{g}(\Omega v, w) \text{ for all } v, w \in TM.$$

Since ω is nondegenerate the map Ω must be invertible. Furthermore, since Ω is clearly anti-symmetric with respect to g the map $-\Omega \circ \Omega = -\Omega^2$ must be symmetric and positive definite. From linear algebra applied fiberwise we know that there must be a positive symmetric square root for $-\Omega^2$. Denote this by $P = \sqrt{-\Omega^2}$. Finite dimensional spectral theory also tell us that $P\Omega = \Omega P$. Now let $J = \Omega P^{-1}$ and notice that

$$J^2 = (\Omega P^{-1})(\Omega P^{-1}) = \Omega^2 P^{-2} = -\Omega^2\Omega^{-2} = -\operatorname{id}.$$

∎

One consequence of this result is that there must be characteristic class obstructions to the existence of a symplectic structure on a manifolds. In fact, if M, ω is a symplectic manifold then it is certainly almost symplectic and so there is an almost complex structure J on M. The tangent bundle is then a complex vector bundle with J giving the action of multiplication by $\sqrt{-1}$ on each fiber T_pM. Denote the resulting complex vector bundle by TM^J and then consider the total Chern class

$$c(TM^J) = c_n(TM^J) + ... + c_1(TM^J) + 1.$$

Here $c_i(TM^J) \in H^{2i}(M, \mathbb{Z})$. Recall that with the orientation given by ω^n the top class $c_n(TM^J)$ is the Euler class $e(TM)$ of TM. Now for the real bundle TM we have the total Pontrijagin class

$$p(TM) = p_n(TM) + ... + p_1(TM) + 1$$

which are related to the Chern classes by the Whitney sum

$$\begin{aligned} p(TM) &= c(TM^J) \oplus c(TM^{-J}) \\ &= (c_n(TM^J) + ... + c_1(TM^J) + 1)((-1)^n c_n(TM^J) - +... + c_1(TM^J) + 1) \end{aligned}$$

where TM^{-J} is the complex bundle with $-J$ giving the multiplication by $\sqrt{-1}$. We have used the fact that

$$c_i(TM^{-J}) = (-1)^i c_i(TM^J).$$

Now the classes $p_k(TM)$ are invariants of the diffeomorphism class of M an so can be considered constant over all possible choices of J. In fact, from the above relations one can deduce a quadratic relation that must be satisfied:

$$p_k(TM) = c_k(TM^J)^2 - 2c_{k-1}(TM^J)c_{k+1}(TM^J) + \cdots + (-1)^k 2c_{2k}(TM^J).$$

Now this places a restriction on what manifolds might have almost complex structures and hence a restriction on having an almost symplectic structure. Of course some manifolds might have an almost symplectic structure but still have no symplectic structure.

Definition 24.6 *A positive definite real bilinear form h on an almost complex manifold M, J is will be called Hermitian metric or J-metric if h is J invariant. In this case h is the real part of a Hermitian form on the complex vector bundle TM, J given by*

$$\langle v, w\rangle = h(v, w) + ih(Jv, w)$$

Definition 24.7 *A diffeomorphism $\phi : M, J, h \to M, J, h$ is called a Hermitian isometry if and only if $T\phi \circ J = J \circ T\phi$ and*

$$h(T\phi v, T\phi w) = h(v, w).$$

A group action $\rho : G \times M \to M$ is called a Hermitian action if $\rho(g, .)$ is a Hermitian isometry for all g. In this case, we have for every $p \in M$ a the representation $d\rho_p : H_p \to \mathrm{Aut}(T_pM, J_p)$ of the isotropy subgroup H_p given by

$$d\rho_p(g)v = T_p\rho_g \cdot v.$$

Definition 24.8 *Let M, J be a complex manifold and ω a symplectic structure on M. The manifold is called a* ***Kähler manifold*** *if $h(v, w) := \omega(v, Jw)$ is positive definite.*

Equivalently we can define a ***Kähler manifold as*** *a complex manifold M, J with Hermitian metric h with the property that the nondegenerate 2-form $\omega(v, w) := h(v, Jw)$ is closed.*

Thus we have the following for a Kähler manifold:

1. A complex structure J,
2. A J-invariant positive definite bilinear form b,
3. A Hermitian form $\langle v, w\rangle = h(v, w) + ih(Jv, w)$.
4. A symplectic form ω with the property that $\omega(v, w) = h(v, Jw)$.

Of course if M, J is a complex manifold with Hermitian metric h then $\omega(v, w) := h(v, Jw)$ automatically gives a nondegenerate 2-form; the question is whether it is closed or not. Mumford's criterion is useful for this purpose:

Theorem 24.3 (Mumford) *Let $\rho : G \times M \to M$ be a smooth Lie group action by Hermitian isometries. For $p \in M$ let H_p be the isometry subgroup of the point p. If $J_p \in d\rho_p(H_p)$ for every p then we have that ω defined by $\omega(v, w) := h(v, Jw)$ is closed.*

Proof. It is easy to see that since ρ preserves both h and J it also preserves ω and $d\omega$. Thus for any given $p \in M$, we have

$$d\omega(d\rho_p(g)u, d\rho_p(g)v, d\rho_p(g)w) = d\omega(u, v, w)$$

for all $g \in H_p$ and all $u, v, w \in T_pM$. By assumption there is a $g_p \in H_p$ with $J_p = d\rho_p(g_p)$. Thus with this choice the previous equation applied twice gives

$$\begin{aligned} d\omega(u, v, w) &= d\omega(J_pu, J_pv, J_pw) \\ &= d\omega(J_p^2u, J_p^2v, J_p^2w) \\ &= d\omega(-u, -v, -w) = -d\omega(u, v, w) \end{aligned}$$

so $d\omega = 0$ at p which was an arbitrary point so $d\omega = 0$. ■

Since a Kähler manifold is a posteriori a Riemannian manifold it has associated with it the Levi-Civita connection ∇. In the following we view J as an element of $\mathfrak{X}(M)$.

Theorem 24.4 *For a Kähler manifold M, J, h with associated symplectic form ω we have that*

$$d\omega = 0 \quad \text{if and only if} \quad \nabla J = 0.$$

24.5 Symplectic musical isomorphisms

Since a symplectic form ω on a manifold M is nondegenerate we have a map

$$\omega_\flat : TM \to T^*M$$

given by $\omega_\flat(X_p)(v_p) = \omega(X_p, v_p)$ and the inverse $\omega^\sharp$ is such that

$$\iota_{\omega^\sharp(\alpha)}\omega = \alpha$$

or

$$\omega(\omega^\sharp(\alpha_p), v_p) = \alpha_p(v_p)$$

Let check that $\omega^\sharp$ really is the inverse. (one could easily be off by a sign in this business.) We have

$$\begin{aligned} \omega_\flat(\omega^\sharp(\alpha_p))(v_p) &= \omega(\omega^\sharp(\alpha_p), v_p) = \alpha_p(v_p) \text{ for all } v_p \\ &\implies \omega_\flat(\omega^\sharp(\alpha_p)) = \alpha_p. \end{aligned}$$

Notice that $\omega^\sharp$ induces a map on sections also denoted by $\omega^\sharp$ with inverse $\omega_\flat : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$.

Notation 24.1 *Let us abbreviate $\omega^\sharp(\alpha)$ to $\sharp\alpha$ and $\omega_\flat(v)$ to $\flat v$.*

24.6 Darboux's Theorem

Lemma 24.2 (Darboux's theorem) *On a $2n$- manifold (M, ω) with a closed 2-form ω with $\omega^n \neq 0$ (for instance if (M, ω) is symplectic) there exists a sub-atlas consisting of charts called symplectic charts (canonical coordinates) characterized by the property that the expression for ω in such a chart is*

$$\omega_U = \sum_{i=1}^{n} dx^i \wedge dx^{i+n}$$

and so in particular M must have even dimension $2n$.

Remark 24.1 *Let us agree that the canonical coordinates can be written (x^i, y_i) instead of (x^i, x^{i+n}) when convenient.*

Remark 24.2 *It should be noticed that if x^i, y_i is a symplectic chart then $\sharp dx^i$ must be such that*

$$\sum_{r=1}^{n} dx^r \wedge dy^r(\sharp dx^i, \frac{\partial}{\partial x^j}) = \delta^i_j$$

but also

$$\sum_{r=1}^{n} dx^r \wedge dy^r(\sharp dx^i, \frac{\partial}{\partial x^j}) = \sum_{r=1}^{n} \left(dx^r(\sharp dx) dy^r(\frac{\partial}{\partial x^j}) - dy^r(\sharp dx^i) dx^r(\frac{\partial}{\partial x^j}) \right)$$
$$= -dy^j(\sharp dx^i)$$

and so we conclude that $\sharp dx^i = -\frac{\partial}{\partial y^i}$ *and similarly* $\sharp dy^i = \frac{\partial}{\partial x^i}$.

Proof. We will use induction and follow closely the presentation in [**?**]. Assume the theorem is true for symplectic manifolds of dimension $2(n-1)$. Let $p \in M$. Choose a function y^1 on some open neighborhood of p such that $dy_1(p) \neq 0$. Let $X = \sharp dy_1$ and then X will not vanish at p. We can then choose another function x^1 such that $Xx^1 = 1$ and we let $Y = -\sharp dx^1$. Now since $d\omega = 0$ we can use Cartan's formula to get

$$\mathcal{L}_X \omega = \mathcal{L}_Y \omega = 0.$$

In the following we use the notation $\langle X, \omega \rangle = \iota_X \omega$ (see notation 7.1). Contract ω with the bracket of X and Y :

$$\langle [X,Y], \omega \rangle = \langle \mathcal{L}_X Y, \omega \rangle = \mathcal{L}_X \langle Y, \omega \rangle - \langle Y, \mathcal{L}_X \omega \rangle$$
$$= \mathcal{L}_X(-dx^1) = -d(X(x^1)) = -d1 = 0.$$

Now since ω is nondegenerate this implies that $[X,Y] = 0$ and so there must be a local coordinate system $(x^1, y_1, w^1, ..., w^{2n-2})$ with

$$\frac{\partial}{\partial y_1} = Y$$
$$\frac{\partial}{\partial x^1} = X.$$

In particular, the theorem is true if $n = 1$. Assume the theorem is true for symplectic manifolds of dimension $2(n-1)$. If we let $\omega' = \omega - dx^1 \wedge dy_1$ then since $d\omega' = 0$ and hence

$$\langle X, \omega' \rangle = \mathcal{L}_X \omega' = \langle Y, \omega' \rangle = \mathcal{L}_Y \omega' = 0$$

we conclude that ω' can be expressed as a 2-form in the $w^1, ..., w^{2n-2}$ variables alone. Furthermore,

$$0 \neq \omega^n = (\omega - dx^1 \wedge dy_1)^n$$
$$= \pm n dx^1 \wedge dy_1 \wedge (\omega')^n$$

from which it follows that ω' is the pull-back of a form nondegenerate form ϖ on $\mathbb{R}^{2n-2}$. To be exact if we let the coordinate chart given by $(x^1, y_1, w^1, ..., w^{2n-2})$ by denoted by ψ and let pr be the projection $\mathbb{R}^{2n} = \mathbb{R}^2 \times \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1}$ then $\omega' = (pr \circ \psi)^* \varpi$. Thus the induction hypothesis says that ω' has the form

$\omega' = \sum_{i=2}^{n} dx^i \wedge dy_i$ for some functions x^i, y_i with $i = 2, ..., n$. It is easy to see that the construction implies that in some neighborhood of p the full set of functions x^i, y_i with $i = 1, ..., n$ form the desired symplectic chart. ■

An atlas $\mathcal{A}$ of symplectic charts is called a symplectic atlas. A chart (U, φ) is called compatible with the symplectic atlas $\mathcal{A}$ if for every $(\psi_\alpha, U_\alpha) \in \mathcal{A}$ we have

$$(\varphi \circ \psi^{-1})^* \omega_0 = \omega_0$$

for the canonical symplectic $\omega_{can} = \sum_{i=1}^{n} du^i \wedge du^{i+n}$ defined on $\psi_\alpha(U \cap U_\alpha) \subset \mathbb{R}^{2n}$ using standard rectangular coordinates u^i.

24.7 Poisson Brackets and Hamiltonian vector fields

Definition 24.9 (on forms) *The **Poisson bracket** of two 1-forms is defined to be*

$$\{\alpha, \beta\}_{\pm} = \mp\flat[\sharp\alpha, \sharp\beta]$$

where the musical symbols refer to the maps $\omega^\sharp$ and $\omega_\flat$. This puts a Lie algebra structure on the space of 1-forms $\Omega^1(M) = \mathfrak{X}^*(M)$.

Definition 24.10 (on functions) *The **Poisson bracket** of two smooth functions is defined to be*

$$\{f, g\}_{\pm} = \pm\omega(\sharp df, \sharp dg) = \pm\omega(X_f, X_g)$$

This puts a Lie algebra structure on the space $\mathcal{F}(M)$ of smooth function on the symplectic M. It is easily seen (using $dg = \iota_{X_g}\omega$) that $\{f, g\}_{\pm} = \pm L_{X_g} f = \mp L_{X_f} g$ which shows that $f \mapsto \{f, g\}$ is a derivation for fixed g. The connection between the two Poisson brackets is

$$d\{f, g\}_{\pm} = \{df, dg\}_{\pm}.$$

Let us take canonical coordinates so that $\omega = \sum_{i=1}^{n} dx^i \wedge dy_i$. If $X_p = \sum_{i=1}^{n} dx^i(X) \frac{\partial}{\partial x^i} + \sum_{i=1}^{n} dy_i(X) \frac{\partial}{\partial y_i}$ and $v_p = dx^i(v_p) \frac{\partial}{\partial x^i} + dy_i(v_p) \frac{\partial}{\partial y_i}$ then using the Einstein summation convention we have

$$\begin{aligned}
&\omega_\flat(X)(v_p) \\
&= \omega(dx^i(X) \frac{\partial}{\partial x^i} + dy_i(X) \frac{\partial}{\partial y_i}, dx^i(v_p) \frac{\partial}{\partial x^i} + dy_i(v_p) \frac{\partial}{\partial y_i}) \\
&= (dx^i(X) dy_i - dy_i(X) dx^i)(v_p)
\end{aligned}$$

so we have

Lemma 24.3 $\omega_\flat(X_p) = \sum_{i=1}^{n} dx^i(X) dy_i - dy_i(X) dx^i = \sum_{i=1}^{n} (-dy_i(X) dx^i + dx^i(X) dy_i)$

Corollary 24.1 *If* $\alpha = \sum_{i=1}^n \alpha(\frac{\partial}{\partial x^i})dx^i + \sum_{i=1}^n \alpha(\frac{\partial}{\partial y_i})dy^i$ *then* $\omega^\sharp(\alpha) = \sum_{i=1}^n \alpha(\frac{\partial}{\partial y_i})\frac{\partial}{\partial x^i} - \sum_{i=1}^n \alpha(\frac{\partial}{\partial x^i})\frac{\partial}{\partial y_i}$

An now for the local formula:

Corollary 24.2 $\{f,g\} = \sum_{i=1}^n (\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x^i})$

Proof. $df = \frac{\partial f}{\partial x^i}dx^i + \frac{\partial f}{\partial y_i}dy_i$ and $dg = \frac{\partial g}{\partial x^j}dx^j + \frac{\partial g}{\partial y_i}dy_i$ so $\sharp df = \frac{\partial f}{\partial y_i}\frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial}{\partial y_i}$ and similarly for dg. Thus (using the summation convention again);

$$\begin{aligned}\{f,g\} &= \omega(\sharp df, \sharp dg)\\ &= \omega(\frac{\partial f}{\partial y_i}\frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial}{\partial y_i}, \frac{\partial g}{\partial y_i}\frac{\partial}{\partial x^j} - \frac{\partial g}{\partial x^j}\frac{\partial}{\partial y_i})\\ &= \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x^i}\end{aligned}$$

■

A main point about Poison Brackets is

Theorem 24.5 *f is constant along the orbits of X_g if and only if $\{f,g\} = 0$. In fact,*

$$\tfrac{d}{dt} g \circ \varphi_t^{X_f} = 0 \iff \quad \{f,g\} = 0 \quad \iff \tfrac{d}{dt} f \circ \varphi_t^{X_g} = 0$$

Proof. $\frac{d}{dt} g \circ \varphi_t^{X_f} = (\varphi_t^{X_f})^* L_{X_f} g = (\varphi_t^{X_f})^* \{f,g\}$. Also use $\{f,g\} = -\{g,f\}$.

■

The equations of motion for a Hamiltonian H are

$$\frac{d}{dt} f \circ \varphi_t^{X_H} = \pm\{f \circ \varphi_t^{X_H}, H\}_\pm = \mp\{H, f \circ \varphi_t^{X_H}\}_\pm$$

which is true by the following simple computation

$$\begin{aligned}\frac{d}{dt} f \circ \varphi_t^{X_H} &= \frac{d}{dt}(\varphi_t^{X_H})^* f = (\varphi_t^{X_H})^* L_{X_H} f\\ &= L_{X_H}(f \circ \varphi_t^{X_H}) = \{f \circ \varphi_t^{X_H}, H\}_\pm.\end{aligned}$$

Notation 24.2 *¿From now on we will use only $\{.,.\}_+$ unless otherwise indicated and shall write $\{.,.\}$ for $\{.,.\}_+$.*

Definition 24.11 *A Hamiltonian system is a triple (M, ω, H) where M is a smooth manifold, ω is a symplectic form and H is a smooth function $H : M \to \mathbb{R}$.*

The main example, at least from the point of view of mechanics, is the cotangent bundle of a manifold which is discussed below. From a mechanical point of view the Hamiltonian function controls the dynamics and so is special.

Let us return to the general case of a symplectic manifold M, ω

Definition 24.12 *Now if $H : M \to \mathbb{R}$ is smooth then we define the* ***Hamiltonian vector field*** *X_H with energy function H to be $\omega^\sharp dH$ so that by definition $\iota_{X_H}\omega = dH$.*

Definition 24.13 *A vector field X on M, ω is called a* ***locally Hamiltonian vector field*** *or a* ***symplectic vector field*** *if and only if $L_X\omega = 0$.*

If a symplectic vector field is complete then we have that $(\varphi_t^X)^*\omega$ is defined for all $t \in \mathbb{R}$. Otherwise, for any relatively compact open set U the restriction φ_t^X to U is well defined for all $t \le b(U)$ for some number depending only on U. Thus $(\varphi_t^X)^*\omega$ is defined on U for $t \le b(U)$. Since U can be chosen to contain any point of interest and since M can be covered by relatively compact sets, it will be of little harm to write $(\varphi_t^X)^*\omega$ even in the case that X is not complete.

Lemma 24.4 *The following are equivalent:*

1. *X is* ***symplectic vector field****, i.e. $L_X\omega = 0$*
2. *$\iota_X\omega$ is closed*
3. *$(\varphi_t^X)^*\omega = \omega$*
4. *X is locally a Hamiltonian vector field.*

Proof. (1)$\Longleftrightarrow$ (4) by the Poincaré lemma. Next, notice that $L_X\omega = d \circ \iota_X\omega + \iota_X \circ d\omega = d \circ \iota_X\omega$ so we have (2)$\Longleftrightarrow$(1). The implication (2)$\Longleftrightarrow$(3) follows from Theorem 2.8. ∎

Proposition 24.2 *We have the following easily deduced facts concerning Hamiltonian vector fields:*

1. The H is constant along integral curves of X_H
2. The flow of X_H is a local symplectomorphism. That is $\varphi_t^{X_H\,*}\omega = \omega$

Notation 24.3 *Denote the set of all Hamiltonian vector fields on M, ω by $\mathcal{H}(\omega)$ and the set of all symplectic vector fields by $\mathcal{SP}(\omega)$*

Proposition 24.3 *The set $\mathcal{SP}(\omega)$ is a Lie subalgebra of $\mathfrak{X}(M)$. In fact, we have $[\mathcal{SP}(\omega), \mathcal{SP}(\omega)] \subset \mathcal{H}(\omega) \subset \mathfrak{X}(M)$.*

Proof. Let $X, Y \in \mathcal{SP}(\omega)$. Then

$$\begin{aligned}
[X,Y]\lrcorner\omega &= \mathcal{L}_X Y\lrcorner\omega = \mathcal{L}_X(Y\lrcorner\omega) - Y\lrcorner\mathcal{L}_X\omega \\
&= d(X\lrcorner Y\lrcorner\omega) + X\lrcorner d(Y\lrcorner\omega) - 0 \\
&= d(X\lrcorner Y\lrcorner\omega) + 0 + 0 \\
&= -d(\omega(X,Y)) = -X_{\omega(X,Y)}\lrcorner\omega
\end{aligned}$$

and since ω in nondegenerate we have $[X, Y] = X_{-\omega(X,Y)} \in \mathcal{H}(\omega)$. ∎

24.8 Configuration space and Phase space

Consider the cotangent bundle of a manifold Q with projection map

$$\pi : T^*Q \to Q$$

and define the **canonical 1-form** $\theta \in T^*(T^*Q)$ by

$$\theta : v_{\alpha_p} \mapsto \alpha_p(T\pi \cdot v_{\alpha_p})$$

where $\alpha_p \in T_p^*Q$ and $v_{\alpha_p} \in T_{\alpha_p}(T_p^*Q)$. In local coordinates this reads

$$\theta_0 = \sum p_i dq^i.$$

Then $\omega_{T^*Q} = -d\theta$ is a symplectic form that in natural coordinates reads

$$\omega_{T^*Q} = \sum dq^i \wedge dp_i$$

Lemma 24.5 *θ is the unique 1-form such that for any $\beta \in \Omega^1(Q)$ we have*

$$\beta^*\theta = \beta$$

where we view β as $\beta : Q \to T^*Q$.

Proof: $\beta^*\theta(v_q) = \theta|_{\beta(q)}(T\beta \cdot v_q) = \beta(q)(T\pi \circ T\beta \cdot v_q) = \beta(q)(v_q)$ since $T\pi \circ T\beta = T(\pi \circ \beta) = T(\text{id}) = \text{id}$.

The cotangent lift T^*f of a diffeomorphism $f : Q_1 \to Q_2$ is defined by the commutative diagram

$$\begin{array}{ccc} T^*Q_1 & \xleftarrow{T^*f} & T^*Q_2 \\ \downarrow & & \downarrow \\ Q_1 & \xrightarrow{f} & Q_2 \end{array}$$

and is a symplectic map; i.e. $(T^*f)^*\omega_0 = \omega_0$. In fact, we even have $(T^*f)^*\theta_0 = \theta_0$.

The triple (T^*Q, ω_{T^*Q}, H) is a Hamiltonian system for any choice of smooth function. The most common form for H in this case is $\frac{1}{2}K + V$ where K is a Riemannian metric that is constructed using the mass distribution of the bodies modeled by the system and V is a smooth potential function which, in a conservative system, depends only on $\mathbf{q}$ when viewed in natural cotangent bundle coordinates q^i, p_i.

Now we have $\sharp dg = \frac{\partial g}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial g}{\partial q^i}\frac{\partial}{\partial p_i}$ and introducing the $\pm$ notation one more time we have

$$\begin{aligned} \{f, g\}_\pm &= \pm\omega_{T^*Q}(\sharp df, \sharp dg) = \pm df(\sharp dg) = \pm df(\frac{\partial g}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial g}{\partial q^i}\frac{\partial}{\partial p_i}) \\ &= \pm(\frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}) \\ &= \pm(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}) \end{aligned}$$

Thus letting

$$\varphi_t^{X_H}(q_0^1, ..., q_0^n, p_0^1, ..., p_0^n) = (q^1(t), ..., q^n(t), p^1(t), ..., p^n(t))$$

the equations of motions read

$$\begin{aligned}\frac{d}{dt}f(q(t), p(t)) &= \frac{d}{dt}f \circ \varphi_t^{X_H} = \{f \circ \varphi_t^{X_H}, H\} \\ &= \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q^i}.\end{aligned}$$

Where we have abbreviated $f \circ \varphi_t^{X_H}$ to just f. In particular, if $f = q^i$ and $f = p_i$ then

$$\begin{aligned}\dot{q}^i(t) &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i(t) &= -\frac{\partial H}{\partial q^i}\end{aligned}$$

which should be familiar.

24.9 Transfer of symplectic structure to the Tangent bundle

Case I: a (pseudo) Riemannian manifold

If Q, g is a (pseudo) Riemannian manifold then we have a map $\mathrm{g}^\flat : TQ \to T^*Q$ defined by

$$\mathrm{g}^\flat(v)(w) = \mathrm{g}(v, w)$$

and using this we can define a symplectic form ϖ_0 on TQ by

$$\varpi_0 = \left(\mathrm{g}^\flat\right)^* \omega$$

(Note that $d\varpi_0 = d(\mathrm{g}^{\flat *}\omega) = \mathrm{g}^{\flat *} d\omega = 0$.) In fact, ϖ_0 is exact since ω is exact:

$$\begin{aligned}\varpi_0 &= \left(\mathrm{g}^\flat\right)^* \omega \\ &= \left(\mathrm{g}^\flat\right)^* d\theta = d\left(\mathrm{g}^{\flat *}\theta\right).\end{aligned}$$

Let us write $\Theta_0 = \mathrm{g}^{\flat *}\theta$. Locally we have

$$\begin{aligned}\Theta_0(x, v)(v_1, v_2) &= \mathrm{g}_x(v, v_1) \text{ or} \\ \Theta_0 &= \sum \mathrm{g}_{ij}\dot{q}^i dq^j\end{aligned}$$

and also

$$\begin{aligned}&\varpi_0(x,v)((v_1,v_2),((w_1,w_2))) \\ &= \mathrm{g}_x(w_2,v_1) - \mathrm{g}_x(v_2,w_1) + D_x\mathrm{g}_x(v,v_1)\cdot w_1 - D_x\mathrm{g}_x(v,w_1)\cdot v_1\end{aligned}$$

which in classical notation (and for finite dimensions) looks like

$$\varpi_h = \mathrm{g}_{ij}dq^i \wedge d\dot{q}^j + \sum \frac{\partial \mathrm{g}_{ij}}{\partial q^k}\dot{q}^i dq^j \wedge dq^k$$

Case II: Transfer of symplectic structure by a Lagrangian function.

Definition 24.14 *Let $L : TQ \to Q$ be a Lagrangian on a manifold Q. We say that L is **regular** or **non-degenerate** at $\xi \in TQ$ if in any canonical coordinate system $(q,\dot{q})$ whose domain contains ξ, the matrix*

$$\left[\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}(q(\xi),\dot{q}(\xi))\right]$$

*is non-degenerate. L is called **regular** or **nondegenerate** if it is regular at all points in TQ.*

We will need the following general concept:

Definition 24.15 *Let $\pi_E : E \to M$ and $\pi_F : F \to M$ be two vector bundles. A map $L : E \to F$ is called a **fiber preserving map** if the following diagram commutes:*

$$\begin{array}{ccccc} E & & \xrightarrow{L} & & F \\ & & & & \\ \pi_E & \searrow & & \swarrow & \pi_F \\ & & M & & \end{array}$$

We do not require that the map L be linear on the fibers and so in general L is not a vector bundle morphism.

Definition 24.16 *If $L : E \to F$ is a fiber preserving map then if we denote the restriction of L to a fiber E_p by L_p define the **fiber derivative***

$$\mathbf{F}L : E \to \mathrm{Hom}(E,F)$$

by $\mathbf{F}L : e_p \mapsto Df|_p(e_p)$ for $e_p \in E_p$.

In our application of this concept, we take F to be the trivial bundle $Q \times \mathbb{R}$ over Q so $\mathrm{Hom}(E,F) = \mathrm{Hom}(E,\mathbb{R}) = T^*Q$.

Lemma 24.6 *A Lagrangian function $L : TQ \to \mathbb{R}$ gives rise to a fiber derivative $\mathbf{F}L : TQ \to T^*Q$. The Lagrangian is nondegenerate if and only if $\mathbf{F}L$ is a diffeomorphism.*

Definition 24.17 *The form ϖ_L is defined by*

$$\varpi_L = (\mathbf{F}L)^* \omega$$

Lemma 24.7 *ω_L is a symplectic form on TQ if and only if L is nondegenerate (i.e. if $\mathbf{F}L$ is a diffeomorphism).*

Observe that we can also define $\theta_L = (\mathbf{F}L)^* \theta$ so that $d\theta_L = d(\mathbf{F}L)^* \theta = (\mathbf{F}L)^* d\theta = (\mathbf{F}L)^* \omega = \varpi_L$ so we see that ω_L is exact (and hence closed a required for a symplectic form).

Now in natural coordinates we have

$$\varpi_L = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j} dq^i \wedge dq^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} dq^i \wedge d\dot{q}^j$$

as can be verified using direct calculation.

The following connection between the transferred forms ϖ_L and ϖ_0 and occasionally not pointed out in some texts.

Theorem 24.6 *Let V be a smooth function on a Riemannian manifold M, h. If we define a Lagrangian by $L = \frac{1}{2}h - V$ then the Legendre transformation $\mathbf{F}L :: TQ \to T^*Q$ is just the map $g^\flat$ and hence $\varpi_L = \varpi_h$.*

Proof. We work locally. Then the Legendre transformation is given by

$$\begin{array}{l} q^i \mapsto q^i \\ \dot{q}^i \mapsto \frac{\partial L}{\partial \dot{q}^i} \end{array} .$$

But since $L(\dot{\mathbf{q}}, \dot{\mathbf{q}}) = \frac{1}{2}\mathrm{g}(\dot{\mathbf{q}}, \dot{\mathbf{q}}) - V(q)$ we have $\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial}{\partial \dot{q}^i}\frac{1}{2}\mathrm{g}_{kl}\dot{q}^l\dot{q}^k = \mathrm{g}_{il}\dot{q}^l$ which together with $q^i \mapsto q^i$ is the coordinate expression for $\mathrm{g}^\flat$:

$$\begin{array}{l} q^i \mapsto q^i \\ \dot{q}^i \mapsto \mathrm{g}_{il}\dot{q}^l \end{array}$$

∎

24.10 Coadjoint Orbits

Let G be a Lie group and consider $\mathrm{Ad}^\dagger : G \to GL(\mathfrak{g}^*)$ and the corresponding coadjoint action as in example 24.5. For every $\xi \in \mathfrak{g}^*$ we have a Left invariant 1-form on G defined by

$$\theta^\xi = \xi \circ \omega_G$$

where ω_G is the canonical $\mathfrak{g}$-valued 1-form (the Maurer-Cartan form). Let the G_ξ be the isotropy subgroup of G for a point $\xi \in \mathfrak{g}^*$ under the coadjoint action. Then it is standard that orbit $G \cdot \xi$ is canonically diffeomorphic to the orbit space G/G_ξ and the map $\phi_\xi : g \mapsto g \cdot \xi$ is a submersion onto . Then we have

Theorem 24.7 *There is a unique symplectic form* Ω^{ξ} *on* $G/G_{\xi} \cong G \cdot \xi$ *such that* $\phi_{\xi}^{*}\Omega^{\xi} = d\theta^{\xi}$.

Proof: If such a form as Ω^{ξ} exists as stated then we must have

$$\Omega^{\xi}(T\phi_{\xi}.v, T\phi_{\xi}.w) = d\theta^{\xi}(v, w) \text{ for all } v, w \in T_g G$$

We will show that this in fact *defines* Ω^{ξ} as a symplectic form on the orbit $G \cdot \xi$. First of all notice that by the structure equations for the Maurer-Cartan form we have for $v, w \in T_e G = \mathfrak{g}$

$$\begin{aligned} d\theta^{\xi}(v, w) &= \xi(d\omega_G(v, w)) = \xi(\omega_G([v, w])) \\ &= \xi(-[v, w]) = \mathrm{ad}^{\dagger}(v)(\xi)(w) \end{aligned}$$

¿From this we see that

$$\mathrm{ad}^{\dagger}(v)(\xi) = 0 \Longleftrightarrow v \in \mathrm{Null}(\left. d\theta^{\xi}\right|_e)$$

where $\mathrm{Null}(\left. d\theta^{\xi}\right|_e) = \{v \in \mathfrak{g} : \left. d\theta^{\xi}\right|_e (v, w) \text{ for all } w \in \mathfrak{g}\}$. On the other hand, $G_{\xi} = \ker\{g \longmapsto \mathrm{Ad}_g^{\dagger}(\xi)\}$ so $\mathrm{ad}^{\dagger}(v)(\xi) = 0$ if and only if $v \in T_e G_{\xi} = \mathfrak{g}_{\xi}$.

Now notice that since $d\theta^{\xi}$ is left invariant we have that $\mathrm{Null}(\left. d\theta^{\xi}\right|_g) = TL_g(\mathfrak{g}_{\xi})$ which is the tangent space to the coset gG_{ξ} and which is also $\ker \left. T\phi_{\xi}\right|_g$. Thus we conclude that

$$\mathrm{Null}(\left. d\theta^{\xi}\right|_g) = \ker \left. T\phi_{\xi}\right|_g .$$

It follows that we have a natural isomorphism

$$T_{g\cdot\xi}(G \cdot \xi) = \left. T\phi_{\xi}\right|_g (T_g G) \approx T_g G/(TL_g(\mathfrak{g}_{\xi}))$$

Another view: Let the vector field on $G \cdot \xi$ corresponding to $v, w \in \mathfrak{g}$ generated by the action be denoted by $v^{\dagger}$and $w^{\dagger}$. Then we have $\Omega^{\xi}(\xi)(v^{\dagger}, w^{\dagger}) := \xi(-[v, w])$ at $\xi \in G \cdot \xi$ and then extend to the rest of the points of the orbit by equivariance:

$$\Omega^{\xi}(g \cdot \xi)(v^{\dagger}, w^{\dagger}) = \overset{\dagger}{\underset{g}{\mathrm{Ad}}}(\xi(-[v, w]))$$

24.11 The Rigid Body

In what follows we will describe the rigid body rotating about one of its points in three different versions. The basic idea is that we can represent the configuration space as a subset of $\mathbb{R}^{3N}$ with a very natural kinetic energy function. But this space is also isomorphic to the rotation group $SO(3)$ and we can transfer the kinetic energy metric over to $SO(3)$ and then the evolution of the system is given by geodesics in $SO(3)$ with respect to this metric. Next we take advantage of the fact that the tangent bundle of $SO(3)$ is trivial to transfer the setup over to a trivial bundle. But there are two natural ways to do this and we explore the relation between the two.

24.11.1 The configuration in $\mathbb{R}^{3N}$

Let us consider a rigid body to consist of a set of point masses located in $\mathbb{R}^3$ at points with position vectors $\mathbf{r}_1(t), ...\mathbf{r}_N(t)$ at time t. Thus $\mathbf{r}_i = (x_1, x_2, x_3)$ is the coordinates of the i-th point mass. Let $m_1, ..., m_N$ denote the masses of the particles. To say that this set of point masses is rigid is to say that the distances $|\mathbf{r}_i - \mathbf{r}_j|$ are constant for each choice of i and j. Let us assume for simplicity that the body is in a state of uniform rectilinear motion so that by re-choosing our coordinate axes if necessary we can assume that the there is one of the point masses at the origin of our coordinate system at all times. Now the set of all possible configurations is some submanifold of $\mathbb{R}^{3N}$ which we denote by M. Let us also assume that at least 3 of the masses, say those located at $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_2$ are situated so that the position vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_2$ form a basis of $\mathbb{R}^3$. For convenience let $\mathbf{r}$ and $\dot{\mathbf{r}}$ be abbreviations for $(\mathbf{r}_1(t), ..., \mathbf{r}_N(t))$ and $(\dot{\mathbf{r}}_1(t), ..., \dot{\mathbf{r}}_N(t))$.The correct kinetic energy for the system of particles forming the rigid body is $\frac{1}{2}K(\dot{\mathbf{r}}, \dot{\mathbf{r}})$ where the kinetic energy metric K is

$$K(\mathbf{v}, \mathbf{w}) = m_1 \mathbf{v}_1 \cdot \mathbf{w}_1 + \cdots + m_N \mathbf{v}_N \cdot \mathbf{w}_N.$$

Since there are no other forces on the body other than those that constrain the body to be rigid the Lagrangian for M is just $\frac{1}{2}K(\dot{\mathbf{r}}, \dot{\mathbf{r}})$ and the evolution of the point in M representing the body is a geodesic when we use as Hamiltonian K and the symplectic form pulled over from T^*M as described previously.

24.11.2 Modelling the rigid body on $S\mathrm{O}(3)$

Let $\mathbf{r}_1(0), ...\mathbf{r}(0)_N$ denote the initial positions of our point masses. Under these condition there is a unique matrix valued function $g(t)$ with values in $S\mathrm{O}(3)$ such that $\mathbf{r}_i(t) = g(t)\mathbf{r}_i(0)$. Thus the motion of the body is determined by the curve in $S\mathrm{O}(3)$ given by $t \mapsto g(t)$. In fact, we can map $S\mathrm{O}(3)$ to the set of all possible configurations of the points making up the body in a 1-1 manner by letting $\mathbf{r}_1(0) = \xi_1, ...\mathbf{r}(0)_N = \xi_N$ and mapping $\Phi : g \mapsto (g\xi_1, ..., g\xi_N) \in M \subset \mathbb{R}^{3N}$. If we use the map Φ to transfer this over to $TS\mathrm{O}(3)$ we get

$$k(\xi, \upsilon) = K(T\Phi \cdot \xi, T\Phi \cdot \upsilon)$$

for $\xi, \upsilon \in TS\mathrm{O}(3)$. Now k is a Riemannian metric on $S\mathrm{O}(3)$ and in fact, k is a left invariant metric:

$$k(\xi, \upsilon) = k(TL_g\xi, TL_g\upsilon) \text{ for all } \xi, \upsilon \in TS\mathrm{O}(3).$$

Exercise 24.2 *Show that* k *really is left invariant. Hint: Consider the map* $\mu_{g_0} : (\mathbf{v_1}, \cdots, \mathbf{v_N}) \mapsto (g_0\mathbf{v_1}, \cdots, g_0\mathbf{v_N})$ *for* $g_0 \in S\mathrm{O}(3)$ *and notice that* $\mu_{g_0} \circ \Phi = \Phi \circ L_{g_0}$ *and hence* $T\mu_{g_0} \circ T\Phi = T\Phi \circ TL_{g_0}$.

Now by construction, the Riemannian manifolds M, K and $S\mathrm{O}(3)$, k are isometric. Thus the corresponding path $g(t)$ in $S\mathrm{O}(3)$ is a geodesic with respect to the left invariant metric k. Our Hamiltonian system is now $(TS\mathrm{O}(3), \Omega_k, k)$ where Ω_k is the Legendre transformation of the canonical symplectic form Ω on $T^*S\mathrm{O}(3)$

24.11.3 The trivial bundle picture

Recall that we the Lie algebra of $SO(3)$ is the vector space of skew-symmetric matrices $\mathfrak{su}(3)$. We have the two trivializations of the tangent bundle $T\mathrm{SO}(3)$ given by

$$\begin{aligned}\mathrm{triv}_L(v_g) &= (g, \omega_G(v_g)) = (g, g^{-1}v_g)\\ \mathrm{triv}_R(v_g) &= (g, \omega_G(v_g)) = (g, v_g g^{-1})\end{aligned}$$

with inverse maps $\mathrm{SO}(3) \times \mathfrak{so}(3) \to T\mathrm{SO}(3)$ given by

$$\begin{aligned}(g, B) &\mapsto TL_g B\\ (g, B) &\mapsto TR_g B\end{aligned}$$

Now we should be able to represent the system in the trivial bundle $\mathrm{SO}(3) \times \mathfrak{so}(3)$ via the map $\mathrm{triv}_L(v_g) = (g, \omega_G(v_g)) = (g, g^{-1}v_g)$. Thus we let k_0 be the metric on $\mathrm{SO}(3) \times \mathfrak{so}(3)$ coming from the metric k. Thus by definition

$$\mathrm{k}_0((g, v), (g, w)) = \mathrm{k}(TL_g v, TL_g w) = \mathrm{k}_e(v, w)$$

where $v, w \in \mathfrak{so}(3)$ are skew-symmetric matrices.

24.12 The momentum map and Hamiltonian actions

Remark 24.3 *In this section all Lie groups will be assumed to be connected.*

Suppose that (a connected Lie group) G acts on M, ω as a group of symplectomorphisms.

$$\sigma : G \times M \to M$$

Then we say that σ is a **symplectic G-action** . Since G acts on M we have for every $v \in \mathfrak{g}$ the fundamental vector field $X^v = v^\sigma$. The fundamental vector field will be symplectic (locally Hamiltonian). Thus every one-parameter group g^t of G induces a symplectic vector field on M. Actually, it is only the infinitesimal action that matters at first so we define

Definition 24.18 *Let M be a smooth manifold and let $\mathfrak{g}$ be the Lie algebra of a connected Lie group G. A linear map $\sigma' : v \mapsto X^v$ from $\mathfrak{g}$ into $\mathfrak{X}(M)$ is called a **$\mathfrak{g}$-action** if*

$$\begin{aligned}[X^v, X^w] &= -X^{[v,w]} \text{ or}\\ [\sigma'(v), \sigma'(w)] &= -\sigma'([v, w]).\end{aligned}$$

*If M, ω is symplectic and the $\mathfrak{g}$-action is such that $\mathcal{L}_{X^v}\omega = 0$ for all $v \in \mathfrak{g}$ we say that the action is a **symplectic g-action**.*

Definition 24.19 *Every symplectic action $\sigma : G \times M \to M$ induces a $\mathfrak{g}$-action $d\sigma$ via*

$$d\sigma : v \mapsto X^v$$
$$\text{where } X^v(x) = \left.\frac{d}{dt}\right|_0 \sigma(\exp(tv), x).$$

In some cases, we may be able to show that for all v the symplectic field X^v is a full fledged Hamiltonian vector field. In this case associated to each $v \in \mathfrak{g}$ there is a Hamiltonian function $J_v = J_{X^v}$ with corresponding Hamiltonian vector field equal to X^v and J_v is determined up to a constant by $X^v = \sharp dJ_{X^v}$. Now $\iota_{X^v}\omega$ is always closed since $d\iota_{X^v}\omega = \mathcal{L}_{X^v}\omega$. When is it possible to define J_v for every $v \in \mathfrak{g}$?

Lemma 24.8 *Given a symplectic $\mathfrak{g}$-action $\sigma' : v \mapsto X^v$ as above, there is a linear map $v \mapsto J_v$ such that $X^v = \sharp dJ_v$ for every $v \in \mathfrak{g}$ if and only if $\iota_{X^v}\omega$ is exact for all $v \in \mathfrak{g}$.*

Proof. If $H_v = H_{X^v}$ exists for all v then $dJ_{X^v} = \omega(X^v, .) = \iota_{X^v}\omega$ for all v so $\iota_{X^v}\omega$ is exact for all $v \in \mathfrak{g}$. Conversely, if for every $v \in \mathfrak{g}$ there is a smooth function h_v with $dh_v = \iota_{X^v}\omega$ then $X^v = \sharp dh_v$ so h_v is Hamiltonian for X^v. Now let $v_1, ..., v_n$ be a basis for $\mathfrak{g}$ and define $J_{v_i} = h_{v_i}$ and extend linearly. ■

Notice that the property that $v \mapsto J_v$ is linear means that we can define a map $J : M \to \mathfrak{g}^*$ by
$$J(x)(v) = J_v(x)$$
and this is called a **momentum map** .

Definition 24.20 *A symplectic G-action σ (resp. $\mathfrak{g}$-action σ') on M such that for every $v \in \mathfrak{g}$ the vector field X^v is a Hamiltonian vector field on M is called a **Hamiltonian G-action** (resp. Hamiltonian $\mathfrak{g}$-action).*

We can thus associate to every Hamiltonian action at least one momentum map-this being unique up to an additive constant.

Example 24.6 *If G acts on a manifold Q by diffeomorphisms then G lifts to an action on the cotangent bundle T^*M which is automatically symplectic. In fact, because $\omega_0 = d\theta_0$ is exact the action is also a Hamiltonian action. The Hamiltonian function associated to an element $v \in \mathfrak{g}$ is given by*

$$J_v(x) = \theta_0 \left(\left.\frac{d}{dt}\right|_0 \exp(tv) \cdot x \right).$$

Definition 24.21 *If G (resp. $\mathfrak{g}$) acts on M in a symplectic manner as above such that the action is Hamiltonian and such that we may choose a momentum map J such that*
$$J_{[v,w]} = \{J_v, J_w\}$$
*where $J_v(x) = J(x)(v)$ then we say that the action is a **strongly Hamiltonian** G-action (resp. $\mathfrak{g}$-action).*

Example 24.7 *The action of example 24.6 is strongly Hamiltonian.*

We would like to have a way to measure of whether a Hamiltonian action is strong or not. Essentially we are just going to be using the difference $J_{[v,w]} - \{J_v, J_w\}$ but it will be convenient to introduce another view which we postpone until the next section where we study "Poisson structures".

PUT IN THEOREM ABOUT MOMENTUM CONSERVATION!!!!

What is a momentum map in the cotangent case? Pick a fixed point $\alpha \in T^*Q$ and consider the map $\Phi_\alpha : G \to T^*Q$ given by $\Phi_\alpha(g) = g \cdot \alpha = g^{-1*}\alpha$. Now consider the pull-back of the canonical 1-form $\Phi_\alpha^*\theta_0$.

Lemma 24.9 *The restriction* $\Phi_\alpha^*\theta_0|_{\mathfrak{g}}$ *is an element of* $\mathfrak{g}^*$ *and the map* $\alpha \mapsto \Phi_\alpha^*\theta_0|_{\mathfrak{g}}$ *is the momentum map.*

Proof. We must show that $\Phi_\alpha^*\theta_0|_{\mathfrak{g}}(v) = H_v(\alpha)$ for all $v \in \mathfrak{g}$. Does $\Phi_\alpha^*\theta_0|_{\mathfrak{g}}(v)$ live in the right place? Let $g_v^t = \exp(vt)$. Then

$$\begin{aligned}
(T_e\Phi_\alpha v) &= \left.\frac{d}{dt}\right|_0 \Phi_\alpha(\exp(vt)) \\
&= \left.\frac{d}{dt}\right|_0 (\exp(-vt))^*\alpha \\
&\left.\frac{d}{dt}\right|_0 \exp(vt) \cdot \alpha
\end{aligned}$$

We have

$$\begin{aligned}
\Phi_\alpha^*\theta_0|_{\mathfrak{g}}(v) &= \theta_0|_{\mathfrak{g}}(T_e\Phi_\alpha v) \\
= \theta_0(\left.\frac{d}{dt}\right|_0 &\exp(vt) \cdot \alpha) = J_v(\alpha)
\end{aligned}$$

■

Definition 24.22 *Let G act on a symplectic manifold M, ω and suppose that the action is Hamiltonian. A momentum map J for the action is said to be equivariant with respect to the coadjoint action if* $J(g \cdot x) = \mathrm{Ad}^*_{g^{-1}} J(x)$.

Chapter 25

Poisson Geometry

Life is good for only two things, discovering mathematics and teaching mathematics
–Siméon Poisson

25.1 Poisson Manifolds

In this chapter we generalize our study of symplectic geometry by approaching things from the side of a Poisson bracket.

Definition 25.1 *A* ***Poisson structure*** *on an associative algebra $\mathcal{A}$ is a Lie algebra structure with bracket denoted by $\{.,.\}$ such for a fixed $a \in \mathcal{A}$ that the map $x \mapsto \{a, x\}$ is a derivation of the algebra. An associative algebra with a Poisson structure is called a* ***Poisson algebra*** *and the bracket is called a* ***Poisson bracket*** *.*

We have already seen an example of a Poisson structure on the algebra $\mathfrak{F}(M)$ of smooth functions on a symplectic manifold. Namely,

$$\{f, g\} = \omega(\omega^\sharp df, \omega^\sharp dg).$$

By the Darboux theorem we know that we can choose local coordinates $(q^1, ..., q^n, p_1, ..., p_n)$ on a neighborhood of any given point in the manifold. Recall also that in such coordinates we have

$$\omega^\sharp df = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \sum_{i=1}^{n} \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}$$

sometimes called the symplectic gradient. It follows that

$$\sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}\right)$$

Definition 25.2 *A smooth manifold with a Poisson structure on is algebra of smooth functions is called a* ***Poisson manifold****.*

So every symplectic n-manifold gives rise to a Poisson structure. On the other hand, there are Poisson manifolds that are not so by virtue of being a symplectic manifold.

Now if our manifold is finite dimensional then every derivation of $\mathfrak{F}(M)$ is given by a vector field and since $g \mapsto \{f, g\}$ is a derivation there is a corresponding vector field X_f. Since the bracket is determined by these vector field and since vector fields can be defined locally (recall the presheaf $\mathfrak{X}_M$) we see that a Poisson structure is also a locally defined structure. In fact, $U \mapsto \mathfrak{F}_M(U)$ is a presheaf of Poisson algebras.

Now if we consider the map $w : \mathfrak{F}_M \to \mathfrak{X}_M$ defined by $\{f, g\} = w(f) \cdot g$ we see that $\{f, g\} = w(f) \cdot g = -w(g) \cdot f$ and so $\{f, g\}(p)$ depends only on the differentials df, dg of f and g. Thus we have a tensor $B(.,.) \in \Gamma \bigwedge^2 TM$ such that $B(df, dg) = \{f, g\}$. In other words, $B_p(.,.)$ is a symmetric bilinear map $T_p^*M \times T_p^*M \to \mathbb{R}$. Now any such tensor gives a bundle map $B^\sharp : T^*M \mapsto T^{**}M = TM$ by the rule $B^\sharp(\alpha)(\beta) = B(\beta, \alpha)$ for $\beta, \alpha \in T_p^*M$ and any $p \in M$. In other words, $B(\beta, \alpha) = \beta(B^\sharp(\alpha))$ for all $\beta \in T_p^*M$ and arbitrary $p \in M$. The 2-vector B is called the Poisson tensor for the given Poisson structure. B is also sometimes called a cosymplectic **structure** for reasons that we will now explain.

If M, ω is a symplectic manifold then the map $\omega_\flat : TM \to T^*M$ can be inverted to give a map $\omega^\sharp : T^*M \to TM$ and then a form $W \in \bigwedge^2 TM$ defined by $\omega^\sharp(\alpha)(\beta) = W(\beta, \alpha)$ (here again β, α must be in the same fiber). Now this form can be used to define a Poisson bracket by setting $\{f, g\} = W(df, dg)$ and so W is the corresponding Poisson tensor. But notice that

$$\begin{aligned}\{f, g\} &= W(df, dg) = \omega^\sharp(dg)(df) = df(\omega^\sharp(dg)) \\ &= \omega(\omega^\sharp df, \omega^\sharp dg)\end{aligned}$$

which is just the original Poisson bracket defined in the symplectic manifold M, ω.

Given a Poisson manifold $M, \{.,.\}$ we can always define $\{.,.\}_-$ by $\{f, g\}_- = \{g, f\}$. Since we some times refer to a Poisson manifold $M, \{.,.\}$ by referring just to the space we will denote M with the opposite Poisson structure by M^-.

A Poisson map is map $\phi : M, \{.,.\}_1 \to N, \{.,.\}_2$ is a smooth map such that $\phi^*\{f, g\} = \{\phi^* f, \phi^* g\}$ for all $f, g \in \mathfrak{F}(M)$.

For any subset S of a Poisson manifold let S_0 be the set of functions from $\mathfrak{F}(M)$ that vanish on S. A submanifold S of a Poisson manifold $M, \{.,.\}$ is called **coisotropic** if S_0 closed under the Poisson bracket. A Poisson manifold is called symplectic if the Poisson tensor B is non-degenerate since in this case we can use $B^\sharp$ to define a symplectic form on M. A Poisson manifold admits a (singular) foliation such that the leaves are symplectic. By a theorem of A. Weinstien we can locally in a neighborhood of a point p find a coordinate system

(q^i, p_i, w^i) centered at p and such that

$$B = \sum_{i=1}^{k} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2} \sum_{i,j} a^{ij}() \frac{\partial}{\partial w^i} \wedge \frac{\partial}{\partial w^j}$$

where the smooth functions depend only on the w's. vanish at p. Here k is the dimension of the leave through p. The rank of the map $B^\sharp$ on T_p^*M is k.

Now to give a typical example let $\mathfrak{g}$ be a Lie algebra with bracket $[.,.]$ and $\mathfrak{g}^*$ its dual. Choose a basis $e_1, ..., e_n$ of $\mathfrak{g}$ and the corresponding dual basis $\epsilon^1, ..., \epsilon^n$ for $\mathfrak{g}^*$. With respect to the basis $e_1, ..., e_n$ we have

$$[e_i, e_j] = \sum C^k_{ij} e_k$$

where C^k_{ij} are the structure constants.

For any functions $f, g \in \mathfrak{F}(\mathfrak{g}^*)$ we have that df_α, dg_α are linear maps $\mathfrak{g}^* \to \mathbb{R}$ where we identify $T_\alpha \mathfrak{g}^*$ with $\mathfrak{g}^*$. This means that df_α, dg_α can be considered to be in $\mathfrak{g}$ by the identification $\mathfrak{g}^{**} = \mathfrak{g}$. Now define the $\pm$ Poisson structure on $\mathfrak{g}^*$ by

$$\{f, g\}_\pm(\alpha) = \pm\alpha([df_\alpha, dg_\alpha])$$

Now the basis $e_1, ..., e_n$ is a coordinate system y on $\mathfrak{g}^*$ by $y_i(\alpha) = \alpha(e_i)$.

Proposition 25.1 *In terms of this coordinate system the Poisson bracket just defined is*

$$\{f, g\}_\pm = \pm \sum_{i=1}^{n} B_{ij} \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial y_j}$$

where $B_{ij} = \sum C^k_{ij} y_k$.

Proof. We suppress the $\pm$ and compute:

$$\begin{aligned}
\{f, g\} = [df, dg] &= [\sum \frac{\partial f}{\partial y_i} dy_i, \sum \frac{\partial g}{\partial y_j} dy_j] \\
&= \sum \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial y_j} [dy_i, dy_j] = \sum \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial y_j} \sum C^k_{ij} y_k \\
&= \sum_{i=1}^{n} B_{ij} \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial y_j}
\end{aligned}$$

∎

Appendix A

Topological Spaces

In this section we briefly introduce the basic notions from point set topology together with some basic examples. We include this section only as a review and a reference since we expect that the reader should already have a reasonable knowledge of point set topology. In the Euclidean coordinate plane $\mathbb{R}^n$ consisting of all n-tuples of real numbers $(x_1, x_2, ...x_n)$ we have a natural notion of distance between two points. The formula for the distance between two points $p_1 = (x_1, x_2, ...x_n)$ and $p_2 = (y_1, y_2, ..., y_n)$ is simply

$$d(p_1, p_2) = \sqrt{\sum (x_i - y_i)^2}. \tag{A.1}$$

The set of all points of distance less than ϵ from a given point p_0 in the plain is denoted $B(p_0, \epsilon)$, i.e.

$$B(p_0, \epsilon) = \{p \in \mathbb{R}^n : d(p, p_0) < \epsilon\}. \tag{A.2}$$

The set $B(p_0, \epsilon)$ is call the **open ball** of radius ϵ and center p_0. A subset S of $\mathbb{R}^2$ is called **open** if every one of its points is the center of an open ball completely contained inside S. The set of all open subsets of the plane has the property that the union of any number of open sets is still open and the intersection of any finite number of open sets is still open. The abstraction of this situation leads to the concept of a topological space.

Definition A.1 *A set X together with a family $\mathfrak{T}$ of subsets of X is called a topological space if the family $\mathfrak{T}$ has the following three properties.*

1. *$X \in \mathfrak{T}$ and $\emptyset \in \mathfrak{T}$.*
2. *If U_1 and U_2 are both in $\mathfrak{T}$ then $U_1 \cap U_2 \in \mathfrak{T}$ also.*
3. *If $\{U_\alpha\}_{\alpha \in A}$ is any sub-family of $\mathfrak{T}$ indexed by a set A then the union $\bigcup_{\alpha \in A} U_\alpha$ is also in $\mathfrak{T}$.*

In the definition above the family of subsets $\mathfrak{T}$ is called a **topology** on X and the sets in $\mathfrak{T}$ are called **open** sets. The compliment $U^c := X \setminus U$ of an open set U is called **closed** set. The reader is warned that a generic set may be neither open nor closed. Also, some subsets of X might be both open and closed (consider X itself and the empty set). A topology $\mathfrak{T}_2$ is said to be **finer** than a topology $\mathfrak{T}_1$ if $\mathfrak{T}_1 \subset \mathfrak{T}_2$ and in this case we also say that $\mathfrak{T}_1$ is **coarser** than $\mathfrak{T}_2$. We also say that the topology $\mathfrak{T}_1$ is **weaker** than $\mathfrak{T}_2$ and that $\mathfrak{T}_2$ is **stronger** than $\mathfrak{T}_1$.

Neither one of these topologies is generally very interesting but we shall soon introduce much richer topologies. A fact worthy of note in this context is the fact that if $X, \mathfrak{T}$ is a topological space and $S \subset X$ then S inherits a topological structure from X. Namely, a topology on S (called the **relative topology**) is given by

$$\mathfrak{T}_S = \{\text{all sets of the form } S \cap T \text{ where } T \in \mathfrak{T}\} \tag{A.3}$$

In this case we say that S is a topological subspace of X.

Definition A.2 *A map between topological spaces $f : X \to Y$ is said to be* ***continuous at*** *$p \in X$ if for any open set O containing $f(p)$ there is an open set U containing $p \in X$ such that $f(U) \subset O$. A map $f : X \to Y$ is said to be* ***continuous*** *if it is continuous at each point $p \in X$.*

Proposition A.1 *$f : X \to Y$ is continuous if and only if $f^{-1}(O)$ is open for every open set $O \subset Y$.*

Definition A.3 *A subset of a topological space is called* ***closed*** *if it is the compliment of an open set.*

Closed sets enjoy properties complementary to those of open sets:

1. The whole space X and the empty set $\emptyset$ are both closed.

2. The intersection of any family of closed sets is a closed set.

3. The union of a finite number of closed sets is closed.

Since the intersection of closed sets is again a closed every set, the subset $S \subset X$ is contained in a closed set which is the smallest of all closed sets containing S. This **closure** $\overline{S}$ is the intersection of all closed subsets containing S:

$$\overline{S} = \bigcap_{S \subset F} F.$$

Similarly, the **interior** of a set S is the largest open set contained in S and is denoted by $\overset{\circ}{S}$. A point $p \in S \subset X$ is called an **interior point** of S if there is an open set containing p and contained in S. The interior of S is just the set of all its interior points. It may be shown that $\overset{\circ}{S} = (\overline{S^c})^c$

Definition A.4 *The (topological)* ***boundary*** *of a set $S \subset X$ is $\partial S := \overline{S} \cap \overline{S^c}$ and*

We say that a set $S \subset X$ is **dense** in X if $\overline{S} = X$.

Definition A.5 *A subset of a topological space $X, \mathfrak{T}$ is called* ***clopen*** *if it is both open and closed.*

Definition A.6 *A topological space X is called* ***connected*** *if it is not the union of two proper clopen set. Here, proper means not X or $\emptyset$. A topological space X is called* ***path connected*** *if for every pair of points $p, q \in X$ there is a continuous map $c : [a, b] \to X$ (a path) such that $c(a) = q$ and $c(b) = p$. (Here $[a, b] \subset \mathbb{R}$ is endowed with the relative topology inherited from the topology on $\mathbb{R}$.)*

Example A.1 *The unit sphere S^2 is a topological subspace of the Euclidean space $\mathbb{R}^3$.*

Let X be a set and $\{\mathfrak{T}_\alpha\}_{\alpha \in A}$ any family of topologies on X indexed by some set A. The the intersection

$$\mathfrak{T} = \bigcap_{\alpha \in A} \mathfrak{T}_\alpha$$

is a topology on X. Furthermore, $\mathfrak{T}$ is coarser that every $\mathfrak{T}_\alpha$.

Given any family $\mathfrak{F}$ of subsets of X there exists a weakest (coarsest) topology containing all sets of $\mathfrak{F}$. We will denote this topology by $\mathfrak{T}(\mathfrak{F})$.

One interesting application of this is the following; Given a family of maps $\{f_\alpha\}$ from a set S to a topological space $Y, \mathfrak{T}_Y$ there is a coarsest topology on S such that all of the maps f_α are continuous. This topology will be denoted $\mathfrak{T}_{\{f_\alpha\}}$ and is called the topology generated by the family of maps $\{f_\alpha\}$.

Definition A.7 *If X and Y are topological spaces then we define the* ***product topology*** *on $X \times Y$ as the topology generated by the projections $pr_1 : X \times Y \to X$ and $pr_2 : X \times Y \to Y$.*

Definition A.8 *If $\pi : X \to Y$ is a surjective map where X is a topological space but Y is just a set. Then the* ***quotient topology*** *is the topology generated by the map π. In particular, if $A \subset X$ we may form the set of equivalence classes X/A where $x \sim y$ if both are in A or they are equal. The the map $x \mapsto [x]$ is surjective and so we may form the quotient topology on X/A.*

Let X be a topological space and $x \in X$. A family of subsets $\mathcal{B}_x$ all of which contain x is called an **open neighborhood base** at x if every open set containing x contains (as a superset) an set from $\mathcal{B}_x$. If X has a countable open base at each $x \in X$ we call X **first countable**.

A subfamily $\mathcal{B}$ is called a **base** for a topology $\mathfrak{T}$ on X if the topology $\mathfrak{T}$ is exactly the set of all unions of elements of $\mathcal{B}$. If X has a countable base for its given topology we say that X is a **second countable** topological space.

By considering balls of rational radii and rational centers one can see that $\mathbb{R}^n$ is first and second countable.

A.0.1 Separation Axioms

Another way to classify topological spaces is according to the following scheme:

(**Separation Axioms**)

A topological space $X, \mathfrak{T}$ is called a T_0 space if given $x, y \in X$, $x \neq y$, there exists either an open set containing x, but not y or the other way around (We don't get to choose which one).

A topological space $X, \mathfrak{T}$ is called T_1 if whenever given any x,$y \in X$ there is an open set containing x but not y (and the other way around;we get to do it either way).

A topological space $X, \mathfrak{T}$ is called T_2 or **Hausdorff** if whenever given any two points x, $y \in X$ there are disjoint open sets U_1 and U_2 with $x \in U_1$ and $y \in U_2$.

A topological space $X, \mathfrak{T}$ is called T_3 or **regular** if whenever given a closed set $F \subset X$ and a point $x \in X\backslash F$ there are disjoint open sets U_1 and U_2 with $x \in U_1$ and $F \subset U_2$

A topological space $X, \mathfrak{T}$ is called T_4 or **normal** if given any two disjoint closed subsets of X, say F_1 and F_2, there are two disjoint open sets U_1 and U_2 with $F_1 \subset U_1$ and $F_2 \subset U_2$.

Lemma A.1 (Urysohn) *Let X be normal and $F, G \subset X$ closed subsets with $F \cap G = \emptyset$. Then there exists a continuous function $f : X \to [0, 1] \subset \mathbb{R}$ such that $f(F) = 0$ and $f(G) = 1$.*

A open cover of topological space X (resp. subset $S \subset X$) a collection of open subsets of X, say$\{U_\alpha\}$, such that $X = \bigcup U_\alpha$ (resp. $S \subset \bigcup U_\alpha$). For example the set of all open disks of radius $\epsilon > 0$ in the plane covers the plane. A finite cover consists of only a finite number of open sets.

Definition A.9 *A topological space X is called **compact** if every open cover of X can be reduced to a finite open cover by eliminating some (possibly an infinite number) of the open sets of the cover. A subset $S \subset X$ is called **compact** if it is compact as a topological subspace (i.e. with the relative topology).*

Proposition A.2 *The continuous image of a compact set is compact.*

A.0.2 Metric Spaces

If the set X has a notion of distance attached to it then we can get an associated topology. This leads to the notion of a metric space.

A set X together with a function $d : X \times X \to \mathbb{R}$ is called a **metric space** if

$d(x, x) \geq 0$ for all $x \in X$

$d(x, y) = 0$ if and only if $x = y$

$d(x, z) \leq d(x, y) + d(y, z)$ for any $x, y, z \in X$ (*this is called the triangle inequality*).

The function d is called a **metric** or a **distance function**.

Imitating the situation in the plane we can define the notion of an open ball $B(p_0, \epsilon)$ with center p_0 and radius ϵ. Now once we have the metric then we have a topology; we define a subset of a metric space X, d to be open if every point of S is an interior point where a point $p \in S$ is called an **interior point** of S if there is some ball $B(p, \epsilon)$ with center p and (sufficiently small) radius $\epsilon > 0$ completely contained in S. The family of all of these metrically defined open sets forms a topology on X which we will denote by $\mathfrak{T}_d$. It is easy to see that any $B(p, \epsilon)$ is open according to our definition.

If $f : X, d \to Y, \rho$ is a map of metric spaces then f is continuous at $x \in X$ if and only if for every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if $d(x', x) < \delta(\epsilon)$ then $\rho(f(x'), f(x)) < \epsilon$.

Definition A.10 *A sequence of elements $x_1, x_2, ...$ of a metric space X, d is said to* ***converge*** *to p if for every $\epsilon > 0$ there is an $N(\epsilon) > 0$ such that if $k > N(\epsilon)$ then $x_k \in B(p, \epsilon)$. A sequence $x_1, x_2, ...$ is called a* ***Cauchy sequence*** *is for every $\epsilon > 0$ there is an $N(\epsilon) > 0$ such that if $k, l > N(\epsilon)$ then $d(x_k, x_l) < \epsilon$. A metric space X, d is said to be* ***complete*** *if every Cauchy sequence also converges.*

A map $f : X, d \to Y, \rho$ of metric spaces is continuous at $x \in X$ if and only if for every sequence x_i converging to x, the sequence $y_i := f(x_i)$ converges to $f(x)$.

A.1 Attaching Spaces and Quotient Topology

Suppose that we have a topological space X and a surjective set map $f : X \to S$ onto some set S. We may endow S with a natural topology according to the following recipe. A subset $U \subset S$ is defined to be open if and only if $f^{-1}(U)$ is an open subset of X. This is particularly useful when we have some equivalence relation on X which allows us to consider the set of equivalence classes $X/\sim$. In this case we have the canonical map $\varrho : X \to X/\sim$ that takes $x \in X$ to its equivalence class $[x]$. The quotient topology is then given as before by the requirement that $U \subset S$ is open if and only if and only if $\varrho^{-1}(U)$ is open in X. A common application of this idea is the identification of a subspace to a point. Here we have some subspace $A \subset X$ and the equivalence relation is given by the following two requirements:

$$\begin{array}{ll} \text{If } x \in X \backslash A & \text{then } x \sim y \text{ only if } x = y \\ \text{If } x \in A & \text{then } x \sim y \text{ for any } y \in A \end{array}$$

In other words, every element of A is identified with every other element of A. We often denote this space by X/A.

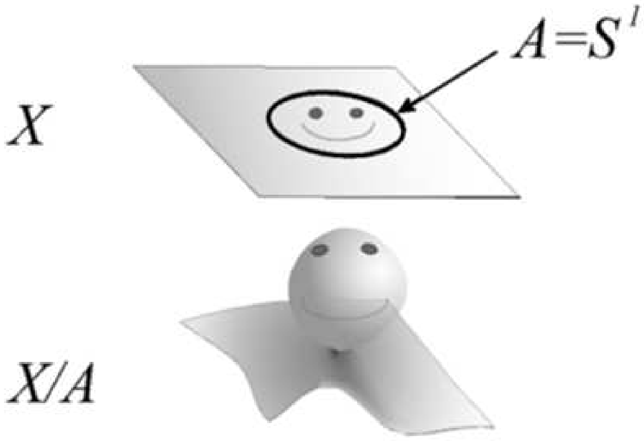
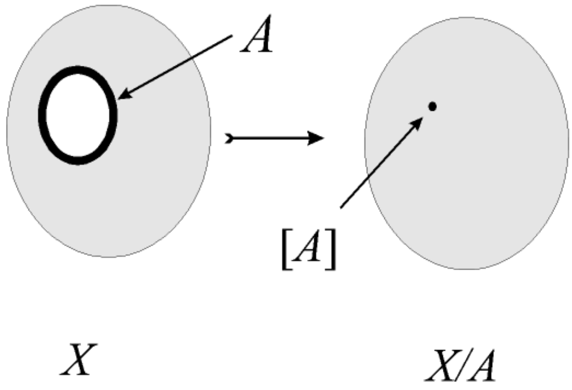


Figure A.1: creation of a "hole"

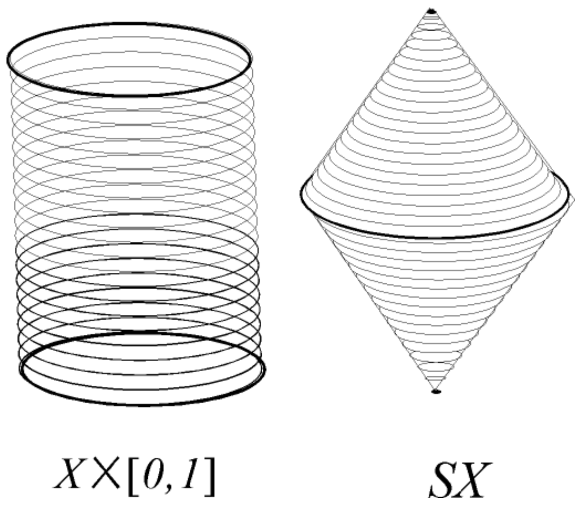


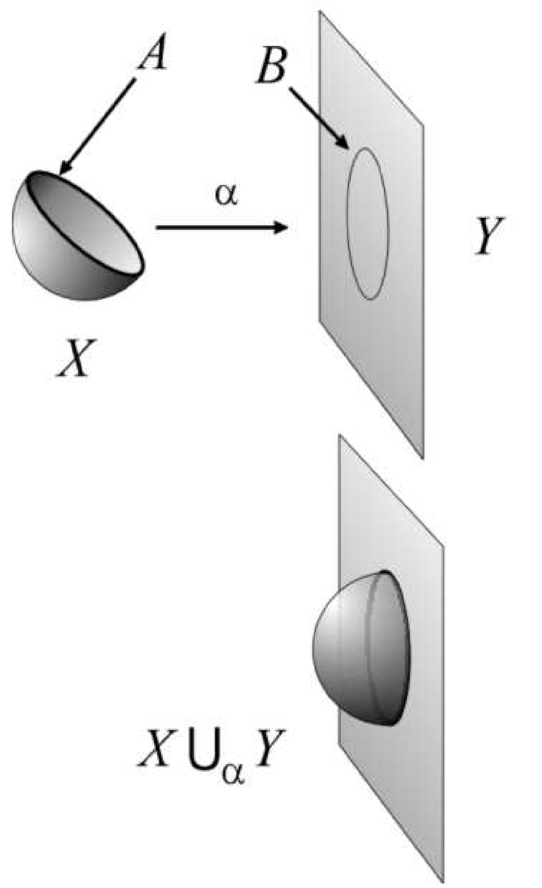
A hole is removed by identification

It is not difficult to verify that if X is Hausdorff (resp. normal) and A is closed then X/A is Hausdorff (resp. normal). The identification of a subset to a point need not simplify the topology but may also complicate the topology as shown in the figure.

An important example of this construction is the suspension. If X is a topological space then we define its suspension SX to be $(X \times [0,1])/A$ where $A := (X \times \{0\}) \cup (X \times \{1\})$. For example it is easy to see that $SS^1 \cong S^2$. More generally, $SS^{n-1} \cong S^n$.

Consider two topological spaces X and Y and subset $A \subset X$ a closed subset. Suppose that we have a map $\alpha : A \to B \subset Y$. Using this map we may define an equivalence relation on the disjoint union $X \bigsqcup Y$ that is given by requiring that $x \sim \alpha(x)$ for $x \in A$. The resulting topological space is denoted $X \cup_\alpha Y$.


X×[0,1]
SX


A
B
α
Y
X
X ∪α Y

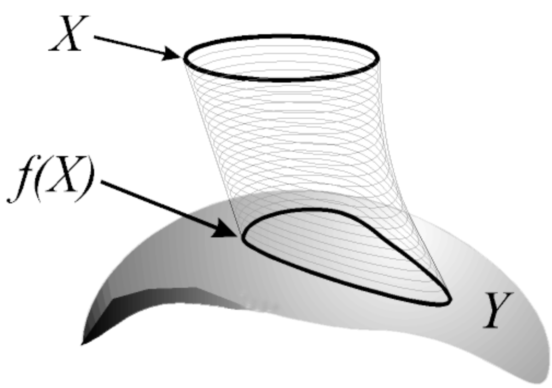


Figure A.2: Mapping Cylinder

Attaching a 2-cell

Another useful construction is that of a the mapping cylinder of a map $f : X \to Y$. First we transfer the map to a map on the base $X \times \{0\}$ of the cylinder $X \times I$ by

$$f(x, 0) := f(x)$$

and then we form the quotient $Y \cup_f (X \times I)$. We denote this quotient by M_f and call it the mapping cylinder of f.

A.2 Topological Vector Spaces

We shall outline some of the basic definitions and theorems concerning topological vector spaces.

Definition A.11 *A **topological vector space (TVS)** is a vector space* V *with a Hausdorff topology such that the addition and scalar multiplication operations are (jointly) continuous.*

Definition A.12 *Recall that a neighborhood of a point p in a topological space is a subset that has a nonempty interior containing p. The set of all neighborhoods that contain a point* x *in a topological vector space* V *is denoted* $\mathcal{N}(x)$.

The families $\mathcal{N}(x)$ for various x satisfy the following neighborhood axioms

1. Every set that contains a set from $\mathcal{N}(x)$ is also a set from $\mathcal{N}(x)$
2. If N_i is a family of sets from $\mathcal{N}(x)$ then $\bigcap_i N_i \in \mathcal{N}(x)$
3. Every $N \in \mathcal{N}(x)$ contains x

4. If $V \in \mathcal{N}(x)$ then there exists $W \in \mathcal{N}(x)$ such that for all $y \in W$, $V \in \mathcal{N}(y)$.

Conversely, let X be some set. If for each $x \in X$ there is a family $\mathcal{N}(x)$ of subsets of X that satisfy the above neighborhood axioms then there is a uniquely determined topology on X for which the families $\mathcal{N}(x)$ are the neighborhoods of the points x. For this a subset $U \subset X$ is open if and only if for each $x \in U$ we have $U \in \mathcal{N}(x)$.

Definition A.13 *A sequence x_n in a TVS is call a Cauchy sequence if and only if for every neighborhood U of 0 there is a number N_U such that $x_l - x_k \in U$ for all $k, l \geq N_U$.*

Definition A.14 *A relatively nice situation is when V has a norm that induces the topology. Recall that a **norm** is a function $\|\| : v \mapsto \|v\| \in \mathbb{R}$ defined on V such that for all $v, w \in \mathsf{V}$ we have*

1. $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$,
2. $\|v + w\| \leq \|v\| + \|w\|$,
3. $\|\alpha v\| = |\alpha| \, \|v\|$ for all $\alpha \in \mathbb{R}$.

   In this case we have a metric on V given by $\text{dist}(v, w) := \|v - w\|$. A **seminorm** is a function $\|\| : v \mapsto \|v\| \in \mathbb{R}$ such that 2) and 3) hold but instead of 1) we require only that $\|v\| \geq 0$.

Definition A.15 *A **normed space** V is a TVS that has a metric topology given by a norm. That is the topology is generated by the family of all open balls*

$$B_{\mathsf{V}}(x, r) := \{x \in \mathsf{V} : \|x\| < r\}.$$

Definition A.16 *A linear map $\ell : \mathsf{V} \to \mathsf{W}$ between normed spaces is called bounded if and only if there is a constant C such that for all $v \in \mathsf{V}$ we have $\|\ell v\|_{\mathsf{W}} \leq C \|v\|_{\mathsf{V}}$. If ℓ is bounded then the smallest such constant C is*

$$\|\ell\| := \sup \frac{\|\ell v\|_{\mathsf{W}}}{\|v\|_{\mathsf{V}}} = \sup\{\|\ell v\|_{\mathsf{W}} : \|v\|_{\mathsf{V}} \leq 1\}$$

The set of all bounded linear maps $\mathsf{V} \to \mathsf{W}$ is denoted $\mathcal{B}(\mathsf{V},\mathsf{W})$. The vector space $\mathcal{B}(\mathsf{V},\mathsf{W})$ is itself a normed space with the norm given as above.

Definition A.17 *A **locally convex topological vector space** V is a TVS such that it's topology is generated by a family of seminorms $\{\|.\|_\alpha\}_\alpha$. This means that we give V the weakest topology such that all $\|.\|_\alpha$ are continuous. Since we have taken a TVS to be Hausdorff we require that the family of seminorms is **sufficient** in the sense that for each $x \in \mathsf{V}$ we have $\bigcap\{x : \|x\|_\alpha = 0\} = \emptyset$. A locally convex topological vector space is sometimes called a locally convex space and so we abbreviate the latter to **LCS**.*

Example A.2 *Let Ω be an open set in $\mathbb{R}^n$ or any manifold. For each $x \in \Omega$ define a seminorm ρ_x on $C(\Omega)$ by $\rho_x(f) = f(x)$. This family of seminorms makes $C(\Omega)$ a topological vector space. In this topology convergence is pointwise convergence. Also, $C(\Omega)$ is not complete with this TVS structure.*

Definition A.18 *An LCS that is complete (every Cauchy sequence converges) is called a **Frechet space**.*

Definition A.19 *A complete normed space is called a **Banach space**.*

Example A.3 *Suppose that X, μ is a σ-finite measure space and let $p \geq 1$. The set $L^p(X, \mu)$ of all with respect to measurable functions $f : X \to \mathbb{C}$ such that $\int |f|^p \, d\mu \leq \infty$ is a Banach space with the norm $\|f\| := \left(\int |f|^p \, d\mu\right)^{1/p}$. Technically functions equal almost everywhere $d\mu$ must be identified.*

Example A.4 *The space $C_b(\Omega)$ of bounded continuous functions on Ω is a Banach space with norm given by $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$.*

Example A.5 *Once again let Ω be an open subset of $\mathbb{R}^n$. For each compact $K \subset\subset \Omega$ we have a seminorm on $C(\Omega)$ defined by $f \mapsto \|f\|_K := \sup_{x\in K} |f(x)|$. The corresponding convergence is the uniform convergence on compact subsets of Ω. It is often useful to notice that the same topology can be obtained by using $\|f\|_{K_i}$ obtained from a countable sequence of nested compact sets $K_1 \subset K_2 \subset ...$ such that*

$$\bigcup K_n = \Omega.$$

Such a sequence is called an exhaustion of Ω.

If we have topological vector space V and a closed subspace S, then we can form the quotient V/S. The quotient can be turned in to a normed space by introducing as norm

$$\|[x]\|_{\mathrm{V/S}} := \inf_{v\in[x]} \|v\|.$$

If S is not closed then this only defines a seminorm.

Theorem A.1 *If* V *is Banach space and a closed subspace* S *a closed (linear) subspace then* V/S *is a Banach space with the above defined norm.*

Proof. Let x_n be a sequence in V such that $[x_n]$ is a Cauchy sequence in V/S. Choose a subsequence such that $\|[x_n] - [x_{n+1}]\| \leq 1/2^n$ for $n = 1, 2, ...$ Setting s_1 equal to zero we find $s_2 \in \mathrm{S}$ such that $\|x_1 - (x_2 + s_2)\|$ and continuing inductively define a sequence s_i such that such that $\{x_n + s_n\}$ is a Cauchy sequence in V. Thus there is an element $y \in \mathrm{V}$ with $x_n + s_n \to y$. But since the quotient map is norm decreasing the sequence $[x_n + s_n] = [x_n]$ must also converge;

$$[x_n] \to [y].$$

■

Remark A.1 *It is also true that if S is closed and* V/S *is a Banach space then so is* V.

A.2.1 Hilbert Spaces

Definition A.20 *A Hilbert space $\mathcal{H}$ is a complex vector space with a Hermitian inner product $\langle .,. \rangle$. A Hermitian inner product is a bilinear form with the following properties:*
1) $\langle v, w \rangle = \overline{\langle v, w \rangle}$
2) $\langle v, \alpha w_1 + \beta w_2 \rangle = \alpha \langle v, w_1 \rangle + \beta \langle v, w_2 \rangle$
3) $\langle v, v \rangle \geq 0$ *and* $\langle v, v \rangle = 0$ *only if* $v = 0$.

One of the most fundamental properties of a Hilbert space is the projection property.

Theorem A.2 *If K is a convex, closed subset of a Hilbert space $\mathcal{H}$, then for any given $x \in \mathcal{H}$ there is a unique element $p_K(x) \in \mathcal{H}$ which minimizes the distance $\|x - y\|$ over $y \in K$. That is*

$$\|x - p_K(x)\| = \inf_{y \in K} \|x - y\| .$$

If K is a closed linear subspace then the map $x \mapsto p_K(x)$ is a bounded linear operator with the projection property $p_K^2 = p_K$.

Definition A.21 *For any subset $S \in \mathcal{H}$ we have the orthogonal compliment $S^\perp$ defined by*

$$S^\perp = \{x \in \mathcal{H} : \langle x, s \rangle = 0 \textit{ for all } s \in S\}.$$

$S^\perp$ is easily seen to be a linear subspace of $\mathcal{H}$. Since $\ell_s : x \mapsto \langle x, s \rangle$ is continuous for all s and since

$$S^\perp = \cap_s \ell_s^{-1}(0)$$

we see that $S^\perp$ is closed. Now notice that since by definition

$$\|x - P_s x\|^2 \leq \|x - P_s x - \lambda s\|^2$$

for any $s \in S$ and any real λ we have $\|x - P_s x\|^2 \leq \|x - P_s x\|^2 - 2\lambda \langle x - P_s x, s \rangle + \lambda^2 \|s\|^2$. Thus we see that $p(\lambda) := \|x - P_s x\|^2 - 2\lambda \langle x - P_s x, s \rangle + \lambda^2 \|s\|^2$ is a polynomial in λ with a minimum at $\lambda = 0$. This forces $\langle x - P_s x, s \rangle = 0$ and so we see that $x - P_s x$. From this we see that any $x \in \mathcal{H}$ can be written as $x = x - P_s x + P_s x = s + s^\perp$. On the other hand it is easy to show that $S^\perp \cap S = 0$. Thus we have $\mathcal{H} = S \oplus S^\perp$ for any closed linear subspace $S \subset \mathcal{H}$. In particular the decomposition of any x as $s + s^\perp \in S \oplus S^\perp$ is unique.

Appendix B

The Language of Category Theory

Category theory provides a powerful means of organizing our thinking in mathematics. Some readers may be put off by the abstract nature of category theory. To such readers I can only say that it is not really difficult to catch on to the spirit of category theory and the payoff in terms of organizing mathematical thinking is considerable. I encourage these readers to give it a chance. In any case, it is not strictly necessary for the reader to be completely at home with category theory before going further into the book. In particular, physics and engineering students may not be used to this kind of abstraction and should simply try to gradually get used to the language of categories. Feel free to defer reading this appendix on Category theory until it seems necessary

Roughly speaking, category theory is an attempt at clarifying structural similarities that tie together different parts of mathematics. A category has "objects" and "morphisms". The prototypical category is just the category **Set** which has for its objects ordinary *sets* and for its morphisms *maps* between sets. The most important category for differential geometry is what is sometimes called the "smooth category" consisting of smooth manifolds and smooth maps. (The definition of these terms given in the text proper but roughly speaking smooth means differentiable.)

Now on to the formal definition of a category.

Definition B.1 *A* ***category*** $\mathfrak{C}$ *is a collection of objects* $\mathrm{Ob}(\mathfrak{C}) = \{X, Y, Z, ...\}$ *and for every pair of objects* X, Y *a set* $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$ *called the set of* ***morphisms*** *from* X *to* Y*. The family of all such morphisms will be denoted* $\mathrm{Mor}(\mathfrak{C})$*. In addition, a category is required to have a* ***composition*** *law which is defined as a map* $\circ : \mathrm{Hom}_{\mathfrak{C}}(X, Y) \times \mathrm{Hom}_{\mathfrak{C}}(Y, Z) \to \mathrm{Hom}_{\mathfrak{C}}(X, Z)$ *such that for every three objects* $X, Y, Z \in Obj(\mathfrak{C})$ *the following axioms hold:*

Axiom B.1 (Cat1) $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$ *and* $\mathrm{Hom}_{\mathfrak{C}}(Z, W)$ *are disjoint unless* $X = Z$ *and* $Y = W$ *in which case* $\mathrm{Hom}_{\mathfrak{C}}(X, Y) = \mathrm{Hom}_{\mathfrak{C}}(Z, W)$.

Axiom B.2 (Cat2) *The composition law is associative:* $f\circ(g\circ h)=(f\circ g)\circ h$.

Axiom B.3 (Cat3) *Each set of morphisms of the form* $\mathrm{Hom}_{\mathfrak{C}}(X,X)$ *must contain a necessarily element* id_X, *the identity element, such that* $f\circ\mathrm{id}_X=f$ *for any* $f\in\mathrm{Hom}_{\mathfrak{C}}(X,Y)$ *(and any* Y*), and* $\mathrm{id}_X\circ f=f$ *for any* $f\in\mathrm{Hom}_{\mathfrak{C}}(Y,X)$.

Notation B.1 *A morphism is sometimes written using an arrow. For example, if* $f\in\mathrm{Hom}_{\mathfrak{C}}(X,Y)$ *we would indicate this by writing* $f:X\to Y$ *or by* $X\xrightarrow{f}Y$.

The notion of category is typified by the case where the objects are sets and the morphisms are maps between the sets. In fact, subject to putting extra structure on the sets and the maps, this will be almost the only type of category we shall need to talk about. On the other hand there are plenty of interesting categories of this type. Examples include the following.

1. **Grp**: The objects are groups and the morphisms are group homomorphisms.
2. **Rng** : The objects are rings and the morphisms are ring homomorphisms.
3. $\mathbf{Lin}_{\mathbb{F}}$: The objects are vector spaces over the field $\mathbb{F}$ and the morphisms are linear maps. This category is referred to as the linear category or the vector space category
4. **Top**: The objects are topological spaces and the morphisms are continuous maps.
5. $\mathbf{Man}^r$**:** The category of C^r-differentiable manifolds and C^r-maps: One of the main categories discussed in this book. This is also called the smooth or differentiable category especially when $r=\infty$.

Notation B.2 *If for some morphisms* $f_i:X_i\to Y_i$, $(i=1,2)$*,* $g_X:X_1\to X_2$ *and* $g_Y:Y_1\to Y_2$ *we have* $g_Y\circ f_1=f_2\circ g_X$ *then we express this by saying that the following diagram "commutes":*

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "g_X"'] & Y_1 \arrow[d, "g_Y"] \\ X_2 \arrow[r, "f_2"'] & Y_2 \end{tikzcd}$$

Similarly, if $h\circ f=g$ *we say that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, "h"] \\ & Z \end{tikzcd}$$

commutes. More generally, tracing out a path of arrows in a diagram corresponds to composition of morphisms and to say that such a diagram commutes is to say that the compositions arising from two paths of arrows that begin and end at the same object are equal.

Definition B.2 *Suppose that $f : X \to Y$ is a morphism from some category $\mathfrak{C}$. If f has the property that for any two (parallel) morphisms $g_1, g_2 : Z \to X$ we always have that $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$, i.e. if f is "left cancellable", then we call f a* ***monomorphism****. Similarly, if $f : X \to Y$ is "right cancellable" we call f an* ***epimorphism****. A morphism that is both a monomorphism and an epimorphism is called an* ***isomorphism.*** *If the category needs to be specified then we talk about a* ***$\mathfrak{C}$-monomorphism, $\mathfrak{C}$-epimorphism*** *and so on).*

In some cases we will use other terminology. For example, an isomorphism in the smooth category is called a **diffeomorphism**. In the linear category, we speak of linear maps and linear isomorphisms. Morphisms which comprise $\mathrm{Hom}_{\mathfrak{C}}(X, X)$ are also called endomorphisms and so we also write $\mathrm{End}_{\mathfrak{C}}(X) := \mathrm{Hom}_{\mathfrak{C}}(X, X)$. The set of all isomorphisms in $\mathrm{Hom}_{\mathfrak{C}}(X, X)$ is sometimes denoted by $\mathrm{Aut}_{\mathfrak{C}}(X)$ and these morphisms are called **automorphisms**.

We single out the following: In many categories like the above we can form a sort of derived category that uses the notion of pointed space and pointed map. For example, we have the "pointed topological category" . A pointed topological space is an topological space X together with a distinguished point p. Thus a typical object in the pointed topological category would be written (X, p). A morphism $f : (X, p) \to (W, q)$ is a continuous map such that $f(p) = q$.

B.0.2 Functors

A **functor** $\mathcal{F}$ is a pair of maps both denoted by the same letter $\mathcal{F}$ that map objects and morphisms from one category to those of another

$$\mathcal{F} : \mathrm{Ob}(\mathfrak{C}_1) \to \mathrm{Ob}(\mathfrak{C}_2)$$
$$\mathcal{F} : \mathrm{Mor}(\mathfrak{C}_1) \to \mathrm{Mor}(\mathfrak{C}_2)$$

such that composition and identity morphisms are respected: This means that for a morphism $f : X \to Y$, the morphism

$$\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$$

is a morphism in the second category and we must have

1. $\mathcal{F}(\mathrm{id}_{\mathfrak{C}_1}) = \mathrm{id}_{\mathfrak{C}_2}$
2. If $f : X \to Y$ and $g : Y \to Z$ then $\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$, $\mathcal{F}(g) : \mathcal{F}(Y) \to \mathcal{F}(Z)$ and
$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

Example B.1 *Let* $\mathbf{Lin}_{\mathbb{R}}$ *be the category whose objects are real vector spaces and whose morphisms are real linear maps. Similarly, let* $\mathbf{Lin}_{\mathbb{C}}$ *be the category of complex vector spaces with complex linear maps. To each real vector space* V *we can associate the complex vector space* $\mathbb{C}\otimes_{\mathbb{R}}\mathrm{V}$ *called the complexification of* V *and to each linear map of real vector spaces* $\ell:\mathrm{V}\to\mathrm{W}$ *we associate the complex extension* $\ell_{\mathbb{C}}:\mathbb{C}\otimes_{\mathbb{R}}\mathrm{V}\to\mathbb{C}\otimes_{\mathbb{R}}\mathrm{W}$*. Here,* $\mathbb{C}\otimes_{\mathbb{R}}\mathrm{V}$ *is easily thought of as the vector space* V *where now complex scalars are allowed. Elements of* $\mathbb{C}\otimes_{\mathbb{R}}\mathrm{V}$ *are generated by elements of the form* $c\otimes v$ *where* $c\in\mathbb{C}$*,* $v\in\mathrm{V}$ *and we have* $i(c\otimes v)=ic\otimes v$ *where* $i=\sqrt{-1}$*. The map* $\ell_{\mathbb{C}}:\mathbb{C}\otimes_{\mathbb{R}}\mathrm{V}\to\mathbb{C}\otimes_{\mathbb{R}}\mathrm{W}$ *is defined by the requirement* $\ell_{\mathbb{C}}(c\otimes v)=c\otimes\ell v$*. Now the assignments*

$$\ell\mapsto\ell_{\mathbb{C}}$$
$$\mathrm{V}\mapsto\mathbb{C}\otimes_{\mathbb{R}}\mathrm{V}$$

define a functor from $\mathbf{Lin}_{\mathbb{R}}$ *to* $\mathbf{Lin}_{\mathbb{C}}$*.*

Remark B.1 *In practice, complexification amounts to simply allowing complex scalars. For instance, we might just write cv instead of* $c\otimes v$*.*

Actually, what we have defined here is a **covariant functor**. A **contravariant functor** is defined similarly except that the order of composition is reversed so that instead of **Funct2** above we would have $\mathcal{F}(g\circ f)=\mathcal{F}(f)\circ\mathcal{F}(g)$. An example of a contravariant functor is the dual vector space functor which is a functor from the category of vector spaces $\mathbf{Lin}_{\mathbb{R}}$ to itself which sends each space V to its dual V^* and each linear map to its dual (or transpose). Under this functor a morphism

$$\mathrm{V}\ \overset{L}{\to}\ \mathrm{W}$$

is sent to the morphism

$$\mathrm{V}^*\ \overset{L^*}{\leftarrow}\ \mathrm{W}^*$$

Notice the arrow reversal.

Remark B.2 *One of the most important functors for our purposes is the* ***tangent functor*** *defined in section* **??**. *Roughly speaking this functor replaces differentiable maps and spaces by their linear parts.*

Example B.2 *Consider the category of real vector spaces and linear maps. To every vector space* V *we can associate the dual of the dual* V^{**}*. This is a* covariant *functor which is the composition of the dual functor with itself:*

$$\begin{array}{ccccc} V & & W^* & & V^{**} \\ A\downarrow & \mapsto & A^*\downarrow & \mapsto & A^{**}\downarrow \\ W & & V^* & & W^{**} \end{array}$$

B.0.3 Natural transformations

Now suppose we have two functors

$$\begin{aligned} \mathcal{F}_1 &: Ob(\mathfrak{C}_1) \to Ob(\mathfrak{C}_2) \\ \mathcal{F}_1 &: Mor(\mathfrak{C}_1) \to Mor(\mathfrak{C}_2) \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_2 &: Ob(\mathfrak{C}_1) \to Ob(\mathfrak{C}_2) \\ \mathcal{F}_2 &: Mor(\mathfrak{C}_1) \to Mor(\mathfrak{C}_2) \end{aligned}$$

A natural transformation $\mathcal{T}$ from $\mathcal{F}_1$ to $\mathcal{F}_2$ is an assignment to each object X of $\mathfrak{C}_1$ a morphism $\mathcal{T}(X) : \mathcal{F}_1(X) \to \mathcal{F}_2(X)$ such that for every morphism $f : X \to Y$ of $\mathfrak{C}_1$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{F}_1(X) \arrow[r, "\mathcal{T}(X)"] \arrow[d, "\mathcal{F}_1(f)"'] & \mathcal{F}_2(X) \arrow[d, "\mathcal{F}_2(f)"] \\ \mathcal{F}_1(Y) \arrow[r, "\mathcal{T}(Y)"'] & \mathcal{F}_2(Y) \end{tikzcd}$$

A common first example is the natural transformation ι between the identity functor $I : \mathbf{Lin}_{\mathbb{R}} \to \mathbf{Lin}_{\mathbb{R}}$ and the double dual functor $** : \mathbf{Lin}_{\mathbb{R}} \to \mathbf{Lin}_{\mathbb{R}}$:

$$\begin{tikzcd} \mathrm{V} \arrow[r, "\iota(\mathrm{V})"] \arrow[d, "f"'] & \mathrm{V}^{**} \arrow[d, "f^{**}"] \\ \mathrm{W} \arrow[r, "\iota(\mathrm{W})"'] & \mathrm{W}^{**} \end{tikzcd} .$$

The map $\mathrm{V} \to \mathrm{V}^{**}$ sends a vector to a linear function $\widetilde{v} : \mathrm{V}^* \to \mathbb{R}$ defined by $\widetilde{v}(\alpha) := \alpha(v)$ (the hunter becomes the hunted so to speak). If there is an inverse natural transformation $\mathcal{T}^{-1}$ in the obvious sense, then we say that $\mathcal{T}$ is a natural isomorphism and for any object $X \in \mathfrak{C}_1$ we say that $\mathcal{F}_1(X)$ is naturally isomorphic to $\mathcal{F}_2(X)$. The natural transformation just defined is easily checked to have an inverse so is a natural isomorphism. The point here is not just that V is isomorphic to V^{**} in the category $\mathbf{Lin}_{\mathbb{R}}$ but that the isomorphism exhibited is natural. It works for all the spaces V in a uniform way that involves no special choices. This is to be contrasted with the fact that V is isomorphic to V^* where the construction of such an isomorphism involves an arbitrary choice of a basis.

Appendix C

Calculus

I never could make out what those damn dots meant.
Lord Randolph Churchill

C.1 Derivative

Modern differential geometry is based on the theory of differentiable manifolds- a natural extension of multivariable calculus. Multivariable calculus is said to be done on (or in) an n-dimensional coordinate space $\mathbb{R}^n$ (also called variously "Euclidean space" or sometimes "Cartesian space". We hope that the great majority of readers will be comfortable with standard multivariable calculus. A reader who felt the need for a review could do no better than to study the classic book "Calculus on Manifolds" by Michael Spivak. This book does multivariable calculus[1] in a way suitable for modern differential geometry. It also has the virtue of being short. On the other hand, calculus easily generalizes from $\mathbb{R}^n$ to Banach spaces (a nice class of infinite dimensional vector spaces). We will recall a few definitions and facts from functional analysis and then review highlights from differential calculus while simultaneously generalizing to Banach spaces.

A **topological vector space** over $\mathbb{R}$ is a vector space V with a topology such that vector addition and scalar multiplication are continuous. This means that the map from $\mathsf{V} \times \mathsf{V}$ to V given by $(v_1, v_2) \mapsto v_1 + v_2$ and the map from $\mathbb{R} \times \mathsf{V}$ to V given by $(a, v) \mapsto av$ are continuous maps. Here we have given $\mathsf{V} \times \mathsf{V}$ and $\mathbb{R} \times \mathsf{V}$ the product topologies.

Definition C.1 *A map between topological vector spaces which is both a continuous linear map and which has a continuous linear inverse is called a* ***toplinear isomorphism****.*

A toplinear isomorphism is then just a linear isomorphism which is also a homeomorphism.

[1] Despite the title, most of Spivak's book is about calculus rather than manifolds.

We will be interested in topological vector spaces which get their topology from a norm function:

Definition C.2 *A **norm** on a real vector space* V *is a map* $\|.\| : \mathsf{V} \to \mathbb{R}$ *such that the following hold true:*
i) $\|v\| \geq 0$ *for all* $v \in \mathsf{V}$ *and* $\|v\| = 0$ *only if* $v = 0$.
ii) $\|av\| = |a| \|v\|$ *for all* $a \in \mathbb{R}$ *and all* $v \in \mathsf{V}$.
iii) If $v_1, v_2 \in \mathsf{V}$*, then* $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ *(triangle inequality). A vector space together with a norm is called a **normed vector space**.*

Definition C.3 *Let* E *and* F *be normed spaces. A linear map* $A : \mathsf{E} \longrightarrow \mathsf{F}$ *is said to be bounded if*

$$\|A(v)\| \leq C \|v\|$$

for all $v \in \mathsf{E}$*. For convenience, we have used the same notation for the norms in both spaces. If* $\|A(v)\| = \|v\|$ *for all* $v \in \mathsf{E}$ *we call* A *an **isometry**. If* A *is a toplinear isomorphism which is also an isometry we say that* A *is an **isometric isomorphism.***

It is a standard fact that a linear map between normed spaces is bounded if and only if it is continuous.

The standard norm for $\mathbb{R}^n$ is given by $\left\|(x^1, ..., x^n)\right\| = \sqrt{\sum_{i=1}^n (x^i)^2}$. Imitating what we do in $\mathbb{R}^n$ we can define a distance function for a normed vector space by letting $\operatorname{dist}(v_1, v_2) := \|v_2 - v_1\|$. The distance function gives a topology in the usual way. The convergence of a sequence is defined with respect to the distance function. A sequence $\{v_i\}$ is said to be a **Cauchy sequence** if given any $\varepsilon > 0$ there is an N such that $\operatorname{dist}(v_n, v_m) = \|v_n - v_m\| < \varepsilon$ whenever $n, m > N$. In $\mathbb{R}^n$ every Cauchy sequence is a convergent sequence. This is a good property with many consequences.

Definition C.4 *A normed vector space with the property that every Cauchy sequence converges is called a complete normed vector space or a **Banach space**.*

Note that if we restrict the norm on a Banach space to a closed subspace then that subspace itself becomes a Banach space. This is not true unless the subspace is closed.

Consider two Banach spaces V and W. A continuous map $A : \mathsf{V} \to \mathsf{W}$ which is also a linear isomorphism can be shown to have a continuous linear inverse. In other words, A is a toplinear isomorphism [A,B,R].

Even though some aspects of calculus can be generalized without problems for fairly general spaces, the most general case that we shall consider is the case of a Banach space.

What we have defined are real normed vector spaces and real Banach spaced but there is also the easily defined notion of complex normed spaces and complex Banach spaces. In functional analysis the complex case is central but for calculus it is the real Banach spaces that are central. Of course, every complex Banach space is also a real Banach space in an obvious way. For simplicity and definiteness all normed spaces and Banach spaces in this chapter will be real Banach spaces as defined above. Given two normed spaces V and W with norms $\|.\|_1$ and $\|.\|_2$ we can form a normed space from the Cartesian product $\mathsf{V}\times\mathsf{W}$ by using the norm $\|(v,w)\| := \max\{\|v\|_1, \|w\|_2\}$. The vector space structure on $\mathsf{V}\times\mathsf{W}$ is that of the (outer) direct sum.

Two norms on V, say $\|.\|'$and $\|.\|''$ are equivalent if there exist positive constants c and C such that

$$c\,\|x\|' \leq \|x\|'' \leq C\,\|x\|'$$

for all $x\in\mathsf{V}$. There are many norms for $\mathsf{V}\times\mathsf{W}$ equivalent to that given above including

$$\|(v,w)\|' := \sqrt{\|v\|_1^2 + \|w\|_2^2}$$

and also

$$\|(v,w)\|'' := \|v\|_1 + \|w\|_2\,.$$

If V and W are Banach spaces then so is $\mathsf{V}\times\mathsf{W}$ with either of the above norms. The topology induced on $\mathsf{V}\times\mathsf{W}$ by any of these equivalent norms is exactly the product topology.

Let W_1 and W_2 be subspaces of a Banach space V. We write $\mathsf{W}_1+\mathsf{W}_2$ to indicate the subspace

$$\{v\in\mathsf{V} : v = w_1 + w_2 \text{ for } w_1\in\mathsf{W}_1 \text{ and } w_2\in\mathsf{W}_2\}$$

If $\mathsf{V}=\mathsf{W}_1+\mathsf{W}_2$ then any $v\in V$ can be written as $v=w_1+w_2$ for $w_1\in\mathsf{W}_1$ and $w_2\in\mathsf{W}_2$. If furthermore $\mathsf{W}_1\cap\mathsf{W}_2=\emptyset$ then then this decomposition is unique and we say that W_1 and W_2 are complementary. Now unless a subspace is closed it will itself not be a Banach space and so if we are given a closed subspace W_1 of V then it is ideal if there can be found a subspace W_2 which is complementary to W_1 and which is *also closed*. In this case we write $\mathsf{V}=\mathsf{W}_1\oplus\mathsf{W}_2$. One can use the closed graph theorem to prove following.

Theorem C.1 *If* W_1 *and* W_2 *are complementary closed subspaces of a Banach space* V *then there is a toplinear isomorphism* $\mathsf{W}_1\times\mathsf{W}_2\cong\mathsf{V}$ *given by*

$$(w_1,w_2)\longleftrightarrow w_1+w_2.$$

When it is convenient, we can identify $\mathsf{W}_1\oplus\mathsf{W}_2$ with $\mathsf{W}_1\times\mathsf{W}_2$.

Let E be a Banach space and $\mathsf{W}\subset\mathsf{E}$ a *closed* subspace. If there is a *closed* complementary subspace W' say that W is a **split subspace** of E. The reason why it is important for a subspace to be split is because then we can use the isomorphism $\mathsf{W}\times\mathsf{W}'\cong\mathsf{W}\oplus\mathsf{W}'$. This will be an important technical consideration in the sequel.

Definition C.5 (Notation) *We will denote the set of all continuous (bounded) linear maps from a normed space* E *to a normed space* F *by* $L(\mathsf{E},\mathsf{F})$. *The set of all continuous linear isomorphisms from* E *onto* F *will be denoted by* $Gl(\mathsf{E},\mathsf{F})$. *In case,* $\mathsf{E}=\mathsf{F}$ *the corresponding spaces will be denoted by* $\mathfrak{gl}(\mathsf{E})$ *and* $Gl(\mathsf{E})$.

$Gl(\mathsf{E})$ is a group under composition and is called the **general linear group**.

Definition C.6 *Let V_i, $i = 1, ..., k$ and W be normed spaces. A map $\mu : \mathsf{V}_1 \times \cdots \times \mathsf{V}_k \to \mathsf{W}$ is called **multilinear** (k-multilinear) if for each i, $1 \le i \le k$ and each fixed $(w_1, ..., \widehat{w_i}, ..., w_k) \in \mathsf{V}_1 \times \cdots \times \widehat{\mathsf{V}_1} \times \cdots \times \mathsf{V}_k$ we have that the map*

$$v \mapsto \mu(w_1, ..., \underset{i-th\ slot}{v}, ..., w_k),$$

*obtained by fixing all but the i-th variable, is a linear map. In other words, we require that μ be $\mathbb{R}$- linear in each slot separately. A multilinear map $\mu : \mathsf{V}_1 \times \cdots \times \mathsf{V}_k \to \mathsf{W}$ is said to be **bounded** if and only if there is a constant C such that*

$$\|\mu(v_1, v_2, ..., v_k)\|_{\mathsf{W}} \le C \|v_1\|_{\mathsf{E}_1} \|v_2\|_{\mathsf{E}_2} \cdots \|v_k\|_{\mathsf{E}_k}$$

for all $(v_1, ..., v_k) \in \mathsf{E}_1 \times \cdots \times \mathsf{E}_k$.

Now $\mathsf{V}_1 \times \cdots \times \mathsf{V}_k$ is a normed space in several equivalent ways just in the same way that we defined before for the case $k = 2$. The topology is the product topology.

Proposition C.1 *A multilinear map $\mu : \mathsf{V}_1 \times \cdots \times \mathsf{V}_k \to \mathsf{W}$ is bounded if and only if it is continuous.*

Proof. ($\Leftarrow$) We shall simplify by letting $k = 2$. Let (a_1, a_2) and (v_1, v_2) be elements of $\mathsf{E}_1 \times \mathsf{E}_2$ and write

$$\begin{aligned} &\mu(v_1, v_2) - \mu(a_1, a_2) \\ &= \mu(v_1 - a_1, v_2) + \mu(a_1, v_2 - a_2). \end{aligned}$$

We then have

$$\begin{aligned} &\|\mu(v_1, v_2) - \mu(a_1, a_2)\| \\ &\le C \|v_1 - a_1\| \|v_2\| + C \|a_1\| \|v_2 - a_2\| \end{aligned}$$

and so if $\|(v_1, v_2) - (a_1, a_2)\| \to 0$ then $\|v_i - a_i\| \to 0$ and we see that

$$\|\mu(v_1, v_2) - \mu(a_1, a_2)\| \to 0.$$

(Recall that $\|(v_1, v_2)\| := \max\{\|v_1\|, \|v_2\|\}$).

($\Rightarrow$) Start out by assuming that μ is continuous at $(0, 0)$. Then for $r > 0$ sufficiently small, $(v_1, v_2) \in B((0, 0), r)$ implies that $\|\mu(v_1, v_2)\| \le 1$ so if for $i = 1, 2$ we let

$$z_i := \frac{r v_i}{\|v_1\|_i + \epsilon} \quad \text{for some } \epsilon > 0$$

then $(z_1, z_2) \in B((0, 0), r)$ and $\|\mu(z_1, z_2)\| \le 1$. The case $(v_1, v_2) = (0, 0)$ is trivial so assume $(v_1, v_2) \ne (0, 0)$. Then we have

$$\begin{aligned} \mu(z_1, z_2) &= \mu(\frac{r v_1}{\|v_1\| + \epsilon}, \frac{r v_2}{\|v_2\| + \epsilon}) \\ &= \frac{r^2}{(\|v_1\| + \epsilon)(\|v_2\| + \epsilon)} \mu(v_1, v_2) \le 1 \end{aligned}$$

and so $\mu(v_1, v_2) \le r^{-2}(\|v_1\| + \epsilon)(\|v_2\| + \epsilon)$. Now let $\epsilon \to 0$ to get the result. ∎

Notation C.1 *The set of all bounded multilinear maps* $\mathsf{E}_1 \times \cdots \times \mathsf{E}_k \to \mathsf{W}$ *will be denoted by* $L(\mathsf{E}_1, ..., \mathsf{E}_k; \mathsf{W})$*. If* $\mathsf{E}_1 = \cdots = \mathsf{E}_k = \mathsf{E}$ *then we write* $L^k(\mathsf{E}; \mathsf{W})$ *instead of* $L(\mathsf{E}, ..., \mathsf{E}; \mathsf{W})$

Notation C.2 *For linear maps* $T : \mathsf{V} \to \mathsf{W}$ *we sometimes write* $T \cdot v$ *instead of* $T(v)$ *depending on the notational needs of the moment. In fact, a particularly useful notational device is the following: Suppose we have map* $A : X \to L(\mathsf{V}; \mathsf{W})$*. Then* $A(x)(\ v)$ *makes sense but we may find ourselves in a situation where* $A|_x\ v$ *is even more clear. This latter notation suggests a family of linear maps* $\{\ A|_x\}$ *parameterized by* $x \in X$*.*

Definition C.7 *A multilinear map* $\mu : \mathsf{V} \times \cdots \times \mathsf{V} \to \mathsf{W}$ *is called* ***symmetric*** *if for any* $v_1, v_2, ..., v_k \in \mathsf{V}$ *we have that*

$$\mu(v_{\sigma(1)}, v_{\sigma(2)}, ..., v_{\sigma(k)}) = \mu(v_1, v_2, ..., v_k)$$

for all permutations σ *on the letters* $\{1, 2, ..., k\}$*. Similarly,* μ *is called* ***skew-symmetric*** *or* ***alternating if***

$$\mu(v_{\sigma(1)}, v_{\sigma(2)}, ..., v_{\sigma(k)}) = sgn(\sigma)\mu(v_1, v_2, ..., v_k)$$

for all permutations σ*. The set of all bounded symmetric (resp. skew-symmetric) multilinear maps* $\mathsf{V} \times \cdots \times \mathsf{V} \to \mathsf{W}$ *is denoted* $L^k_{sym}(\mathsf{V}; \mathsf{W})$ *(resp.* $L^k_{skew}(\mathsf{V}; \mathsf{W})$ *or* $L^k_{alt}(\mathsf{V}; \mathsf{W})$*).*

Now if W is complete, that is, if W is a Banach space then the space $L(\mathsf{V}, \mathsf{W})$ is a Banach space in its own right with norm given by

$$\|A\| = \sup_{v \in \mathsf{V}, v \neq 0} \frac{\|A(v)\|_{\mathsf{W}}}{\|v\|_{\mathsf{V}}} = \sup\{\|A(v)\|_{\mathsf{W}} : \|v\|_{\mathsf{V}} = 1\}.$$

Similarly, the spaces $L(\mathsf{E}_1, ..., \mathsf{E}_k; \mathsf{W})$ are also Banach spaces normed by

$$\|\mu\| := \sup\{\|\mu(v_1, v_2, ..., v_k)\|_{\mathsf{W}} : \|v_i\|_{\mathsf{E}_i} = 1 \text{ for } i = 1, .., k\}$$

There is a natural linear bijection $L(\mathsf{V}, L(\mathsf{V}, \mathsf{W})) \cong L^2(\mathsf{V}, \mathsf{W})$ given by $T \mapsto \iota\ T$ where

$$(\iota T)(v_1)(v_2) = T(v_1, v_2)$$

and we identify the two spaces and write T instead of $\iota\ T$. We also have $L(\mathsf{V}, L(\mathsf{V}, L(\mathsf{V}, \mathsf{W})) \cong L^3(\mathsf{V}; \mathsf{W})$ and in general $L(\mathsf{V}, L(\mathsf{V}, L(\mathsf{V}, ..., L(\mathsf{V}, \mathsf{W}))..) \cong L^k(\mathsf{V}; \mathsf{W})$ etc. It is also not hard to show that the isomorphism above is continuous and norm preserving, that is, ι is an isometric isomorphism.

We now come the central definition of differential calculus.

Definition C.8 *A map* $f : \mathsf{V} \supset U \to \mathsf{W}$ *between normed spaces and defined on an open set* $U \subset \mathsf{V}$ *is said to be* ***differentiable at*** $p \in U$ *if and only if there is a bounded linear map* $A_p \in L(\mathsf{V}, \mathsf{W})$ *such that*

$$\lim_{\|h\| \to 0} \frac{\|f(p + h) - f(p) - A_p \cdot h\|}{\|h\|} = 0$$

Proposition C.2 *If A_p exists for a given function f then it is unique.*

Proof. Suppose that A_p and B_p both satisfy the requirements of the definition. That is the limit in question equals zero. For $p + h \in U$ we have

$$\begin{aligned} A_p \cdot h - B_p \cdot h &= (f(p+h) - f(p) - A_p \cdot h) \\ &\quad - (f(p+h) - f(p) - B_p \cdot h)\,. \end{aligned}$$

Dividing by $\|h\|$ and taking the limit as $\|h\| \to 0$ we get

$$\|A_p h - B_p h\| \,/\, \|h\| \to 0$$

Now let $h \neq 0$ be arbitrary and choose $\epsilon > 0$ small enough that $p + \epsilon h \in U$. Then we have

$$\|A_p(\epsilon h) - B_p(\epsilon \mathsf{h})\| \,/\, \|\epsilon \mathsf{h}\| \to 0.$$

But by linearity $\|A_p(\epsilon h) - B_p(\epsilon h)\| \,/\, \|\epsilon h\| = \|A_p h - B_p h\| \,/\, \|h\|$ which doesn't even depend on ϵ so in fact $\|A_p h - B_p h\| = 0$. ■

If a function f is differentiable at p the linear map A_p which exists by definition and is unique by the above result will be denoted by $Df(p)$. The linear map $Df(p)$ is called the **derivative** of f at p. We will also use the notation $Df|_p$ or sometimes $f'(p)$. We often write $Df|_p \cdot h$ instead of $Df(p)(h)$.

It is not hard to show that the derivative of a constant map is constant and the derivative of a (bounded) linear map is the very same linear map.

If we are interested in differentiating "in one direction" then we may use the natural notion of directional derivative. A map $f : \mathsf{V} \supset U \to \mathsf{W}$ has a directional derivative $D_h f$ at p in the direction h if the following limit exists:

$$(D_h f)(p) := \lim_{\varepsilon \to 0} \frac{f(p + \varepsilon h) - f(p)}{\varepsilon}$$

In other words, $D_h f(p) = \frac{d}{dt}\big|_{t=0} f(p+th)$. But a function may have a directional derivative in every direction (at some fixed p), that is, for every $h \in \mathsf{V}$ and yet still not be differentiable at p in the sense of definition **??**.

Notation C.3 *The directional derivative is written as $(D_h f)(p)$ and, in case f is actually differentiable at p, this is equal to $Df|_p\, h = Df(p) \cdot h$ (the proof is easy). Note carefully that $D_x f$ should not be confused with $Df|_x$.*

Let us now restrict our attention to complete normed spaces. From now on $\mathsf{V}, \mathsf{W}, \mathsf{E}$ etc. will refer to Banach spaces.

If it happens that a map $f : U \subset \mathsf{V} \to \mathsf{W}$ is differentiable for all p throughout some open set U then we say that f is differentiable on U. We then have a map $Df : U \subset \mathsf{V} \to L(\mathsf{V}, \mathsf{W})$ given by $p \mapsto Df(p)$. This map is called the derivative of f. If this map itself is differentiable at some $p \in \mathsf{V}$ then its derivative at p is denoted $DDf(p) = D^2 f(p)$ or $D^2 f\big|_p$ and is an element of $L(\mathsf{V}, L(\mathsf{V}, \mathsf{W})) \cong L^2(\mathsf{V}; \mathsf{W})$ which is called the second derivative at p. If in turn $D^2 f\big|_p$ exist for all p throughout U then we have a map $D^2 f : U \to L^2(\mathsf{V}; \mathsf{W})$ called the second derivative. Similarly, we may inductively define $D^k f\big|_p \in L^k(\mathsf{V};\mathsf{W})$ and $D^k f : U \to L^k(\mathsf{V};\mathsf{W})$ whenever f is sufficiently nice that the process can be iterated appropriately (also see **??** below).

Definition C.9 *We say that a map $f : U \subset \mathsf{V} \to \mathsf{W}$ is C^r-differentiable on U if $D^r f\big|_p \in L^r(\mathsf{V},\mathsf{W})$ exists for all $p \in U$ and if $D^r f$ is continuous as map $U \to L^r(\mathsf{V},\mathsf{W})$. If f is C^r-differentiable on U for all $r > 0$ then we say that f is C^∞ or **smooth** (on U).*

To complete the notation we let C^0 indicate mere continuity.

The reader should not find it hard to see that a bounded multilinear map is C^∞.

Definition C.10 *A bijection f between open sets $U_\alpha \subset \mathsf{V}$ and $U_\beta \subset \mathsf{W}$ is called a C^r-**diffeomorphism** if and only if f and f^{-1} are both C^r-differentiable (on U_α and U_β respectively). If $r = \infty$ then we simply call f a diffeomorphism.*

Definition C.11 *Let U be open in V. A map $f : U \to \mathsf{W}$ is called a **local** C^r **diffeomorphism** if and only if for every $p \in U$ there is an open set $U_p \subset U$ with $p \in U_p$ such that $f|_{U_p} : U_p \to f(U_p)$ is a C^r-diffeomorphism.*

We will sometimes think of the derivative of a curve [2] $c : I \subset \mathbb{R} \to \mathsf{E}$ at $t_0 \in I$, as a velocity vector and so we are identifying $Dc|_{t_0} \in L(\mathbb{R},\mathsf{E})$ with $Dc|_{t_0} \cdot 1 \in \mathsf{E}$. Here the number 1 is playing the role of the unit vector in $\mathbb{R}$. Especially in this context we write the velocity vector using the notation $\dot{c}(t_0)$.

It will be useful to define an integral for maps from an interval $[a,b]$ into a Banach space V. First we define the integral for step functions. A function f on an interval $[a,b]$ is a **step function** if there is a partition $a = t_0 < t_1 < \cdots < t_k = b$ such that f is constant, with value say f_i, on each subinterval $[t_i, t_{i+1})$. The set of step functions so defined is a vector space. We define the integral of a step function f over $[a,b]$ by

$$\int_{[a,b]} f := \sum_{i=0}^{k-1} f(t_i)\Delta t_i$$

where $\Delta t_i := t_{i+1} - t_i$. One checks that the definition is independent of the partition chosen. Now the set of all step functions from $[a,b]$ into V is a linear subspace of the Banach space $\mathcal{B}(a,b,\mathsf{V})$ of all bounded functions of $[a,b]$ into V and the integral is a linear map on this space. The norm on $\mathcal{B}(a,b,\mathsf{V})$ is given by $\|f\| = \sup_{a\le t\le b} \|f(t)\|$. If we denote the closure of the space of step functions in this Banach space by $\bar{\mathcal{S}}(a,b,\mathsf{V})$ then we can extend the definition of the integral to $\bar{\mathcal{S}}(a,b,\mathsf{V})$ by continuity since on step functions f we have

$$\left|\int_{[a,b]} f\right| \le (b-a)\,\|f\|_\infty .$$

[2] We will often use the letter I to denote a generic (usually open) interval in the real line.

In the limit, this bound persists and so is valid for all $f \in \bar{\mathcal{S}}(a, b, \mathsf{V})$. This integral is called the **Cauchy-Bochner** integral and is a bounded linear map $\bar{\mathcal{S}}(a, b, \mathsf{V}) \to \mathsf{V}$. It is important to notice that $\bar{\mathcal{S}}(a, b, \mathsf{V})$ contains the continuous functions $C([a, b], \mathsf{V})$ because such may be uniformly approximated by elements of $\mathcal{S}(a, b, \mathsf{V})$ and so we can integrate these functions using the Cauchy-Bochner integral.

Lemma C.1 *If* $\ell : \mathsf{V} \to \mathsf{W}$ *is a bounded linear map of Banach spaces then for any* $f \in \bar{\mathcal{S}}(a, b, \mathsf{V})$ *we have*

$$\int_{[a,b]} \ell \circ f = \ell \left(\int_{[a,b]} f \right)$$

Proof. This is obvious for step functions. The general result follows by taking a limit for a sequence of step functions converging to f in $\bar{\mathcal{S}}(a, b, \mathsf{V})$. ■

The following is a version of the **mean value theorem**:

Theorem C.2 *Let* V *and* W *be Banach spaces. Let* $c : [a, b] \to \mathsf{V}$ *be a* C^1-*map with image contained in an open set* $U \subset \mathsf{V}$. *Also, let* $f : U \to \mathsf{W}$ *be a* C^1 *map. Then*

$$f(c(b)) - f(c(a)) = \int_0^1 Df(c(t)) \cdot c'(t)dt.$$

If $c(t) = (1 - t)x + ty$ *then*

$$f(y) - f(x) = \int_0^1 Df(c(t))dt \cdot (y - x).$$

Notice that in the previous theorem we have $\int_0^1 Df(c(t))dt \in L(\mathsf{V}, \mathsf{W})$.

A subset U of a Banach space (or any vector space) is said to be convex if it has the property that whenever x and y are contained in U then so are all points of the line segment $l_{xy} := \{(1 - t)x + ty : 0 \leq t \leq 1\}$.

Corollary C.1 *Let* U *be a convex open set in a Banach space* V *and* $f : U \to \mathsf{W}$ *a* C^1 *map into another Banach space* W. *Then for any* $x, y \in U$ *we have*

$$\|f(y) - f(x)\| \leq C_{x,y} \|y - x\|$$

where $C_{x,y}$ *is the supremum over all values taken by* f *on point of the line segment* l_{xy} *(see above).*

Let $f : U \subset \mathsf{E} \to \mathsf{F}$ be a map and suppose that we have a splitting $\mathsf{E} = \mathsf{E}_1 \times \mathsf{E}_2$. Let (x, y) denote a generic element of $\mathsf{E}_1 \times \mathsf{E}_2$. Now for every $(a, b) \in U \subset \mathsf{E}_1 \times \mathsf{E}_2$ the partial maps $f_{a,} : y \mapsto f(a, y)$ and $f_{,b} : x \mapsto f(x, b)$ are defined in some neighborhood of b (resp. a). Notice the logical placement of commas in this notation. We define the partial derivatives, when they exist, by $D_2 f(a, b) := Df_{a,}(b)$ and $D_1 f(a, b) := Df_{,b}(a)$. These are, of course, linear maps.

$$D_1 f(a, b) : \mathsf{E}_1 \to \mathsf{F}$$
$$D_2 f(a, b) : \mathsf{E}_2 \to \mathsf{F}$$

Remark C.1 *It is useful to notice that if we consider that maps $\iota_{a,} : x \mapsto (a, x)$ and $\iota_{,b} : x \mapsto (x, a)$ then $D_2 f(a, b) = D(f \circ \iota_{a,})(b)$ and $D_1 f(a, b) = D(f \circ \iota_{,b})(a)$.*

The partial derivative can exist even in cases where f might not be differentiable in the sense we have defined. This is a slight generalization of the point made earlier: f might be differentiable only in certain directions with out being fully differentiable in the sense of **??**. On the other hand, we have

Proposition C.3 *If f has continuous partial derivatives $D_i f(x, y) : \mathsf{E}_i \to \mathsf{F}$ near $(x, y) \in \mathsf{E}_1 \times \mathsf{E}_2$ then $Df(x, y)$ exists and is continuous. In this case, we have for $\mathrm{v} = (\mathrm{v}_1, \mathrm{v}_2)$,*

$$
\begin{aligned}
&Df(x, y) \cdot (\mathrm{v}_1, \mathrm{v}_2) \\
&= D_1 f(x, y) \cdot \mathrm{v}_1 + D_2 f(x, y) \cdot \mathrm{v}_2.
\end{aligned}
$$

Remark C.2 *The reader will surely not be confused if we also use notation such as $\partial_1 f$, $\partial_x f$ or the traditional $\frac{\partial}{\partial x}$ instead of $D_1 f$.*

Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$ be a map that is differentiable at $a = (a^1, ..., a^n) \in \mathbb{R}^n$. The map f is given by m functions $f^i : U \to \mathbb{R}$, $1 \leq i \leq m$. The above proposition have an obvious generalization to the case where we decompose the Banach space into more than two factors as in $\mathbb{R}^m = \mathbb{R} \times \cdots \times \mathbb{R}$ and we find that if all partials $\frac{\partial f^i}{\partial x^j}$ are continuous in U then f is C^1.

With respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, the derivative is given by an $n \times m$ matrix called the **Jacobian** matrix:

$$
J_a(f) := \begin{pmatrix}
\frac{\partial f^1}{\partial x^1}(a) & \frac{\partial f^1}{\partial x^2}(a) & \cdots & \frac{\partial f^1}{\partial x^n}(a) \\
\frac{\partial f^2}{\partial x^1}(a) & & & \frac{\partial f^2}{\partial x^n}(a) \\
\vdots & & \ddots & \\
\frac{\partial f^m}{\partial x^1}(a) & & & \frac{\partial f^m}{\partial x^n}(a)
\end{pmatrix}.
$$

The rank of this matrix is called the rank of f at a. If $n = m$ then the Jacobian is a square matrix and $\det(J_a(f))$ is called the **Jacobian determinant** at a. If f is differentiable near a then it follows from the inverse mapping theorem proved below that if $\det(J_a(f)) \neq 0$ then there is some open set containing a on which f has a differentiable inverse. The Jacobian of this inverse at $f(x)$ is the inverse of the Jacobian of f at x.

C.1.1 Chain Rule, Product rule and Taylor's Theorem

Theorem C.3 (Chain Rule) *Let U_1 and U_2 be open subsets of Banach spaces E_1 and E_2 respectively. Suppose we have continuous maps composing as*

$$U_1 \xrightarrow{f} U_2 \xrightarrow{g} \mathsf{E}_3$$

where E_3 is a third Banach space. If f is differentiable at p and g is differentiable at $f(p)$ then the composition is differentiable at p and $D(g \circ f) = Dg(f(p)) \circ Dg(p)$. In other words, if $v \in \mathsf{E}_1$ then

$$D(g \circ f)\big|_p \cdot v = Dg\big|_{f(p)} \cdot (Df\big|_p \cdot v).$$

Furthermore, if $f \in C^r(U_1)$ and $g \in C^r(U_2)$ then $g \circ f \in C^r(U_1)$.

Proof. Let us use the notation $O_1(v)$, $O_2(v)$ etc. to mean functions such that $O_i(v) \to 0$ as $\|v\| \to 0$. Let $y = f(p)$. Since f is differentiable at p we have $f(p+h) = y + Df\big|_p \cdot h + \|h\|\, O_1(h) := y + \Delta y$ and since g is differentiable at y we have $g(y + \Delta y) = Dg\big|_y \cdot (\Delta y) + \|\Delta y\|\, O_2(\Delta y)$. Now $\Delta y \to 0$ as $h \to 0$ and in turn $O_2(\Delta y) \to 0$ hence

$$\begin{aligned}
g \circ f(p+h) &= g(y + \Delta y) \\
&= Dg\big|_y \cdot (\Delta y) + \|\Delta y\|\, O_2(\Delta y) \\
&= Dg\big|_y \cdot (Df\big|_p \cdot h + \|h\|\, O_1(h)) + \|h\|\, O_3(h) \\
&= Dg\big|_y \cdot Df\big|_p \cdot h + \|h\|\; Dg\big|_y \cdot O_1(h) + \|h\|\, O_3(h) \\
&= Dg\big|_y \cdot Df\big|_p \cdot h + \|h\|\, O_4(h)
\end{aligned}$$

which implies that $g \circ f$ is differentiable at p with the derivative given by the promised formula.

Now we wish to show that $f, g \in C^r$ $r \geq 1$ implies that $g \circ f \in C^r$ also. The bilinear map defined by composition $comp : L(\mathsf{E}_1, \mathsf{E}_2) \times L(\mathsf{E}_2, \mathsf{E}_3) \to L(\mathsf{E}_1, \mathsf{E}_3)$ is bounded . Define a map

$$m_{f,g} : p \mapsto (Dg(f(p)), Df(p))$$

which is defined on U_1. Consider the composition $comp \circ m_{f,g}$. Since f and g are at least C^1 this composite map is clearly continuous. Now we may proceed inductively. Consider the $r - th$ statement:

composition of C^r maps are C^r

Suppose f and g are C^{r+1} then Df is C^r and $Dg \circ f$ is C^r by the inductive hypothesis so that $m_{f,g}$ is C^r. A bounded bilinear functional is C^∞. Thus $comp$ is C^∞ and by examining $comp \circ m_{f,g}$ we see that the result follows. ∎

The following lemma is useful for calculations and may be used without explicit mention:

Lemma C.2 *Let $f : U \subset \mathsf{V} \to \mathsf{W}$ be twice differentiable at $x_0 \in U \subset \mathsf{V}$; then the map $D_v f : x \mapsto Df(x) \cdot v$ is differentiable at x_0 and its derivative at x_0 is given by*

Proof. The map $D_v f : x \mapsto Df(x) \cdot v$ is decomposed as the composition

$$x \overset{Df}{\mapsto} Df\big|_x \overset{R^v}{\mapsto} Df\big|_x \cdot v$$

where $R^v : L(\mathsf{V},\mathsf{W}) \mapsto \mathsf{W}$ is the map $(A,b) \mapsto A \cdot b$. The chain rule gives

$$\begin{aligned} D(D_v f)(x_0) \cdot h &= DR^v(\left.Df\right|_{x_0}) \cdot \left.D(Df)\right|_{x_0} \cdot h) \\ &= DR^v(Df(x_0)) \cdot (D^2 f(x_0) \cdot h). \end{aligned}$$

But R^v is linear and so $DR^v(y) = R^v$ for all y. Thus

$$\begin{aligned} \left.D(D_v f)\right|_{x_0} \cdot h &= R^v(D^2 f(x_0) \cdot h) \\ &= (D^2 f(x_0) \cdot h) \cdot v = D^2 f(x_0)(h,v). \end{aligned}$$

$$\left.D(D_v f)\right|_{x_0} \cdot h = D^2 f(x_0)(h,v).$$

■

Theorem C.4 *If $f : U \subset \mathsf{V} \to \mathsf{W}$ is twice differentiable on U such that $D^2 f$ is continuous, i.e. if $f \in C^2(U)$ then $D^2 f$ is symmetric:*

$$D^2 f(p)(w,v) = D^2 f(p)(v,w).$$

More generally, if $D^k f$ exists and is continuous then $D^k f\ (p) \in L^k_{sym}(\mathsf{V};\mathsf{W})$.

Proof. Let $p \in U$ and define an affine map $A : \mathbb{R}^2 \to \mathsf{V}$ by $A(s,t) := p + sv + tw$. By the chain rule we have

$$\frac{\partial^2 (f \circ A)}{\partial s \partial t}(0) = D^2(f \circ A)(0) \cdot (\mathbf{e}_1, \mathbf{e}_2) = D^2 f(p) \cdot (v,w)$$

where $\mathbf{e}_1, \mathbf{e}_2$ is the standard basis of $\mathbb{R}^2$. Thus it suffices to prove that

$$\frac{\partial^2 (f \circ A)}{\partial s \partial t}(0) = \frac{\partial^2 (f \circ A)}{\partial t \partial s}(0).$$

In fact, for any $\ell \in \mathsf{V}^*$ we have

$$\frac{\partial^2 (\ell \circ f \circ A)}{\partial s \partial t}(0) = \ell \left(\frac{\partial^2 (f \circ A)}{\partial s \partial t} \right)(0)$$

and so by the Hahn-Banach theorem it suffices to prove that $\frac{\partial^2 (\ell \circ f \circ A)}{\partial s \partial t}(0) = \frac{\partial^2 (\ell \circ f \circ A)}{\partial t \partial s}(0)$ which is the standard 1-variable version of the theorem which we assume known. The result for $D^k f$ is proven by induction. ■

Theorem C.5 *Let $\varrho \in L(\mathsf{F}_1, \mathsf{F}_2; \mathsf{W})$ be a bilinear map and let $f_1 : U \subset \mathsf{E} \to \mathsf{F}_1$ and $f_2 : U \subset \mathsf{E} \to \mathsf{F}_2$ be differentiable (resp. $C^r, r \geq 1$) maps. Then the composition $\varrho(f_1, f_2)$ is differentiable (resp. $C^r, r \geq 1$) on U where $\varrho(f_1, f_2) : x \mapsto \varrho(f_1(x), f_2(x))$. Furthermore,*

$$\left.D\varrho\right|_x (f_1, f_2) \cdot v = \varrho(\left.Df_1\right|_x \cdot v, f_2(x)) + \varrho(f_1(x), \left.Df_2\right|_x \cdot v).$$

In particular, if F is an Banach algebra with product $\star$ and $f_1 : U \subset \mathsf{E} \to \mathsf{F}$ and $f_2 : U \subset \mathsf{E} \to \mathsf{F}$ then $f_1 \star f_2$ is defined as a function and

$$D(f_1 \star f_2) \cdot v = (Df_1 \cdot v) \star (f_2) + (Df_1 \cdot v) \star (Df_2 \cdot v).$$

Recall that for a fixed x, higher derivatives $D^p f|_x$ are symmetric multilinear maps. For the following let $(y)^k$ denote $\underbrace{(y, y, ..., y)}_{k\text{-times}}$. With this notation we have the following version of Taylor's theorem.

Theorem C.6 (Taylor's theorem) *Given Banach spaces* V *and* W, *a* C^r *function* $f : U \to \mathsf{W}$ *and a line segment* $t \mapsto (1-t)x + ty$ *contained in* U, *we have that* $t \mapsto D^p f(x + ty) \cdot (y)^p$ *is defined and continuous for* $1 \leq p \leq k$ *and*

$$f(x+y) = f(x) + \frac{1}{1!}\, Df\big|_x \cdot y + \frac{1}{2!}\, D^2 f\big|_x \cdot (y)^2 + \cdots + \frac{1}{(k-1)!}\, D^{k-1} f\big|_x \cdot (y)^{\times(k-1)}$$
$$+ \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} D^k f(x+ty) \cdot (y)^k dt$$

The proof is by induction and follows the usual proof closely. See [A,B,R]. The point is that we still have an integration by parts formula coming from the product rule and we still have the fundamental theorem of calculus.

C.1.2 Local theory of differentiable maps

Inverse Mapping Theorem

The main reason for restricting our calculus to Banach spaces is that the inverse mapping theorem holds for Banach spaces and there is no simple and general inverse mapping theory on more general topological vector spaces. The so called hard inverse mapping theorems such as that of Nash and Moser require special estimates and are constructed to apply only in a very controlled situation.

Definition C.12 *Let* E *and* F *be Banach spaces. A map will be called a* C^r ***diffeomorphism near*** p *if there is some open set* $U \subset \operatorname{dom}(f)$ *containing* p *such that* $f|_U : U \to f(U)$ *is a* C^r *diffeomorphism onto an open set* $f(U)$. *If* f *is a* C^r *diffeomorphism near* p *for all* $p \in U = \operatorname{dom}(f)$ *then we say that* f *is a* ***local*** C^r ***diffeomorphism***.

Definition C.13 *Let* X, d_1 *and* Y, d_2 *be metric spaces. A map* $f : X \to Y$ *is said to be* ***Lipschitz continuous*** *(with constant* k*) if there is a* $k > 0$ *such that* $d(f(x_1), f(x_2)) \leq kd(x_1, x_2)$ *for all* $x_1, x_2 \in X$. *If* $0 < k < 1$ *the map is called a* ***contraction mapping*** *(with constant* k*) or is said to be* k***-contractive***.

The following technical result has numerous applications and uses the idea of iterating a map. **Warning**: For this theorem f^n will denote the $n-$fold composition $f \circ f \circ \cdots \circ f$ rather than a product.

Proposition C.4 (Contraction Mapping Principle) *Let* F *be a closed subset of a complete metric space* (M, d). *Let* $f : F \to F$ *be a* k*-contractive map such that*

$$d(f(x), f(y)) \leq kd(x, y)$$

for some fixed $0 \leq k < 1$. *Then*

1) there is exactly one $x_0 \in F$ such that $f(x_0) = x_0$. Thus x_0 is a fixed point for f. Furthermore,

2) for any $y \in F$ the sequence $y_n := f^n(y)$ converges to the fixed point x_0 with the error estimate $d(y_n, x_0) \leq \frac{k^n}{1-k} d(y_1, x_0)$.

Proof. Let $y \in F$. By iteration

$$d(f^n(y), f^{n-1}(y)) \leq kd(f^{n-1}(y), f^{n-2}(y)) \leq \cdots \leq k^{n-1} d(f(y), y)$$

as follows:

$$\begin{aligned} d(f^{n+j+1}(y), f^n(y)) &\leq d(f^{n+j+1}(y), f^{n+j}(y)) + \cdots + d(f^{n+1}(y), f^n(y)) \\ &\leq (k^{j+1} + \cdots + k) d(f^n(y), f^{n-1}(y)) \\ &\leq \frac{k}{1-k} d(f^n(y), f^{n-1}(y)) \\ &\frac{k^n}{1-k} d(f^1(y), y)) \end{aligned}$$

>From this, and the fact that $0 \leq k < 1$, one can conclude that the sequence $f^n(y) = x_n$ is Cauchy. Thus $f^n(y) \to x_0$ for some x_0 which is in F since F is closed. On the other hand,

$$x_0 = \lim_{n \to 0} f^n(y) = \lim_{n \to 0} f(f^{n-1}(y)) = f(x_0)$$

by continuity of f. Thus x_0 is a fixed point. If u_0 where also a fixed point then

$$d(x_0, u_0) = d(f(x_0), f(u_0)) \leq kd(x_0, u_0)$$

which forces $x_0 = u_0$. The error estimate in (2) of the statement of the theorem is left as an easy exercise. ■

Remark C.3 *Note that a Lipschitz map f may not satisfy the hypotheses of the last theorem even if $k < 1$ since U is not a complete metric space unless $U = \mathsf{E}$.*

Definition C.14 *A continuous map $f : U \to \mathsf{E}$ such that $L_f := \mathrm{id}_U - f$ is injective has a not necessarily continuous inverse G_f and the invertible map $R_f := \mathrm{id}_{\mathsf{E}} - G_f$ will be called the* ***resolvent operator*** *for f.*

The resolvent is a term that is usually used in the context of linear maps and the definition in that context may vary slightly. Namely, what we have defined here would be the resolvent of $\pm L_f$. Be that as it may, we have the following useful result.

Theorem C.7 *Let E be a Banach space. If $f : \mathsf{E} \to \mathsf{E}$ is continuous map that is Lipschitz continuous with constant k where $0 \leq k < 1$, then the resolvent R_f exists and is Lipschitz continuous with constant $\frac{k}{1-k}$.*

Proof. Consider the equation $x - f(x) = y$. We claim that for any $y \in \mathsf{E}$ this equation has a unique solution. This follows because the map $F : \mathsf{E} \to \mathsf{E}$ defined by $F(x) = f(x) + y$ is k-contractive on the complete normed space E as a result of the hypotheses. Thus by the contraction mapping principle there is a unique x fixed by F which means a unique x such that $f(x) + y = x$. Thus the inverse G_f exists and is defined on all of E. Let $R_f := \mathrm{id}_{\mathsf{E}} - G_f$ and choose $y_1, y_2 \in \mathsf{E}$ and corresponding unique x_i , $i = 1, 2$ with $x_i - f(x_i) = y_i$. We have

$$\begin{aligned}
\|R_f(y_1) - R_f(y_2)\| &= \|f(x_1) - f(x_2)\| \\
&\leq k \|x_1 - x_2\| \leq \\
&\leq k \|y_1 - R_f(y_1) - (y_2 - R_f(y_2))\| \leq \\
&\leq k \|y_1 - y_2\| + \|R_f(y_1) - R_f(y_2)\| .
\end{aligned}$$

Solving this inequality we get

$$\|R_f(y_1) - R_f(y_2)\| \leq \frac{k}{1-k} \|y_1 - y_2\| .$$

■

Lemma C.3 *The space $Gl(\mathsf{E}, \mathsf{F})$ of continuous linear isomorphisms is an open subset of the Banach space $L(\mathsf{E}, \mathsf{F})$. In particular, if $\|\mathrm{id} - A\| < 1$ for some $A \in Gl(\mathsf{E})$ then $A^{-1} = \lim_{N\to\infty} \sum_{n=0}^{N} (\mathrm{id} - A)^n$.*

Proof. Let $A_0 \in GL(\mathsf{E}, \mathsf{F})$. The map $A \mapsto A_0^{-1} \circ A$ is continuous and maps $GL(\mathsf{E}, \mathsf{F})$ onto $GL(\mathsf{E}, \mathsf{F})$. If follows that we may assume that $\mathsf{E} = \mathsf{F}$ and that $A_0 = \mathrm{id}_{\mathsf{E}}$. Our task is to show that elements of $\mathrm{L}(\mathsf{E}, \mathsf{E})$ close enough to id_{E} are in fact elements of $GL(\mathsf{E})$. For this we show that

$$\|\mathrm{id} - A\| < 1$$

implies that $A \in GL(\mathsf{E})$. We use the fact that the norm on $\mathrm{L}(\mathsf{E}, \mathsf{E})$ is an algebra norm. Thus $\|A_1 \circ A_2\| \leq \|A_1\| \|A_2\|$ for all $A_1, A_2 \in \mathrm{L}(\mathsf{E}, \mathsf{E})$. We abbreviate id by "1" and denote $\mathrm{id} - A$ by Λ. Let $\Lambda^2 := \Lambda \circ \Lambda$, $\Lambda^3 := \Lambda \circ \Lambda \circ \Lambda$ and so forth. We now form a Neumann series :

$$\begin{aligned}
\pi_0 &= 1 \\
\pi_1 &= 1 + \Lambda \\
\pi_2 &= 1 + \Lambda + \Lambda^2 \\
&\vdots \\
\pi_n &= 1 + \Lambda + \Lambda^2 + \cdots + \Lambda^n.
\end{aligned}$$

By comparison with the Neumann series of real numbers formed in the same way using $\|A\|$ instead of A we see that $\{\pi_n\}$ is a Cauchy sequence since $\|\Lambda\| = \|\mathrm{id} - A\| < 1$. Thus $\{\pi_n\}$ is convergent to some element ρ. Now we have $(1 - \Lambda)\pi_n = 1 - \Lambda^{n+1}$ and letting $n \to \infty$ we see that $(1 - \Lambda)\rho = 1$ or in other words, $A\rho = 1$. ■

Lemma C.4 *The map $\mathcal{I} : Gl(\mathsf{E},\mathsf{F}) \to Gl(\mathsf{E},\mathsf{F})$ given by taking inverses is a C^∞ map and the derivative of $\mathcal{I} : g \mapsto g^{-1}$ at some $g_0 \in Gl(\mathsf{E},\mathsf{F})$ is the linear map given by the formula: $D\mathcal{I}|_{g_0} : A \mapsto -g_0^{-1}Ag_0^{-1}$.*

Proof. Suppose that we can show that the result is true for $g_0 = \mathrm{id}$. Then pick any $h_0 \in GL(\mathsf{E},\mathsf{F})$ and consider the isomorphisms $L_{h_0} : GL(\mathsf{E}) \to GL(\mathsf{E},\mathsf{F})$ and $R_{h_0^{-1}} : GL(\mathsf{E}) \to GL(\mathsf{E},\mathsf{F})$ given by $\phi \mapsto h_0\phi$ and $\phi \mapsto \phi h_0^{-1}$ respectively. The map $g \mapsto g^{-1}$ can be decomposed as

$$g \overset{L_{h_0^{-1}}}{\mapsto} h_0^{-1} \circ g \overset{\mathrm{inv}_{\mathsf{E}}}{\mapsto} (h_0^{-1} \circ g)^{-1} \overset{R_{h_0^{-1}}}{\mapsto} g^{-1}h_0h_0^{-1} = g^{-1}.$$

Now suppose that we have the result at $g_0 = \mathrm{id}$ in $GL(\mathsf{E})$. This means that $D\mathrm{inv}_{\mathsf{E}}|_{h_0} : A \mapsto -A$. Now by the chain rule we have

$$\begin{aligned}
(D\,\mathrm{inv}|_{h_0}) \cdot \mathrm{A} &= D(R_{h_0^{-1}} \circ \mathrm{inv}_{\mathsf{E}} \circ L_{h_0^{-1}}) \cdot \mathrm{A} \\
&= \left(R_{h_0^{-1}} \circ D\,\mathrm{inv}_{\mathsf{E}}|_{\mathrm{id}} \circ L_{h_0^{-1}}\right) \cdot \mathrm{A} \\
&= R_{h_0^{-1}} \circ (-\mathrm{A}) \circ L_{h_0^{-1}} = -h_0^{-1}\mathrm{A}h_0^{-1}
\end{aligned}$$

so the result is true for an arbitrary $h_0 \in GL(\mathsf{E},\mathsf{F})$. Thus we are reduced to showing that $D\mathrm{inv}_{\mathsf{E}}|_{\mathrm{id}} : A \mapsto -A$. The definition of derivative leads us to check that the following limit is zero.

$$\lim_{\|\mathrm{A}\|\to 0} \frac{\left\|(\mathrm{id}+\mathrm{A})^{-1} - (\mathrm{id})^{-1} - (-\mathrm{A})\right\|}{\|\mathrm{A}\|}.$$

Note that for small enough $\|\mathrm{A}\|$, the inverse $(\mathrm{id}+A)^{-1}$ exists and so the above limit makes sense. By our previous result (13) the above difference quotient becomes

$$\begin{aligned}
&\lim_{\|\mathrm{A}\|\to 0} \frac{\left\|(\mathrm{id}+\mathrm{A})^{-1} - \mathrm{id}+\mathrm{A}\right\|}{\|\mathrm{A}\|} \\
&= \lim_{\|\mathrm{A}\|\to 0} \frac{\|\sum_{n=0}^{\infty}(\mathrm{id}-(\mathrm{id}+\mathrm{A}))^n - \mathrm{id}+\mathrm{A}\|}{\|\mathrm{A}\|} \\
&= \lim_{\|\mathrm{A}\|\to 0} \frac{\|\sum_{n=0}^{\infty}(-\mathrm{A})^n - \mathrm{id}+\mathrm{A}\|}{\|\mathrm{A}\|} \\
&= \lim_{\|\mathrm{A}\|\to 0} \frac{\|\sum_{n=2}^{\infty}(-\mathrm{A})^n\|}{\|\mathrm{A}\|} \le \lim_{\|\mathrm{A}\|\to 0} \frac{\sum_{n=2}^{\infty}\|\mathrm{A}\|^n}{\|\mathrm{A}\|} \\
&= \lim_{\|\mathrm{A}\|\to 0} \sum_{n=1}^{\infty}\|\mathrm{A}\|^n = \lim_{\|\mathrm{A}\|\to 0} \frac{\|\mathrm{A}\|}{1-\|\mathrm{A}\|} = 0.
\end{aligned}$$

■

Theorem C.8 (Inverse Mapping Theorem) *Let E and F be Banach spaces and $f : U \to \mathsf{F}$ be a C^r mapping defined an open set $U \subset \mathsf{E}$. Suppose that*

$x_0 \in U$ and that $f'(x_0) = Df|_x : \mathsf{E} \to \mathsf{F}$ is a continuous linear isomorphism. Then there exists an open set $V \subset U$ with $x_0 \in V$ such that $f : V \to f(V) \subset \mathsf{F}$ is a C^r-diffeomorphism. Furthermore the derivative of f^{-1} at y is given by $Df^{-1}\big|_y = (Df|_{f^{-1}(y)})^{-1}$.

Proof. By considering $(Df|_x)^{-1} \circ f$ and by composing with translations we may as well just assume from the start that $f : \mathsf{E} \to \mathsf{E}$ with $x_0 = 0$, $f(0) = 0$ and $Df|_0 = \mathrm{id}_E$. Now if we let $g = x - f(x)$, then $Dg|_0 = 0$ and so if $r > 0$ is small enough then

$$\| Dg|_x \| < \frac{1}{2}$$

for $x \in B(0, 2r)$. The mean value theorem now tells us that $\|g(x_2) - g(x_1)\| \leq \frac{1}{2}\|x_2 - x_1\|$ for $x_2, x_1 \in \overline{B}(0, r)$ and that $g(\overline{B}(0, r)) \subset \overline{B}(0, r/2)$. Let $y_0 \in \overline{B}(0, r/2)$. It is not hard to show that the map $c : x \mapsto y_0 + x - f(x)$ is a contraction mapping $c : \overline{B}(0, r) \to \overline{B}(0, r)$ with constant $\frac{1}{2}$. The contraction mapping principle **??** says that c has a unique fixed point $x_0 \in \overline{B}(0, r)$. But $c(x_0) = x_0$ just translates to $y_0 + x_0 - f(x_0) = x_0$ and then $f(x_0) = y_0$. So x_0 is the unique element of $\overline{B}(0, r)$ satisfying this equation. But then since $y_0 \in \overline{B}(0, r/2)$ was an arbitrary element of $\overline{B}(0, r/2)$ it follows that the restriction $f : \overline{B}(0, r/2) \to f(\overline{B}(0, r/2))$ is invertible. But f^{-1} is also continuous since

$$\begin{aligned}
\left\|f^{-1}(y_2) - f^{-1}(y_1)\right\| &= \|x_2 - x_1\| \\
&\leq \|f(x_2) - f(x_1)\| + \|g(x_2) - g(x_1)\| \\
&\leq \|f(x_2) - f(x_1)\| + \frac{1}{2}\|x_2 - x_1\| \\
&= \|y_2 - y_1\| + \frac{1}{2}\left\|f^{-1}(y_2) - f^{-1}(y_1)\right\|
\end{aligned}$$

Thus $\left\|f^{-1}(y_2) - f^{-1}(y_1)\right\| \leq 2\|y_2 - y_1\|$ and so f^{-1} is continuous. In fact, f^{-1} is also differentiable on $B(0, r/2)$. To see this let $f(x_2) = y_2$ and $f(x_1) = y_1$ with $x_2, x_1 \in \overline{B}(0, r)$ and $y_2, y_1 \in \overline{B}(0, r/2)$. The norm of $Df(x_1))^{-1}$ is bounded (by continuity) on $\overline{B}(0, r)$ by some number B. Setting $x_2 - x_1 = \Delta x$ and $y_2 - y_1 = \Delta y$ and using $(Df(x_1))^{-1}Df(x_1) = \mathrm{id}$ we have

$$\begin{aligned}
&\left\|f^{-1}(y_2) - f^{-1}(y_1) - (Df(x_1))^{-1} \cdot \Delta y\right\| \\
&= \left\|\Delta x - (Df(x_1))^{-1}(f(x_2) - f(x_1))\right\| \\
&= \left\|\{(Df(x_1))^{-1}Df(x_1)\}\Delta x - \{(Df(x_1))^{-1}Df(x_1)\}(Df(x_1))^{-1}(f(x_2) - f(x_1))\right\| \\
&\leq B\|Df(x_1)\Delta x - (f(x_2) - f(x_1))\| \leq o(\Delta x) = o(\Delta y) \text{ (by continuity).}
\end{aligned}$$

Thus $Df^{-1}(y_1)$ exists and is equal to $(Df(x_1))^{-1} = (Df(f^{-1}(y_1)))^{-1}$. A simple argument using this last equation shows that $Df^{-1}(y_1)$ depends continuously on y_1 and so f^{-1} is C^1. The fact that f^{-1} is actually C^r follows from a simple induction argument that uses the fact that Df is C^{r-1} together with lemma **??**. This last step is left to the reader. ∎

Corollary C.2 *Let $U \subset \mathsf{E}$ be an open set. Suppose that $f : U \to \mathsf{F}$ is differentiable with $Df(p) : \mathsf{E} \to \mathsf{F}$ a (bounded) linear isomorphism for each $p \in U$. Then f is a local diffeomorphism.*

Example C.1 *Consider the map $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$\phi(x, y) := (x^2 - y^2, 2xy)$$

The derivative is given by the matrix

$$\begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

which is invertible for all $(x, y) \neq (0, 0)$. Thus, by the inverse mapping theorem, the restriction of ϕ to a sufficiently small open disk centered at any point but the origin will be a diffeomorphism. We may say that the restriction $\phi|_{\mathbb{R}^2 \setminus \{(0,0)\}}$ is a local diffeomorphism. However, notice that $\phi(x, y) = \phi(-x, -y)$ so generically ϕ is a 2-1 map and in particular is not a (global) diffeomorphism.

The next theorem is basic for differentiable manifold theory.

Theorem C.9 (Implicit Mapping Theorem) *Let $\mathsf{E}_1, \mathsf{E}_2$ and W be Banach spaces and $O \subset \mathsf{E}_1 \times \mathsf{E}_2$ open. Let $f : O \to \mathsf{W}$ be a C^rmapping such that $f(x_0, y_0) = 0$. If $D_2 f_{(x_0,y_0)} : \mathsf{E}_2 \to \mathsf{W}$ is a continuous linear isomorphism then there exists an open sets $U_1 \subset \mathsf{E}_1$ and $U_2 \subset \mathsf{E}_2$ such that $U_1 \times U_2 \subset O$ with $x_0 \in U_0$ and unique C^r mapping $g : U_1 \to U_2$ with $g(x_0) = y_0$ such that for all $(x, y) \in U_1 \times U_2$*

$$f(x, y) = 0 \text{ if and only if } y = g(x)$$

The function g in the theorem satisfies $f(x, g(x)) = 0$ which says that graph of g is contained in $(U_1 \times U_2) \cap f^{-1}(0)$ but the conclusion of the theorem is stronger since it says that in fact the graph of g is exactly equal to $(U_1 \times U_2) \cap f^{-1}(0)$.

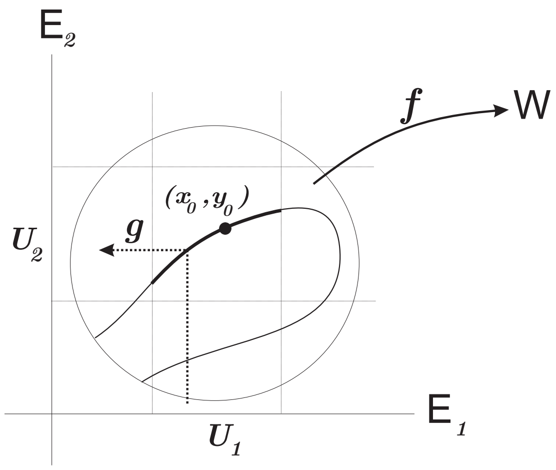
Proof of the implicit mapping theorem. Let $F : O \to \mathsf{E}_1 \times \mathsf{W}$ be defined by

$$F(x, y) = (x, f(x, y)).$$

Notice that $DF|_{(x_0,y_0)}$ has the form

$$\begin{bmatrix} id & 0 \\ D_1 f(x_0, y_0) & D_2 f(x_0, y_0) \end{bmatrix}$$

and it is easily seen that this is a toplinear isomorphism from $\mathsf{E}_1 \times \mathsf{E}_2$ to $\mathsf{E}_1 \times \mathsf{W}$. Thus by the inverse mapping theorem there is an open set $O' \subset O$ containing (x_0, y_0) such that F restricted to O' is a diffeomorphism. Now take open sets U_1 and U_2 so that $(x_0, y_0) \in U_1 \times U_2 \subset O'$ and let $\psi := F| (U_1 \times U_2)$. Then ψ is a diffeomorphism and being a restriction of F we have $\psi(x, y) = (x, f(x, y))$



for all $(x, y) \in U_1 \times U_2$. Now ψ^{-1} must have the form $\psi^{-1}(x, w) = (x, h(x, w))$ where $h : \psi(U_1 \times U_2) \to U_2$. Note that $\psi(U_1 \times U_2) = U_1 \times h(U_1 \times U_2)$.

Let $g(x) := h(x, 0)$. Then $(x, 0) = \psi \circ \psi^{-1}(x, 0) = \psi \circ (x, h(x, 0)) = (x, f(x, h(x, 0)))$ so that in particular $0 = f(x, h(x, 0)) = f(x, g(x))$ from which we now see that $\mathrm{graph}(g) \subset (U_1 \times U_2) \cap f^{-1}(0)$.

We now show that $(U_1 \times U_2) \cap f^{-1}(0) \subset \mathrm{graph}(g)$. Suppose that for some $(x, y) \in U_1 \times U_2$ we have $f(x, y) = 0$. Then $\psi(x, y) = (x, 0)$ and so

$$\begin{aligned}(x, y) &= \psi^{-1} \circ \psi(x, y) \\ &= \psi^{-1}(x, 0) = (x, h(x, 0)) \\ &= (x, g(x))\end{aligned}$$

from which we see that $y = g(x)$ and thus $(x, y) \in \mathrm{graph}(g)$. ■

The simplest situation is that of a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(a, b) = 0$ and $D_2f(a, b) \neq 0$. Then the implicit mapping theorem gives a function g so that $f(x, g(x)) = 0$ for all x sufficiently near a. Note, however, the following exercise:

Exercise C.1 *Find a function $f : \mathbb{R}^2 \to \mathbb{R}$ with $D_2f(0, 0) = 0$ and a continuous function g with $f(x, g(x)) = 0$ for all x sufficiently near a. Thus we see that the implicit mapping theorem gives sufficient but not necessary conditions for the existence of a function g with the property $f(x, g(x)) = 0$.*

In the case of a map $f : U \to V$ between open subsets of Euclidean spaces (say $\mathbb{R}^n$ and $\mathbb{R}^m$) we have the notion of **rank at** $p \in U$ which is just the rank of the linear map $Df(p) : \mathbb{R}^n \to \mathbb{R}^m$.

The local behavior of a differentiable map near a point is to a large extent indicated by the behavior of the derivative at that point. We now turn to a series of results with that theme. First, a bit more about the linear situation.

Definition C.15 *We say that a continuous linear map $A_1 : \mathsf{E}_1 \to \mathsf{F}_1$ is* ***equivalent*** *to a map $A_2 : \mathsf{E}_2 \to \mathsf{F}_2$ if there are continuous linear isomorphisms $\alpha : \mathsf{E}_1 \to \mathsf{E}_2$ and $\beta : \mathsf{F}_1 \to \mathsf{F}_2$ such that $A_2 = \beta \circ A_1 \circ \alpha^{-1}$.*

Definition C.16 *Let $A : \mathsf{E} \to \mathsf{F}$ be an surjective continuous linear map. We say that A is a* ***splitting linear surjection*** *if there is a Banach space E_1 with $\mathsf{E} \cong \mathsf{E}_1 \times \mathsf{F}$ and if A is equivalent to the projection $pr_2 : (x, y) \mapsto y$.*

$$\begin{array}{ccc} \mathsf{E} & \to & \mathsf{F} \\ \alpha\downarrow & & \downarrow\beta \\ \mathsf{E}_1 \times \mathsf{F} & \to & \mathsf{F} \end{array}$$

Lemma C.5 *If $A : \mathsf{E} \to \mathsf{F}$ is a splitting surjection then there is a linear isomorphism $\delta : \mathsf{F} \times \mathsf{E}_2 \to \mathsf{E}$ such that $A \circ \delta : \mathsf{E}_1 \times \mathsf{F} \to \mathsf{F}$ is the projection $(x, y) \mapsto y$.*

Proof. By definition there exist isomorphisms $\alpha : \mathsf{E} \to \mathsf{E}_1 \times \mathsf{F}$ and $\beta : \mathsf{F} \to \mathsf{F}$ such that $\beta \circ A \circ \alpha^{-1}$ is the projection $pr_2 : \mathsf{E}_1 \times \mathsf{F} \to \mathsf{F}$. We write pr_2 as follows

$$\begin{aligned} pr_2 &= \beta^{-1} \circ pr_2 \circ (\mathrm{id}_{\mathsf{E}_1}, \beta) \\ &= \beta^{-1} \circ \left(\beta \circ A \circ \alpha^{-1}\right) \circ (\mathrm{id}_{\mathsf{E}_1}, \beta) \\ &= A \circ \alpha^{-1} \circ (\mathrm{id}_{\mathsf{E}_1}, \beta) := A \circ \delta \end{aligned}$$

and so $\delta = \alpha^{-1} \circ (\mathrm{id}_{\mathsf{E}_1}, \beta)$ does the job. ∎

Definition C.17 *Let $A : \mathsf{E} \to \mathsf{F}$ be an injective continuous map. We say that A is a* ***splitting linear injection*** *if there is a Banach space F_2 with $\mathsf{F} \cong \mathsf{E} \times \mathsf{F}_2$ and if A is equivalent to the linear injection $\mathsf{E} \to \mathsf{E} \times \mathsf{F}_2$ defined by $x \mapsto (x, 0)$.*

$$\begin{array}{ccc} \mathsf{E} & \to & \mathsf{F} \\ \alpha\downarrow & & \downarrow\beta \\ \mathsf{E} & \to & \mathsf{E} \times \mathsf{F}_2 \end{array}$$

Lemma C.6 *If $A : \mathsf{E} \to \mathsf{F}$ is a splitting linear injection as above then there exists a linear isomorphism $\delta : \mathsf{F} \to \mathsf{E} \times \mathsf{F}_2$ such that $\delta \circ A : \mathsf{E} \to \mathsf{E} \times \mathsf{F}_2$ is the injection $x \mapsto (x, 0)$.*

Proof. By definition there are isomorphisms $\alpha : \mathsf{E} \to \mathsf{E}$ and $\beta : \mathsf{F} \to \mathsf{E} \times \mathsf{F}_2$ such that $\beta \circ A \circ \alpha^{-1}$ is the injection $\mathrm{inj}_1 : \mathsf{E} \to \mathsf{E} \times \mathsf{F}_2$. We have

$$\begin{aligned} &\mathrm{inj}_1 = (\alpha^{-1} \times \mathrm{id}_{\mathsf{F}_2}) \circ \mathrm{inj}_1 \circ \alpha \\ &(\alpha^{-1} \times \mathrm{id}_{\mathsf{F}_2}) \circ \left(\beta \circ A \circ \alpha^{-1}\right) \circ \alpha \\ &= (\alpha^{-1} \times \mathrm{id}_{\mathsf{F}_2}) \circ \beta \circ A \\ &= \delta \circ A \end{aligned}$$

∎

If A is a splitting linear injection as above it easy to see that there are closed subspaces F_1 and F_2 of F such that $\mathsf{F} = \mathsf{F}_1 \oplus \mathsf{F}_2$ and such that A maps E isomorphically onto F_1 .

Local (nonlinear) case.

Let X, Y be topological spaces. When we write $f :: \mathsf{X} \to \mathsf{Y}$ we imply only that f is defined on some open set in X. We shall employ a similar use of the symbol "$::$" when talking about continuous maps between (open subsets of) topological spaces in general. If we wish to indicate that f is defined near $p \in \mathsf{X}$ and that $f(p) = q$ we will used the pointed category notation together with the symbol "$::$":

$$f :: (\mathsf{X}, p) \to (\mathsf{Y}, q)$$

We will refer to such a map as a **pointed local map**. Of course, every map $f : U \to V$ determines a pointed local map $f :: (\mathsf{X}, p) \to (\mathsf{Y}, f(p))$ for every $p \in U$. Notice that we use the same symbol f for the pointed map. This is a convenient abuse of notation and allows us to apply some of the present terminology to maps without explicitly mentioning pointed maps. Local maps may be composed with the understanding that the domain of the composite map may become smaller: If $f :: (\mathsf{X}, p) \to (\mathsf{Y}, q)$ and $g :: (\mathsf{Y}, q) \to (\mathsf{G}, z)$ then $g \circ f :: (\mathsf{X}, p) \to (\mathsf{G}, z)$ and the domain of $g \circ f$ will be a non-empty open set containing p. Also, we will say that two such maps $f_1 :: (\mathsf{X}, p) \to (\mathsf{Y}, q)$ and $f_2 :: (\mathsf{X}, p) \to (\mathsf{Y}, q)$ are **equal near** p if there is an open set O with $p \in O \subset \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$ such that the restrictions of these maps to O are equal:

$$f_1|_O = f_2|_O$$

in this case will simply write "$f_1 = f_2$ (near p)". We also say that f_1 and f_2 have the same **germ** at p.

Definition C.18 *Let $f_1 :: (\mathsf{E}_1, p_1) \to (\mathsf{F}_1, q_1)$ be a pointed local map and $f_2 :: (\mathsf{E}_2, p_2) \to (\mathsf{F}_2, q_2)$ another such local map. We say that f_1 and f_2 are **(locally) equivalent** if there exist local diffeomorphisms $\alpha :: \mathsf{E}_1 \to \mathsf{E}_2$ and $\beta :: \mathsf{F}_1 \to \mathsf{F}_2$ such that $f_1 = \alpha \circ f_2 \circ \beta^{-1}$ (near p_1) or equivalently if $f_2 = \beta^{-1} \circ f_1 \circ \alpha$ (near p_2).*

$$\begin{array}{ccc} (\mathsf{E}_1, p_1) & \xrightarrow{f_1} & (\mathsf{F}_1, q_1) \\ \alpha \downarrow & & \downarrow \beta \\ (\mathsf{E}_2, p_2) & \xrightarrow{f_2} & (\mathsf{F}_2, q_2) \end{array}$$

Definition C.19 *Let $f :: (\mathsf{E}, p) \to (\mathsf{F}, q)$ be a pointed local map. We say that f is a **locally splitting surjection** or **local submersion** (at p) if there is a Banach space E_1 $\mathsf{E} \cong \mathsf{E}_1 \times \mathsf{F}$ and if f is locally equivalent (at p) to the projection $pr_2 : \mathsf{E}_1 \times \mathsf{F} \to \mathsf{F}$.*

Lemma C.7 *If f is a locally splitting surjection as above there are open sets $U_1 \times U_2 \subset \mathsf{F} \times \mathsf{E}_2$ and $V \subset \mathsf{F}$ together with a local diffeomorphism $\varphi : U_1 \times U_2 \subset \mathsf{E}_1 \times \mathsf{F} \to V \subset \mathsf{F}$ such that $f \circ \varphi(x, y) = y$ for all $(x, y) \in U_1 \times U_2$.*

Proof. *This is the local (nonlinear) version of lemma **??** and is proved just as easily.* ■

Definition C.20 *Let $f :: (\mathsf{E}, p) \to (\mathsf{F}, q)$ be a pointed local map. We say that f is a **locally splitting injection** (at p) or **local immersion** (at p) if there exists a Banach space F_2 with $\mathsf{F} \cong \mathsf{E} \times \mathsf{F}_2$ and if f is locally equivalent to the injection* $\mathrm{inj}_1 :: (\mathsf{E}, 0) \to (\mathsf{E} \times \mathsf{F}_2, 0)$.

As a complement to lemma **??** we have

Lemma C.8 *If f is a locally splitting injection at p as above there is an open set U_1 containing p and local diffeomorphism $\varphi : U_1 \subset \mathsf{F} \to U_2 \subset \mathsf{E} \times \mathsf{F}_2$ and such that $\varphi \circ f(x) = (x, 0)$ for all $x \in U_1$.*

The easy proof is similar to the proof of **??**.

Theorem C.10 (local submersion) *Let $f :: (\mathsf{E}, p) \to (\mathsf{F}, q)$ be a local map. If $Df|_p : \mathsf{E} \to \mathsf{F}$ is a splitting surjection then $f :: (\mathsf{E}, p) \to (\mathsf{F}, q)$ is a local submersion.*

Proof. Composing with translations if necessary we may assume that $p = 0$ and $q = 0$. By assumption there are toplinear isomorphisms α and β and the following commutative diagram:

$$\begin{array}{ccc} \mathsf{E} & \overset{Df(0)}{\to} & \mathsf{F} \\ \alpha \downarrow & & \downarrow \beta \\ \mathsf{E}_1 \times \mathsf{F} & \overset{pr_2}{\to} & \mathsf{F} \end{array}$$

Now consider that map $\bar{f} := \beta \circ f \circ \alpha^{-1}$. Notice that by the chain rule $D\bar{f}\big|_{(0,0)} = \beta \circ Df(0) \circ \alpha^{-1} = pr_2$. We consider now the partial derivative $D_2\bar{f}(0,0) : \mathsf{F} \to \mathsf{F}$. Now recall remark **??** and notice that inj_2 is the same as what we there called $\iota_{0,}$. We have that $D_2\bar{f}(0,0) = D\left(\bar{f} \circ \mathrm{inj}_2\right)\big|_0 = D\bar{f}\big|_{(0,0)} \circ \mathrm{inj}_2 = id_{\mathsf{F}}$. We see from this that $\bar{f}$ satisfies the hypotheses of the implicit function theorem. Let ψ, h be as the proof of that theorem. Then $(x, w) = \psi \circ \psi^{-1}(x, w) = \psi(x, h(x, w)) = (x, \bar{f}(x, h(x, w)))$

$w = \bar{f}(x, h(x, w)) = \bar{f} \circ \psi^{-1}(x, w)$. So $\bar{f} \circ \psi^{-1}$ is locally equal to the projection pr_2 but also $\bar{f} \circ \psi^{-1}$ is locally equivalent to f: $\bar{f} \circ \psi^{-1} = \beta \circ f \circ \alpha^{-1} \circ \psi^{-1} = \beta \circ f \circ (\psi \circ \alpha)^{-1}$. ∎

The finite dimensional version says that if $f :: (\mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ is a local map with rank k near 0. Then there are diffeomorphisms $g_1 :: (\mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R}^n \times \mathbb{R}^k, 0)$ and $g_2 :: (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$ such that near 0 the map

$$g_2 \circ f \circ g_1^{-1} :: (\mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R}^n, 0)$$

is just the projection $(x, y) \mapsto y$.

Theorem C.11 (local immersion) *Let $f :: (\mathsf{E}, p) \to (\mathsf{F}, q)$ be a local map. If $Df|_p : \mathsf{E} \to \mathsf{F}$ is a splitting injection then $f :: (\mathsf{E}, p) \to (\mathsf{F}, q)$ is a local immersion at p.*

Proof. We leave the details as an exercise. Hint: Show that we may as well assume that $\mathsf{F} = \mathsf{E} \times \mathsf{F}_2$ and that the image of $Df|_p$ is $\mathsf{E} \times \{0\}$. One then consider the map $F : \mathsf{E} \times \mathsf{F}_2 \to \mathsf{E} \times \mathsf{F}_2$ given by

$$(x, y) \mapsto f(x) + (0, y).$$

Now use the inverse function theorem. Also see [A,B,R]. ■

The finite dimensional version says that if $f :: \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^k$ is a map of constant rank n in some neighborhood of $0 \in \mathbb{R}^n$, then there is $g_1 :: \mathbb{R}^n \to \mathbb{R}^n$ with $g_1(0) = 0$, and a $g_2 :: \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^k$ such that $g_2 \circ f \circ g_1^{-1}$ is just given by $x \mapsto (x, 0) \in \mathbb{R}^n \times \mathbb{R}^k$.

If the reader thinks about what is meant by local immersion and local submersion they will realize that in each case the derivative map $Df(p)$ has full rank. That is, the rank of the Jacobian matrix in either case is a big as the dimensions of the spaces involved will allow. Now rank is only semicontinuous and this is what makes full rank extend from points out onto neighborhoods so to speak. On the other hand, we can get more general maps into the picture if we explicitly assume that the rank is locally constant. We will state the following theorem only for the finite dimensional case. However there is a way to formulate and prove a version for infinite dimensional Banach spaces that can be found in [A,B,R].

Theorem C.12 (The Rank Theorem) *Let $f : (\mathbb{R}^n, p) \to (\mathbb{R}^m, q)$ be a local map such that Df has constant rank r in an open set containing p. Then there are local diffeomorphisms $g_1 : (\mathbb{R}^n, p) \to (\mathbb{R}^n, 0)$ and $g_2 : (\mathbb{R}^m, q) \to (\mathbb{R}^m, 0)$ such that $g_2 \circ f \circ g_1^{-1}$ is a local diffeomorphism near 0 with the form*

$$(x^1, ..., x^n) \mapsto (x^1, ..., x^r, 0, ..., 0).$$

Proof. Without loss of generality we may assume that $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ and that (reindexing) the $r \times r$ matrix

$$\left(\frac{\partial f^j}{\partial x^j}\right)_{1 \le i,j \le r}$$

is nonsingular in an open ball centered at the origin of $\mathbb{R}^n$. Now form a map $g_1(x^1, ...x^n) = (f^1(x), ..., f^r(x), x^{r+1}, ..., x^n)$. The Jacobian matrix of g_1 has the block matrix form

$$\begin{bmatrix} \left(\frac{\partial f^i}{\partial x^j}\right) & \\ 0 & I_{n-r} \end{bmatrix}$$

which has nonzero determinant at 0 and so by the inverse mapping theorem g_1 must be a local diffeomorphism near 0. Restrict the domain of g_1 to this possibly smaller open set. It is not hard to see that the map $f \circ g_1^{-1}$ is of the form $(z^1, ..., z^n) \mapsto (z^1, ..., z^r, \gamma^{r+1}(z), ..., \gamma^m(z))$ and so has Jacobian matrix of the form

$$\begin{bmatrix} I_r & 0 \\ * & \left(\frac{\partial \gamma^i}{\partial x^j}\right) \end{bmatrix}.$$

Now the rank of $\left(\frac{\partial\gamma^i}{\partial x^j}\right)_{r+1\leq i\leq m,\ r+1\leq j\leq n}$ must be zero near 0 since the rank$(f) =$ rank$(f\circ h^{-1}) = r$ near 0. On the said (possibly smaller) neighborhood we now define the map $g_2 : (\mathbb{R}^m, q) \to (\mathbb{R}^m, 0)$ by

$$(y^1, ..., y^m) \mapsto (y^1, ..., y^r, y^{r+1} - \gamma^{r+1}(y_*, 0), ..., y^m - \gamma^m(y_*, 0))$$

where $(y_*, 0) = (y^1, ..., y^r, 0, ..., 0)$. The Jacobian matrix of g_2 has the form

$$\begin{bmatrix} I_r & 0 \\ * & I \end{bmatrix}$$

and so is invertible and the composition $g_2 \circ f \circ g_1^{-1}$ has the form

$$\begin{aligned} z &\overset{f\circ g_1^{-1}}{\mapsto} (z_*, \gamma_{r+1}(z), ..., \gamma_m(z)) \\ &\overset{g_2}{\mapsto} (z_*, \gamma_{r+1}(z) - \gamma_{r+1}(z_*, 0), ..., \gamma_m(z) - \gamma_m(z_*, 0)) \end{aligned}$$

where $(z_*, 0) = (z^1, ..., z^r, 0, ..., 0)$. It is not difficult to check that $g_2 \circ f \circ g_1^{-1}$ has the required form near 0. ■

This theorem gives theorems C.11 and C.10 except that the projection would be on the first factor which is not a significant difference. The proof is similar in spirit to the proof of the implicit function theorem and can be found in [Bro Jan].

C.1.3 The Tangent Bundle of an Open Subset of a Banach Space

Later on we will define the notion of a tangent space and tangent bundle for a differentiable manifold. In fact, we will give several alternative definitions. Here we give a somewhat preliminary definition that applies to the case of an open set U in a Banach space.

Definition C.21 *Let E be a Banach space and $U \subset \mathsf{E}$ an open subset. A **tangent vector** at $x \in U$ is a pair (x, v) where $v \in \mathsf{E}$. The **tangent space** at $x \in U$ is defined to be $T_xU := T_x\mathsf{E} := \{x\} \times \mathsf{E}$ and the tangent bundle TU over U is the union of the tangent spaces and so is just $TU = U \times \mathsf{E}$.*

If we use the dual space E^* we get the notions of cotangent vector, cotangent space and cotangent bundle over U denoted T^*U.

In the current setting, it is often not necessary to distinguish between T_xU and E since we can often tell from context that an element $v \in \mathsf{E}$ is to be interpreted as based at some point $x \in U$. For instance a vector field in this setting is just a map $X : U \to \mathsf{E}$ but where $X(x)$ should be thought of as based at x.

Definition C.22 *If $f : U \to \mathsf{F}$ is a C^r map into a Banach space F then the **tangent map** $Tf : TU \to T\mathsf{F}$ is defined by*

$$Tf \cdot (x, v) = (f(x), Df(x) \cdot v).$$

The map takes the tangent space $T_xU = T_x\mathsf{E}$ linearly into the tangent space $T_{f(x)}\mathsf{F}$ for each $x \in U$. The projection onto the first factor is written $\tau_U : TU = U \times \mathsf{E} \to U$ and given by $\tau_U(x, v) = x$. We also have a projection $\pi_U : T^*U = U \times \mathsf{E}^* \to U$ defined similarly.

If $f : U \to V$ is a diffeomorphism of open sets U and V in E and F respectively then Tf is a diffeomorphism that is linear on the fibers and such that we have a commutative diagram:

$$\begin{array}{cccc} TU = & U \times \mathsf{E} & \stackrel{Tf}{\to} & V \times \mathsf{F} = TV \\ & \tau_U \downarrow & & \downarrow \tau_V \\ & U & \to & V \\ & & f & \end{array}$$

The pair is an example of what is called a **local bundle map**. . Note that here the maps τ_U and τ_V are nothing but the projections onto first factors.

The chain rule looks much better if we use the tangent map:

Theorem C.13 *Let U_1 and U_2 be open subsets of Banach spaces E_1 and E_2 respectively. Suppose we have differentiable (resp. $C^r, r \geq 1$) maps composing as*

$$U_1 \xrightarrow{f} U_2 \xrightarrow{g} \mathsf{E}_3$$

where E_3 is a third Banach space. Then the composition $g \circ f$ is differentiable (resp. $C^r, r \geq 1$) and $T(g \circ f) = Tg \circ Tf$

$$\begin{array}{ccccc} TU_1 & \stackrel{Tf}{\to} & TU_2 & \stackrel{Tg}{\to} & T\mathsf{E}_3 \\ \tau_{U_1} \downarrow & & \tau_{U_2} \downarrow & & \downarrow \tau_{E_3} \\ U_1 & \stackrel{f}{\to} & U_2 & \stackrel{g}{\to} & \mathsf{E}_3 \end{array}$$

Notation C.4 *(and convention) There are three ways to express the "differential/derivative" of a differentiable map $f : U \subset \mathsf{E} \to \mathsf{F}$. These are depicted in figure* **??***.*

1. *The first is just $Df : \mathsf{E} \to \mathsf{F}$ or more precisely $Df|_x : \mathsf{E} \to \mathsf{F}$ for any point $x \in U$.*

2. *This one is new for us. It is common but not completely standard :*

   $$dF : TU \to \mathsf{F}$$

   This is just the map $(x, v) \to Df|_x v$. We will use this notation also in the setting of maps from manifolds into vector spaces where there is a canonical trivialization of the tangent bundle of the target manifold (all of these terms will be defined). The most overused symbol for various "differentials" is d.

3. *Lastly, we have the tangent map $Tf : TU \to T\mathsf{F}$ which we defined above. This is the one that generalizes to manifolds without problems.*

C.1.4 Extrema

A real valued function f on a topological space X is continuous at $x_0 \in X$ if for every $\epsilon > 0$ there is a open set U_ϵ containing x_0 such that

$$U_\epsilon \subset \{x : f(x) > f(x_0) - \epsilon\} \cap \{x : f(x) < f(x_0) + \epsilon\}.$$

Baire introduced the notion of semicontinuity by essentially using only one of the intersected sets above. Namely, f is lower semicontinuous at x_0 if for every $\epsilon > 0$ there is a open set U_ϵ containing x_0 such that

$$U_\epsilon \subset \{x : f(x) > f(x_0) - \epsilon\}.$$

Of course there is the symmetrical notion of upper semicontinuity. Lower semicontinuity is appropriately introduced in connection with the search for x such that $f(x)$ is a (local) minimum. Since replacing f by $-f$ interchanges upper and lower semicontinuity and also maxima and minima it we be sufficient to limit the discussion to minima. If the topological space is Hausdorff then we can have a simple and criterion for the existence of a minimum:

Theorem C.14 *Let M be Hausdorff and $f : M \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function such that there exists a number $c < \infty$ such that $M_{f,c} := \{x \in M : f(x) \leq c\}$ is nonempty and sequentially compact then there exists a minimizing $x_0 \in M$:*

$$f(x_0) = \inf_{x \in M} f(x).$$

This theorem is a sort of generalization of the theorem that a continuous function on a compact set achieve a minimum (and maximum). We need to include semicontinuous functions since even some of the most elementary examples for geometric minimization problems naturally involve functionals that are only semicontinuous for very natural topologies on the set M.

Now if M is a convex set in some vector space then if $f : M \to \mathbb{R}$ is strictly convex on M (meaning that $0 < t < 1 \implies f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2)$) then a simple argument shows that f can have at most one minimizer x_0 in M. We are thus led to a wish list in our pursuit of a minimum for a function. We hope to simultaneously have

1. M convex,
2. $f : M \to \mathbb{R}$ strictly convex,
3. f lower semicontinuous,
4. there exists $c < \infty$ so that $M_{f,c}$ is nonempty and sequentially compact.

Just as in elementary calculus, a differentiable function f has an extrema at a point x_0 only if x_0 is a boundary point of the domain of f or, if x_0 is an interior point, $Df(x_0) = 0$. In case $Df(x_0) = 0$ for some interior point x_0, we may not jump to the conclusion that f achieves an extrema at x_0. As expected, there is a second derivative test that may help in this case:

Theorem C.15 *Let $f : U \subset \mathsf{E} \to \mathsf{F}$ be twice differentiable at $x_0 \in U$ (assume U is open). If f has a local extrema at x_0 then $D^2 f(x_0)(v, v) \geq 0$ for all $v \in \mathsf{E}$. If $D^2 f(x_0)$ is a nondegenerate bilinear form then if $D^2 f(x_0)(v, v) > 0$ (resp. < 0) then f achieves a minimum (resp. maximum) at x_0.*

In practice we may have a choice of several possible sets M, topologies for M or the vector space contains M and so on. But then we must strike a balance since a topology with more open sets has more lower semicontinuous functions while one with less open sets means more sequentially compact sets.

C.2 Problem Set

1. Find the matrix that represents (with respects to standard bases) the derivative the map $f : \mathbb{R}^n \to \mathbb{R}^m$ given by

   a) $f(x) = Ax$ for an $m \times n$ matrix A.

   b) $f(x) = x^t Ax$ for an $n \times n$ matrix A (here $m = 1$).

   c) $f(x) = x^1 x^2 \cdots x^n$ (here $m = 1$).

2. Find the derivative of the map $F : L^2([0, 1]) \to L^2([0, 1])$ given by

$$F[f](x) = \int_0^1 k(x, y) \left[f(y)\right]^2 dy$$

   where $k(x, y)$ is a bounded continuous function on $[0, 1] \times [0, 1]$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and define $F : C[0, 1] \to C[0, 1]$ by

$$F(g) := f \circ g$$

   Show that F is differentiable and $DF|_g : C[0, 1] \to C[0, 1]$ is the linear map given by $\left(DF|_g \cdot u\right)(t) = f'(g(t)) \cdot u(t)$.

4. Let $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be C^∞ and define

$$S[c] = \int_0^1 L(c(t), c'(t), t)dt$$

   which is defined on the Banach space B of all C^1 curves $c : [0, 1] \to R^n$ with $c(0) = 0$ and $c(1) = 0$ and with the norm $\|c\| = \sup_{t \in [0,1]}\{|c(t)| + |c'(t)|\}$. Find a function $g_c : [0, 1] \to R^n$ such that

$$DS|_c \cdot b = \int_0^1 \langle g_c(t), b(t)\rangle dt$$

   or in other words,

$$DS|_c \cdot b = \int_0^1 \sum_{i=1}^n g_c^i(t) b^i(t) dt.$$

5. In the last problem, if we had not insisted that $c(0) = 0$ and $c(1) = 0$, but rather that $c(0) = x_0$ and $c(1) = x_1$, then the space wouldn't even have been a vector space let alone a Banach space. But this fixed endpoint family of curves is exactly what is usually considered for functionals of this type. Anyway, convince yourself that this is not a serious problem by using the notion of an affine space (like a vector space but no origin and only differences are defined.). Is the tangent space of the this space of fixed endpoint curves a Banach space?

   Hint: If we choose a fixed curve c_0 which is the point in the Banach space at which we wish to take the derivative then we can write $\mathcal{B}_{\vec{x}_0\vec{x}_1} = \mathcal{B} + c_0$ where

$$\begin{aligned}\mathcal{B}_{\vec{x}_0\vec{x}_1} &= \{c : c(0) = \vec{x}_0 \text{ and } c(1) = \vec{x}_1\} \\ \mathcal{B} &= \{c : c(0) = 0 \text{ and } c(1) = 0\}\end{aligned}$$

   Then we have $T_{c_0}\mathcal{B}_{\vec{x}_0\vec{x}_1} \cong \mathcal{B}$. Thus we should consider $DS|_{c_0} : \mathcal{B} \to \mathcal{B}$.

6. Let $\mathrm{Fl}_t(.)$ be defined by $\mathrm{Fl}_t(x) = (t+1)x$ for $t \in (-1/2, 1/2)$ and $x \in \mathbb{R}^n$. Assume that the map is jointly C^1 in both variable. Find the derivative of

$$f(t) = \int_{D(t)} (tx)^2 \, dx$$

   at $t = 0$, where $D(t) := \mathrm{Fl}_t(D)$ the image of the disk $D = \{|x| \leq 1\}$.

   Hint: Find the Jacobian $J_t := \det[DFl_t(x)]$ and then convert the integral above to one just over $D(0) = D$.

7. Let $M_{n\times n}(\mathbb{R})$ be the vector space of $n \times n$ matrices with real entries and let $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ be the determinant map. The derivative at the identity element I should be a linear map $D\det(I) : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$. Show that $D\det(I)\cdot B = Tr(B)$. More generally, show that $D\det(A)\cdot B = Tr((\mathrm{cof}\, A)^t\, B)$ where $\mathrm{cof}\, A$ is the matrix of cofactors of A.

   What is $\frac{\partial}{\partial x_{ij}} \det X$ where $X = (x_{ij})$?

8. Let $A : U \subset \mathsf{E} \to L(\mathsf{F}, \mathsf{F})$ be a C^r map and define $F : U \times \mathsf{F} \to \mathsf{F}$ by $F(u, f) := A(u)f$. Show that F is also C^r.

9. Show that if F is any closed subset of $\mathbb{R}^n$ there is a C^∞-function f whose zero set $\{x : f(x) = 0\}$ is exactly F.

10. Let U be an open set in $\mathbb{R}^n$. For $f \in C^k(U)$ and $S \subset U$ a compact set, let $\|f\|_k^S := \sum_{|\alpha|\leq k} \sup_{x\in S} \left|\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x)\right|$. a) Show that (1) $\|rf\|_k^S = |r|\, \|f\|_k^S$ for any $r \in \mathbb{R}$, (2) $\|f_1 + f_2\|_k^S \leq \|f_1\|_k^S + \|f_2\|_k^S$ for any $f_1, f_2 \in C^k(U)$, (3) $\|fg\|_k^S \leq \|f\|_k^S\, \|g\|_k^S$ for $f, g \in C^k(U)$.

    b) Let $\{K_i\}$ be a compact subsets of U such that $U = \bigcup_i K_i$. Show that $d(f, g) := \sum_{i=0}^\infty \frac{1}{2^i} \frac{\|f-g\|_k^{K_i}}{1+\|f-g\|_k^{K_i}}$ defines a complete metric space structure on $C^k(U)$.

11. Let E and F be real Banach spaces. A function $f : \mathsf{E} \to \mathsf{F}$ is said to be homogeneous of degree k if $f(rx) = rf(x)$ for all $r \in \mathbb{R}$ and $x \in \mathsf{E}$. Show that if f is homogeneous of degree k and is differentiable, then $Df(v) \cdot v = kf(v)$.

12. Show that the implicit mapping theorem implies the inverse mapping theorem. Hint: Consider $g(x, y) = f(x) - y$ for $f : U \to \mathsf{F}$.

Appendix D

Modules and Multilinearity

A module is an algebraic object that shows up quite a bit in differential geometry and analysis (at least implicitly). A module is a generalization of a vector space where the algebraic field $\mathbb{F}$ is replaced by a ring or an algebra over a field. The modules that occur in differential are almost always finitely generated projective modules over the algebra of C^r functions and these correspond to the spaces of C^r sections of vector bundles. We give the abstract definitions but we ask the reader to constantly keep in mind two cases. The first is just the vector spaces which are the fibers of vector bundles. In this case the ring in the definition below is the field $\mathbb{F}$ (the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$) and the module is just a vector space. The second case, already mentioned, is where the ring is the algebra $C^r(M)$ and the module is the set of C^r sections of a vector bundle.

As we have indicated, a module is similar to a vector space with the differences stemming from the use of elements of a ring R as the scalars rather than the field of complex $\mathbb{C}$ or real numbers $\mathbb{R}$. For an element v of a module V, one still has $1\mathrm{v} = \mathrm{v}$, $0\mathrm{v} = 0$ and $-1\mathrm{v} = -\mathrm{v}$. Of course, every vector space is also a module since the latter is a generalization of the notion of vector space. We also have maps between modules, the module homomorphisms (see definition D.2 below), which make the class of modules and module homomorphism into a category.

Definition D.1 *Let* R *be a ring. A left* R-***module*** *(or a left* ***module*** *over* R*) is an abelian group* $(\mathrm{V}, +)$ *together with an operation* $\mathsf{R} \times \mathrm{V} \to \mathrm{V}$ *written* $(a, v) \mapsto av$ *such that*

1) $(a + b)v = av + bv$ *for all* $a, b \in \mathsf{R}$ *and all* $v \in \mathrm{V}$,

2) $a(v_1 + v_2) = av_1 + av_2$ *for all* $a \in \mathsf{R}$ *and all* $v_2, v_1 \in \mathrm{V}$.

A right R-***module*** *is defined similarly with the multiplication of the right so that*

1) $v(a + b) = va + vb$ *for all* $a, b \in \mathsf{R}$ *and all* $v \in \mathrm{V}$,

2) $(v_1 + v_2)a = v_1a + v_2a$ *for all* $a \in \mathsf{R}$ *and all* $v_2, v_1 \in \mathrm{V}$. **.**

If the ring is commutative (the usual case for us) then we may write $av = va$

and consider any right module as a left module and vice versa. Even if the ring is not commutative we will usually stick to left modules and so we drop the reference to "left" and refer to such as R-modules.

Remark D.1 *We shall often refer to the elements of* R *as* ***scalars****.*

Example D.1 *An abelian group* $(A, +)$ *is a* $\mathbb{Z}$ *module and a* $\mathbb{Z}$*-module is none other than an abelian group. Here we take the product of* $n \in \mathbb{Z}$ *with* $x \in A$ *to be* $nx := x + \cdots + x$ *if* $n \geq 0$ *and* $nx := -(x + \cdots + x)$ *if* $n < 0$ *(in either case we are adding* $|n|$ *terms).*

Example D.2 *The set of all* $m \times n$ *matrices with entries being elements of a commutative ring* R *(for example real polynomials) is an* R*-module.*

Definition D.2 *Let* V_1 *and* V_2 *be modules over a ring* R*. A map* $L : \mathrm{V}_1 \to \mathrm{V}_2$ *is called* ***module homomorphism*** *or* ***linear map*** *if*

$$L(av_1 + bv_2) = aL(v_1) + bL(v_2).$$

By analogy with the case of vector spaces, which module theory includes, we often characterize a module homomorphism L *by saying that* L *is* ***linear over*** R*.*

Example D.3 *The set of all module homomorphisms of a module* V *(lets say over a commutative ring* R*) onto another module* M *is also a module in its own right and is denoted* $\mathrm{Hom}_{\mathsf{R}}(\mathrm{V}, \mathrm{M})$ *or* $L_{\mathsf{R}}(\mathrm{V}, \mathrm{M})$ *(we mainly use the latter).*

Example D.4 *Let* V *be a vector space and* $\ell : \mathrm{V} \to \mathrm{V}$ *a linear operator. Using this one operator we may consider* V *as a module over the ring of polynomials* $\mathbb{R}[t]$ *by defining the "scalar" multiplication by*

$$p(t)v := p(\ell)v$$

for $p \in \mathbb{R}[t]$*.*

Since the ring is usually fixed we often omit mention of the ring. In particular, we often abbreviate $L_{\mathsf{R}}(\mathrm{V}, \mathrm{W})$ to $L(\mathrm{V}, \mathrm{W})$. Similar omissions will be made without further mention.

Remark D.2 *If the modules are infinite dimensional topological vector spaces such as Banach space then we must distinguish between the bounded linear maps and simply linear maps. In the topological vector space setting* $L(\mathsf{E}; \mathsf{F})$ *would normally denote bounded linear maps.*

A **submodule** is defined in the obvious way as a subset $S \subset \mathrm{V}$ that is closed under the operations inherited from V so that S itself is a module. The intersection of all submodules containing a subset $A \subset \mathrm{V}$ is called the **submodule generated by** A and is denoted $\langle A \rangle$. A is called a **generating set**. If $\langle A \rangle = \mathrm{V}$ for a *finite* set A, then we say that V is **finitely generated**.

Let S be a submodule of V and consider the quotient abelian group V/S consisting of cosets, that is, sets of the form $[v] := v + S = \{v + x : x \in S\}$ with addition given by $[v] + [w] = [v + w]$. We define a scalar multiplication by elements of the ring R by $a[v] := [av]$ respectively. In this way, V/S is a module called a **quotient module**.

Many of the operations that exist for vector spaces have analogues in the module category. For example, the direct sum of modules is defined in the obvious way. Also, for any module homomorphism $L : \mathrm{V}_1 \to \mathrm{V}_2$ we have the usual notions of kernel and image:

$$\ker L = \{v \in \mathrm{V}_1 : L(v) = 0\}$$
$$\mathrm{img}(L) = L(\mathrm{V}_1) = \{w \in \mathrm{V}_2 : w = Lv \text{ for some } v \in \mathrm{V}_1\}.$$

These are submodules of V_1 and V_2 respectively.

On the other hand, modules are generally not as simple to study as vector spaces. For example, there are several notions of dimension. The following notions for a vector space all lead to the same notion of dimension. For a completely general module these are all potentially different notions:

1. Length n of the longest chain of submodules

$$0 = \mathrm{V}_n \subsetneq \cdots \subsetneq \mathrm{V}_1 \subsetneq \mathrm{V}$$

2. The cardinality of the largest linearly independent set (see below).

3. The cardinality of a basis (see below).

For simplicity in our study of dimension, let us now assume that R is commutative.

Definition D.3 *A set of elements $e_1, ..., e_k$ of a module are said to be* ***linearly dependent*** *if there exist ring elements $r_1, ..., r_k \in \mathsf{R}$ not all zero, such that $r_1 w_1 + \cdots + r_k w_k = 0$. Otherwise, they are said to be* ***linearly independent****. We also speak of the set $\{e_1, ..., e_k\}$ as being a* ***linearly independent set****.*

So far so good but it is important to realize that just because $e_1, ..., e_k$ are linearly dependent doesn't mean that we may write each of these e_i as a linear combination of the others. It may even be that some single element e forms a linearly dependent set since there may be a nonzero r such that $re = 0$ (such a e is said to have **torsion**).

If a linearly independent set $\{e_1, ..., e_k\}$ is maximal in size then we say that the module has **rank** k. Another strange possibility is that a maximal linearly independent set may not be a generating set for the module and hence may not be a basis in the sense to be defined below. The point is that although for an arbitrary $w \in \mathrm{V}$ we must have that $\{e_1, ..., e_k\} \cup \{w\}$ is linearly dependent and hence there must be a nontrivial expression $rw + r_1 e_1 + \cdots + r_k e_k = 0$, it does **not** follow that we may solve for w since r may not be an invertible element of the ring (i.e. it may not be a **unit**).

Definition D.4 *If B is a generating set for a module V such that every element of V has a* unique *expression as a* finite R*-linear combination of elements of B then we say that B is a **basis** for V.*

Definition D.5 *If an R-module has a basis, then it is referred to as a **finitely generated free module**. If this basis is finite we indicate this by refering to the module as a **finitely generated free module**.*

It turns out that just as for vector spaces the cardinality of a basis for a finitely generated free module V is the same as that of every other basis for V. If a module over a (commutative) ring R has a basis then the number of elements in the basis is called the **dimension** and must in this case be the same as the rank (the size of a maximal linearly independent set). Thus a finitely generated free moduel is also called a finite dimensional free module.

Exercise D.1 *Show that every finitely generated R-module is the homomorphic image of a finitely generated free module.*

If R is a field then every module is free and is a vector space by definition. In this case, the current definitions of dimension and basis coincide with the usual ones.

The ring R is itself a free R-module with standard basis given by $\{1\}$. Also, $\mathsf{R}^n := \mathsf{R} \times \cdots \times \mathsf{R}$ is a finitely generated free module with standard basis $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ where, as usual $\mathbf{e}_i := (0, ..., 1, ..., 0)$; the only nonzero entry being in the i-th position. Up to isomorphism, these account for all finitely generated free modules: If a module V is free with basis $e_1, ..., e_n$ then we have an isomorphism $\mathsf{R}^n \cong \mathrm{V}$ given by

$$(r_1, ..., r_n) \mapsto r_1 e_1 + \cdots + r_n e_n.$$

As we mentioned, $C^r(M)-$modules are of particular interest. But the ring $C^r(M)$ is also a vector space over $\mathbb{R}$ and this means we have one more layer of structure:

Definition D.6 *Let k be a commutative ring, for example a field such as $\mathbb{R}$ or $\mathbb{C}$. A ring A is called a **k-algebra** if there is a ring homomorphism $\mu : \mathsf{k} \to \mathsf{A}$ such that the image $\mu(\mathsf{k})$ consists of elements that commute with everything in A. In particular, A is a module over k.*

Often we have $\mathsf{k} \subset \mathsf{A}$ and μ is inclusion.

Example D.5 *The ring $C^r(M)$ is an $\mathbb{R}$-algebra.*

Because of this example we shall consider A-modules where A is an algebra over some k. In this context the elements of A are still called scalars but the elements of k will be referred to as **constants**.

Example D.6 *The set $\mathfrak{X}_M(U)$ of vector fields defined on an open set $U$is a vector space over $\mathbb{R}$ but it is also a module over the $\mathbb{R}$-algebra $C^\infty(U)$. So for all $X, Y \in \mathfrak{X}_M(U)$ and all $f, g \in C^\infty(U)$ we have*

1. $f(X+Y) = fX + fY$
2. $(f+g)X = fX + gX$
3. $f(gX) = (fg)X$

Similarly, $\mathfrak{X}_M^*(U) = \Gamma(U, T^*M)$ is also a module over $C^\infty(U)$ that is naturally identified with the module dual $\mathfrak{X}_M(U)^*$ by the pairing $(\theta, X) \mapsto \theta(X)$. Here $\theta(X) \in C^\infty(U)$ and is defined by $p \mapsto \theta_p(X_p)$. The set of all vector fields $\mathcal{Z} \subset \mathfrak{X}(U)$ that are zero at a fixed point $p \in U$ is a submodule in $\mathfrak{X}(U)$. If $U, (x^1, ..., x^n)$ is a coordinate chart then the set of vector fields

$$\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}$$

is a basis (over $C^\infty(U)$) for the module $\mathfrak{X}(U)$ and so it is a finitely generated free module. Similarly,

$$dx^1, ..., dx^n$$

is a basis for $\mathfrak{X}_M^*(U)$. It is important to realize that if U is not the domain of a coordinate chart then it may be that $\mathfrak{X}_M(U)$ and $\mathfrak{X}_M(U)^*$ have no basis. In particular, we should not expect $\mathfrak{X}(M)$ to have a basis in the general case.

The sections of any vector bundle over a manifold M form a $C^\infty(M)$-module denoted $\Gamma(E)$. Let $E \to M$ be a trivial vector bundle of finite rank n. Then there exists a basis of vector fields $e_1, ..., e_n$ for the module $\Gamma(E)$. Thus for any section X there exist unique functions f^i such that

$$X = f^i e_i \text{ (summation convention)}$$

In fact, since E is trivial, we may as well assume that $E = M \times \mathbb{R}^n \overset{pr_1}{\to} M$. Then for any basis $u_1, ..., u_n$ for $\mathbb{R}^n$ we may take

$$e_i(x) := (x, u_i).$$

(The e_i form a "local frame field").

Definition D.7 *Let* V_i*,* $i = 1, ..., k$ *and* W *be modules over a ring* R*. A map* $\mu : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{W}$ *is called* ***multilinear*** *(k-multilinear) if for each* i*,* $1 \le i \le k$ *and each fixed* $(v_1, ..., \widehat{v_i}, ..., v_k) \in \mathrm{V}_1 \times \cdots \times \widehat{\mathrm{V}_1} \times \cdots \times \mathrm{V}_k$ *we have that the map*

$$v \mapsto \mu(v_1, ..., \underset{i-th}{v}, ..., v_k),$$

obtained by fixing all but the i-th variable, is a module homomorphism. In other words, we require that μ *be* R*- linear in each slot separately. The set of all multilinear maps* $\mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{W}$ *is denoted* $L_{\mathsf{R}}(\mathrm{V}_1, ..., \mathrm{V}_k; \mathrm{W})$*. If* $\mathrm{V}_1 = \cdots = \mathrm{V}_k = \mathrm{V}$ *then we abbreviate this to* $L^k_{\mathsf{R}}(\mathrm{V}; \mathrm{W})$*.*

If R is commutative then the space of multilinear maps $L_{\mathsf{R}}(\mathrm{V}_1, ..., \mathrm{V}_k; \mathrm{W})$ is itself an R-module in a fairly obvious way. For $a, b \in \mathsf{R}$ and $\mu_1, \mu_2 \in L_{\mathsf{R}}(\mathrm{V}_1, ..., \mathrm{V}_k; \mathrm{W})$ then $a\mu_1 + b\mu_2$ is defined in the usual way.

Let us agree to use the following abbreviation: $\mathrm{V}^k = \mathrm{V} \times \cdots \times \mathrm{V}$ ($k-$fold Cartesian product).

Definition D.8 *The dual of an* $\mathsf{R}-$*module* V *is the module* $\mathrm{V}^* := L_{\mathsf{R}}(\mathrm{V}, \mathsf{R})$ *of all* R*-linear functionals on* V.

Any element of V can be thought of as an element of $\mathrm{V}^{**} := L_{\mathsf{R}}(\mathrm{V}^*, \mathsf{R})$ according to $w(\alpha) := \alpha(w)$. This provides a map $\mathrm{V} \hookrightarrow \mathrm{V}^{**}$ and if this map is an isomorphism then we say that V is **reflexive** .

If V is reflexive then we are free to identify V with V^{**}.

Exercise D.2 *Show that if* V *is a finitely generated free module with finite dimension then* V *is reflexive.*

We sometimes write $w \lrcorner(\alpha) = \langle w, \alpha \rangle = \langle \alpha, w \rangle$.

For completeness we include the definition of a projective module but what is important for us is that the finitely generated projective modules over $C^\infty(M)$ correspond to spaces of sections of smooth vector bundles. These modules and not necessarily free but are reflexive and have many other good properties such as being "locally free".

Definition D.9 *A module* V *is projective if, whenever* V *is a quotient of a module* W *there exists a module* U *such that the direct sum* $\mathrm{V} \oplus \mathrm{U}$ *is isomorphic to* W.

Suppose now that we have two R-modules V_1 and V_2. Consider the category $\mathcal{C}_{\mathrm{V}_1 \times \mathrm{V}_2}$ whose objects are bilinear maps $\mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{W}$ where W varies over all R-modules but V_1 and V_2 are fixed. A morphism from, say $\mu_1 : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{W}_1$ to $\mu_2 : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{W}_2$ is defined to be a map $\ell : \mathrm{W}_1 \to \mathrm{W}_2$ such that the diagram

$$\begin{tikzcd} & \mathrm{W}_1 \arrow[d, "\ell"] \\ \mathrm{V}_1 \times \mathrm{V}_2 \arrow[ur, "\mu_1"] \arrow[dr, "\mu_2"'] & \\ & \mathrm{W}_2 \end{tikzcd}$$

commutes.

Now we come to a main point: Suppose that there is an R-module $\mathrm{T}_{\mathrm{V}_1, \mathrm{V}_2}$ together with a bilinear map $\otimes : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{T}_{\mathrm{V}_1, \mathrm{V}_2}$ that has the following universal property for this category: For every bilinear map $\mu : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{W}$ there is a unique linear map $\widetilde{\mu} : \mathrm{T}_{\mathrm{V}_1, \mathrm{V}_2} \to \mathrm{W}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{V}_1 \times \mathrm{V}_2 \arrow[r, "\mu"] \arrow[d, "\otimes"] & \mathrm{W} \\ \mathrm{T}_{\mathrm{V}_1, \mathrm{V}_2} \arrow[ur, "\widetilde{\mu}"'] & \end{tikzcd}$$

If such a pair $(\mathrm{T}_{\mathrm{V}_1, \mathrm{V}_2}, \otimes)$ exists with this property then it is unique up to isomorphism in $\mathcal{C}_{\mathrm{V}_1 \times \mathrm{V}_2}$. In other words, if $\widehat{\otimes} : \mathrm{V}_1 \times \mathrm{V}_2 \to \widehat{\mathrm{T}}_{\mathrm{V}_1, \mathrm{V}_2}$ is another object with this universal property then there is a module isomorphism $\mathrm{T}_{\mathrm{V}_1, \mathrm{V}_2} \cong$

$\widehat{\mathrm{T}}_{\mathrm{V}_1,\mathrm{V}_2}$ such that the following diagram commutes:

$$\begin{array}{ccc} & & \mathrm{T}_{\mathrm{V}_1,\mathrm{V}_2} \\ & \nearrow^{\otimes} & \\ \mathrm{V}_1 \times \mathrm{V}_2 & & \cong\downarrow \\ & \searrow^{\widehat{\otimes}} & \\ & & \widehat{\mathrm{T}}_{\mathrm{V}_1,\mathrm{V}_2} \end{array}$$

We refer to any such universal object as a tensor product of V_1 and V_2. We will indicate the construction of a specific tensor product that we denote by $\mathrm{V}_1 \otimes \mathrm{V}_2$ with corresponding map $\otimes : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{V}_1 \otimes \mathrm{V}_2$. This will be *the* tensor product. The idea is simple: We let $\mathrm{V}_1 \otimes \mathrm{V}_2$ be the set of all linear combinations of symbols of the form $v_1 \otimes v_2$ for $v_1 \in \mathrm{V}_1$ and $v_2 \in \mathrm{V}_2$, subject to the relations

$$\begin{aligned} (v_1 + v_2) \otimes v &= v_1 \otimes v + v_2 \otimes v \\ v \otimes (v_1 + v_2) &= v \otimes v_1 + v \otimes v_2 \\ r\,(v_1 \otimes v_2) &= rv_1 \otimes v_2 = v_1 \otimes rv_2 \end{aligned}$$

The map $\otimes$ is then simply $\otimes : (v_1, v_2) \to v_1 \otimes v_2$. More generally, we seek a universal object for k-multilinear maps $\mu : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{W}$.

Definition D.10 *A module* $\mathrm{T} = \mathrm{T}_{\mathrm{V}_1,\cdots,\mathrm{V}_k}$ *together with a multilinear map* $\otimes : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{T}$ *is called* ***universal for*** k***-multilinear maps on*** $\mathrm{V}_1 \times \cdots \times \mathrm{V}_k$ *if for every multilinear map* $\mu : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{W}$ *there is a unique linear map* $\widetilde{\mu} : \mathrm{T} \to \mathrm{W}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{V}_1 \times \cdots \times \mathrm{V}_k & \xrightarrow{\mu} & \mathrm{W} \\ \otimes\downarrow & \nearrow_{\widetilde{\mu}} & \\ \mathrm{T} & & \end{array},$$

i.e. we must have $\mu = \widetilde{\mu} \circ \otimes$. *If such a universal object exists it will be called a* ***tensor product*** *of* $\mathrm{V}_1, \ldots, \mathrm{V}_k$ *and the module itself* $\mathrm{T} = \mathrm{T}_{\mathrm{V}_1,\cdots,\mathrm{V}_k}$ *is also referred to as a tensor product of the modules* $\mathrm{V}_1, \ldots, \mathrm{V}_k$.

The tensor product is again unique up to isomorphism. The usual specific realization of the tensor product of modules $\mathrm{V}_1, \ldots, \mathrm{V}_k$ is, roughly, the set of all linear combinations of symbols of the form $v_1 \otimes \cdots \otimes v_k$ subject to the obvious multilinear relations:

$$(v_1 \otimes \cdots \otimes av_i \otimes \cdots \otimes v_k = a(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k)$$

and

$$\begin{aligned} &(v_1 \otimes \cdots \otimes (v_i + w_i) \otimes \cdots \otimes v_k) \\ &= v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_k + v_1 \otimes \cdots \otimes w_i \otimes \cdots \otimes v_k \end{aligned}$$

This space is denoted $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$ which we also write as $\bigotimes_{i=1}^k \mathrm{V}_i$. Also, we will use $\mathrm{V}^{\otimes k}$ to denote $\mathrm{V} \otimes \cdots \otimes \mathrm{V}$ ($k-$fold tensor product of V). The associated map $\otimes : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$ is simply

$$\otimes : (v_1, ..., v_k) \mapsto v_1 \otimes \cdots \otimes v_k$$

To be a bit more pedantic, we may take $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k := F(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k)/\sim$ where $F(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k)$ is the finitely generated free module on the *set* $\mathrm{V}_1 \times \cdots \times \mathrm{V}_k$ and and the equivalence relation "$\sim$" is genererated by the relations

$$(v_1, ..., av_i, ..., v_k) = a(v_1, ..., v_i, ..., v_k)$$

and

$$\begin{aligned} &(v_1, ..., (v_i + w_i), ..., v_k) \\ &= (v_1, ..., v_i, ..., v_k) + (v_1, ..., w_i, ..., v_k) \end{aligned}$$

Each element $(v_1, ..., v_k)$ of the set $\mathrm{V}_1 \times \cdots \times \mathrm{V}_k$ is naturally identified with a generator of the finitely generated free module $F(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k)$ and we have the obvious injection $\mathrm{V}_1 \times \cdots \times \mathrm{V}_k \hookrightarrow F(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k)$. Its equivalence class is denoted $v_1 \otimes \cdots \otimes v_k$ and the map $\otimes$ is then the composition $\mathrm{V}_1 \times \cdots \times \mathrm{V}_k \hookrightarrow F(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k) \to F(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k)/\sim$.

Proposition D.1 *If $f : \mathrm{V}_1 \to \mathrm{W}_1$ and $f : \mathrm{V}_1 \to \mathrm{W}_1$ are module homomorphisms then their is a unique homomorphism, the tensor product $f \otimes g : \mathrm{V}_1 \otimes \mathrm{V}_2 \to \mathrm{W}_1 \otimes \mathrm{W}_2$ which has the characterizing properties that $f \otimes g$ be linear and that $f \otimes g(v_1 \otimes v_2) = (fv_1) \otimes (gv_2)$ for all $v_1 \in \mathrm{V}_1, v_2 \in \mathrm{V}_2$. Similarly, if $f_i : \mathrm{V}_i \to \mathrm{W}_i$ we may obtain $\otimes_i f_i : \bigotimes_{i=1}^k \mathrm{V}_i \to \bigotimes_{i=1}^k \mathrm{W}_i$.*

Proof. exercise. ∎

Definition D.11 *Elements of $\bigotimes_{i=1}^k \mathrm{V}_i$ that may be written as $\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_k$ for some v_i, are called **decomposable** .*

Remark D.3 *It is clear from our specific realization of $\bigotimes_{i=1}^k \mathrm{V}_i$ that element in the image of $\otimes : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \bigotimes_{i=1}^k \mathrm{V}_i$ span $\bigotimes_{i=1}^k \mathrm{V}_i$. I.e. decomposable elements span the space.*

Exercise D.3 *Not all elements are decomposable but the decomposable elements generate $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$.*

It may be that the V_i may be modules over more that one ring. For example, any complex vector space is a module over both $\mathbb{R}$ and $\mathbb{C}$. Also, the module of smooth vector fields $\mathfrak{X}_M(U)$ is a module over $C^\infty(U)$ and a module (actually a vector space) over $\mathbb{R}$. Thus it is sometimes important to indicate the ring involved and so we write the tensor product of two R-modules V and W as $\mathrm{V} \otimes_{\mathsf{R}} \mathrm{W}$. For instance, there is a big difference between $\mathfrak{X}_M(U) \otimes_{C^\infty(U)} \mathfrak{X}_M(U)$ and $\mathfrak{X}_M(U) \otimes_{\mathbb{R}} \mathfrak{X}_M(U)$.

Lemma D.1 *There are the following natural isomorphisms:*

1) $(\mathrm{V}\otimes \mathrm{W})\otimes \mathrm{U} \cong \mathrm{V}\otimes(\mathrm{W}\otimes \mathrm{U}) \cong \mathrm{V}\otimes \mathrm{W}\otimes \mathrm{U}$ *and under these isomorphisms* $(v\otimes w)\otimes u \longleftrightarrow v\otimes(w\otimes u) \longleftrightarrow v\otimes w\otimes u$.

2) $\mathrm{V}\otimes \mathrm{W} \cong \mathrm{W}\otimes \mathrm{V}$ *and under this isomorphism* $v\otimes w \longleftrightarrow w\otimes v$.

Proof. We prove (1) and leave (2) as an exercise.

Elements of the form $(v\otimes w)\otimes u$ generate $(\mathrm{V}\otimes \mathrm{W})\otimes \mathrm{U}$ so any map that sends $(v\otimes w)\otimes u$ to $v\otimes(w\otimes u)$ for all v,w,u must be unique. Now we have compositions

$$(\mathrm{V}\times \mathrm{W})\times \mathrm{U} \overset{\otimes\times \mathrm{id}_{\mathrm{U}}}{\to} (\mathrm{V}\otimes \mathrm{W})\times \mathrm{U} \overset{\otimes}{\to} (\mathrm{V}\otimes \mathrm{W})\otimes \mathrm{U}$$

and

$$\mathrm{V}\times(\mathrm{W}\times \mathrm{U}) \overset{\mathrm{id}_{\mathrm{U}}\times\otimes}{\to} (\mathrm{V}\times \mathrm{W})\otimes \mathrm{U} \overset{\otimes}{\to} \mathrm{V}\otimes(\mathrm{W}\otimes \mathrm{U}).$$

It is a simple matter to check that these composite maps have the same universal property as the map $\mathrm{V}\times \mathrm{W}\times \mathrm{U} \overset{\otimes}{\to} \mathrm{V}\otimes \mathrm{W}\otimes \mathrm{U}$. The result now follows from the existence and essential uniqueness results proven so far (E.1 and E.1). ■

We shall use the first isomorphism and the obvious generalizations to identify $\mathrm{V}_1\otimes\cdots\otimes \mathrm{V}_k$ with all legal parenthetical constructions such as $(((\mathrm{V}_1\otimes \mathrm{V}_2)\otimes\cdots\otimes \mathrm{V}_j)\otimes\cdots)\otimes \mathrm{V}_k$ and so forth. In short, we may construct $\mathrm{V}_1\otimes\cdots\otimes \mathrm{V}_k$ by tensoring spaces two at a time. In particular we assume the isomorphisms (as identifications)

$$(\mathrm{V}_1\otimes\cdots\otimes \mathrm{V}_k)\otimes(\mathrm{W}_1\otimes\cdots\otimes \mathrm{W}_k) \cong \mathrm{V}_1\otimes\cdots\otimes \mathrm{V}_k\otimes \mathrm{W}_1\otimes\cdots\otimes \mathrm{W}_k$$

where $(v_1\otimes\cdots\otimes v_k)\otimes(w_1\otimes\cdots\otimes w_k)$ maps to $v_1\otimes\cdots\otimes v_k\otimes w_1\otimes\cdots\otimes w_k$.

Proposition D.2 *We have natural isomorphisms*

$$\mathrm{V}\otimes\mathsf{R} \cong \mathrm{V} \cong \mathsf{R}\otimes \mathrm{V}$$

given on decomposible elements as $v\otimes r \mapsto v \mapsto r\otimes v$.

The proof is left to the reader. The following proposition gives a basic and often used isomorphism:

Proposition D.3 *For* $\mathsf{R}-$*modules* $\mathrm{W},\mathrm{V},\mathrm{U}$ *we have*

$$L_{\mathsf{R}}(\mathrm{W}\otimes \mathrm{V},\mathrm{U}) \cong L(\mathrm{W},\mathrm{V};\mathrm{U})$$

More generally,

$$L_{\mathsf{R}}(\mathrm{W}_1\otimes\cdots\otimes \mathrm{W}_k,\mathrm{U}) \cong L(\mathrm{W}_1,...,\mathrm{W}_k;\mathrm{U})$$

Proof. This is more or less just a restatement of the universal property of $\mathrm{W}\otimes \mathrm{V}$. One should check that this association is indeed an isomorphism. ■

Exercise D.4 *Show that if* W *is free with basis* $(f_1,...,f_n)$ *then* W^* *is also free and has a dual basis* $(f^1,...,f^n)$*, that is,* $f^i(f_j)=\delta^i_j$.

Theorem D.1 *If* $\mathrm{V}_1, ..., \mathrm{V}_k$ *are free* $\mathsf{R}-$*modules and if* $(e_1^j, ..., e_{n_j}^j)$ *is a basis for* V_j *then set of all decomposable elements of the form* $e_{i_1}^1 \otimes \cdots \otimes e_{i_k}^k$ *form a basis for* $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$.

Proof. We prove this for the case of $k = 2$. the general case is similar. We wish to show that if $(e_1, ..., e_{n_1})$ is a basis for V_1 and $(f_1, ..., f_{n_2})$ is a basis for V_2 then $\{e_i \otimes f_j\}$ is a basis for $\mathrm{V}_1 \otimes \mathrm{V}_2$. Define $\phi_{lk} : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathsf{R}$ by $\phi_{lk}(e_i, f_j) = \delta_i^l \delta_j^k 1$ where 1 is the identity in R and

$$\delta_i^l \delta_j^k 1 := \begin{cases} 1 \quad \text{if} \quad (l,k) = (i,j) \\ \\ 0 \qquad\quad \text{otherwise} \end{cases}$$

Extend this definition bilinearly. These maps are linearly independent in $L(\mathrm{W}, \mathrm{V}; \mathsf{R})$ since if $\sum_{lk} a_{lk}\phi_{lk} = 0$ in R then for any i, j we have

$$\begin{aligned} 0 &= \sum_{lk} a_{lk}\phi_{lk}(e_i, f_j) = \sum_{lk} a_{lk}\delta_i^l \delta_j^k 1 \\ &= a_{ij} \end{aligned}$$

Thus $\dim(\mathrm{V}_1 \otimes \mathrm{V}_2) = \dim L(\mathrm{W}, \mathrm{V}; \mathsf{R}) \geq n_1 n_2$. On the other hand, $\{v_i \otimes w_j\}$ spans the set of all decomposable elements and hence the whole space $\mathrm{V}_1 \otimes \mathrm{V}_2$ so that $\dim(\mathrm{V}_1 \otimes \mathrm{V}_2) \leq n_1 n_2$ and it follows that $\{v_i \otimes w_j\}$ is a basis. ■

Proposition D.4 *There is a unique* R*-module map* $\iota : L(\mathrm{V}_1, \mathrm{W}_1) \otimes \cdots \otimes L(\mathrm{V}_k, \mathrm{W}_k) \to L(\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k, \mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_k)$ *such that if* $f_1 \otimes \cdots \otimes f_k$ *is a (decomposable) element of* $L(\mathrm{V}_1, \mathrm{W}_1) \otimes \cdots \otimes L(\mathrm{V}_k, \mathrm{W}_k)$ *then*

$$\iota(f_1 \otimes \cdots \otimes f_k)(v_1 \otimes \cdots \otimes v_k) = f_1(v_1) \cdots f_k(v_k).$$

If the modules are all fintely generated and free then this is an isomorphism.

Proof. If such a map exists, it must be unique since the decomposable elements span $L(\mathrm{V}_1, \mathrm{W}_1) \otimes \cdots \otimes L(\mathrm{V}_k, \mathrm{W}_k)$. To show existence we define a multilinear map

$$\vartheta : \mathrm{V}_1^* \times \cdots \times \mathrm{V}_k^* \times \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_k$$

by the recipe

$$(f_1, ..., f_k, v_1, ..., v_k) \mapsto f_1(v_1) \otimes \cdots \otimes f_k(v_k).$$

By the universal property there must be a linear map

$$\widetilde{\vartheta} : \mathrm{V}_1^* \otimes \cdots \otimes \mathrm{V}_k^* \otimes \mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k \to \mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_k$$

such that $\widetilde{\vartheta} \circ \otimes = \vartheta$ where $\otimes$ is the universal map. Now define

$$\begin{aligned} &\iota(f_1 \otimes \cdots \otimes f_k)(v_1 \otimes \cdots \otimes v_k) \\ &:= \widetilde{\vartheta}(f_1 \otimes \cdots \otimes f_k \otimes v_1 \otimes \cdots \otimes v_k). \end{aligned}$$

The fact, that ι is an isomorphism in case the V_i are all free follows easily from exercise D.4 and theorem D.1. ■

Since $\mathsf{R} \otimes \mathsf{R} = \mathsf{R}$ we obtain

Corollary D.1 *There is a unique* R*-module map* $\iota : \mathrm{V}_1^* \otimes \cdots \otimes \mathrm{V}_k^* \to (\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k)^*$ *such that if* $\alpha_1 \otimes \cdots \otimes \alpha_k$ *is a (decomposable) element of* $\mathrm{V}_1^* \otimes \cdots \otimes \mathrm{V}_k^*$ *then*

$$\iota(\alpha_1 \otimes \cdots \otimes \alpha_k)(v_1 \otimes \cdots \otimes v_k) = \alpha_1(v_1) \cdots \alpha_k(v_k).$$

If the modules are all finitely generated and free then this is an isomorphism.

Corollary D.2 *There is a unique module map* $\iota_0 : \mathrm{W} \otimes \mathrm{V}^* \to L(\mathrm{V}, \mathrm{W})$ *such that if* $v \otimes \beta$ *is a (decomposable) element of* $\mathrm{W} \otimes \mathrm{V}^*$ *then*

$$\iota_0(w \otimes \alpha)(v) = \alpha(v)w$$

If V *and* W *are finitely generated free modules then this is an isomorphism.*

Proof. If we associate to every $w \in \mathrm{W}$ the map $w^{map} \in L(\mathsf{R}, \mathrm{W})$ given by $w^{map}(r) := rw$ then we get an isomorphism $\mathrm{W} \cong L(\mathsf{R}, \mathrm{W})$. Use this and then compose

$$\begin{aligned} \mathrm{W} \otimes \mathrm{V}^* &\to L(\mathsf{R}, \mathrm{W}) \otimes L(\mathrm{V}, \mathsf{R}) \\ &\to L(\mathsf{R} \otimes \mathrm{V}, \mathrm{W} \otimes \mathsf{R}) \cong L(\mathrm{V}, \mathrm{W}) \end{aligned}$$

$$\begin{aligned} \mathrm{V}^* \otimes \mathrm{W} &\to L(\mathrm{V}, \mathsf{R}) \otimes L(\mathsf{R}, \mathrm{W}) \\ &\to L(\mathrm{V} \otimes \mathsf{R}, \mathsf{R} \otimes \mathrm{W}) \cong L(\mathrm{V}, \mathrm{W}) \end{aligned}$$

■

By combining Corollary D.1 with Proposition D.3 and taking $\mathrm{U} = \mathsf{R}$ we obtain

Corollary D.3 *There is a unique* R*-module map* $\iota : \mathrm{V}_1^* \otimes \cdots \otimes \mathrm{V}_k^* \to L(\mathrm{V}_1, ..., \mathrm{V}_k; \mathsf{R})$ *such that if* $\alpha_1 \otimes \cdots \otimes \alpha_k$ *is a (decomposable) element of* $\mathrm{V}_1^* \otimes \cdots \otimes \mathrm{V}_k^*$ *then*

$$\iota(\alpha_1 \otimes \cdots \otimes \alpha_k)(v_1, ..., v_k) = \alpha_1(v_1) \cdots \alpha_k(v_k).$$

If the finitely generated free modules are all free then this is an isomorphism.

Appendix E

Modules, Multilinear Algebra

A great many people think they are thinking when they are merely rearranging their prejudices.

-William James (1842-1910)

Synopsis: Multilinear maps, tensors, tensor fields.

The set of all vector fields on a manifold is a vector space over the real numbers but not only can we add vector fields and scale by numbers but we may also scale by smooth functions. We say that the vector fields form a module over the ring of smooth functions. A module is similar to a vector space with the differences stemming from the use of elements of a ring R as the scalars rather than the field of complex $\mathbb{C}$ or real numbers $\mathbb{R}$. For a module, one still has $1w = w$, $0w = 0$ and $-1w = -w$. Of course, every vector space is also a module since the latter is a generalization of the notion of vector space.

Definition E.1 *Let* R *be a ring. A left* R-***module*** *(or a left* ***module*** *over* R*) is an abelian group* $W, +$ *together with an operation* $\mathsf{R} \times W \to W$ *written* $(a, w) \mapsto aw$ *such that*

1) $(a + b)w = aw + bw$ *for all* $a, b \in \mathsf{R}$ *and all* $w \in W$*,*

2) $a(w_1 + w_2) = aw_1 + aw_2$ *for all* $a \in \mathsf{R}$ *and all* $w_2, w_1 \in W$*.*

A right R-***module*** *is defined similarly with the multiplication of the right so that*

1) $w(a + b) = wa + wb$ *for all* $a, b \in \mathsf{R}$ *and all* $w \in W$*,*

2) $(w_1 + w_2)a = w_1 a + w_2 a$ *for all* $a \in \mathsf{R}$ *and all* $w_2, w_1 \in W$*.* **.**

If the ring is commutative (the usual case for us) then we may right $aw = wa$ and consider any right module as a left module and vice versa. Even if the ring is not commutative we will usually stick to left modules and so we drop the reference to "left" and refer to such as R-modules.

Remark E.1 *We shall often refer to the elements of* R *as* ***scalars****.*

Example E.1 *An abelian group $A, +$ is a $\mathbb{Z}$ module and a $\mathbb{Z}$-module is none other than an abelian group. Here we take the product of $n \in \mathbb{Z}$ with $x \in A$ to be $nx := x + \cdots + x$ if $n \geq 0$ and $nx := -(x + \cdots + x)$ if $n < 0$ (in either case we are adding $|n|$ terms).*

Example E.2 *The set of all $m \times n$ matrices with entries being elements of a commutative ring R (for example real polynomials) is an R-module.*

Example E.3 *The module of all module homomorphisms of a module W onto another module M is a module and is denoted $\mathrm{Hom}_{\mathsf{R}}(\mathrm{W}, \mathrm{M})$ or $L_{\mathsf{R}}(\mathrm{W}, \mathrm{M})$.*

Example E.4 *Let V be a vector space and $\ell : \mathrm{V} \to \mathrm{V}$ a linear operator. Using this one operator we may consider V as a module over the ring of polynomials $\mathbb{R}[t]$ by defining the "scalar" multiplication by*

$$p(t)v := p(\ell)v$$

for $p \in \mathbb{R}[t]$.

Since the ring is usually fixed we often omit mention of the ring. In particular, we often abbreviate $L_{\mathsf{R}}(\mathrm{W}, \mathrm{M})$ to $L(\mathrm{W}, \mathrm{M})$. Similar omissions will be made without further mention. Also, since every real (resp. complex) Banach space E is a vector space and hence a module over $\mathbb{R}$ (resp. $\mathbb{C}$) we must distinguish between the bounded linear maps which we have denoted up until now as $L(\mathsf{E}; \mathsf{F})$ and the linear maps that would be denoted the same way in the context of modules. Our convention will be the following:

Definition E.2 ((convention)) *In case the modules in question are presented as infinite dimensional topological vector spaces, say E and F we will let $L(\mathsf{E}; \mathsf{F})$ continue to mean the space of bounded linear operator unless otherwise stated.*

A **submodule** is defined in the obvious way as a subset $S \subset \mathrm{W}$ that is closed under the operations inherited from W so that S itself is a module. The intersection of all submodules containing a subset $A \subset \mathrm{W}$ is called the **submodule generated by** A and is denoted $\langle A \rangle$ and A is called a **generating set**. If $\langle A \rangle = \mathrm{W}$ for a finite set A, then we say that W is finitely generated.

Let S be a submodule of W and consider the quotient abelian group W/S consisting of cosets, that is sets of the form $[v] := v + S = \{v + x : x \in S\}$ with addition given by $[v] + [w] = [v + w]$. We define a scalar multiplication by elements of the ring R by $a[v] := [av]$ respectively. In this way, W/S is a module called a quotient module.

Definition E.3 *Let W_1 and W_2 be modules over a ring R. A map $L : W_1 \to W_2$ is called* ***module homomorphism*** *if*

$$L(aw_1 + bw_2) = aL(w_1) + bL(w_2).$$

By analogy with the case of vector spaces, which module theory includes, we often characterize a module homomorphism L by saying that L is ***linear over*** R.

A real (resp. complex) vector space is none other than a module over the field of real numbers $\mathbb{R}$ (resp. complex numbers $\mathbb{C}$). In fact, most of the modules we encounter will be either vector spaces or spaces of sections of some vector bundle.

Many of the operations that exist for vector spaces have analogues in the modules category. For example, the direct sum of modules is defined in the obvious way. Also, for any module homomorphism $L : \mathrm{W}_1 \to \mathrm{W}_2$ we have the usual notions of kernel and image:

$$\ker L = \{v \in \mathrm{W}_1 : L(v) = 0\}$$
$$\operatorname{img}(L) = L(\mathrm{W}_1) = \{w \in \mathrm{W}_2 : w = Lv \text{ for some } v \in \mathrm{W}_1\}.$$

These are submodules of W_1 and W_2 respectively.

On the other hand, modules are generally not as simple to study as vector spaces. For example, there are several notions of dimension. The following notions for a vector space all lead to the same notion of dimension. For a completely general module these are all potentially different notions:

1. Length of the longest chain of submodules
$$0 = \mathrm{W}_n \subsetneq \cdots \subsetneq \mathrm{W}_1 \subsetneq \mathrm{W}$$
2. The cardinality of the largest linearly independent set (see below).
3. The cardinality of a basis (see below).

For simplicity in our study of dimension, let us now assume that R is commutative.

Definition E.4 *A set of elements $w_1, ..., w_k$ of a module are said to be* ***linearly dependent*** *if there exist ring elements $r_1, ..., r_k \in \mathsf{R}$ not all zero, such that $r_1 w_1 + \cdots + r_k w_k = 0$. Otherwise, they are said to be* ***linearly independent****. We also speak of the set $\{w_1, ..., w_k\}$ as being a* ***linearly independent set****.*

So far so good but it is important to realize that just because $w_1, ..., w_k$ are linearly independent doesn't mean that we may write each of these w_i as a linear combination of the others. It may even be that some element w forms a linearly dependent set since there may be a nonzero r such that $rw = 0$ (such a w is said to have **torsion**).

If a linearly independent set $\{w_1, ..., w_k\}$ is maximal in size then we say that the module has **rank** k. Another strange possibility is that a maximal linearly independent set may not be a generating set for the module and hence may not be a basis in the sense to be defined below. The point is that although for an arbitrary $w \in \mathrm{W}$ we must have that $\{w_1, ..., w_k\} \cup \{w\}$ is linearly dependent and hence there must be a nontrivial expression $rw + r_1 w_1 + \cdots + r_k w_k = 0$, it does **not** follow that we may solve for w since r may not be an invertible element of the ring (i.e. it may not be a **unit**).

Definition E.5 *If B is a generating set for a module* W *such that every element of* W *has a unique expression as a* finite R*-linear combination of elements of B then we say that B is a* ***basis*** *for* W.

Definition E.6 *If an* R*-module has a basis then it is referred to as a* ***finitely generated free module****.*

If a module over a (commutative) ring R has a basis then the number of elements in the basis is called the **dimension** and must in this case be the same as the rank (the size of a maximal linearly independent set). If a module W is free with basis $w_1, ..., w_n$ then we have an isomorphism $\mathsf{R}^n \cong \mathrm{W}$ given by

$$(r_1, ..., r_n) \mapsto r_1 w_1 + \cdots + r_n w_n.$$

Exercise E.1 *Show that every finitely generated* R*-module is the homomorphic image of a finitely generated free module.*

It turns out that just as for vector spaces the cardinality of a basis for a finitely generated free module W is the same as that of every other basis for W. The cardinality of any basis for a finitely generated free module W is called the dimension of W. If R is a field then every module is free and is a vector space by definition. In this case, the current definitions of dimension and basis coincide with the usual ones.

The ring R is itself a free R-module with standard basis given by $\{1\}$. Also, $\mathsf{R}^n := \mathsf{R} \times \cdots \times \mathsf{R}$ is a finitely generated free module with standard basis $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ where, as usual $\mathbf{e}_i := (0, ..., 1, ..., 0)$; the only nonzero entry being in the i-th position.

Definition E.7 *Let* k *be a commutative ring, for example a field such as* $\mathbb{R}$ *or* $\mathbb{C}$*. A ring* A *is called a* k***-algebra*** *if there is a ring homomorphism* $\mu : \mathsf{k} \to \mathsf{R}$ *such that the image* $\mu(\mathsf{k})$ *consists of elements that commute with everything in* A*. In particular,* A *is a module over* k*.*

Example E.5 *The ring* $\mathcal{C}^\infty_M(U)$ *is an* $\mathbb{R}$*-algebra.*

We shall have occasion to consider A-modules where A is an algebra over some k. In this context the elements of A are still called scalars but the elements of $\mathsf{k} \subset \mathsf{A}$ will be referred to as **constants**.

Example E.6 *For an open set $U \subset M$ the set vector fields $\mathfrak{X}_M(U)$ is a vector space over $\mathbb{R}$ but it is also a module over the $\mathbb{R}$-algebra $\mathcal{C}^\infty_M(U)$. So for all $X, Y \in \mathfrak{X}_M(U)$ and all $f, g \in \mathcal{C}^\infty_M(U)$ we have*

1. $f(X + Y) = fX + fY$
2. $(f + g)X = fX + gX$
3. $f(gX) = (fg)X$

Similarly, $\mathfrak{X}_M^*(U) = \Gamma(U, T^*M)$ is also a module over $\mathcal{C}_M^\infty(U)$ that is naturally identified with the module dual $\mathfrak{X}_M(U)^*$ by the pairing $(\theta, X) \mapsto \theta(X)$. Here $\theta(X) \in \mathcal{C}_M^\infty(U)$ and is defined by $p \mapsto \theta_p(X_p)$. The set of all vector fields $\mathcal{Z} \subset \mathfrak{X}(U)$ that are zero at a fixed point $p \in U$ is a submodule in $\mathfrak{X}(U)$. If $U, (x^1, ..., x^n)$ is a coordinate chart then the set of vector fields

$$\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}$$

is a basis (over $\mathcal{C}_M^\infty(U)$) for the module $\mathfrak{X}(U)$. Similarly,

$$dx^1, ..., dx^n$$

is a basis for $\mathfrak{X}_M^*(U)$. It is important to realize that if U is not a coordinate chart domain then it may be that $\mathfrak{X}_M(U)$ and $\mathfrak{X}_M(U)^*$ have no basis. In particular, we should not expect $\mathfrak{X}(M)$ to have a basis in the general case.

Example E.7 *The sections of any vector bundle over a manifold M form a $C^\infty(M)$-module denoted $\Gamma(E)$. Let $E \to M$ be a trivial vector bundle of finite rank n. Then there exists a basis of vector fields $E_1, ..., E_n$ for the module $\Gamma(E)$. Thus for any section X there exist unique functions f^i such that*

$$X = \sum f^i E_i.$$

In fact, since E is trivial we may as well assume that $E = M \times \mathbb{R}^n \stackrel{pr_1}{\to} M$. Then for any basis $e_1, ..., e_n$ for $\mathbb{R}^n$ we may take

$$E_i(x) := (x, e_i).$$

Definition E.8 *Let V_i, $i = 1, ..., k$ and W be modules over a ring R. A map $\mu : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{W}$ is called* ***multilinear*** *(k-multilinear) if for each i, $1 \le i \le k$ and each fixed $(w_1, ..., \widehat{w_i}, ..., w_k) \in \mathrm{V}_1 \times \cdots \times \widehat{\mathrm{V}_1} \times \cdots \times \mathrm{V}_k$ we have that the map*

$$v \mapsto \mu(w_1, ..., \underset{i-th}{v}, ..., w_k),$$

obtained by fixing all but the i-th variable, is a module homomorphism. In other words, we require that μ be R- linear in each slot separately. The set of all multilinear maps $\mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{W}$ is denoted $L_{\mathsf{R}}(\mathrm{V}_1, ..., \mathrm{V}_k; \mathrm{W})$. If $\mathrm{V}_1 = \cdots = \mathrm{V}_k = \mathrm{V}$ then we abbreviate this to $L^k_{\mathsf{R}}(\mathrm{V}; \mathrm{W})$.

The space of multilinear maps $L_{\mathsf{R}}(\mathrm{V}_1, ..., \mathrm{V}_k; \mathrm{W})$ is itself an R-module in a fairly obvious way. For $a, b \in \mathsf{R}$ and $\mu_1, \mu_2 \in L_{\mathsf{R}}(\mathrm{V}_1, ..., \mathrm{V}_k; \mathrm{W})$ then $a\mu_1 + b\mu_2$ is defined in the usual way.

Purely algebraic results

In this section we intend all modules to be treated strictly as modules. Thus we do not require multilinear maps to be bounded. In particular, $L_{\mathsf{R}}(\mathrm{V}_1, ..., \mathrm{V}_k; \mathrm{W})$ does not refer to bounded multilinear maps even if the modules are coincidentally Banach spaces. We shall comment on how thing look in the Banach space category in a later section.

Definition E.9 *The dual of an* $\mathsf{R}-$*module* W *is the module* $\mathrm{W}^* := \mathrm{Hom}_{\mathsf{R}}(\mathrm{W},\mathsf{R})$ *of all* R*-linear functionals on* W.

Any element of W can be though of as an element of $\mathrm{W}^{**} := \mathrm{Hom}_{\mathsf{R}}(\mathrm{W}^*,\mathsf{R})$ according to $w(\alpha) := \alpha(w)$. This provides a map $\mathrm{W} \hookrightarrow \mathrm{W}^{**}$ an if this map is an isomorphism then we say that W is **reflexive** .

If W is reflexive then we are free to identify W with W^{**}.

Exercise E.2 *Show that if* W *is a free with finite dimension then* W *is reflexive. We sometimes write* $w\lrcorner(\alpha) = \langle w, \alpha\rangle = \langle \alpha, w\rangle$.

There is a bit of uncertainty about how to use the word "tensor". On the one hand, a tensor is a certain kind of multilinear mapping. On the other hand, a tensor is an element of a tensor product (defined below) of several copies of a module and its dual. For finite dimensional vector spaces these two viewpoints turn out to be equivalent as we shall see but since we are also interested in infinite dimensional spaces we must make a terminological distinction. We make the following slightly nonstandard definition:

Definition E.10 *Let* V *and* W *be* R*-modules. A* W*-valued* $({}^{r}{}_{s})$*-**tensor map** on* V *is a multilinear mapping of the form*

$$\Lambda : \underbrace{\mathrm{V}^* \times \mathrm{V}^* \cdots \times \mathrm{V}^*}_{r-\text{times}} \times \underbrace{\mathrm{V} \times \mathrm{V} \times \cdots \times \mathrm{V}}_{s-\text{times}} \to \mathrm{W}.$$

The set of all tensor maps into W will be denoted $T^{r}{}_{s}(\mathrm{V};\mathrm{W})$. Similarly, a W-valued $({}_{s}{}^{r})$-**tensor map** on V is a multilinear mapping of the form

$$\Lambda : \underbrace{\mathrm{V} \times \mathrm{V} \times \cdots \times \mathrm{V}}_{s-\text{times}} \times \underbrace{\mathrm{V}^* \times \mathrm{V}^* \cdots \times \mathrm{V}^*}_{r-\text{times}} \to \mathrm{W}$$

and the corresponding space of all such is denoted $T_{s}{}^{r}(\mathrm{V};\mathrm{W})$.

There is, of course, a natural isomorphism $T_{s}{}^{r}(\mathrm{V};\mathrm{W}) \cong T^{r}{}_{s}(\mathrm{V};\mathrm{W})$ induced by the map $\mathrm{V}^s \times \mathrm{V}^{*r} \cong \mathrm{V}^{*r} \times \mathrm{V}^s$ given on homogeneous elements by $v \otimes \omega \mapsto \omega \otimes v$. (Warning) In the presence of an inner product there is another possible isomorphism here given by $v \otimes \omega \mapsto \flat v \otimes \sharp\omega$. **This map is a "transpose" map and just as we do not identify a matrix with its transpose we do not generally identify individual elements under this isomorphism.**

Remark E.2 *The reader may have wondered about the possibly of multilinear maps were the covariant and contravariant variables are interlaced such as* $\Upsilon : \mathrm{V} \times \mathrm{V}^* \times \mathrm{V} \times \mathrm{V}^* \times \mathrm{V}^* \to \mathrm{W}$. *Of course, such things exist and this example would be an element of what we might denote by* $T_{1}{}^{1}{}_{1}{}^{2}(\mathrm{V};\mathrm{W})$. *But we can agree to associate to each such object an unique element of* $T^{3}{}_{2}(\mathrm{V};\mathrm{W})$ *by simple keeping the order among the covariant variable and among the contravariant variable but shifting all covariant variables to the left of the contravariant variables. Some authors have insisted on the need to avoid this consolidation for reasons which we will show to be unnecessary below.*

Notation E.1 *For the most part we shall be needing only* $T^{r}{}_{s}(\mathrm{V};\mathrm{W})$ *and so we agree abbreviate this to* $T^r_s(\mathrm{V};\mathrm{W})$ *and call the elements* (r,s)*-**tensor maps**. So by convention*

$$T^r_s(\mathrm{V};\mathrm{W}) := T^{r}{}_{s}(\mathrm{V};\mathrm{W})$$
$$\textit{but}$$
$$T^r_s(\mathrm{V};\mathrm{W}) \neq T_{r}{}^{s}(\mathrm{V};\mathrm{W}).$$

Elements of $T^r_0(\mathrm{V})$ are said to be of **contravariant type** and of degree r and those in $T^0_s(\mathrm{V})$ are of **covariant type** (and degree s). If $r,s>0$ then the elements of $T^r_s(\mathrm{V})$ are called **mixed tensors (of tensors of mixed type)** with contravariant degree r and covariant degree s.

Remark E.3 *An* $\mathbb{R}$ *-valued* (r,s)*-**tensor map** is usually just called an* (r,s)*-**tensor** but as we mentioned above, there is another common meaning for this term which is equivalent in the case of finite dimensional vector spaces. The word* tensor *is also used to mean "tensor field" (defined below). The context will determine the proper meaning.*

Remark E.4 *The space* $T^r_s(\mathrm{V};\mathsf{R})$ *is sometimes denoted by* $T^r_s(\mathrm{V};\mathsf{R})$ *(or even* $T^r_s(\mathrm{V})$ *in case* $\mathsf{R}=\mathbb{R}$*) but we reserve this notation for another space defined below which is canonically isomorphic to* $T^r_s(\mathrm{V};\mathsf{R})$ *in case* V *is free with finite dimension.*

Definition E.11 *If* $\Upsilon_1 \in T^{k_1}_{l_1}(\mathrm{V};\mathsf{R})$ *and* $\Upsilon_2 \in T^{k_2}_{l_2}(\mathrm{V};\mathsf{R})$ *then* $\Upsilon_1\otimes\Upsilon_2 \in T^{k_1+k_2}_{l_1+l_2}(\mathrm{V};\mathsf{R})$ *where*

$$\begin{aligned}&(\Upsilon_1\otimes\Upsilon_2)(\theta_1,...,\theta_{k_1+k_2},\mathrm{v}_1,...,\mathrm{v}_{l_1+l_2})\\&=\Upsilon_1(\theta_1,...,\theta_{k_1},\mathrm{v}_1,,...,\mathrm{v}_{l_1})\Upsilon_2(\theta_{k_1+1},...,\theta_{k_2},\mathrm{v}_{l_1+1},,...,\mathrm{v}_{l_2}).\end{aligned}$$

Remark E.5 *We call this type of tensor product the **"map" tensor product** in case we need to distinguish it from the tensor product defined below.*

Now suppose that V is free with finite dimension n. Then there is a basis $f_1,...,f_n$ for V with dual basis $f^1,...,f^n$. Now we have $\mathrm{V}^* = T^0_1(\mathrm{V};\mathsf{R})$. Also, we may consider $f_i \in \mathrm{V}^{**} = T^0_1(\mathrm{V}^*;\mathsf{R})$ and then, as above, take tensor products to get elements of the form

$$f_{i_1}\otimes\cdots\otimes f_{i_r}\otimes f^{j_1}\otimes\cdots\otimes f^{j_s}.$$

These are multilinear maps in $T^r_s(\mathrm{V};\mathsf{R})$ by definition:

$$\begin{aligned}&(f_{i_1}\otimes\cdots\otimes f_{i_r}\otimes f^{j_1}\otimes\cdots\otimes f^{j_s})(\alpha_1,...,\alpha_r,v_1,...,v_s)\\&=\alpha_1(f_{i_1})\cdots\alpha_r(f_{i_r})f^{j_1}(v_1)\cdots f^{j_s}(v_s).\end{aligned}$$

There are $n^s n^r$ such maps that together form a basis for $T^r_s(\mathrm{V};\mathsf{R})$ which is therefore also free. Thus we may write any tensor map $\Upsilon\in T^r_s(\mathrm{V};\mathsf{R})$ as a sum

$$\Upsilon = \sum \Upsilon^{i_1\ldots i_r}{}_{j_1,\ldots j_s}\, f_{i_1}\otimes\cdots\otimes f_{i_r}\otimes f^{j_1}\otimes\cdots\otimes f^{j_s}$$

and the scalars $\Upsilon^{i_1 \ldots i_r}{}_{j_1,\ldots j_s} \in \mathsf{R}$

We shall be particularly interested in the case where all the modules are real (or complex) vector spaces such as the tangent space at a point on a smooth manifold. As we mention above, we will define a space $T^r_s(\mathrm{V};\mathsf{R})$ for each (r,s) that is canonically isomorphic to $T^r_s(\mathrm{V};\mathsf{R})$. There will be a product $\otimes T^r_s(\mathrm{V};\mathsf{R}) \times T^p_q(\mathrm{V};\mathsf{R}) \to T^{r+p}_{s+q}(\mathrm{V};\mathsf{R})$ for these spaces also and this will match up with the current definition under the canonical isomorphism.

Example E.8 *The inner product (or "dot product") on the Euclidean vector space $\mathbb{R}^n$ given for vectors $\tilde{\mathrm{v}} = (v_1, ..., v_n)$ and $\tilde{\mathrm{w}} = (w_1, ..., w_n)$ by*

$$(\vec{v}, \vec{w}) \mapsto \langle \vec{v}, \vec{w} \rangle = \sum_{i=1}^{n} v_i w_i$$

*is 2-multilinear (more commonly called **bilinear**).*

Example E.9 *For any n vectors $\tilde{\mathrm{v}}_1, ..., \tilde{\mathrm{v}}_n \in \mathbb{R}^n$ the determinant of $(\tilde{\mathrm{v}}_1, ..., \tilde{\mathrm{v}}_n)$ is defined by considering $\tilde{\mathrm{v}}_1, ..., \tilde{\mathrm{v}}_n$ as columns and taking the determinant of the resulting $n \times n$ matrix. The resulting function is denoted* $\det(\tilde{\mathrm{v}}_1, ..., \tilde{\mathrm{v}}_n)$ *and is n-multilinear.*

Example E.10 *Let $\mathfrak{X}(M)$ be the $C^\infty(M)$-module of vector fields on a manifold M and let $\mathfrak{X}^*(M)$ be the $C^\infty(M)$-module of 1-forms on M. The map $\mathfrak{X}^*(M) \times \mathfrak{X}(M) \to C^\infty(M)$ given by $(\alpha, X) \mapsto \alpha(X) \in C^\infty(M)$ is clearly multilinear (bilinear) over $C^\infty(M)$.*

Suppose now that we have two R-modules V_1 and V_2. Let us construct a category $\mathcal{C}_{\mathrm{V}_1 \times \mathrm{V}_2}$ whose objects are bilinear maps $\mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{W}$ where W varies over all R-modules but V_1 and V_2 are fixed. A morphism from, say $\mu_1 : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{W}_1$ to $\mu_2 : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{W}_2$ is defined to be a map $\ell : \mathrm{W}_1 \to \mathrm{W}_2$ such that the diagram

$$\begin{tikzcd} & \mathrm{W}_1 \arrow[dd, "\ell"] \\ \mathrm{V}_1 \times \mathrm{V}_2 \arrow[ur, "\mu_1"] \arrow[dr, "\mu_2"'] & \\ & \mathrm{W}_2 \end{tikzcd}$$

commutes. Suppose that there is an R-module $\mathrm{T}_{\mathrm{V}_1,\mathrm{V}_2}$ together with a bilinear map $\mathrm{t} : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{T}_{\mathrm{V}_1,\mathrm{V}_2}$ that has the following universal property for this category: For every bilinear map $\mu : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{W}$ there is a unique linear map $\widetilde{\mu} : \mathrm{T}_{\mathrm{V}_1,\mathrm{V}_2} \to \mathrm{W}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{V}_1 \times \mathrm{V}_2 \arrow[r, "\mu"] \arrow[d, "\mathrm{t}"] & \mathrm{W} \\ \mathrm{T}_{\mathrm{V}_1,\mathrm{V}_2} \arrow[ur, "\widetilde{\mu}"'] & \end{tikzcd}$$

If such a pair $\mathrm{T}_{\mathrm{V}_1,\mathrm{V}_2}, \mathrm{t}$ exists with this property then it is unique up to isomorphism in $\mathcal{C}_{\mathrm{V}_1 \times \mathrm{V}_2}$. In other words, if $\widehat{\mathrm{t}} : \mathrm{V}_1 \times \mathrm{V}_2 \to \widehat{\mathrm{T}}_{\mathrm{V}_1,\mathrm{V}_2}$ is another object with this universal property then there is a module isomorphism $\mathrm{T}_{\mathrm{V}_1,\mathrm{V}_2} \cong \widehat{\mathrm{T}}_{\mathrm{V}_1,\mathrm{V}_2}$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathrm{T}_{\mathrm{V}_1,\mathrm{V}_2} \arrow[dd, "\cong"'] \\ \mathrm{V}_1 \times \mathrm{V}_2 \arrow[ur, "\mathrm{t}"] \arrow[dr, "\widehat{\mathrm{t}}"'] & \\ & \widehat{\mathrm{T}}_{\mathrm{V}_1,\mathrm{V}_2} \end{tikzcd}$$

We refer to any such universal object as a tensor product of V_1 and V_2. We will construct a specific tensor product that we denote by $\mathrm{V}_1 \otimes \mathrm{V}_2$ with the corresponding map denoted by $\otimes : \mathrm{V}_1 \times \mathrm{V}_2 \to \mathrm{V}_1 \otimes \mathrm{V}_2$. More generally, we seek a universal object for k-multilinear maps $\mu : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{W}$.

Definition E.12 *A module* $\mathrm{T} = \mathrm{T}_{\mathrm{V}_1,\cdots,\mathrm{V}_k}$ *together with a multilinear map* $\mathrm{u} : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{T}$ *is called* ***universal for*** k***-multilinear maps on*** $\mathrm{V}_1 \times \cdots \times \mathrm{V}_k$ *if for every multilinear map* $\mu : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{W}$ *there is a unique linear map* $\widetilde{\mu} : \mathrm{T} \to \mathrm{W}$ *such that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \arrow[r, "\mu"] \arrow[d, "\mathrm{u}"'] & \mathrm{W} \\ \mathrm{T} \arrow[ur, "\widetilde{\mu}"'] & \end{tikzcd},$$

i.e. we must have $\mu = \widetilde{\mu} \circ \mathrm{u}$. *If such a universal object exists it will be called a* ***tensor product*** *of* $\mathrm{V}_1, \ldots, \mathrm{V}_k$ *and the module itself* $\mathrm{T} = \mathrm{T}_{\mathrm{V}_1,\cdots,\mathrm{V}_k}$ *is also referred to as a tensor product of the modules* $\mathrm{V}_1, \ldots, \mathrm{V}_k$.

Lemma E.1 *If two modules* $\mathrm{T}_1, \mathrm{u}_1$ *and* $\mathrm{T}_2, \mathrm{u}_2$ *are both universal for* k*-multilinear maps on* $\mathrm{V}_1 \times \cdots \times \mathrm{V}_k$ *then there is an isomorphism* $\Phi : \mathrm{T}_1 \to \mathrm{T}_2$ *such that* $\Phi \circ \mathrm{u}_1 = \mathrm{u}_2$*;*

$$\begin{tikzcd} & \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \arrow[dl, "\mathrm{u}_1"'] \arrow[dr, "\mathrm{u}_2"] & \\ \mathrm{T}_1 \arrow[rr, "\Phi"] & & \mathrm{T}_2 \end{tikzcd}.$$

Proof. By the assumption of universality, there are maps u_1 and u_2 such that $\Phi \circ \mathrm{u}_1 = \mathrm{u}_2$ and $\bar{\Phi} \circ \mathrm{u}_2 = \mathrm{u}_1$. We thus have $\bar{\Phi} \circ \Phi \circ \mathrm{u}_1 = \mathrm{u}_1 = \mathrm{id}$ and by the uniqueness part of the universality of u_1 we must have $\bar{\Phi} \circ \Phi = \mathrm{id}$ or $\bar{\Phi} = \Phi^{-1}$. ∎

We now show the existence of a tensor product. The specific tensor product of modules $\mathrm{V}_1, \ldots, \mathrm{V}_k$ that we construct will be denoted by $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$ and the corresponding map will be denoted by

$$\otimes^k : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k.$$

We start out by introducing the notion of a finitely generated free module on an arbitrary set. If S is just some set, then we may consider the set $F_{\mathsf{R}}(S)$ all finite formal linear combinations of elements of S with coefficients from R.

For example, if $a, b, c \in \mathsf{R}$ and $s_1, s_2, s_3 \in S$ then $as_1 + bs_2 + cs_3$ is such a formal linear combination. In general, an element of $F_{\mathsf{R}}(S)$ will be of the form

$$\sum_{s \in S} a_s s$$

where the coefficients a_s are elements of R and all but finitely many are 0. Thus the sums involved are always finite. Addition of two such expressions and multiplication by elements of R are defined in the obvious way;

$$b \sum_{s \in S} a_s s = \sum_{s \in S} b a_s s$$
$$\sum_{s \in S} a_s s + \sum_{s \in S} b_s s = \sum_{s \in S} (a_s + b_s) s.$$

This is all just a way of speaking of functions $a() : S \to \mathsf{R}$ with finite support. It is also just a means of forcing the element of our arbitrary set to be the "basis elements" of a modules. The resulting module $F_{\mathsf{R}}(S)$ is called the **finitely generated free module generated by** S. For example, the set of all formal linear combinations of the set of symbols $\{\mathbf{i}, \mathbf{j}\}$ over the real number ring, is just a 2 dimensional vector space with basis $\{\mathbf{i}, \mathbf{j}\}$.

Let $\mathrm{V}_1, \cdots, \mathrm{V}_k$ be modules over R and let $\mathrm{F}_{\mathsf{R}}(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k)$ denote the set of all formal linear combinations of elements of the Cartesian product $\mathrm{V}_1 \times \cdots \times \mathrm{V}_k$. For example

$$3(v_1, w) - 2(v_2, w) \in \mathrm{F}_{\mathsf{R}}(\mathrm{V}, \mathrm{W})$$

but it is **not true** that $3(v_1, w) - 2(v_2, w) = 3(v_1 - 2v_2, w)$ since (v_1, w), (v_2, w) and $(v_1 - 2v_2, w)$ are linearly independent by definition. We now define a submodule of $\mathrm{F}_{\mathsf{R}}(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k)$. Consider the set B of all elements of $\mathrm{F}_{\mathsf{R}}(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k)$ that have one of the following two forms:

1.
$$(\mathrm{v}_1, ..., a\mathrm{v}_i, \cdots, \mathrm{v}_k) - a(\mathrm{v}_1, ..., \mathrm{v}_i, \cdots, \mathrm{v}_k)$$
for some $a \in \mathsf{R}$ and some $1 \le i \le k$ and some $(\mathrm{v}_1, ..., \mathrm{v}_i, \cdots, \mathrm{v}_k) \in \mathrm{V}_1 \times \cdots \times \mathrm{V}_k$.

2.
$$(\mathrm{v}_1, ..., \mathrm{v}_i + \mathrm{w}_i, \cdots, \mathrm{v}_k) - (\mathrm{v}_1, ..., \mathrm{v}_i, \cdots, \mathrm{v}_k) - (\mathrm{v}_1, ..., \mathrm{w}_i, \cdots, \mathrm{v}_k)$$
for some $1 \le i \le k$ and some choice of $(\mathrm{v}_1, ..., \mathrm{v}_i, \cdots, \mathrm{v}_k) \in \mathrm{V}_1 \times \cdots \times \mathrm{V}_k$ and $\mathrm{w}_i \in \mathrm{V}_i$.

We now define $\langle B \rangle$ to be the submodule generated by B and then define the tensor product $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$ of the spaces $\mathrm{V}_1, \cdots, \mathrm{V}_k$ to be the quotient module $\mathrm{F}_{\mathsf{R}}(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k)/\langle B \rangle$. Let

$$\pi : \mathrm{F}_{\mathsf{R}}(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k) \to \mathrm{F}_{\mathsf{R}}(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k)/\langle B \rangle$$

be the quotient map and define $\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_k$ to be the image of $(\mathrm{v}_1, \cdots, \mathrm{v}_k) \in \mathrm{V}_1 \times \cdots \times \mathrm{V}_k$ under this quotient map. The quotient is the tensor space we were looking for

$$\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k := \mathrm{F}_{\mathsf{R}}(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k)/\langle B \rangle.$$

To get our universal object we need to define the corresponding map. The map we need is just the composition

$$\mathrm{V}_1 \times \cdots \times \mathrm{V}_k \hookrightarrow \mathrm{F}_{\mathsf{R}}(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k) \to \mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k.$$

We denote this map by $\otimes^k : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$. Notice that $\otimes^k(\mathrm{v}_1, \cdots, \mathrm{v}_k) = \mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_k$. By construction, we have the following facts:

1.
$$\mathrm{v}_1 \otimes \cdots \otimes a\mathrm{v}_i \otimes \cdots \otimes \mathrm{v}_k = a\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_i \otimes \cdots \otimes \mathrm{v}_k$$

   for any $a \in \mathsf{R}$, any $i \in \{1, 2, ..., k\}$ and any $(\mathrm{v}_1, ..., \mathrm{v}_i, \cdots, \mathrm{v}_k) \in \mathrm{V}_1 \times \cdots \times \mathrm{V}_k$.

2.
$$\begin{aligned} &\mathrm{v}_1 \otimes \cdots \otimes (\mathrm{v}_i + \mathrm{w}_i) \otimes \cdots \otimes \mathrm{v}_k \\ &= \mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_i \otimes \cdots \otimes \mathrm{v}_k + \mathrm{v}_1 \otimes \cdots \otimes \mathrm{w}_i \otimes \cdots \otimes \mathrm{v}_k \end{aligned}$$

   any $i \in \{1, 2, ..., k\}$ and for all choices of $\mathrm{v}_1, ..., \mathrm{v}_k$ and w_i.

   Thus $\otimes^k$ is multilinear.

Definition E.13 *The elements in the image of π, that is, elements that may be written as $\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_k$ for some v_i, are called **decomposable** .*

Exercise E.3 *Not all elements are decomposable but the decomposable elements generate $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$.*

It may be that the V_i may be modules over more that one ring. For example, any complex vector space is a module over both $\mathbb{R}$ and $\mathbb{C}$. Also, the module of smooth vector fields $\mathfrak{X}_M(U)$ is a module over $C^\infty(U)$ and a module (actually a vector space) over $\mathbb{R}$. Thus it is sometimes important to indicate the ring involved and so we write the tensor product of two R-modules V and W as $\mathrm{V} \otimes_{\mathsf{R}} \mathrm{W}$. For instance, there is a big difference between $\mathfrak{X}_M(U) \otimes_{C^\infty(U)} \mathfrak{X}_M(U)$ and $\mathfrak{X}_M(U) \otimes_{\mathbb{R}} \mathfrak{X}_M(U)$.

Now let $\otimes^k : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to \mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$ be natural map defined above which is the composition of the set injection $\mathrm{V}_1 \times \cdots \times \mathrm{V}_k \hookrightarrow \mathrm{F}_{\mathsf{R}}(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k)$ and the quotient map $\mathrm{F}_{\mathsf{R}}(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k) \to \mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$. We have seen that this map actually turns out to be a multilinear map.

Theorem E.1 *Given modules* $\mathrm{V}_1, ..., \mathrm{V}_k$*, the space* $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$ *together with the map* $\otimes^k$*has the following universal property:*

For any k*-multilinear map* $\mu : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to W$*, there is a unique linear map* $\widetilde{\mu} : \mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k \to W$ *called the universal map, such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{V}_1 \times \cdots \times \mathrm{V}_k & \xrightarrow{\mu} & W \\ \otimes^k \downarrow & \nearrow_{\widetilde{\mu}} & \\ \mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k & & \end{array} .$$

Thus the pair $(\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k, \otimes^k)$ is universal for k-multilinear maps on $\mathrm{V}_1 \times \cdots \times \mathrm{V}_k$ and by E.1 if T, u is any other universal pair for k-multilinear map we have $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k \cong \mathrm{T}$. The module $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$ is called *the* **tensor product** of $\mathrm{V}_1, ..., \mathrm{V}_k$.

Proof. Suppose that $\mu : \mathrm{V}_1 \times \cdots \times \mathrm{V}_k \to W$ is multilinear. Since, $\mathrm{F}_{\mathsf{R}}(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k)$ is free there is a unique linear map $M : \mathrm{F}_{\mathsf{R}}(\mathrm{V}_1 \times \cdots \times \mathrm{V}_k) \to W$ such that $M((\mathrm{v}_1, ..., \mathrm{v}_k)) = \mu(\mathrm{v}_1, ..., \mathrm{v}_k)$. Clearly, this map is zero on $\langle B \rangle$ and so there is a factorization $\widetilde{\mu}$ of M through $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$. Thus we always have

$$\widetilde{\mu}(\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_k) = M((\mathrm{v}_1, ..., \mathrm{v}_k)) = \mu(\mathrm{v}_1, ..., \mathrm{v}_k).$$

It is easy to see that $\widetilde{\mu}$ is unique since a linear map is determined by its action on generators (the decomposable elements generate $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$) ■

Lemma E.2 *We have the following natural isomorphisms:*

1) $(\mathrm{V} \otimes \mathrm{W}) \otimes \mathrm{U} \cong \mathrm{V} \otimes (\mathrm{W} \otimes \mathrm{U}) \cong \mathrm{V} \otimes (\mathrm{W} \otimes \mathrm{U})$ *and under these isomorphisms* $(\mathrm{v} \otimes \mathrm{w}) \otimes \mathrm{u} \longleftrightarrow \mathrm{v} \otimes (\mathrm{w} \otimes \mathrm{u}) \longleftrightarrow \mathrm{v} \otimes \mathrm{w} \otimes \mathrm{u}$.

2) $\mathrm{V} \otimes \mathrm{W} \cong \mathrm{W} \otimes \mathrm{V}$ *and under this isomorphism* $\mathrm{v} \otimes \mathrm{w} \longleftrightarrow \mathrm{w} \otimes \mathrm{v}$.

Proof. We prove (1) and leave (2) as an exercise.

Elements of the form $(\mathrm{v} \otimes \mathrm{w}) \otimes \mathrm{u}$ generate $(\mathrm{V} \otimes \mathrm{W}) \otimes \mathrm{U}$ so any map that sends $(\mathrm{v} \otimes \mathrm{w}) \otimes \mathrm{u}$ to $\mathrm{v} \otimes (\mathrm{w} \otimes \mathrm{u})$ for all $\mathrm{v}, \mathrm{w}, \mathrm{u}$ must be unique. Now we have compositions

$$(\mathrm{V} \times \mathrm{W}) \times \mathrm{U} \overset{\otimes \times \mathrm{id}_{\mathrm{U}}}{\to} (\mathrm{V} \otimes \mathrm{W}) \times \mathrm{U} \overset{\otimes}{\to} (\mathrm{V} \otimes \mathrm{W}) \otimes \mathrm{U}$$

and

$$\mathrm{V} \times (\mathrm{W} \times \mathrm{U}) \overset{\mathrm{id}_{\mathrm{U}} \times \otimes}{\to} (\mathrm{V} \times \mathrm{W}) \otimes \mathrm{U} \overset{\otimes}{\to} \mathrm{V} \otimes (\mathrm{W} \otimes \mathrm{U}).$$

It is a simple matter to check that these composite maps have the same universal property as the map $\mathrm{V} \times \mathrm{W} \times \mathrm{U} \overset{\otimes}{\to} \mathrm{V} \otimes \mathrm{W} \otimes \mathrm{U}$. The result now follows from the existence and essential uniqueness results proven so far (E.1 and E.1). ■

We shall use the first isomorphism and the obvious generalizations to identify $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$ with all legal parenthetical constructions such as $(((\mathrm{V}_1 \otimes \mathrm{V}_2) \otimes \cdots \otimes \mathrm{V}_j) \otimes \cdots) \otimes \mathrm{V}_k$ and so forth. In short, we may construct $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$ by tensoring spaces two at a time. In particular we assume the isomorphisms

$$(\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k) \otimes (\mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_k) \cong \mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k \otimes \mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_k$$

which map $(\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_k) \otimes (\mathrm{w}_1 \otimes \cdots \otimes \mathrm{w}_k)$ to $\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_k \otimes \mathrm{w}_1 \otimes \cdots \otimes \mathrm{w}_k$.

Consider the situation where we have module homomorphisms $h_i : \mathrm{W}_i \to \mathrm{V}_i$ for $1 \leq i \leq m$. We may then define a map $T(h_1, ..., h_m) : \mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_m \to \mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_m$ (by using the universal property again) so that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{W}_1 \times \cdots \times \mathrm{W}_m & \stackrel{h_1 \times ... \times h_m}{\to} & \mathrm{V}_1 \times \cdots \times \mathrm{V}_m \\ \otimes^k \downarrow & & \otimes^k \downarrow \\ \mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_m & \stackrel{T(h_1,...,h_m)}{\to} & \mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_m \end{array}.$$

This is functorial in the sense that

$$T(h_1, ..., h_m) \circ T(g_1, ..., g_m) = T(h_1 \circ g_1, ..., h_m \circ g_m)$$

and $T(\mathrm{id}, ..., \mathrm{id}) = \mathrm{id}$. Also, $T(h_1, ..., h_m)$ has the following effect on decomposable elements:

$$T(h_1, ..., h_m)(\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_m) = h_1(\mathrm{v}_1) \otimes \cdots \otimes h_m(\mathrm{v}_m).$$

Now we could jump the gun a bit and use the notation $h_1 \otimes \cdots \otimes h_m$ for $T(h_1, ..., h_m)$ but is this the same thing as the element of $\mathrm{Hom}_{\mathsf{R}}(\mathrm{W}_1, \mathrm{V}_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathsf{R}}(\mathrm{W}_m, \mathrm{V}_m)$ which must be denoted the same way? The answer is that in general, these are distinct objects. On the other hand, there is little harm done if context determines which of the two possible meanings we are invoking. Furthermore, we shall see than in many cases, the two meanings actually do coincide.

The following proposition give a basic and often used isomorphism:

Proposition E.1 *For* $\mathsf{R}-$*modules* $\mathrm{W}, \mathrm{V}, \mathrm{U}$ *we have*

$$\mathrm{Hom}_{\mathsf{R}}(\mathrm{W} \otimes \mathrm{V}, \mathrm{U}) \cong L(\mathrm{W}, \mathrm{V}; \mathrm{U})$$

More generally,

$$\mathrm{Hom}_{\mathsf{R}}(\mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_k, \mathrm{U}) \cong L(\mathrm{W}_1, ..., \mathrm{W}_k; \mathrm{U})$$

Proof. This is more or less just a restatement of the universal property of $\mathrm{W} \otimes \mathrm{V}$. One should check that this association is indeed an isomorphism. ∎

Exercise E.4 *Show that if* W *is free with basis* $(f_1, ..., f_n)$ *then* W^* *is also free and has a dual basis* $f^1, ..., f^n$*, that is,* $f^i(f_j) = \delta^i_j$.

Theorem E.2 *If* $\mathrm{V}_1, ..., \mathrm{V}_k$ *are free* $\mathsf{R}-$*modules and if* $(v^j_1, ..., v^j_{n_j})$ *is a basis for* V_j *then set of all decomposable elements of the form* $v^1_{i_1} \otimes \cdots \otimes v^k_{i_k}$ *form a basis for* $\mathrm{V}_1 \otimes \cdots \otimes \mathrm{V}_k$.

Proposition E.2 *There is a unique* R*-module map* $\iota : \mathrm{W}^*_1 \otimes \cdots \otimes \mathrm{W}^*_k \to (\mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_k)^*$ *such that if* $\alpha_1 \otimes \cdots \otimes \alpha_k$ *is a (decomposable) element of* $\mathrm{W}^*_1 \otimes \cdots \otimes \mathrm{W}^*_k$ *then*

$$\iota(\alpha_1 \otimes \cdots \otimes \alpha_k)(w_1 \otimes \cdots \otimes w_k) = \alpha_1(w_1) \cdots \alpha_k(w_k).$$

If the modules are all free then this is an isomorphism.

Proof. If such a map exists, it must be unique since the decomposable elements span $\mathrm{W}_1^* \otimes \cdots \otimes \mathrm{W}_k^*$. To show existence we define a multilinear map

$$\vartheta : \mathrm{W}_1^* \times \cdots \times \mathrm{W}_k^* \times \mathrm{W}_1 \times \cdots \times \mathrm{W}_k \to \mathsf{R}$$

by the recipe

$$(\alpha_1, ..., \alpha_k, w_1, ..., w_k) \mapsto \alpha_1(w_1) \cdots \alpha_k(w_k).$$

By the universal property there must be a linear map

$$\widetilde{\vartheta} : \mathrm{W}_1^* \otimes \cdots \otimes \mathrm{W}_k^* \otimes \mathrm{W}_1 \otimes \cdots \otimes \mathrm{W}_k \to \mathsf{R}$$

such that $\widetilde{\vartheta} \circ u = \vartheta$ where u is the universal map. Now define

$$\begin{aligned} &\iota(\alpha_1 \otimes \cdots \otimes \alpha_k)(w_1 \otimes \cdots \otimes w_k) \\ &:= \widetilde{\vartheta}(\alpha_1 \otimes \cdots \otimes \alpha_k \otimes w_1 \otimes \cdots \otimes w_k). \end{aligned}$$

The fact, that ι is an isomorphism in case the W_i are all free follows easily from exercise D.4 and theorem **??**. ■

Definition E.14 *The k-th tensor power of a module* W *is defined to be*

$$\mathrm{W}^{\otimes k} := \mathrm{W} \otimes \cdots \otimes \mathrm{W}.$$

This module is also denoted $\bigotimes^k(\mathrm{W})$. *We also define the space of* $({}^r{}_s)$-***tensors on*** W *:*

$$\bigotimes{}^r{}_s(\mathrm{W}) := \mathrm{W}^{\otimes r} \otimes (\mathrm{W}^*)^{\otimes s}.$$

Similarly, $\bigotimes{}_s{}^r(\mathrm{W}) := (\mathrm{W}^*)^{\otimes s} \otimes \mathrm{W}^{\otimes r}$ *is the space of* $({}_s{}^r)$-***tensors on*** W.

Again, although we distinguish $\bigotimes{}^r{}_s(\mathrm{W})$ from $\bigotimes{}_s{}^r(\mathrm{W})$ we shall be able to develop things so as to use mostly the space $\bigotimes{}^r{}_s(\mathrm{W})$ and so eventually we take $\bigotimes^r_s(\mathrm{W})$ to mean $\bigotimes{}^r{}_s(\mathrm{W})$.

If $(v_1, ..., v_n)$ is a basis for a module V and $(v^1, ..., v^n)$ the dual basis for V^* then a basis for $\bigotimes{}^r{}_s(\mathrm{V})$ is given by

$$\{v_{i_1} \otimes \cdots \otimes v_{i_r} \otimes v^{j_1} \otimes \cdots \otimes v^{j_s}\}$$

where the index set is the set $\mathcal{I}(r, s, n)$ defined by

$$\mathcal{I}(r, s, n) := \{(i_1, ..., i_r; j_1, ..., j_s) : 1 \leq i_k \leq n \text{ and } 1 \leq j_k \leq n\}.$$

Thus $\bigotimes{}^r{}_s(\mathrm{V})$ has dimension $n^r n^s$ (where $n = \dim(\mathrm{V})$).

We restate the universal property in this special case of tensors:

Proposition E.3 (Universal mapping property) *Given a module or vector space* V *over* R, *then* $\bigotimes^r_s(\mathrm{V})$ *has associated with it, a map*

$$\otimes_{({}^r{}_s)} : \underbrace{\mathrm{V} \times \mathrm{V} \times \cdots \times \mathrm{V}}_{r} \times \underbrace{\mathrm{V}^* \times \mathrm{V}^* \times \cdots \times \mathrm{V}^*}_{s} \to \bigotimes{}^r{}_s(\mathrm{V})$$

such that for any multilinear map $\Lambda \in T_{r}{}^{s}(\mathrm{V};\mathsf{R})$*;*

$$\Lambda : \underbrace{\mathrm{V} \times \mathrm{V} \times \cdots \times \mathrm{V}}_{r-times} \times \underbrace{\mathrm{V}^* \times \mathrm{V}^* \cdots \times \mathrm{V}^*}_{s-times} \to \mathsf{R}$$

there is a unique linear map $\widetilde{\Lambda} : \bigotimes^{r}{}_{s}(\mathrm{V}) \to \mathsf{R}$ *such that* $\widetilde{\Lambda} \circ \otimes_{\binom{r}{s}} = \Lambda$*. Up to isomorphism, the space* $\bigotimes^{r}{}_{s}(\mathrm{V})$ *is the unique space with this universal mapping property.*

Corollary E.1 *There is an isomorphism* $(\bigotimes^{r}{}_{s}(\mathrm{V}))^* \cong T_{r}{}^{s}(\mathrm{V})$ *given by* $\widetilde{\Lambda} \mapsto \widetilde{\Lambda} \circ \otimes^r_s$*. (**Warning**: Notice that the* $T^{r}{}_{s}(\mathrm{V})$ *occurring here is **not** the default space* $T^{r}{}_{s}(\mathrm{V})$ *that we eventually denote by* $T^r_s(\mathrm{V})$*.*

Corollary E.2 $(\bigotimes^{r}{}_{s}(\mathrm{V}^*))^* = T_{r}{}^{s}(\mathrm{V}^*)$

Now along the lines of the map of proposition E.2 we have a homomorphism

$$\iota^r_{.s} : \bigotimes^{r}{}_{s}(\mathrm{V}) \to T^{r}{}_{s}(\mathrm{V}) \tag{E.1}$$

given by

$$\begin{aligned} &\iota^r_{.s}\big((\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_r \otimes \eta^1 \otimes \cdots .. \otimes \eta^s)\big)(\theta^1, ..., \theta^r, \mathrm{w}_1, .., \mathrm{w}_s) \\ &= \theta_1(\mathrm{v}_1)\theta_2(\mathrm{v}_2) \cdots \theta_l(\mathrm{v}_l)\eta^1(\mathrm{w}_1)\eta^2(\mathrm{w}_2) \cdots \eta^k(\mathrm{w}_k) \end{aligned}$$

$\theta^1, \theta^2, ..., \theta^r \in \mathrm{V}^*$ and $\mathrm{w}_1, \mathrm{w}_2, .., \mathrm{w}_k \in \mathrm{V}$. If V is a finite dimensional finitely generated free module then we have $\mathrm{V} = \mathrm{V}^{**}$. This is the reflexive property.

Definition E.15 *We say that* V *is **totally reflexive** if the homomorphism* **??** *just given is in fact an isomorphism. This happens for finitely generated free modules:*

Proposition E.4 *For a finite dimensional finitely generated free module* V *we have a natural isomorphism* $\bigotimes^{r}{}_{s}(\mathrm{V}) \cong T^{r}{}_{s}(\mathrm{V})$*. The isomorphism is given by the map* $\iota^r_{.s}$ *(see* **??***)*

Proof. Just to get the existence of a natural isomorphism we may observe that

$$\begin{aligned} \bigotimes^{r}{}_{s}(\mathrm{V}) &= \bigotimes^{r}{}_{s}(\mathrm{V}^{**}) = (\bigotimes^{r}{}_{s}(\mathrm{V}^*))^* \\ &= T_{r}{}^{s}(\mathrm{V}^*) = L(\mathrm{V}^{*r}, \mathrm{V}^{**s}; \mathsf{R}) \\ &= L(\mathrm{V}^{*r}, \mathrm{V}^s; \mathsf{R}) := T^{r}{}_{s}(\mathrm{V}). \end{aligned}$$

We would like to take a more direct approach. Since V is free we may take a basis $\{f_1, ..., f_n\}$ and a dual basis $\{f^1, ..., f^n\}$ for V^*. It is easy to see that ι^r_s sends the basis elements of $\bigotimes^{r}{}_{s}(\mathrm{V})$ to basis elements of $T^{r}{}_{s}(\mathrm{V};\mathsf{R})$ as for example

$$\iota^1_1 : f^i \otimes f_j \mapsto f^i \otimes f_j$$

where only the interpretation of the $\otimes$ changes. On the right side $f^i \otimes f_j$ is by definition the multilinear map $f^i \otimes f_j : (\alpha, v) := \alpha\left(f^i\right) f_j(v)$ ■

In the finite dimensional case, we will identify $\bigotimes{}^r{}_s(\mathrm{V})$ with the space $T^r{}_s(\mathrm{V};\mathsf{R}) = L(\mathrm{V}^{r*}, \mathrm{V}^s; \mathsf{R})$ of r, s-multilinear maps. We may freely think of a decomposable tensor $\mathrm{v}_1 \otimes ... \otimes \mathrm{v}_r \otimes \eta^1 \otimes ... \otimes \eta^s$ as a *multilinear map* by the formula

$$(\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_r \otimes \eta^1 \otimes \cdots .. \otimes \eta^s) \cdot (\theta^1, ..., \theta^r, \mathrm{w}_1, .., \mathrm{w}_s)$$
$$= \theta_1(\mathrm{v}_1)\theta_2(\mathrm{v}_2) \cdots \theta_l(\mathrm{v}_l)\eta^1(\mathrm{w}_1)\eta^2(\mathrm{w}_2) \cdots \eta^k(\mathrm{w}_k)$$

$U)$ of smooth vector fields over an open set U in some manifold M. We shall see that for finite dimensional manifolds $T^r_s(\mathfrak{X}(U))$ is naturally isomorphic to the smooth sections of a so called tensor bundle. We take up this important topic shortly.

E.0.1 Contraction of tensors

Consider a tensor of the form $\mathrm{v}_1 \otimes \mathrm{v}_2 \otimes \eta^1 \otimes \eta^2 \in T^2_2(\mathrm{V})$ we can define the 1,1 contraction of $\mathrm{v}_1 \otimes \mathrm{v}_2 \otimes \eta^1 \otimes \eta^2$ as the tensor obtained as

$$C^1_1(\mathrm{v}_1 \otimes \mathrm{v}_2 \otimes \eta^1 \otimes \eta^2) = \eta^1(\mathrm{v}_1)\mathrm{v}_2 \otimes \eta^2.$$

Similarly we can define

$$C^1_2(\mathrm{v}_1 \otimes \mathrm{v}_2 \otimes \eta^1 \otimes \eta^2) = \eta^2(\mathrm{v}_1)\mathrm{v}_2 \otimes \eta^1.$$

In general, we could define C^i_j on "monomials" $\mathrm{v}_1 \otimes \mathrm{v}_2 ... \otimes \mathrm{v}_r \otimes \eta^1 \otimes \eta^2 \otimes ... \otimes \eta^s$ and then extend linearly to all of $T^r_s(\mathrm{V})$. This works fine for V finite dimensional and turns out to give a notion of contraction which is the same a described in the next definition.

Definition E.16 *Let* $\{e_1, ..., e_n\} \subset \mathrm{V}$ *be a basis for* V *and* $\{e^1, ..., e^n\} \subset \mathrm{V}^*$ *the dual basis. If* $\tau \in T^r_s(\mathrm{V})$ *we define* $C^i_j\tau \in T^{r-1}_{s-1}(\mathrm{V})$

$$C^i_j\tau(\theta^1, ..., \theta^{r-1}, \mathrm{w}_1, .., \mathrm{w}_{s-1})$$
$$= \sum_{k=1}^n \tau(\theta^1, ..., \underset{i-th\ position}{e^k}, \ldots, \theta^{r-1}, \mathrm{w}_1, \ldots, \underset{j-th\ position}{e_k}, \ldots, \mathrm{w}_{s-1}).$$

It is easily checked that this definition is independent of the basis chosen. In the infinite dimensional case the sum contraction cannot be defined in general to apply to all tensors. However, we can still define contractions on linear combinations of tensors of the form $\mathrm{v}_1 \otimes \mathrm{v}_2 ... \otimes \mathrm{v}_r \otimes \eta^1 \otimes \eta^2 \otimes ... \otimes \eta^s$ as we did above. Returning to the finite dimensional case, suppose that

$$\tau = \sum \tau^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} e_{i_1} \otimes ... \otimes e_{i_r} \otimes e^{j_1} \otimes ... \otimes e^{j_s}.$$

Then it is easy to check that if we define

$$\tau^{i_1,\ldots,i_{r-1}}_{j_1,\ldots,j_{s-1}} = \sum_{k=1}^{n} \tau^{i_1,\ldots,k,\ldots,i_{r-1}}_{j_1,\ldots,k,\ldots,j_{s-1}}$$

where the upper repeated index k is in the i-th position and the lower occurrence of k is in the j-th position then

$$C^i_j\tau = \sum \tau^{i_1,\ldots,i_{r-1}}_{j_1,\ldots,j_{s-1}} e_{i_1} \otimes \ldots \otimes e_{i_{r-1}} e^{j_1} \otimes \ldots \otimes e^{j_{s-1}}.$$

Even in the infinite dimensional case the following definition makes sense. The contraction process can be repeated until we arrive at a function.

Definition E.17 *Given $\tau \in T^r_s$ and $\sigma = \mathrm{w}_1 \otimes \ldots \otimes \mathrm{w}_l \otimes \eta^1 \otimes \eta^2 \otimes \ldots \otimes \eta^m \in T^l_m$ a simple tensor with $l \le r$ and $m \le s$, we define the contraction against σ by*

$$\begin{aligned}
&\sigma \lrcorner \tau(\alpha_1, \ldots, \alpha_{r-l}, \mathrm{v}_1, \ldots, \mathrm{v}_{r-l}) \\
&:= C(\tau \otimes (\mathrm{w}_1 \otimes \ldots \otimes \mathrm{w}_l \otimes \eta^1 \otimes \eta^2 \otimes \ldots \otimes \eta^m)) \\
&:= \tau(\eta^1, \ldots, \eta^m, \alpha_1, \ldots, \alpha_{r-l}, \mathrm{w}_1, \ldots, \mathrm{w}_l, \mathrm{v}_1, \ldots, \mathrm{v}_{r-l}).
\end{aligned}$$

For a given simple tensor σ we thus have a linear map $\sigma\lrcorner : T^r_s \to T^{r-l}_{s-m}$. For finite dimensional V this can be extended to a bilinear pairing between T^l_m and T^r_s

$$T^l_m(\mathrm{V}) \times T^r_s(\mathrm{V}) \to T^{r-l}_{s-m}(\mathrm{V}).$$

E.0.2 Extended Matrix Notation.

$$\mathbf{A} = \sum A^{i_1\ldots i_r}_{j_1\cdots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}$$

is abbreviated to

$$\mathbf{A} = A^I_J e_I e^J$$

or in matrix notation

$$\mathbf{A} = eAe^*$$

In fact, as a multilinear map we can simply let $\mathbf{v}$ denote an $r-$tuple of vectors from V and θ an $s-$tuple of elements of V^*. Then $\mathbf{A}(\theta, \mathbf{v}) = \theta eAe^*\mathbf{v}$ where for example

$$\theta e = \theta_1(e_{i_1}) \cdots \theta_r(e_{i_r})$$

and

$$e^*\mathbf{v} = e^{j_1}(v_1) \cdots e^{j_s}(v_s)$$

so that

$$\theta eA\varepsilon\mathbf{v} = \sum A^{i_1\ldots i_r}_{j_1\cdots j_s} \theta_1(e_{i_1}) \cdots \theta_r(e_{i_r}) e^{j_1}(v_1) \cdots e^{j_s}(v_s).$$

Notice that if one takes the convention that objects with indices up are "column vectors" and indices down "row vectors" then to get the order of the matrices correctly the repeated indices should read *down then up* going from left to right. So $A^I_J e_I e^J$ should be changed to $e_I A^I_J e^J$ before it can be interpreted as a matrix multiplication.

Remark E.6 *We can also write* $\triangle_I^{K'} A_J^I \triangle_{L'}^J = A_{L'}^{K'}$ *where* $\triangle_{S'}^R = e^R e'_S$.

E.0.3 R−Algebras

Definition E.18 *Let* R *be a commutative ring. An* $\mathsf{R}-$***algebra*** $\mathfrak{A}$ *is an* $\mathsf{R}-$*module that is also a ring with identity* $1_{\mathfrak{A}}$ *where the ring addition and the module addition coincide; and where*

1) $r(a_1a_2) = (ra_1)a_2 = a_1(ra_2)$ *for all* $a_1, a_2 \in \mathfrak{A}$ *and all* $r \in \mathsf{R}$,

2) $(r_1r_2)(a_1a_2) = (r_1a_1)(r_2a_2)$.

If we also have $(a_1a_2)a_3 = a_1(a_2a_3)$ *for all* $a_1, a_2, a_3 \in \mathfrak{A}$ *we call* $\mathfrak{A}$ *an associative* $\mathsf{R}-$*algebra.*

Definition E.19 *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be* $\mathsf{R}-$*algebras. A module homomorphism* $h : \mathfrak{A} \to \mathfrak{B}$ *that is also a ring homomorphism is called an* $\mathsf{R}-$***algebra homomorphism****. Epimorphism, monomorphism and isomorphism are defined in the obvious way.*

If a submodule $\mathfrak{I}$ of an algebra $\mathfrak{A}$ is also a two sided ideal with respect to the ring structure on $\mathfrak{A}$ then $\mathfrak{A}/\mathfrak{I}$ is also an algebra.

Example E.11 *The set of all smooth functions* $C^\infty(U)$ *is an* $\mathbb{R}-$*algebra (*$\mathbb{R}$ *is the real numbers) with unity being the function constantly equal to* 1.

Example E.12 *The set of all complex* $n \times n$ *matrices is an algebra over* $\mathbb{C}$ *with the product being matrix multiplication.*

Example E.13 *The set of all complex* $n \times n$ *matrices with real polynomial entries is an algebra over the ring of polynomials* $\mathbb{R}[x]$.

Definition E.20 *The set of all endomorphisms of an* $\mathsf{R}-$*module* W *is an* $\mathsf{R}-$*algebra denoted* $End_{\mathsf{R}}(\mathrm{W})$ *and called the* ***endomorphism algebra*** *of* W*. Here, the sum and scalar multiplication is defined as usual and the product is composition. Note that for* $r \in \mathsf{R}$

$$r(f \circ g) = (rf) \circ g = f \circ (rg)$$

where $f, g \in End_{\mathsf{R}}(\mathrm{W})$.

Definition E.21 *A* $\mathbb{Z}$*-graded* R*-algebra is an* R*-algebra with a direct sum decomposition* $\mathfrak{A} = \sum_{i \in \mathbb{Z}} \mathfrak{A}_i$ *such that* $\mathfrak{A}_i\mathfrak{A}_j \subset \mathfrak{A}_{i+j}$.

Definition E.22 *Let* $\mathfrak{A} = \sum_{i \in \mathbb{Z}} \mathfrak{A}_i$ *and* $\mathfrak{B} = \sum_{i \in \mathbb{Z}} \mathfrak{B}_i$ *be* $\mathbb{Z}$*-graded algebras. An* R*-algebra homomorphism* $h : \mathfrak{A} \to \mathfrak{B}$ *is a called a* $\mathbb{Z}$*-graded homomorphism if* $h(\mathfrak{A}_i) \subset \mathfrak{B}_i$ *for each* $i \in \mathbb{Z}$.

We now construct the tensor algebra on a fixed $\mathsf{R}-$module W. This algebra is important because is universal in a certain sense and contains the symmetric and alternating algebras as homomorphic images. Consider the following situation: $\mathfrak{A}$ is an $\mathsf{R}-$algebra, W an $\mathsf{R}-$module and $\phi : \mathrm{W} \to \mathfrak{A}$ a module homomorphism. If $h : \mathfrak{A} \to \mathfrak{B}$ is an algebra homomorphism then of course $h \circ \phi : \mathrm{W} \to \mathfrak{B}$ is an $\mathsf{R}-$module homomorphism.

Definition E.23 *Let* W *be an* $\mathsf{R}-$*module. An* $\mathsf{R}-$ *algebra* $\mathfrak{U}$ *together with a map* $\phi : \mathrm{W} \to \mathfrak{U}$ *is called universal with respect to* W *if for any* $\mathsf{R}-$*module homomorphism* $\psi : \mathrm{W} \to \mathfrak{B}$ *there is a unique algebra homomorphism If* $h : \mathfrak{U} \to \mathfrak{B}$ *such that* $h \circ \phi = \psi$.

Again if such a universal object exists it is unique up to isomorphism. We now exhibit the construction of this type of universal algebra. First we define $T^0(\mathrm{W}) := \mathsf{R}$ and $T^1(\mathrm{W}) := \mathrm{W}$. Then we define $T^k(\mathrm{W}) := \mathrm{W}^{k\otimes} = \mathrm{W} \otimes \cdots \otimes \mathrm{W}$. The next step is to form the direct sum $T(\mathrm{W}) := \sum_{i=0}^{\infty} T^i(\mathrm{W})$. In order to make this a $\mathbb{Z}$-graded algebra we define $T^i(\mathrm{W}) := 0$ for $i < 0$ and then define a product on $T(\mathrm{W}) := \sum_{i\in\mathbb{Z}} T^i(\mathrm{W})$ as follows: We know that for $i, j > 0$ there is an isomorphism $\mathrm{W}^{i\otimes} \otimes \mathrm{W}^{j\otimes} \to \mathrm{W}^{(i+j)\otimes}$ and so a bilinear map $\mathrm{W}^{i\otimes} \times \mathrm{W}^{j\otimes} \to \mathrm{W}^{(i+j)\otimes}$ such that

$$\mathrm{w}_1 \otimes \cdots \otimes \mathrm{w}_i \times \mathrm{w}'_1 \otimes \cdots \otimes \mathrm{w}'_j \mapsto \mathrm{w}_1 \otimes \cdots \otimes \mathrm{w}_i \otimes \mathrm{w}'_1 \otimes \cdots \otimes \mathrm{w}'_j.$$

Similarly, we define $T^0(\mathrm{W}) \times \mathrm{W}^{i\otimes} = \mathsf{R} \times \mathrm{W}^{i\otimes} \to \mathrm{W}^{i\otimes}$ by just using scalar multiplication. Also, $\mathrm{W}^{i\otimes} \times \mathrm{W}^{j\otimes} \to 0$ if either i or j is negative. Now we may use the symbol $\otimes$ to denote these multiplications without contradiction and put then together to form an product on $T(\mathrm{W}) := \sum_{i\in\mathbb{Z}} T^i(\mathrm{W})$. It is now clear that $T^i(\mathrm{W}) \times T^j(\mathrm{W}) \mapsto T^i(\mathrm{W}) \otimes T^j(\mathrm{W}) \subset T^{i+j}(\mathrm{W})$ where we make the needed trivial definitions for the negative powers $T^i(\mathrm{W}) = 0$, $i < 0$. Thus $T(\mathrm{W})$ is a graded algebra.

E.0.4 Alternating Multilinear Algebra

In this section we make the simplifying assumption that all of the rings we use will have the following property: The sum of the unity element with itself; $1 + 1$ is invertible. Thus if we use 2 to denote the element $1 + 1$ then we assume the existence of a unique element "1/2" such that $2 \cdot 1/2 = 1$. Thus, in the case of fields, the assumption is that the field is not of characteristic 2. The reader need only worry about two cases:

1. The unity "1" is just the number 1 in some subring of $\mathbb{C}$ (e.g. $\mathbb{R}$ or $\mathbb{Z}$) or
2. the unity "1" refers to some sort of function or section with values in a ring like $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Z}$ that takes on the constant value 1. For example, in the ring $C^\infty(M)$, the unity is just the constant function 1.

Definition E.24 *A* $\mathbb{Z}$*-graded algebra is called skew-commutative (or graded commutative) if for* $a_i \in A_i$ *and* $a_j \in A_j$ *we have*

$$a_i \cdot a_j = (-1)^{kl} a_j \cdot a_i$$

Definition E.25 *A morphism of degree n from a graded algebra* $A = \bigoplus_{i\in\mathbb{Z}} A_i$ *to a graded algebra* $B = \bigoplus_{i\in\mathbb{Z}} B_i$ *is a algebra homomorphism* $h : A \to B$ *such that* $h(A_i) \subset B_{i+n}$.

Definition E.26 *A super algebra is a* $\mathbb{Z}_2$*-graded algebra* $A = A_0 \oplus A_1$ *such that* $A_i \cdot A_j \subset A_{i+j \ mod2}$ *and such that* $a_i \cdot a_j = (-1)^{kl} a_j \cdot a_i$ *for* $i, j \in \mathbb{Z}_2$.

Alternating tensor maps

Definition E.27 *A k-multilinear map $\alpha : \mathrm{V} \times \cdots \times \mathrm{V} \to \mathrm{F}$ is called **alternating** if $\alpha(\mathrm{v}_1, ..., \mathrm{v}_k) = 0$ whenever $\mathrm{v}_i = \mathrm{v}_j$ for some $i \neq j$. The space of all **alternating** k-multilinear maps into F will be denoted by $L^k_{alt}(\mathrm{V}; \mathrm{F})$ or by $L^k_{alt}(\mathrm{V})$ if the ring is either $\mathbb{R}$ or $\mathbb{C}$ and there is no chance of confusion.*

Remark E.7 *Notice that we have moved the k up to make room for the Alt thus $L^k_{alt}(\mathrm{V};\mathsf{R}) \subset L^0_k(\mathrm{V};\mathsf{R})$.*

Thus if $\omega \in L^k_{alt}(\mathrm{V})$, then for any permutation σ of the letters $1, 2, ..., k$ we have

$$\omega(\mathrm{v}_1, \mathrm{v}_2, .., \mathrm{v}_k) = sgn(\sigma)\omega(\mathrm{v}_{\sigma_1}, \mathrm{v}_{\sigma_2}, .., \mathrm{v}_{\sigma_k}).$$

Now given $\omega \in L^r_{alt}(\mathrm{V})$ and $\eta \in L^s_{alt}(\mathrm{V})$ we define their **wedge product** or **exterior product** $\omega \wedge \eta \in L^{r+s}_{alt}(\mathrm{V})$ by the formula

$$\omega \wedge \eta(\mathrm{v}_1, ..., \mathrm{v}_r, \mathrm{v}_{r+1}, ..., \mathrm{v}_{r+s}) := \frac{1}{r!s!} \sum_{\sigma} \mathrm{sgn}(\sigma)\omega(\mathrm{v}_{\sigma_1}, ..., \mathrm{v}_{\sigma_r})\eta(\mathrm{v}_{\sigma_{r+1}}, ..., \mathrm{v}_{\sigma_{r+s}})$$

or by

$$\omega \wedge \eta(\text{“same as above”}) := \sum_{r,s-\text{shuffles } \sigma} \mathrm{sgn}(\sigma)\omega(\mathrm{v}_{\sigma_1}, ..., \mathrm{v}_{\sigma_r})\eta(\mathrm{v}_{\sigma_{r+1}}, ..., \mathrm{v}_{\sigma_{r+s}}).$$

In the latter formula we sum over all permutations such that $\sigma_1 < \sigma_2 < .. < \sigma_r$ and $\sigma_{r+1} < \sigma_{r+2} < .. < \sigma_{r+s}$. This kind of permutation is called an $r, s-$shuffle as indicated in the summation. The most important case is for $\omega, \eta \in L^1_{alt}(\mathrm{V})$ in which case

$$(\omega \wedge \eta)(v, w) = \omega(v)\eta(w) - \omega(w)\eta(v)$$

This is clearly a skew symmetric multi-linear map.

If we use a basis $\varepsilon^1, \varepsilon^2, ..., \varepsilon^n$ for V^* it is easy to show that the set of all elements of the form $\varepsilon^{i_1} \wedge \varepsilon^{i_2} \wedge \cdots \wedge \varepsilon^{i_k}$ with $i_1 < i_2 < ... < i_k$ form a basis for . Thus for any $\omega \in A^k(\mathrm{V})$

$$\omega = \sum_{i_1 < i_2 < ... < i_k} a_{i_1 i_2,..,i_k} \varepsilon^{i_1} \wedge \varepsilon^{i_2} \wedge \cdots \wedge \varepsilon^{i_k}$$

Remark E.8 *In order to facilitate notation we will abbreviate a sequence of k integers, say $i_1, i_2, ..., i_k$, from the set $\{1, 2, ..., \dim(\mathrm{V})\}$ as I and $\varepsilon^{i_1} \wedge \varepsilon^{i_2} \wedge \cdots \wedge \varepsilon^{i_k}$ is written as ε^I. Also, if we require that $i_1 < i_2 < ... < i_k$ we will write $\vec{I}$. We will freely use similar self explanatory notation as we go along with out further comment. For example, the above equation can be written as*

$$\omega = \sum a_{\vec{I}} \varepsilon^{\vec{I}}$$

Lemma E.3 $L^k_{alt}(\mathrm{V}) = 0$ *if* $k > n = \dim(\mathrm{V})$.

Proof. Easy exercise. ■

If one defines $L^0_{alt}(\mathrm{V})$ to be the scalars $\mathbb{K}$ and recalling that $L^1_{alt}(\mathrm{V}) = \mathrm{V}^*$ then the sum

$$L_{alt}(\mathrm{V}) = \bigoplus_{k=0}^{\dim(M)} L^k_{alt}(\mathrm{V})$$

is made into an algebra via the wedge product just defined.

Proposition E.5 *For $\omega \in L^r_{alt}(V)$ and $\eta \in L^s_{alt}(V)$ we have $\omega \wedge \eta = (-1)^{rs}\eta \wedge \omega \in L^{r+s}_{alt}(V)$.*

The Abstract Grassmann Algebra

We wish to construct a space that is universal with respect to alternating multilinear maps. To this end, consider the tensor space $T^k(\mathrm{V}) := \mathrm{V}^{k\otimes}$ and let A be the submodule of $T^k(\mathrm{V})$ generated by elements of the form

$$\mathrm{v}_1 \otimes \cdots \mathrm{v}_i \otimes \cdots \otimes \mathrm{v}_i \cdots \otimes \mathrm{v}_k.$$

In other words, A is generated by decomposable tensors with two (or more) equal factors. We define the space of k-vectors to be

$$\mathrm{V} \wedge \cdots \wedge \mathrm{V} := \bigwedge^k \mathrm{V} := T^k(\mathrm{V})/\mathrm{A}.$$

Let $\mathrm{A}_k : \mathrm{V} \times \cdots \times \mathrm{V} \to T^k(\mathrm{V}) \to \ngeq^k\mathrm{V}$ be the canonical map composed with projection onto $\bigwedge^k \mathrm{V}$. This map turns out to be an alternating multilinear map. We will denote $\mathrm{A}_k(\mathrm{v}_1, ..., \mathrm{v}_k)$ by $\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_k$. The pair $(\bigwedge^k \mathrm{V}, \mathrm{A}_k)$ is universal with respect to alternating k-multilinear maps: Given any alternating k-multilinear map $\alpha : \mathrm{V} \times \cdots \times \mathrm{V} \to \mathrm{F}$, there is a unique linear map $\alpha_\wedge : \bigwedge^k \mathrm{V} \to \mathrm{F}$ such that $\alpha = \alpha_\wedge \circ \mathrm{A}_k$; that is$\bigwedge^k$

$$\begin{array}{ccc} \mathrm{V} \times \cdots \times \mathrm{V} & \xrightarrow{\alpha} & \mathrm{F} \\ \mathrm{A}_k \downarrow & \nearrow_{\alpha_\wedge} & \\ \bigwedge^k \mathrm{V} & & \end{array}$$

commutes. Notice that we also have that $\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_k$ is the image of $\mathrm{v}_1 \otimes \cdots \otimes \mathrm{v}_k$ under the quotient map. Next we define $\bigwedge \mathrm{V} := \sum_{k=0}^{\infty} \bigwedge^k \mathrm{V}$ and impose the multiplication generated by the rule

$$(\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_i) \times (\mathrm{v}'_1 \wedge \cdots \wedge \mathrm{v}'_j) \mapsto \mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_i \wedge \mathrm{v}'_1 \wedge \cdots \wedge \mathrm{v}'_j \in \bigwedge^{i+j} \mathrm{V}.$$

The resulting algebra is called the Grassmann algebra or exterior algebra. If we need to have a $\mathbb{Z}$ grading rather than a $\mathbb{Z}^+$ grading we may define $\bigwedge^k \mathrm{V} := 0$ for $k < 0$ and extend the multiplication in the obvious way.

Notice that since $(\mathrm{v} + \mathrm{w}) \wedge (\mathrm{w} + \mathrm{v}) = 0$, it follows that $\mathrm{v} \wedge \mathrm{w} = -\mathrm{w} \wedge \mathrm{v}$. In fact, any odd permutation of the factors in a decomposable element such as $\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_k$, introduces a change of sign:

$$\begin{aligned} &\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_i \wedge \cdots \wedge \mathrm{v}_j \wedge \cdots \wedge \mathrm{v}_k \\ &= -\mathrm{v}_1 \wedge \cdots \wedge \mathrm{v}_j \wedge \cdots \wedge \mathrm{v}_i \wedge \cdots \wedge \mathrm{v}_k \end{aligned}$$

Lemma E.4 *If* V *is has rank* n, *then* $\bigwedge^k \mathrm{V} = 0$ *for* $k \geq n$. *If* $f_1, ..., f_n$ *is a basis for* V *then the set*

$$\{f_{i_1} \wedge \cdots \wedge f_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$$

is a basis for $\bigwedge^k \mathrm{V}$ *where we agree that* $f_{i_1} \wedge \cdots \wedge f_{i_k} = 1$ *if* $k = 0$.

The following lemma follows easily from the universal property of $\alpha_\wedge : \bigwedge^k \mathrm{V} \to \mathrm{F}$:

Lemma E.5 *There is a natural isomorphism*

$$L^k_{alt}(\mathrm{V}; \mathrm{F}) \cong L(\bigwedge^k \mathrm{V}; \mathrm{F})$$

In particular,

$$L^k_{alt}(\mathrm{V}) \cong \left(\bigwedge^k \mathrm{V}\right)^*.$$

Remark E.9 (Convention) *Let* $\alpha \in L^k_{alt}(\mathrm{V}; \mathrm{F})$. *Because the above isomorphism is so natural it may be taken as an identification and so we sometimes write* $\alpha(v_1, ..., v_k)$ *as* $\alpha(v_1 \wedge \cdots \wedge v_k)$.

Now recall that $\mathrm{V}^* \otimes \cdots \otimes \mathrm{V}^* \cong (\mathrm{V} \otimes \cdots \otimes \mathrm{V})^*$ such that if $\alpha_1 \otimes \cdots \otimes \alpha_k$ is a (decomposable) element of $\mathrm{V}^* \otimes \cdots \otimes \mathrm{V}^*$ then

$$\iota(\alpha_1 \otimes \cdots \otimes \alpha_k)(v_1 \otimes \cdots \otimes v_k) = \alpha_1(v_1) \cdots \alpha_k(v_k).$$

In the finite dimensional case, we have module isomorphisms

$$\bigwedge^k \mathrm{V}^* \cong \left(\bigwedge^k \mathrm{V}\right)^* \cong L^k_{alt}(\mathrm{V})$$

which extends by direct sum to

$$L_{alt}(\mathrm{V}) \cong \bigwedge \mathrm{V}^*$$

This isomorphism is in fact an exterior algebra isomorphism (we have an exterior product defined for both).

The following table summarizes:

Exterior Products	Isomorphisms that hold in finite dimension	Alternating multilinear maps
$\bigwedge^k \mathrm{V}$	$\downarrow$	
$\bigwedge^k \mathrm{V}^* \cong \left(\bigwedge^k \mathrm{V}\right)^*$	$\cong$	$L^k_{alt}(\mathrm{V})$
$\bigwedge \mathrm{V} = \bigoplus_k \bigwedge^k \mathrm{V}$		
$\bigwedge \mathrm{V}^* = \bigoplus_k \bigwedge^k \mathrm{V}^*$	$\overset{\text{graded algebra iso.}}{\cong}$	$A(\mathrm{V}) = \bigoplus_k L^k_{alt}(\mathrm{V})$

E.0.5 Orientation on vector spaces

Let V be a finite dimensional vector space. The set of all ordered bases fall into two classes called orientation classes.

Definition E.28 *Two bases are in the same orientation class if the change of basis matrix from one to the other has positive determinant.*

That is, given two frames (bases) in the same class, say $(f_1, ...f_n)$ and $(\widetilde{f}_1, ...\widetilde{f}_n)$ with

$$\widetilde{f}_i = f_j C_i^j$$

then $\det C > 0$ and we say that the frames determine the same orientation. The relation is easily seen to be an equivalence relation.

Definition E.29 *A choice of one of the two orientation classes of frames for a finite dimensional vector space* V *is called an* ***orientation*** *on* V*. The vector space in then said to be oriented.*

Exercise E.5 *Two frames, say* $(f_1, ..., f_n)$ *and* $(\widetilde{f}_1, ..., \widetilde{f}_n)$ *determine the same orientation on* V *if and only if* $f_1 \wedge ... \wedge f_n = a\ \widetilde{f}_1 \wedge ... \wedge \widetilde{f}_n$ *for some positive real number* $a > 0$.

Exercise E.6 *If* σ *is a permutation on* n *letters* $\{1, 2, ...n\}$ *then* $(f_{\sigma 1}, ..., f_{\sigma n})$ *determine the same orientation if and only if* $sgn(\sigma) = +1$.

A top form $\omega \in L_{alt}^n(\mathrm{V})$ determines an orientation on V by the rule $(f_1, ..., f_n) \sim (\widetilde{f}_1, ..., \widetilde{f}_n)$ if and only if

$$\omega(f_1, ..., f_n) = \omega(\widetilde{f}_1, ..., \widetilde{f}_n).$$

Furthermore, two top forms $\omega_1, \omega_2 \in L_{alt}^n(\mathrm{V})$ determine the same orientation on V if and only if $\omega_1 = a\omega_2$ for some positive real number $a > 0$.

First of all lets get the basic idea down. Think about this: A function defined on the circle with range in the interval $(0, 1)$ can be thought of in terms of its graph. The latter is a subset, a cross section, of the product $S^1 \times (0, 1)$. Now, what would a similar cross section of the Mobius band signify? This can't be the same thing as before since a continuous cross section of the Mobius band would have to cross the center and this need not be so for $S^1 \times (0, 1)$. Such a cross section would have to be a sort of twisted function. The Mobius band (with projection onto its center line) provides us with our first nontrivial example of a fiber bundle. The cylinder $S^1 \times (0, 1)$ with projection onto one the factors is a trivial example. Projection onto S^1 gives a topological line bundle while projection onto the interval is a circle bundle. Often what we call the Mobius band will be the slightly different object that is a twisted version of $S^1 \times \mathbb{R}^1$. Namely, the space obtained by identifying one edge of $[0, 1] \times \mathbb{R}^1$ with the other but with a twist.

A fiber bundle is to be though of as a bundle of -or "parameterized family" of- spaces $E_x = \pi^{-1}(x) \subset E$ called the fibers. Nice bundles have further properties that we shall usually assume without explicit mention. The first one is simply that the spaces are Hausdorff and paracompact. The second one is called local triviality. In order to describe this we need the notion of a bundle map and the ensuing notion of equivalence of general fiber bundles.

Most of what we do here will work either for the general topological category or for the smooth category so we once again employ the conventions of 1.5.1.

Definition E.30 *A general C^r- bundle is a triple $\xi = (E, \pi, X)$ where $\pi : E \to M$ is a surjective C^r-map of C^r-spaces (called the bundle projection). For each $p \in X$ the subspace $E_p := \pi^{-1}(p)$is called the fiber over p. The space E is called the* ***total space*** *and X is the* ***base space****. If $S \subset X$ is a subspace we can always form the restricted bundle (E_S, π_S, S) where $E_S = \pi^{-1}(S)$ and $\pi_S = \pi|_S$ is the restriction.*

Definition E.31 *A C^r-**section** of a general bundle $\pi_E : E \to M$ is a C^r-map $s : M \to E$ such that $\pi_E \circ s = \mathrm{id}_M$. In other words, the following diagram must commute:*

$$\begin{array}{ccc} s & & E \\ & \nearrow & \downarrow & \pi_E \\ M & \to & M \\ & \mathrm{id} & \end{array}.$$

The set of all C^r-sections of a general bundle $\pi_E : E \to M$ is denoted by $\Gamma^k(M, E)$. We also define the notion of a section over an open set U in M is the obvious way and these are denoted by $\Gamma^k(U, E)$.

Notation E.2 *We shall often abbreviate to just $\Gamma(U, E)$ or even $\Gamma(E)$ whenever confusion is unlikely. This is especially true in case $k = \infty$ (smooth case) or $k = 0$ (continuous case).*

Now there are two different ways to treat bundles as a category:

The Category *Bun.*

Actually, we should define the Categories Bun_k; $k = 0, 1, ..., \infty$. The objects of Bun_k are C^k-fiber bundles. We shall abbreviate to just "Bun" in cases where a context has been establish and confusion is unlikely.

Definition E.32 *A morphism from $\mathrm{Hom}_{Bun_k}(\xi_1, \xi_2)$, also called a bundle map from a C^r-fiber bundle $\xi_1 := (E_1, \pi_1, X_1)$ to another fiber bundle $\xi_2 := (E_2, \pi_2, X_2)$ is a pair of C^r-maps $(\bar{f}, f)$ such that the following diagram commutes:*

$$\begin{array}{ccc} E_1 & \xrightarrow{\bar{f}} & E_2 \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

If both maps are C^r-isomorphisms we call the map a (C^r-) bundle isomorphism.

Definition E.33 *Two fiber bundles $\xi_1 := (E_1, \pi_1, X_1)$ and $\xi_2 := (E_2, \pi_2, X_2)$ are **equivalent** in Bun_k or **isomorphic** if there exists a bundle isomorphism from ξ_1 to ξ_2.*

Definition E.34 *Two fiber bundles $\xi_1 := (E_1, \pi_1, X_1)$ and $\xi_2 := (E_2, \pi_2, X_2)$ are said to be **locally equivalent** if for any $y \in E_1$ there is an open set U containing p and a bundle equivalence $(f, \bar{f})$ of the restricted bundles:*

$$\begin{array}{ccc} E_1|U & \stackrel{\bar{f}}{\to} & E_2|f(U) \\ \downarrow & & \downarrow \\ U & \stackrel{f}{\to} & f(U) \end{array}$$

The Category $Bun_k(X)$

Definition E.35 *A morphism from $\mathrm{Hom}_{Bun_k(X)}(\xi_1, \xi_2)$, also called a bundle map over X from a C^r-fiber bundle $\xi_1 := (E_1, \pi_1, X)$ to another fiber bundle $\xi_2 := (E_2, \pi_2, X)$ is a C^r-map $\bar{f}$ such that the following diagram commutes:*

$$\begin{array}{ccccc} E_1 & & \stackrel{\bar{f}}{\to} & & E_2 \\ & \searrow & & \swarrow & \\ & & X & & \end{array}$$

*If both maps are C^r-isomorphisms we call the map a (C^r-) **bundle isomorphism over** X (also called a **bundle equivalence**).*

Definition E.36 *Two fiber bundles $\xi_1 := (E_1, \pi_1, X)$ and $\xi_2 := (E_2, \pi_2, X)$ are **equivalent** in $Bun_k(X)$ or **isomorphic** if there exists a (C^r-) bundle isomorphism over X from ξ_1 to ξ_2.*

By now the reader is no doubt tired of the repetitive use of the index C^r so from now on we will simple refer to space (or manifolds) and maps where the appropriate smoothness C^r will not be explicitly stated unless something only works for a specific value of r.

Definition E.37 *Two fiber bundles $\xi_1 := (E_1, \pi_1, X)$ and $\xi_2 := (E_2, \pi_2, X)$ are said to be **locally equivalent** (over X) if for any $y \in E_1$ there is an open set U containing p and a bundle equivalence $(\bar{f}, f)$ of the restricted bundles:*

$$\begin{array}{ccc} E_1|U & \stackrel{\bar{f}}{\to} & E_2|U \\ \downarrow & & \downarrow \\ U & \stackrel{\mathrm{id}}{\to} & U \end{array}$$

Now for any space X **the trivial bundle** with fiber F is the triple $(X \times F, pr_1, X)$ where pr_1 always denoted the projection onto the first factor. Any bundle over X that is bundle equivalent to $X \times F$ is referred to as a trivial bundle.

We will now add in an extra condition that we will usually need:

Definition E.38 *A (C^r-) fiber bundle $\xi := (E, \pi, X)$ is said to be* ***locally trivial*** *(with fiber F) if every for every $x \in X$ has an open neighborhood U such that $\xi_U := (E_U, \pi_U, U)$ is isomorphic to the trivial bundle $(U \times F, pr_1, U)$. Such a fiber bundle is called a* ***locally trivial fiber bundle****.*

We immediately make the following convention: **All fiber bundles in the book will be assumed to be locally trivial unless otherwise stated.** Once we have the local triviality it follows that each fiber $E_p = \pi^{-1}(p)$ is homeomorphic (in fact, C^r-diffeomorphic) to F.

Notation E.3 *We shall take the liberty of using a variety of notations when talking about bundles most of which are quite common and so the reader may as well get used to them. We sometimes write $F \hookrightarrow E \xrightarrow{\pi} X$ to refer to a fiber bundle with typical fiber F. The notation suggests that F may be embedded into E as one of the fibers. This embedding is not canonical in general.*

A **bundle chart** for a fiber bundle $F \hookrightarrow E \xrightarrow{\pi} X$ is a pair (ϕ, U) where $U \subset M$ is open and $\phi : E|_U \to U \times F$ is a map such that the following diagram commutes:

$$\begin{array}{ccc} E|_U & \xrightarrow{\phi} & U \times F \\ \pi \quad \searrow & & \swarrow \\ & U & \end{array}.$$

Such a map ϕ is also called a **local trivialization**. It follows from the definition that there is a cover of E by **bundle charts** meaning that there are a family of local trivializations $\phi_\alpha : E_{U_\alpha} \to U_\alpha \times F$ such that the open sets U_α cover M. Note that $\phi_\alpha = (\pi, \Phi_\alpha)$ for some smooth map $\Phi_\alpha : E_{U_\alpha} \to F$. It follows that so called **overlap maps** $\phi_\alpha \circ \phi_\beta^{-1} : U_\alpha \cap U_\beta \times F \to U_\alpha \cap U_\beta \times F$ must have the form $\phi_\alpha \circ \phi_\beta^{-1}(x, u) = (x, \phi_{\beta\alpha,x}(u))$ for where $\phi_{\beta\alpha,x} \in Diff^r(F)$ defined for each $x \in U_\alpha \cap U_\beta$. To be explicit, the diffeomorphism $\phi_{\beta\alpha,x}$ arises as follows;

$$y \mapsto (x, y) \mapsto \phi_\alpha \circ \phi_\beta|_{E_y}^{-1}(x, y) = (x, \phi_{\alpha\beta,x}(y)) \mapsto \phi_{\alpha\beta,x}(y).$$

The maps $U_\alpha \cap U_\beta \to Diff^r(F)$ given by $x \mapsto \phi_{\alpha\beta,x}$ are called **transition maps**.

Definition E.39 *Let $F \hookrightarrow E \xrightarrow{\pi} M$ be a (locally trivial) fiber bundle. A cover of E by bundle charts $\{\phi_\alpha, U_\alpha\}$ is called a* ***bundle atlas*** *for the bundle.*

Definition E.40 *It may be that there exists a C^r-group G (a Lie group in the smooth case) and a representation ρ of G in $Diff^r(F)$ such that for each nonempty $U_\alpha \cap U_\beta$ we have $\phi_{\beta\alpha,x} = \rho(g_{\alpha\beta}(x))$ for some C^r-map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. In this case we say that G serves as a* ***structure group*** *for the bundle via the representation ρ. In case the representation is a faithful one then we may as well take G to be a subgroup of $Diff^r(F)$ and then we simply have $\phi_{\beta\alpha,x} = g_{\alpha\beta}(x)$. Alternatively, we may speak in terms of group actions so that G acts on F by diffeomorphisms.*

The maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ must satisfy certain consistency relations:

$$\begin{aligned} g_{\alpha\alpha}(x) &= \mathrm{id} \text{ for } x \in U_\alpha \\ g_{\alpha\beta}(x)g_{\beta\alpha}(x) &= \mathrm{id} \text{ for } x \in U_\alpha \cap U_\beta \\ g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x) &= \mathrm{id} \text{ for } x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned} \tag{E.2}$$

A system of maps $g_{\alpha\beta}$ satisfying these relations is called a **cocycle** for the cover $\{U_\alpha\}$.

Definition E.41 *A fiber bundle $\xi := (F \hookrightarrow E \xrightarrow{\pi} X)$ together with a G action on F is called a G-bundle if there exists a bundle atlas $\{(U_\alpha, \phi_\alpha)\}$ for ξ such that the overlap maps have the form $\phi_\alpha \circ \phi_\beta^{-1}(x, y) = (x, g_{\alpha\beta}(x) \cdot y))$ cocycle $\{g_{\alpha\beta}\}$ for the cover $\{U_\alpha\}$.*

Theorem E.3 *Let G have C^r-action on F and suppose we are given cover $\{U_\alpha\}_{\alpha\in A}$ of a C^r-space M and cocycle $\{g_{\alpha\beta}\}$ for the cover. Then there exists a G-bundle with an atlas $\{(U_\alpha, \phi_\alpha)\}$ satisfying $\phi_\alpha \circ \phi_\beta^{-1}(x, y) = (x, g_{\alpha\beta}(x) \cdot y))$ on any nonempty overlaps $U_\alpha \cap U_\beta$.*

Proof. On the union $\Sigma := \bigcup_\alpha \{\alpha\} \times U_\alpha \times F$ define an equivalence relation such that

$$(\alpha, u, v) \in \{\alpha\} \times U_\alpha \times F$$

is equivalent to $(\beta, x, y) \in \{\beta\} \times U_\beta \times F$ if and only if $u = x$ and $v = g_{\alpha\beta}(x) \cdot y$.

The total space of our bundle is then $E := \Sigma/\sim$. The set Σ is essentially the disjoint union of the product spaces $U_\alpha \times F$ and so has an obvious topology. We then give $E := \Sigma/\sim$ the quotient topology. The bundle projection π_E is induced by $(\alpha, u, v) \mapsto u$. Notice that $\pi_E^{-1}(U_\alpha)$ To get our trivializations we define

$$\phi_\alpha(e) := (u, v) \text{ for } e \in \pi_E^{-1}(U_\alpha)$$

where (u, v) is the unique member of $U_\alpha \times F$ such that $(\alpha, u, v) \in e$. The point here is that $(\alpha, u_1, v_1) \sim (\alpha, u_2, v_2)$ only if $(u_1, v_1) = (u_2, v_2)$. Now suppose $U_\alpha \cap U_\beta \neq \emptyset$. Then for $x \in U_\alpha \cap U_\beta$ the element $\phi_\beta^{-1}(x, y)$ is in $\pi_E^{-1}(U_\alpha \cap U_\beta) = \pi_E^{-1}(U_\alpha) \cap \pi_E^{-1}(U_\beta)$ and so $\phi_\beta^{-1}(x, y) = [(\beta, x, y)] = [(\alpha, u, v)]$.

This means that $x = u$ and $v = g_{\alpha\beta}(x) \cdot y$. From this it is not hard to see that

$$\phi_\alpha \circ \phi_\beta^{-1}(x, y) = (x, g_{\alpha\beta}(x) \cdot y)).$$

We leave the question of the regularity of these maps and the C^r structure to the reader. ∎

An important tool in the study of fiber bundles is the notion of a pull-back bundle. We shall see that construction time and time again. Let $\xi = (F \hookrightarrow E \xrightarrow{\pi} M)$ be a -fiber bundle and suppose we have a -map $f : X \to M$. We want to define a fiber bundle $f^*\xi = (F \hookrightarrow f^*E \to X)$. As a set we have

$$f^*E = \{(x, e) \in X \times E : f(x) = \pi(e)\}.$$

The projection $f^*E \to X$ is the obvious one:$(x, e) \mapsto x \in N$.

Exercise E.7 *Exhibit fiber bundle charts for f^*E.*

Let $\{g_{\alpha\beta}\}$ be a cocycle for some cover $\{U_\alpha\}_{\alpha\in A}$ of M which determines a bundle $\xi = (F \hookrightarrow E \xrightarrow{\pi} M)$. If $f : X \to M$ as above, then $\{g_{\alpha\beta} \circ f\} = \{f^* g_{\alpha\beta}\}$ is a cocycle for the same cover and the bundle determined by this cocycle is (up to isomorphism) none other than the pull-back bundle $f^*\xi$.

$$\{g_{\alpha\beta}\} \rightsquigarrow \xi$$
$$\{f^* g_{\alpha\beta}\} \rightsquigarrow f^*\xi$$

The verification of this is an exercise that is easy but constitutes important experience so the reader should not skip the next exercise:

Exercise E.8 *Verify that above claim.*

Exercise E.9 *Show that if $A \subset M$ is a subspace of the base space of a bundle $\xi = (F \hookrightarrow E \xrightarrow{\pi} M)$ and $\iota : A \hookrightarrow M$ then $\iota^{-1}(\xi)$ is naturally isomorphic to the restricted bundle $\xi_A = (F \hookrightarrow E_A \to A)$.*

An important class of fiber bundles often studied on their own is the vector bundles. Roughly, a vector bundle is a fiber bundle with fibers being vector spaces. More precisely, we make the following definition:

Definition E.42 *A real (or complex) (C^r-) **vector bundle** is a (C^r-) fiber bundle (E, π, X) such that*

(i) Each fiber $E_x := \pi^{-1}(x)$ has the structure of a real (resp. complex) vector space.

*(ii) There exists a cover by bundle charts (U_α, ϕ_α) such that each restriction $\phi_\alpha|_{E_x}$ is a real (resp. complex) vector space isomorphism. We call these **vector bundle charts** or VB-charts..*

Equivalently we can define a vector bundle to be a fiber bundle with typical fiber $\mathbb{F}^n$ and such that the transition maps take values in $Gl(n, \mathbb{F})$.

Exercise E.10 *Show that a locally trivial fiber bundle is a vector bundle if and only if the structure representation is a linear representation. E.42.*

As we indicated above, for a smooth (or C^r , $r > 0$) vector bundle we require that all the maps are smooth (or C^r , $r > 0$) and in particular we require that $f_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl(k, \mathbb{F})$ are all smooth.

The tangent bundle of a manifold is a vector bundle. If the tangent bundle of a manifold is trivial we say that M is **parallelizable**.

Exercise E.11 *Show that a manifold is parallelizable if and only if there are $n - \dim M$ everywhere linearly independent vector fields $X_1, ..., X_n$ defined everywhere on M.*

The set of all vector bundles is a category **Vect**. Once again we need to specify the appropriate morphisms in this category and the correct choice should be obvious. A **vector bundle morphism** (or vector bundle map) between $\xi_1 := (E_1, \pi_1, X_1)$ and $\xi_2 := (E_2, \pi_2, X_2)$ is a bundle map $(\bar{f}, f)$:

$$\begin{array}{ccc} E_1 & \stackrel{\bar{f}}{\rightarrow} & E_2 \\ \downarrow & & \downarrow \\ X_1 & \stackrel{f}{\rightarrow} & X_2 \end{array}$$

that is linear on fibers. That is $\bar{f}\big|_{\pi_1^{-1}(x)}$ is a linear map from $\pi_1^{-1}(x)$ into the fiber $\pi_2^{-1}(x)$. We also have the category $\mathbf{Vect}(X)$ consisting of all vector bundles over the fixed space X. Here the morphisms are bundle maps of the form (F, id_X). Two vector bundles $\xi_1 := (E_1, \pi_1, X)$ and $\xi_2 := (E_2, \pi_2, X)$ over the same space X are **isomorphic** (over X) if there is a bundle isomorphism over X from ξ_1 to ξ_2 that is a linear isomorphism when restricted to each fiber. Such a map is called a vector bundle isomorphism.

In the case of vector bundles the transition maps are given by a representation of a Lie group G as a subgroup of $Gl(n, \mathbb{F})$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ depending on whether the vector bundle is real or complex. More precisely, if $\xi = (\mathbb{F}^k \hookrightarrow E \stackrel{\pi}{\rightarrow} M)$ there is a Lie group homomorphism $\rho : G \rightarrow Gl(k, \mathbb{F})$ such that for some VB-atlas $\{U_\alpha, \phi_\alpha\}$ we have the overlap maps

$$\phi_\alpha \circ \phi_\beta^{-1} : U_\alpha \cap U_\beta \times \mathbb{F}^k \rightarrow U_\alpha \cap U_\beta \times \mathbb{F}^k$$

are given by $\phi_\alpha \circ \phi_\beta^{-1}(x, v) = (x, \rho(g_{\alpha\beta}(x))v)$ for a cocycle $\{g_{\alpha\beta}\}$. In a great many cases, the representation is faithful and we may as well assume that $G \subset Gl(k, \mathbb{F})$ and that the representation is the standard one given by matrix multiplication $v \mapsto gv$. On the other hand we cannot restrict ourselves to this case because of our interest in the phenomenon of spin. A simple observation that gives a hint of what we are talking about is that if $G \subset G(n, \mathbb{R})$ acts on $\mathbb{R}^k$ by matrix multiplication and $h : \widetilde{G} \rightarrow G$ is a covering group homomorphism (or any Lie group homomorphism) then $v \mapsto g \cdot v := h(g)v$ is also action. Put another way, if we define $\rho_h : \widetilde{G} \rightarrow G(n, \mathbb{R})$ by $\rho_h(g) = h(g)v$ then ρ_h is representation of $\widetilde{G}$. The reason we care about this seemingly trivial fact only becomes apparent when we try to globalize this type of lifting as well will see when we study spin structures later on.

To summarize this point we may say that whenever we have a VB-atlas $\{U_\alpha, \phi_\alpha\}$ we have the transition functions $\{\phi_{\alpha\beta}\} = \{x \mapsto \phi_{\beta\alpha,x}\}$ which are given straight from the overlap maps $\phi_\alpha \circ \phi_\beta^{-1}(x, u)$ by $\phi_\alpha \circ \phi_\beta^{-1}(x, u) = (x, \phi_{\beta\alpha,x}(u))$. Of course, the transition functions $\{\phi_{\alpha\beta}\}$ certainly form a cocycle for the cover $\{U_\alpha\}$ but there may be cases when we want a (not necessarily faithful) representation $\rho : G \rightarrow Gl(k, \mathbb{F}^n)$ of some group G not necessarily a subgroup of $Gl(k, \mathbb{F}^n)$ together with some $G-$valued cocycle $\{g_{\alpha\beta}\}$ such that $\phi_{\beta\alpha,x}(u) = \rho(g_{\alpha\beta}(x))$. Actually, we may first have to replace $\{U_\alpha, \phi_\alpha\}$ by a "refinement"; a notion we now define:

Appendix F

Overview of Classical Physics

F.0.6 Units of measurement

In classical mechanics we need units for measurements of length, time and mass. These are called elementary units. WE need to add a measure of electrical current to the list if we want to study electromagnetic phenomenon. Other relevant units in mechanics are derived from these alone. For example, speed has units of length$\times$time^{-1}, volume has units of length$\times$length$\times$length kinetic energy has units of mass$\times$length$\times$length$\times$length$\times$time$^{-1}\times$time^{-1} and so on. A common system, called the SI system uses meters (m), kilograms (km) and seconds (sec) for length, mass and time respectively. In this system, the unit of energy kg$\times$m^{2}sec^{-2} is called a joule. The unit of force in this system is Newtons and decomposes into elementary units as kg$\times$m$\times$sec^{-2}.

F.0.7 Newtons equations

The basic assumptions of Newtonian mechanics can be summarized by saying that the set of all mechanical events M taking place in ordinary three dimensional space is such that we can impose on this set of events a coordinate system called an inertial coordinate system. An inertial coordinate system is first of all a 1-1 correspondence between events and the vector space $\mathbb{R} \times \mathbb{R}^3$ consisting of 4-tuples (t, x, y, z). The laws of mechanics are then described by equations and expressions involving the variables (t, x, y, z) written $(t, \mathbf{x})$ where $\mathbf{x} = (x, y, z)$. There will be many correspondences between the event set and $\mathbb{R} \times \mathbb{R}^3$ but not all are inertial. An inertial coordinate system is picked out by the fact that the equations of physics take on a particularly simple form in such coordinates. Intuitively, the x, y, z variables locate an event in space while t specifies the time of an event. Also, x, y, z should be visualized as determined by measuring against a mutually perpendicular set of three axes and t is measured with respect to some sort of clock with $t = 0$ being chosen arbitrarily according to the demands of the experimental situation. Now we expect that the laws of physics should not prefer any particular such choice of mutually perpendicular axes or choice of starting time. Also, the units of measurement of length and time are conventionally determined by human beings and so the equations of the laws of physics should in some way not depend on this choice in any significant way. Careful consideration along these lines leads to a particular set of "coordinate changes" or transformations which translate among the different inertial coordinate systems. The group of transformations which is chosen for classical (non-relativistic) mechanics is the so called Galilean group Gal.

Definition F.1 *A map $g : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^3$ is called a Galilean transformation if and only if it can be decomposed as a composition of transformations of the following type:*

1. Translation of the origin:
$$(t, \mathbf{x}) \mapsto (t + t_0, \mathbf{x} + \mathbf{x}_0)$$

2. Uniform motion with velocity $\mathbf{v}$:
$$(t, \mathbf{x}) \mapsto (t, \mathbf{x} + t\mathbf{v})$$

3. Rotation of the spatial axes:
$$(t, \mathbf{x}) \mapsto (t, R\mathbf{x})$$
where $R \in O(3)$.

If $(t, \mathbf{x})$ are inertial coordinates then so will $(T, \mathbf{X})$ be inertial coordinates if and only if $(T, \mathbf{X}) = g(t, \mathbf{x})$ for some Galilean transformation. We will take this as given.

The motion of a idealized point mass moving in space is described in an inertial frame $(t, \mathbf{x})$ as a curve $t \mapsto c(t) \in \mathbb{R}^3$ with the corresponding curve $t \mapsto (t, c(t))$ in the (coordinatized) event space $\mathbb{R} \times \mathbb{R}^3$. We often write $\mathbf{x}(t)$ instead of $c(t)$. If we have a system of n particles then we may formally treat this as a single particle moving in an $3n-$dimensional space and so we have a single curve in $\mathbb{R}^{3n}$. Essentially we are concatenating the spatial part of inertial coordinates $\mathbb{R}^{3n} = \mathbb{R}^3 \times \cdots \mathbb{R}^3$ taking each factor as describing a single particle in the system so we take $\mathbf{x} = (x_1, y_1, z_1, ..., x_n, y_n, z_n)$. Thus our new inertial coordinates may be thought of as $\mathbb{R} \times \mathbb{R}^{3n}$. If we have a system of particles it will be convenient to define the momentum vector $\mathbf{p} = (m_1 x_1, m_1 y_1, m_1 z_1, ..., m_n x_n, m_n y_n, m_n z_n) \in \mathbb{R}^{3n}$. In such coordinates, Newton's law for n particles of masses $m_1, ..., m_n$ reads
$$\frac{d^2\mathbf{p}}{dt^2} = \mathbf{F}(\mathbf{x}(t), t)$$
where $t \mapsto \mathbf{x}(t)$ describes the motion of a system of n particles in space as a smooth path in $\mathbb{R}^{3n}$ parameterized by t representing time. The equation has

units of force (Newtons in the SI system). If all bodies involved are taken into account then the force $\mathbf{F}$ cannot depend explicitly on time as can be deduced by the assumption that the form taken by $\mathbf{F}$ must be the same in any inertial coordinate system. We may not always be able to include explicitly all involved bodies and so it may be that our mathematical model will involve a changing force $\mathbf{F}$ exerted on the system from without as it were. As an example consider the effect of the tidal forces on sensitive objects on earth. Also, the example of earths gravity shows that if the earth is not taken into account as one of the particles in the system then the form of $\mathbf{F}$ will not be invariant under all spatial rotations of coordinate axes since now there is a preferred direction (up-down).

F.0.8 Classical particle motion in a conservative field

There are special systems for making measurements that can only be identified in actual practice by interaction with the physical environment. In classical mechanics, a point mass will move in a straight line unless a force is being applied to it. The coordinates in which the mathematical equations describing motion are the simplest are called inertial coordinates (x, y, z, t). If we consider a *single* particle of mass m then Newton's law simplifies to

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}(\mathbf{x}(t), t)$$

The force $\mathbf{F}$ is **conservative** if it doesn't depend on time and there is a potential function $V : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{F}(\mathbf{x}) = -\operatorname{grad} V(\mathbf{x})$. Assume this is the case. Then Newton's law becomes

$$m\frac{d^2}{dt^2}\mathbf{x}(t) + \operatorname{grad} V(\mathbf{x}(t)) = 0.$$

Newton's equations are often written

$$\mathbf{F}(\mathbf{x}(t)) = m\mathbf{a}(t)$$

$\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is the force function and we have taken it to not depend explicitly on time t. The force will be conservative so $\mathbf{F}(\mathbf{x}) = -\operatorname{grad} V(\mathbf{x})$ for some scalar function $V(\mathbf{x})$. The total energy or Hamiltonian function is a function of two vector variables $\mathbf{x}$ and $\mathbf{v}$ given (in this simple situation) by

$$H(\mathbf{x}, \mathbf{v}) = \frac{1}{2}m\left\|\mathbf{v}\right\|^2 + V(\mathbf{x})$$

so that if we plug in $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{v} = \mathbf{x}'(t)$ for the motion of a particle then we get the energy of the particle. Since this is a conservative situation $\mathbf{F}(\mathbf{x}) = -\operatorname{grad} V(\mathbf{x})$ we discover by differentiating and using equation **??** that $\frac{d}{dt}H(\mathbf{x}(t), \mathbf{x}'(t)) = 0$. This says that the total energy is conserved along any path which is a solution to equation **??** as long as $\mathbf{F}(\mathbf{x}) = -\operatorname{grad} V(\mathbf{x})$.

There is a lot of structure that can be discovered by translating the equations of motion into an arbitrary coordinate system (q^1, q^2, q^3) and then extending that to a coordinate system $(q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \dot{q}^3)$ for velocity space $\mathbb{R}^3 \times \mathbb{R}^3$. Here, $\dot{q}^1, \dot{q}^2, \dot{q}^3$ are not derivatives until we compose with a curve $\mathbb{R} \to \mathbb{R}^3$ to get functions of t. Then (and only then) we will take $\dot{q}^1(t), \dot{q}^2(t), \dot{q}^3(t)$ to be the derivatives. Sometimes $(\dot{q}^1(t), \dot{q}^2(t), \dot{q}^3(t))$ is called the **generalized velocity** vector. Its physical meaning depends on the particular form of the generalized coordinates.

In such a coordinate system we have a function $L(\mathbf{q}, \dot{\mathbf{q}})$ called the Lagrangian of the system. Now there is a variational principle that states that if $\mathbf{q}(t)$ is a path which solve the equations of motion and defined from time t_1 to time t_2 then out of all the paths which connect the same points in space at the same times t_1 and t_2 , the one that makes the following action the smallest will be the solution:

$$S(\mathbf{q}(t)) = \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}})dt$$

Now this means that if we add a small variation to $\mathbf{q}$ get another path $\mathbf{q} + \delta\mathbf{q}$ then we calculate formally:

$$\begin{aligned}
\delta S(\mathbf{q}(t)) &= \delta \int_{t_1}^{t_2} L(\mathbf{q}, \dot{\mathbf{q}})dt \\
&\int_{t_1}^{t_2} \left[\delta\mathbf{q}\cdot\frac{\partial}{\partial \mathbf{q}}L(\mathbf{q}, \dot{\mathbf{q}})+\delta\dot{\mathbf{q}}\cdot\frac{\partial}{\partial \dot{\mathbf{q}}}L(\mathbf{q}, \dot{\mathbf{q}})\right] dt \\
&= \int_{t_1}^{t_2} \delta\mathbf{q}\cdot \left(\frac{\partial}{\partial \mathbf{q}}L(\mathbf{q}, \dot{\mathbf{q}}) - \frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{q}}}L(\mathbf{q}, \dot{\mathbf{q}})\right) dt + \left[\delta\mathbf{q}\cdot\frac{\partial}{\partial \dot{\mathbf{q}}}L(\mathbf{q}, \dot{\mathbf{q}})\right]_{t_1}^{t_2}
\end{aligned}$$

If our variation is among those that start and end at the same space-time locations then $\delta\mathbf{q} = \mathbf{0}$ is the end points so the last term vanishes. Now if the path $\mathbf{q}(t)$ is stationary for such variations then $\delta S(\mathbf{q}(t)) = 0$ so

$$\int_{t_1}^{t_2} \delta\mathbf{q}\cdot \left(\frac{\partial}{\partial \mathbf{q}}L(\mathbf{q}, \dot{\mathbf{q}}) - \frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{q}}}L(\mathbf{q}, \dot{\mathbf{q}})\right) dt = 0$$

and since this is true for all such paths we conclude that

$$\frac{\partial}{\partial \mathbf{q}}L(\mathbf{q}, \dot{\mathbf{q}}) - \frac{d}{dt}\frac{\partial}{\partial \dot{\mathbf{q}}}L(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{0}$$

or in indexed scalar form

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0 \text{ for } 1 \leq i \leq 3$$

on a stationary path. This is (these are) the **Euler-Lagrange equation**(s). If $\mathbf{q}$ were just rectangular coordinates and if L were $\frac{1}{2}m\left\|\mathbf{v}\right\|^2 - V(\mathbf{x})$ this turns out to be Newton's equation. Notice, the minus sign in front of the V.

Definition F.2 *For a Lagrangian L we can associate the quantity $E = \sum \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L(\mathbf{q}, \dot{\mathbf{q}})$.*

Let us differentiate E. We get

$$\begin{aligned}
\frac{d}{dt}E &= \frac{d}{dt}\sum \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L(\mathbf{q},\dot{\mathbf{q}}) \\
&= \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}\dot{q}^i - \dot{q}^i\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{d}{dt}L(\mathbf{q},\dot{\mathbf{q}}) \\
&= \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}\dot{q}^i - \dot{q}^i\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\dot{q}^i - \frac{\partial L}{\partial \dot{q}^i}\frac{d}{dt}\dot{q}^i \\
&= 0 \ \text{ by the Euler Lagrange equations.}
\end{aligned} \tag{F.1}$$

Conclusion F.1 *If L does not depend explicitly on time; $\frac{\partial L}{\partial t} = 0$, then the* ***energy*** E ***is conserved ;*** $\frac{dE}{dt} = 0$ *along any solution of the Euler-Lagrange equations..*

But what about spatial symmetries? Suppose that $\frac{\partial}{\partial q^i}L = 0$ for one of the coordinates q^i. Then if we define $p_i = \frac{\partial L}{\partial \dot{q}^i}$ we have

$$\frac{d}{dt}p_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = -\frac{\partial}{\partial q^i}L = 0$$

so p_i is constant along the trajectories of Euler's equations of motion. The quantity $p_i = \frac{\partial L}{\partial \dot{q}^i}$ is called a generalized momentum and we have reached the following

Conclusion F.2 *If $\frac{\partial}{\partial q^i}L = 0$ then p_i is a conserved quantity. This also applies if $\frac{\partial}{\partial \mathbf{q}}L = (\frac{\partial L}{\partial q^1}, ..., \frac{\partial L}{\partial q^n}) = 0$ with the conclusion that the vector $\mathbf{p} = \frac{\partial}{\partial \mathbf{q}}L = (\frac{\partial L}{\partial \dot{q}^1}, ..., \frac{\partial L}{\partial \dot{q}^n})$ is conserved (each component separately).*

Now let us apply this to the case a free particle. The Lagrangian in rectangular inertial coordinates are

$$L(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}m\,|\dot{\mathbf{x}}|^2$$

and this Lagrangian is symmetric with respect to translations $\mathbf{x} \mapsto \mathbf{x} + \mathbf{c}$

$$L(\mathbf{x}+\mathbf{c},\dot{\mathbf{x}}) = L(\mathbf{x},\dot{\mathbf{x}})$$

and so the generalized momentum vector for this is $\mathbf{p} = m\dot{\mathbf{x}}$ each component of which is conserved. This last quantity is actually the usual momentum vector.

Now let us examine the case where the Lagrangian is invariant with respect to rotations about some fixed point which we will take to be the origin of an inertial coordinate system. For instance suppose the potential function $V(\mathbf{x})$ is invariant in the sense that $V(\mathbf{x}) = V(O\mathbf{x})$ for any orthogonal matrix O. The we can take an antisymmetric matrix A and form the family of orthogonal matrices e^{sA}. The for the Lagrangian

$$L(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}m\,|\dot{\mathbf{x}}|^2 - V(\mathbf{x})$$

we have

$$\begin{aligned}\frac{d}{ds}L(e^{sA}\mathbf{x},e^{sA}\dot{\mathbf{x}}) &= \frac{d}{dt}(\frac{1}{2}m\left|e^{sA}\dot{\mathbf{x}}\right|^2 - V(e^{sA}\mathbf{x}))\\ &= \frac{d}{dt}(\frac{1}{2}m\left|\dot{\mathbf{x}}\right|^2 - V(\mathbf{x})) = 0\end{aligned}$$

On the other hand, recall the result of a variation $\delta\mathbf{q}$

$$\int_{t_1}^{t_2}\delta\mathbf{q}\cdot\left(\frac{\partial}{\partial\mathbf{q}}L(\mathbf{q},\dot{\mathbf{q}}) - \frac{d}{dt}\frac{\partial}{\partial\dot{\mathbf{q}}}L(\mathbf{q},\dot{\mathbf{q}})\right)dt + \left[\delta\mathbf{q}\cdot\frac{\partial}{\partial\dot{\mathbf{q}}}L(\mathbf{q},\dot{\mathbf{q}})\right]_{t_1}^{t_2}$$

what we have done is to let $\delta\mathbf{q} = A\mathbf{q}$ since to first order we have $e^{sA}\mathbf{q} = I + sA\mathbf{q}$. But if $\mathbf{q}(t)$ satisfies Euler's equation then the integral above is zero and yet the whole variation is zero too. We are led to conclude that

$$\left[\delta\mathbf{q}\cdot\frac{\partial}{\partial\dot{\mathbf{q}}}L(\mathbf{q},\dot{\mathbf{q}})\right]_{t_1}^{t_2} = 0$$

which in the present case is

$$\begin{aligned}\left[A\mathbf{x}\cdot\frac{\partial}{\partial\dot{\mathbf{x}}}(\frac{1}{2}m\left|\dot{\mathbf{x}}\right|^2 - V(\mathbf{x}))\right]_{t_1}^{t_2} &= 0\\ [mA\mathbf{x}\cdot\dot{\mathbf{x}}]_{t_1}^{t_2} &= 0\end{aligned}$$

for all t_2 and t_1. Thus the quantity $mA\mathbf{x}\cdot\dot{\mathbf{x}}$ is conserved. Let us apply this with A equal to the following in turn

$$A = \begin{bmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

then we get $mA\mathbf{x}\cdot\dot{\mathbf{x}} = m(-x^2, x^1, 0)\cdot(\dot{x}^1, \dot{x}^2, \dot{x}^3) = m(x^1\dot{x}^2 - \dot{x}^1x^2)$ which is the same as $m\dot{\mathbf{x}}\times\mathbf{k} = \mathbf{p}\times\mathbf{k}$ which is called the angular momentum about the $\mathbf{k}$ axis ($\mathbf{k} = (0, 0, 1)$ so this is the z-axis) and is a conserved quantity. To see the point here notice that

$$e^{tA} = \begin{bmatrix} \cos t & -\sin t & 0\\ \sin t & \cos t & 0\\ 0 & 0 & 1\end{bmatrix}$$

is the rotation about the z-axis. We can do the same thing for the other two coordinate axes and in fact it turns out that for any unit vector $\mathbf{u}$ the angular momentum about that axis defined by $\mathbf{p}\times\mathbf{u}$ is conserved.

Remark F.1 *We started with the assumption that L was invariant under all rotations O but if it had only been invariant under counterclockwise rotations about an axis given by a unit vector* $\mathbf{u}$ *then we could still conclude that at least* $\mathbf{p}\times\mathbf{u}$ *is conserved.*

Remark F.2 *Let begin to use the index notation (like q^i, p_i and x^i etc.) a little more since it will make the transition to fields more natural.*

Now we define the Hamiltonian function derived from a given Lagrangian via the formulas

$$H(\mathbf{q}, \mathbf{p}) = \sum p_i \dot{q}^i - L(\mathbf{q}, \dot{\mathbf{q}})$$
$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

where we think of $\dot{\mathbf{q}}$ as depending on $\mathbf{q}$ and $\mathbf{p}$ via the inversion of $p_i = \frac{\partial L}{\partial \dot{q}^i}$. Now it turns out that if $\mathbf{q}(t), \dot{\mathbf{q}}(t)$ satisfy the Euler Lagrange equations for L then $\mathbf{q}(t)$ and $\mathbf{p}(t)$ satisfy the Hamiltonian equations of motion

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}$$
$$\frac{dp^i}{dt} = -\frac{\partial H}{\partial q^i}$$

One of the beauties of this formulation is that if $Q^i = Q^i(q^j)$ are any other coordinates on $\mathbb{R}^3$ and we define $P^i = p^j \frac{\partial Q^i}{\partial q^j}$ then taking $H(..q^i.,..p^i..) = \widetilde{H}(..Q^i..,..P_i..)$ the equations of motion have the same form in the new coordinates. More generally, if Q, P are related to q, p in such a way that the Jacobian matrix J of the coordinate change (on $\mathbb{R}^3 \times \mathbb{R}^3$) is **symplectic**

$$J^t \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

then the equations **??** will hold in the new coordinates. These kind of coordinate changes on the q, p space $\mathbb{R}^3 \times \mathbb{R}^3$ (momentum space) are called canonical transformations. Mechanics is, in the above sense, invariant under canonical transformations.

Next, take any smooth function $f(q, p)$ on momentum space (also called phase space). Such a function is called an observable. Then along any solution curve $(q(t), p(t))$ to Hamilton's equations we get

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial q}\frac{dq}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} \\ &= \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p^i} + \frac{\partial f}{\partial p^i}\frac{\partial H}{\partial q^i} \\ &= [f, H] \end{aligned}$$

where we have introduced the Poisson bracket $[f, H]$ defined by the last equality above. So we also have the equations of motion in the form $\frac{df}{dt} = [f, H]$ for any function f not just the coordinate functions q and p. Later, we shall study a geometry hiding here; **Symplectic geometry**.

Remark F.3 *For any coordinate* $t, \mathbf{x}$ *we will often consider the curve* $(\mathbf{x}(t), \mathbf{x}'(t)) \in \mathbb{R}^{3n} \times \mathbb{R}^{3n}$ *the latter product space being a simple example of a* ***velocity phase space****.*

F.0.9 Some simple mechanical systems

1. As every student of basic physics know the equations of motion for a particle falling freely through a region of space near the earths surface where the force of gravity is (nearly) constant is $\mathbf{x}''(t) = -g\mathbf{k}$ where $\mathbf{k}$ is the usual vertical unit vector corresponding to a vertical z-axis. Integrating twice gives the form of any solution $\mathbf{x}(t) = -\frac{1}{2}gt^2\mathbf{k}+t\mathbf{v}_0 + \mathbf{x}_0$ for constant vectors $\mathbf{x}_0, \mathbf{v}_0 \in \mathbb{R}^3$. We get different motions depending on the initial conditions $(\mathbf{x}_0, \mathbf{v}_0)$. If the initial conditions are right, for example if $\mathbf{v}_0 = 0$ then this is reduced to the one dimensional equation $x''(t) = -g$. The path of a solution with initial conditions (x_0, v_0) is given in phase space as
$$t \mapsto (-\frac{1}{2}gt^2 + tv_0 + x_0, -gt + v_0)$$
and we have shown the phase trajectories for a few initial conditions.

2. A somewhat general 1-dimensional system is given by a Lagrangian of the form
$$L = \frac{1}{2}a(q)\dot{q}^2 - V(q) \tag{F.2}$$
and example of which is the motion of a particle of mass m along a 1-dimensional continuum and subject to a potential $V(x)$. Then the Lagrangian is $L = \frac{1}{2}m\dot{x}^2 - V(x)$. Instead of writing down the Euler-Lagrange equations we can use the fact that $E = \frac{\partial L}{\partial \dot{x}^i}\dot{x}^i - L(x, \dot{x}) = m\dot{x}^2 - (\frac{1}{2}m\dot{x}^2 - V(x)) = \frac{1}{2}m\dot{x}^2 + V(x)$ is conserved. This is the total energy which is traditionally divided into kinetic energy $\frac{1}{2}m\dot{x}^2$ and potential energy $V(x)$. We have $E = \frac{1}{2}m\dot{x}^2 + V(x)$ for some constant. Then
$$\frac{dx}{dt} = \sqrt{\frac{2E - 2V(x)}{m}}$$
and so
$$t = \sqrt{m/2}\int \frac{1}{\sqrt{E - V(x)}} + c.$$
Notice that we must always have $E - V(x) \geq 0$. This means that if $V(x)$ has a single local minimum between some points $x = a$ and $x = b$ where $E - V = 0$, then the particle must stay between $x = a$ and $x = b$ moving back and forth with some time period. What is the time period?.

3. **Central Field.** A central field is typically given by a potential of the form $V(\mathbf{x}) = -\frac{k}{|\mathbf{x}|}$. Thus the Lagrangian of a particle of mass m in this central field is
$$\frac{1}{2}m\,|\dot{\mathbf{x}}|^2 + \frac{k}{|\mathbf{x}|}$$

where we have centered inertial coordinates at the point where the potential has a singularity $\lim_{\mathbf{x}\to 0} V(\mathbf{x}) = \pm\infty$. In cylindrical coordinates (r, θ, z) the Lagrangian becomes

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + \frac{k}{(r^2+z^2)^{1/2}}.$$

We are taking $q^1 = r$, $q^2 = \theta$ and $q^3 = \theta$. But if initially $z = \dot{z} = 0$ then by conservation of angular momentum discussed above the particle stays in the $z = 0$ plane. Thus we are reduced to a study of the two dimensional case:

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}.$$

What are Lagrange's equations? Answer:

$$\begin{aligned} 0 &= \frac{\partial L}{\partial q^1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^1} \\ &= mr\dot{\theta}^2 - \frac{k}{r^2} - m\dot{r}\ddot{r} \end{aligned}$$

and

$$\begin{aligned} 0 &= \frac{\partial L}{\partial q^2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^2} \\ &= -mr^2\dot{\theta}\ddot{\theta}. \end{aligned}$$

The last equation reaffirms that $\dot{\theta} = \omega_0$ is constant. Then the first equation becomes $mr\omega_0^2 - \frac{k}{r^2} - m\dot{r}\ddot{r} = 0$. On the other hand conservation of energy becomes

4.

$$\frac{1}{2}m(\dot{r}^2 + r^2\omega_0^2) + \frac{k}{r} = E_0 = \frac{1}{2}m(\dot{r}_0^2 + r_0^2\omega_0^2) + \frac{k}{r_0} \qquad \text{or}$$
$$\dot{r}^2 + r^2\omega_0^2 + \frac{2k}{mr} = \frac{2E_0}{m}$$

5. A simple oscillating system is given by $\frac{d^2x}{dt^2} = -x$ which has solutions of the form $x(t) = C_1 \cos t + C_2 \sin t$. This is equivalent to the system

$$\begin{aligned} x' &= v \\ v' &= -x \end{aligned}$$

6. Consider a single particle of mass m which for some reason is viewed with respect to rotating frame and an inertial frame (taken to be stationary). The rotating frame $(\mathbf{E}_1(t), \mathbf{E}_2(t), \mathbf{E}_3(t)) = \mathtt{E}$ (centered at the origin of R^3) is related to stationary frame $(e_{1,}e_2, e_3) = \mathtt{e}$ by an orthogonal matrix O :

$$\mathtt{E}(t) = \mathrm{O}(t)\mathtt{e}$$

and the rectangular coordinates relative to these frames are related by

$$\mathbf{x}(t) = \mathrm{O}(t)\mathbf{X}(t)$$

We then have

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathrm{O}(t)\dot{\mathbf{X}} + \dot{\mathrm{O}}(t)\mathbf{X} \\ &= \mathrm{O}(t)(\dot{\mathbf{X}} + \Omega(t)\mathbf{X})\end{aligned}$$

where $\Omega(t) = \mathrm{O}^t(t)\dot{\mathrm{O}}(t)$ is an angular velocity. The reason we have chosen to work with $\Omega(t)$ rather than directly with $\dot{\mathrm{O}}(t)$ will become clearer later in the book. Let us define the operator D_t by $D_t\mathbf{X} = \dot{\mathbf{X}} + \Omega(t)\mathbf{X}$. This is sometimes called the "total derivative". At any rate the equations of motion in the inertial frame is of the form $m\frac{d\mathbf{x}}{dt} = \mathbf{f}(\dot{\mathbf{x}}, \mathbf{x})$. In the moving frame this becomes an equation of the form

$$m\frac{d}{dt}(\mathrm{O}(t)(\dot{\mathbf{X}} + \Omega(t)\mathbf{X})) = \mathbf{f}(\mathrm{O}(t)\mathbf{X}, \mathrm{O}(t)(\dot{\mathbf{X}} + \Omega(t)\mathbf{X}))$$

and in turn

$$\mathrm{O}(t)\frac{d}{dt}(\dot{\mathbf{X}}+\Omega(t)\mathbf{X})+\dot{\mathrm{O}}(t)(\dot{\mathbf{X}}+\Omega(t)\mathbf{X}) = m^{-1}\mathbf{f}(\mathrm{O}(t)\mathbf{X}, \mathrm{O}(t)(\dot{\mathbf{X}}+\Omega(t)\mathbf{X})).$$

Now recall the definition of D_t we get

$$\mathrm{O}(t)(\frac{d}{dt}D_t\mathbf{X} + \Omega(t)D_t\mathbf{X}) = m^{-1}\mathbf{f}(\mathrm{O}(t)\mathbf{X}, \mathrm{O}(t)\mathbf{V})$$

and finally

$$mD_t^2\mathbf{X} = \mathbf{F}(\mathbf{X}, \mathbf{V}) \tag{F.3}$$

where we have defined the **relative velocity** $\mathbf{V} = \dot{\mathbf{X}}+\Omega(t)\mathbf{X}$ and $\mathbf{F}(\mathbf{X}, \mathbf{V})$ is by definition the transformed force $\mathbf{f}(\mathrm{O}(t)\mathbf{X}, \mathrm{O}(t)\mathbf{V})$. The equation we have derived would look the same in any moving frame: It is a covariant expression.

5. Rigid Body We will use this example to demonstrate how to work with the rotation group and it's Lie algebra. The advantage of this approach is that it generalizes to motions in other Lie groups and their algebra's. Let us denote the group of orthogonal matrices of determinant one by SO(3). This is the rotation group. If the Lagrangian of a particle as in the last example is invariant under actions of the orthogonal group so that $L(\mathbf{x}, \dot{\mathbf{x}}) = L(Qx, Q\dot{x})$ for $Q \in \mathrm{SO}(3)$ then the quantity $\ell = \mathbf{x} \times m\dot{\mathbf{x}}$ is constant for the motion of the particle $\mathbf{x} = \mathbf{x}(t)$ satisfying the equations of motion in the inertial frame. The matrix group SO(3) is an example of a Lie group which we study intensively in later chapters. Associated with every Lie group is its Lie algebra which in this case is the set of all anti-symmetric 3x3 matrices denoted $\mathfrak{so}(3)$. There is an interesting correspondence between and $\mathbb{R}^3$ given by

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \leftrightarrows (\omega_1, \omega_2, \omega_3) = \omega$$

Furthermore if we define the bracket for matrices A and B in $\mathfrak{so}(3)$ by $[A, B] = AB - BA$ then under the above correspondence $[A, B]$ corresponds to the cross product. Let us make the temporary convention that if x is an element of $\mathbb{R}^3$ then the corresponding matrix in $\mathfrak{so}(3)$ will be denoted by using the same letter but a new font while lower case refers to the inertial frame and upper to the moving frame:

$$\mathbf{x} \leftrightarrows \mathsf{x} \in \mathfrak{so}(3) \text{ and}$$
$$\mathbf{X} \leftrightarrows \mathsf{X} \ \in \mathfrak{so}(3) \text{ etc.}$$

	$\mathbb{R}^3$		$\mathfrak{so}(3)$
Inertial frame	$\mathbf{x}$	$\leftrightarrows$	x
Moving frame	$\mathbf{X}$	$\leftrightarrows$	X

Then we have the following chart showing how various operations match up:

$$\begin{array}{ccc}
\mathbf{x} = \mathrm{O}\mathbf{X} & \leftrightarrows & \mathsf{x} = \mathsf{O}\mathsf{X}\mathsf{O}^t \\
\mathbf{v}_1 \times \mathbf{v}_2 & \leftrightarrows & [\mathsf{v}_1, \mathsf{v}_2] \\
\mathbf{v} = \dot{\mathbf{x}} & \leftrightarrows & \mathsf{v}{=}\dot{\mathsf{x}} \\
\mathbf{V} = D_t\mathbf{X} = \dot{\mathbf{X}} + \Omega(t)\mathbf{X} & \leftrightarrows & \mathsf{V}{=}D_t\mathsf{X} = \dot{\mathsf{X}}{+}[\Omega(t), \mathsf{X}] \\
\ell = \mathbf{x} \times m\dot{\mathbf{x}} & \leftrightarrows & \mathsf{l} = [\mathsf{x}, m\dot{\mathsf{x}}] \\
\ell = \mathrm{O}\mathbf{L} & \leftrightarrows & \mathsf{l} = \mathsf{O}\mathsf{L}\mathsf{O}^t = [\mathsf{V}, \Omega(t)] \\
D_t\mathbf{L} = \dot{\mathbf{L}}{+}\Omega(t) \times \mathbf{L} & \leftrightarrows & D_t\mathsf{L} = \dot{\mathsf{L}} + [\Omega(t), \mathsf{L}]
\end{array}$$

and so on. Some of the quantities are actually defined by their position in this chart. In any case, let us differentiate $\mathbf{l} = \mathbf{x} \times m\dot{\mathbf{x}}$ and use the equations of motion to get

$$\begin{aligned}\frac{d\mathbf{l}}{dt} &= \mathbf{x} \times m\dot{\mathbf{x}} \\ &= \mathbf{x} \times m\ddot{\mathbf{x}} + \mathbf{0} \\ &= \mathbf{x} \times \mathbf{f}.\end{aligned}$$

But we have seen that if the Lagrangian (and hence the force $\mathbf{f}$) is invariant under rotations that $\frac{d\mathbf{l}}{dt} = 0$ along any solution curve. Let us examine this case. We have $\frac{d\mathbf{l}}{dt} = 0$ and in the moving frame $D_t\mathbf{L} = \dot{\mathbf{L}} + \Omega(t)\mathbf{L}$. Transferring the equations over to our $\mathfrak{so}(3)$ representation we have $D_t\mathsf{L} = \dot{\mathsf{L}} + [\Omega(t), \mathsf{L}] = 0$. Now if our particle is rigidly attached to the rotating frame, that is, if $\dot{\mathbf{x}} = 0$ then $\dot{\mathsf{X}} = 0$ and $\mathsf{V} = [\Omega(t), \mathsf{X}]$ so

$$\mathsf{L} = m[\mathsf{X}, [\Omega(t), \mathsf{X}]].$$

In Lie algebra theory the map $\mathsf{v} \mapsto [\mathsf{x}, \mathsf{v}] = -[\mathsf{v}, \mathsf{x}]$ is denoted $\mathrm{ad}(\mathsf{x})$ and is linear. With this notation the above becomes

$$\mathsf{L} = -m\, \mathrm{ad}(\mathsf{X})\Omega(t).$$

The map $I : \mathsf{X} \mapsto -m\,\mathrm{ad}(\mathsf{X})\Omega(t) = I(\mathsf{X})$ is called the momentum operator. Suppose now that we have k particles of masses $m_1, m_2, ...m_2$ each at rigidly attached to the rotating frame and each giving quantities $\mathsf{x}_i, \mathsf{X}_i$ etc. Then to total angular momentum is $\sum I(\mathsf{X}_i)$. Now if we have a continuum of mass with mass density ρ in a moving region B_t (a rigid body) then letting $\mathsf{X}_{\mathbf{u}}(t)$ denote path in $\mathfrak{so}(3)$ of the point of initially at $\mathbf{u} \in B_0 \in \mathbb{R}^3$ then we can integrate to get the total angular momentum at time t;

$$\mathsf{L}_{tot}(t) = -\int_B \mathrm{ad}(\mathsf{X}_{\mathbf{u}}(t))\Omega(t) d\rho(\mathbf{u})$$

which is a conserved quantity.

F.0.10 The Basic Ideas of Relativity

F.0.11 Variational Analysis of Classical Field Theory

In field theory we study functions $\phi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^k$. We use variables $\phi(x^0, x^1, x^2, x^3) = \phi(t, x, y, z)$ A Lagrangian density is a function $\mathcal{L}(\phi, \partial\phi)$ and then the Lagrangian would be

$$L(\phi, \partial\phi) = \int_{V\subset\mathbb{R}^3} \mathcal{L}(\phi, \partial\phi) d^3x$$

and the action is

$$S = \int \int_{V\subset\mathbb{R}^3} \mathcal{L}(\phi, \partial\phi) d^3x dt = \int_{V\times I\subset\mathbb{R}^4} \mathcal{L}(\phi, \partial\phi) d^4x$$

What has happened is that the index i is replaced by the space variable $\vec{x} = (x^1, x^2, x^3)$ and we have the following translation

$$
\begin{array}{lll}
i & \rightarrowtail\rightarrowtail\rightarrowtail & \vec{x} \\
q & \rightarrowtail\rightarrowtail\rightarrowtail & \phi \\
q^i & \rightarrowtail\rightarrowtail\rightarrowtail & \phi(.,\vec{x}) \\
q^i(t) & \rightarrowtail\rightarrowtail\rightarrowtail & \phi(t,\vec{x}) = \phi(x) \\
p^i(t) & \rightarrowtail\rightarrowtail\rightarrowtail & \partial_t\phi(t,\vec{x}) + \nabla_{\vec{x}}\phi(t,\vec{x}) = \partial\phi(x) \\
L(q,p) & \rightarrowtail\rightarrowtail\rightarrowtail & \int_{V\subset\mathbb{R}^3} \mathcal{L}(\phi,\partial\phi)d^3x \\
S = \int L(\mathbf{q},\dot{\mathbf{q}})dt & \rightarrowtail\rightarrowtail\rightarrowtail & S = \int\int \mathcal{L}(\phi,\partial\phi)d^3x dt
\end{array}
$$

where $\partial\phi = (\partial_0\phi, \partial_1\phi, \partial_2\phi, \partial_3\phi)$. So in a way, the mechanics of classical massive particles is classical field theory on the space with three points which is the set $\{1, 2, 3\}$. Or we can view field theory as infinitely many particle systems indexed by points of space. In other words, a system with an infinite number of degrees of freedom.

Actually, we have only set up the formalism of scalar fields and have not, for instance, set things up to cover internal degrees of freedom like spin. However, we will discuss spin later in this text. Let us look at the formal variational calculus of field theory. We let $\delta\phi$ be a variation which we might later assume to vanish on the boundary of some region in space-time $U = I \times V \subset \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$. In general, we have

$$
\begin{aligned}
\delta S &= \int_U \left(\delta\phi \frac{\partial\mathcal{L}}{\partial\phi} + \partial_\mu\delta\phi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) d^4x \\
&= \int_U \partial_\mu \left(\delta\phi \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) d^4x + \int_U \delta\phi \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) d^4x
\end{aligned}
$$

Now the first term would vanish by the divergence theorem if $\delta\phi$ vanished on the boundary ∂U. If ϕ were a field that were stationary under such variations then

$$
\delta S = \int_U \delta\phi \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right) d^4x = 0
$$

for all $\delta\phi$ vanishing on ∂U so we can conclude that Lagrange's equation holds for ϕ stationary in this sense and vice versa:

$$
\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0
$$

These are the field equations.

F.0.12 Symmetry and Noether's theorem for field theory

Now an interesting thing happens if the Lagrangian density is invariant under some set of transformations. Suppose that $\delta\phi$ is an infinitesimal "internal" symmetry of the Lagrangian density so that $\delta S(\delta\phi) = 0$ even though $\delta\phi$ does not vanish on the boundary. Then if ϕ is already a solution of the field equations then

$$0 = \delta S = \int_U \partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) d^4x$$

for all regions U. This means that $\partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$ so if we define $j^\mu = \delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$ we get

$$\partial_\mu j^\mu = 0$$

or

$$\frac{\partial}{\partial t} j^0 = -\nabla \cdot \overrightarrow{\mathbf{j}}$$

where $\overrightarrow{\mathbf{j}} = (j^1, j^2, j^3)$ and $\nabla \cdot \overrightarrow{\mathbf{j}} = \operatorname{div}(\overrightarrow{\mathbf{j}})$ is the spatial divergence. This looks like some sort of conservation.. Indeed, if we define the total charge at any time t by

$$Q(t) = \int j^0 d^3x$$

the assuming $\overrightarrow{\mathbf{j}}$ shrinks to zero at infinity then the divergence theorem gives

$$\begin{aligned} \frac{d}{dt} Q(t) &= \int \frac{\partial}{\partial t} j^0 d^3x \\ &= -\int \nabla \cdot \overrightarrow{\mathbf{j}} d^3x = 0 \end{aligned}$$

so the charge $Q(t)$ is a conserved quantity. Let $Q(U, t)$ denote the total charge inside a region U. The charge inside any region U can only change via a flux through the boundary:

$$\begin{aligned} \frac{d}{dt} Q(U, t) &= \int_U \frac{\partial}{\partial t} j^0 d^3x \\ &= \int_{\partial U} \overrightarrow{\mathbf{j}} \cdot \mathbf{n} dS \end{aligned}$$

which is a kind of “local conservation law”. To be honest the above discussion only takes into account so called internal symmetries. An example of an internal symmetry is given by considering a curve of linear transformations of $\mathbb{R}^k$ given as matrices $C(s)$ with $C(0) = I$. Then we vary ϕ by $C(s)\phi$ so that $\delta\phi = \frac{d}{ds}\big|_0 C(s)\phi = C'(0)\phi$. Another possibility is to vary the underlying space so that $C(s, .)$ is now a curve of transformations of $\mathbb{R}^4$ so that if $\phi_s(x) = \phi(C(s, x))$ is a variation of fields then we must take into account the fact that the domain of integration is also varying:

$$L(\phi_s, \partial\phi_s) = \int_{U_s \subset \mathbb{R}^4} \mathcal{L}(\phi_s, \partial\phi_s) d^4x$$

We will make sense of this later.

F.0.13 Electricity and Magnetism

Up until now it has been mysterious how any object of matter could influence any other. It turns out that most of the forces we experience as middle sized objects pushing and pulling on each other is due to a single electromagnetic force. Without the help of special relativity there appears to be two forces; electric and magnetic. Elementary particles that carry electric charges such as electrons or protons, exert forces on each other by means of a field. In a particular Lorentz frame, the electromagnetic field is described by a skew-symmetric matrix of functions called the electromagnetic field tensor:

$$(F_{\mu\nu}) = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}.$$

Where we also have the forms $F_\mu^\nu = \Lambda^{s\nu} F_{\mu s}$ and $F^{\mu\nu} = \Lambda^{s\mu} F_s^\nu$. This tensor can be derived from a potential $\mathsf{A} = (A_0, A_1, A_2, A_3)$ by $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$. The contravariant form of the potential is $(A_0, -A_1, -A_2, -A_3)$ is a four vector often written as

$$\mathsf{A} = (\phi, \overrightarrow{\mathbf{A}}).$$

The action for a charged particle in an electromagnetic field is written in terms of A in a manifestly invariant way as

$$\int_a^b -mcd\tau - \frac{e}{c} A_\mu dx^\mu$$

so writing $\mathsf{A} = (\phi, \overrightarrow{\mathbf{A}})$ we have

$$S = \int_a^b (-mc\frac{d\tau}{dt} - e\phi(t) + \overrightarrow{\mathbf{A}} \cdot \frac{d\tilde{\mathbf{x}}}{dt})dt$$

so in a given frame the Lagrangian is

$$L(\tilde{\mathbf{x}}, \frac{d\tilde{\mathbf{x}}}{dt}, t) = -mc^2\sqrt{1 - (v/c)^2} - e\phi(t) + \overrightarrow{\mathbf{A}} \cdot \frac{d\tilde{\mathbf{x}}}{dt}.$$

Remark F.4 *The system under study is that of a particle in a field and does not describe the dynamics of the field itself. For that we would need more terms in the Lagrangian.*

This is a time dependent Lagrangian because of the $\phi(t)$ term but it turns out that one can re-choose A so that the new $\phi(t)$ is zero and yet still have $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$. This is called change of gauge. Unfortunately, if we wish to express things in such a way that a constant field is given by a constant potential then we cannot make this choice. In any case, we have

$$L(\overrightarrow{\mathbf{x}}, \frac{d\overrightarrow{\mathbf{x}}}{dt}, t) = -mc^2\sqrt{1 - (v/c)^2} - e\phi + \overrightarrow{\mathbf{A}} \cdot \frac{d\overrightarrow{\mathbf{x}}}{dt}$$

and setting $\overrightarrow{\mathbf{v}}=\frac{d\tilde{\mathbf{x}}}{dt}$ and $\left|\overrightarrow{\mathbf{v}}\right| = v$ we get the follow form for energy

$$\overrightarrow{\mathbf{v}}\cdot\frac{\partial}{\partial\overrightarrow{\mathbf{v}}}L(\tilde{\mathbf{x}},\overrightarrow{\mathbf{v}},t) - L(\tilde{\mathbf{x}},\overrightarrow{\mathbf{v}},t) = \frac{mc^2}{\sqrt{1-(v/c)^2}} + e\phi.$$

Now this is not constant with respect to time because $\frac{\partial L}{\partial t}$ is not identically zero. On the other hand, this make sense from another point of view; the particle is interacting with the field and may be picking up energy from the field.

The Euler-Lagrange equations of motion turn out to be

$$\frac{d\tilde{\mathbf{p}}}{dt} = e\tilde{\mathbf{E}} + \frac{e}{c}\overrightarrow{\mathbf{v}} \times \tilde{\mathbf{B}}$$

where $\tilde{\mathbf{E}} = -\frac{1}{c}\frac{\partial\tilde{\mathbf{A}}}{\partial t} - \operatorname{grad}\phi$ and $\tilde{\mathbf{B}} = curl\tilde{\mathbf{A}}$ are the electric and magnetic parts of the field respectively. This decomposition into electric and magnetic parts is an artifact of the choice of inertial frame and may be different in a different frame. Now the momentum $\tilde{\mathbf{p}}$ is $\frac{m\overrightarrow{\mathbf{v}}}{\sqrt{1-(v/c)^2}}$ but a speeds $v << c$ this becomes nearly equal to $m\mathbf{v}$ so the equations of motion of a charged particle reduce to

$$m\frac{d\overrightarrow{\mathbf{v}}}{dt} = e\tilde{\mathbf{E}} + \frac{e}{c}\overrightarrow{\mathbf{v}} \times \tilde{\mathbf{B}}.$$

Notice that is the particle is not moving, or if it is moving parallel the magnetic field $\tilde{\mathbf{B}}$ then the second term on the right vanishes.

The electromagnetic field equations.

We have defined the 3-vectors $\tilde{\mathbf{E}} = -\frac{1}{c}\frac{\partial\tilde{\mathbf{A}}}{\partial t} - \operatorname{grad}\phi$ and $\tilde{\mathbf{B}} = \operatorname{curl}\tilde{\mathbf{A}}$ but since the curl of a gradient is zero it is easy to see that $\operatorname{curl}\tilde{\mathbf{E}} = -\frac{1}{c}\frac{\partial\tilde{\mathbf{B}}}{\partial t}$. Also, from $\tilde{\mathbf{B}} = \operatorname{curl}\tilde{\mathbf{A}}$ we get $\operatorname{div}\tilde{\mathbf{B}} = \mathbf{0}$. This easily derived pair of equations is the first two of the four famous Maxwell's equations. Later we will see that the electromagnetic field tensor is really a differential 2-form F and these two equations reduce to the statement that the (exterior) derivative of F is zero:

$$dF = 0$$

Exercise F.1 *Apply Gauss's theorem and stokes theorem to the first two Maxwell's equations to get the integral forms. What do these equations say physically?*

One thing to notice is that these two equations do not determine $\frac{\partial}{\partial t}\tilde{\mathbf{E}}$.

Now we have not really written down a action or Lagrangian that includes terms that represent the field itself. When that part of the action is added in we get

$$S = \int_a^b (-mc - \frac{e}{c}A_\mu\frac{dx^\mu}{d\tau})d\tau + a\int_V F^{\nu\mu}F_{\nu\mu}dx^4$$

where in so called Gaussian system of units the constant a turns out to be $\frac{-1}{16\pi c}$. Now in a particular Lorentz frame and recalling 16.5 we get $= a\int_V F^{\nu\mu}F_{\nu\mu}dx^4 = \frac{1}{8\pi}\int_V \left|\tilde{\mathbf{E}}\right|^2 - \left|\tilde{\mathbf{B}}\right|^2 dtdxdydz$.

In order to get a better picture in mind let us now assume that there is a continuum of charged particle moving through space and that volume density of charge at any given moment in space-time is ρ so that if $dxdydz = dV$ then ρdV is the charge in the volume dV. Now we introduce the four vector $\rho\mathsf{u} = \rho(d\mathsf{x}/d\tau)$ where u is the velocity 4-vector of the charge at (t,x,y,z). Now recall that $\rho d\mathsf{x}/d\tau = \frac{d\tau}{dt}(\rho, \rho\overrightarrow{\mathbf{v}}) = \frac{d\tau}{dt}(\rho, \tilde{\mathbf{j}}) = \mathsf{j}$. Here $\tilde{\mathbf{j}} = \rho\overrightarrow{\mathbf{v}}$ is the charge current density as viewed in the given frame a vector field varying smoothly from point to point. Write $\mathsf{j} = (j^0, j^1, j^2, j^3)$.

Assuming now that the particle motion is determined and replacing the discrete charge e be the density we have applying the variational principle with the region $U = [a,b] \times V$ says

$$\begin{aligned} 0 &= -\delta\left(\int_V \int_a^b \frac{\rho dV}{c} dV A_\mu \frac{dx^\mu}{d\tau} d\tau + a\int_U F^{\nu\mu} F_{\nu\mu} dx^4\right) \\ &= -\delta\left(\frac{1}{c}\int_U j^\mu A_\mu + aF^{\nu\mu}F_{\nu\mu}dx^4\right) \end{aligned}$$

Now the Euler-Lagrange equations become

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0$$

where $\mathcal{L}(A_\mu, \partial_\mu A_\eta) = \frac{\rho}{c} A_\mu \frac{dx^\mu}{dt} + aF^{\nu\mu}F_{\nu\mu}$ and $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$. If one is careful to remember that $\partial_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu}$ is to be treated as an independent variable one cane arrive at some complicated looking equations and then looking at the matrix 16.5 we can convert the equations into statements about the fields $\tilde{\mathbf{E}}$, $\tilde{\mathbf{B}}$,and $(\rho, \tilde{\mathbf{j}})$. We will not carry this out since we later discover a much more efficient formalism for dealing with the electromagnetic field. Namely, we will use differential forms and the Hodge star operator. At any rate the last two of Maxwell's equations read

$$\begin{aligned} \operatorname{curl} \tilde{\mathbf{B}} &= 0 \\ \operatorname{div} \tilde{\mathbf{E}} &= 4\pi\rho. \end{aligned}$$

F.0.14 Quantum Mechanics

Appendix G

Calculus on Banach Spaces

Mathematics is not only real, but it is the only reality. That is that the entire universe is made of matter is obvious. And matter is made of particles. It's made of electrons and neutrons and protons. So the entire universe is made out of particles. Now what are the particles made out of? They're not made out of anything. The only thing you can say about the reality of an electron is to cite its mathematical properties. So there's a sense in which matter has completely dissolved and what is left is just a mathematical structure.

Gardner on Gardner: JPBM Communications Award Presentation. Focus-The Newsletter of the Mathematical Association of America v. 14, no. 6, December 1994.

G.0.15 Differentiability

For simplicity and definiteness all Banach spaces in this section will be real Banach spaces. First, the reader will recall that a linear map on a normed space, say $A : \mathsf{V}_1 \to \mathsf{V}_2$, is bounded if and only it is continuous at one and therefore any point in V_1. Given two Banach spaces V_1 and V_2 with norms $\|.\|_1$ and $\|.\|_2$ we can form a Banach space from the Cartesian product $\mathsf{V}_1 \times \mathsf{V}_2$ by using the norm $\|(v,u)\| := \max\{\|v\|_1, \|u\|_2\}$. There are many equivalent norms for $\mathsf{V}_1 \times \mathsf{V}_2$ including

$$\|(v,u)\|' := \sqrt{\|v\|_1^2 + \|u\|_2^2}$$
$$\|(v,u)\|'' := \|v\|_1 + \|u\|_2 .$$

Recall that two norms on V, say $\|.\|'$and $\|.\|''$ are equivalent if there exist positive constants c and C such that

$$c\,\|x\|' \leq \|x\|'' \leq C\,\|x\|'$$

for all $x \in \mathsf{V}$. Also, if V is a Banach space and W_1 andW_2 are closed subspaces such that $\mathsf{W}_1 \cap \mathsf{W}_2 = \{0\}$ and such that every $v \in \mathsf{V}$ can be written uniquely in the form $v = w_1 + w_2$ where $w_1 \in \mathsf{W}_1$ and $w_2 \in \mathsf{W}_2$ then we write $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.

In this case there is the natural continuous linear isomorphism $\mathsf{W}_1 \times \mathsf{W}_2 \cong \mathsf{W}_1 \oplus \mathsf{W}_2$ given by

$$(w_1, w_2) \longleftrightarrow w_1 + w_2.$$

When it is convenient, we can identify $\mathsf{W}_1 \oplus \mathsf{W}_2$ with $\mathsf{W}_1 \times \mathsf{W}_2$ and in this case we hedge our bets, as it were, and write $w_1 + w_2$ for either (w_1, w_2) or $w_1 + w_2$ and let the context determine the precise meaning if it matters. Under the representation (w_1, w_2) we need to specify what norm we are using and there is more than one natural choice. We take $\|(w_1, w_2)\| := \max\{\|w_1\|, \|w_2\|\}$ but equivalent norms include, for example, $\|(w_1, w_2)\|_2 := \sqrt{\|w_1\|^2 + \|w_2\|^2}$ which is a good choice if the spaces happen to be Hilbert spaces.

Let E be a Banach space and $\mathsf{W} \subset \mathsf{E}$ a *closed* subspace. We say that W is **complemented**if there is a *closed* subspace W' such that $\mathsf{E} = \mathsf{W} \oplus \mathsf{W}'$. We also say that W is a **split subspace** of E.

Definition G.1 (Notation) *We will denote the set of all continuous (bounded) linear maps from a Banach space* E *to a Banach space* F *by* $L(\mathsf{E}, \mathsf{F})$. *The set of all continuous linear isomorphisms from* E *onto* F *will be denoted by* $GL(\mathsf{E}, \mathsf{F})$. *In case,* $\mathsf{E} = \mathsf{F}$ *the corresponding spaces will be denoted by* $\mathfrak{gl}(\mathsf{E})$ *and* $GL(\mathsf{E})$. *Here* $GL(\mathsf{E})$ *is a group under composition and is called the general linear group .*

Definition G.2 *Let* V_i, $i = 1, ..., k$ *and* W *be Banach spaces. A map* $\mu : \mathsf{V}_1 \times \cdots \times \mathsf{V}_k \to \mathsf{W}$ *is called multilinear (k-multilinear) if for each* i, $1 \leq i \leq k$ *and each fixed* $(w_1, ..., \widehat{w_i}, ..., w_k) \in \mathsf{V}_1 \times \cdots \times \widehat{\mathsf{V}_1} \times \cdots \times \mathsf{V}_k$ *we have that the map*

$$v \mapsto \mu(w_1, ..., \underset{i-th}{v}, ..., w_{k-1}),$$

obtained by fixing all but the i-th variable, is a bounded linear map. In other words, we require that μ *be* R- *linear in each slot separately.*

A multilinear map $\mu : \mathsf{V}_1 \times \cdots \times \mathsf{V}_k \to \mathsf{W}$ is said to be **bounded** if and only if there is a constant C such that

$$\|\mu(v_1, v_2, ..., v_k)\|_{\mathsf{W}} \leq C \|v_1\|_{\mathsf{E}_1} \|v_2\|_{\mathsf{E}_2} \cdots \|v_k\|_{\mathsf{E}_k}$$

for all $(v_1, ..., v_k) \in \mathsf{E}_1 \times \cdots \times \mathsf{E}_k$.

Notation G.1 *The set of all bounded multilinear maps* $\mathsf{E}_1 \times \cdots \times \mathsf{E}_k \to \mathsf{W}$ *will be denoted by* $L(\mathsf{E}_1, ..., \mathsf{E}_k; \mathsf{W})$. *If* $\mathsf{E}_1 = \cdots = \mathsf{E}_k = \mathsf{E}$ *then we write* $L^k(\mathsf{E}; \mathsf{W})$ *instead of* $L(\mathsf{E}, ..., \mathsf{E}; \mathsf{W})$

Definition G.3 *Let* E *be a Hilbert space with inner product denoted by* $\langle ., . \rangle$. *Then* $O(\mathsf{E})$ *denotes the group of linear isometries from* E *onto itself. That is, the bijective linear maps* $\Phi : \mathsf{E} \to \mathsf{E}$ *such that* $\langle \Phi v, \Phi w \rangle = \langle v, w \rangle$ *for all* $v, w \in \mathsf{E}$. *The group* $O(\mathsf{E})$ *is called the* ***orthogonal group*** *(or sometimes the* ***Hilbert group*** *in the infinite dimensional case).*

Notation G.2 *For linear maps $T : \mathsf{V} \to \mathsf{W}$ we sometimes write $T \cdot v$ instead of $T(v)$ depending on the notational needs of the moment. In fact, a particularly useful notational device is the following: Suppose we have map $A : X \to L(\mathsf{V};\mathsf{W})$. Then $A(x)\cdot\ v$ makes sense but if instead of A the map needed to be indexed or something then things would get quite crowded. All in all it is sometimes better to write $A|_x\ v$. In fact, if we do this then $A|_x\,(v)$is also clear.*

Definition G.4 *A (bounded) multilinear map $\mu : \mathsf{V} \times \cdots \times \mathsf{V} \to \mathsf{W}$ is called* ***symmetric*** *(resp.* ***skew-symmetric*** *or* ***alternating****) if and only if for any $v_1, v_2, ..., v_k \in \mathsf{V}$ we have that*

$$\begin{aligned} \mu(v_1, v_2, ..., v_k) &= K(v_{\sigma 1}, v_{\sigma 2}, ..., v_{\sigma k}) \\ \textit{resp. } \mu(v_1, v_2, ..., v_k) &= sgn(\sigma)\mu(v_{\sigma 1}, v_{\sigma 2}, ..., v_{\sigma k}) \end{aligned}$$

for all permutations σ on the letters $\{1, 2, ..., k\}$. The set of all bounded symmetric (resp. skew-symmetric) multilinear maps $\mathsf{V} \times \cdots \times \mathsf{V} \to \mathsf{W}$ is denoted $L^k_{sym}(\mathsf{V};\mathsf{W})$ (resp. $L^k_{skew}(\mathsf{V};\mathsf{W})$ or $L^k_{alt}(\mathsf{V};\mathsf{W})$).

Now the space $L(\mathsf{V}, \mathsf{W})$ is a Banach space in its own right with the norm

$$\|l\| = \sup_{v \in \mathsf{V}} \frac{\|l(v)\|_{\mathsf{W}}}{\|v\|_{\mathsf{V}}} = \sup\{\|l(v)\|_{\mathsf{W}} : \|v\|_{\mathsf{V}} = 1\}.$$

The spaces $L(\mathsf{E}_1, ..., \mathsf{E}_k; \mathsf{W})$ are also Banach spaces normed by

$$\|\mu\| := \sup\{\|\mu(v_1, v_2, ..., v_k)\|_{\mathsf{W}} : \|v_i\|_{\mathsf{E}_i} = 1 \text{ for } i = 1, .., k\}$$

Proposition G.1 *A k-multilinear map $\mu \in L(\mathsf{E}_1, ..., \mathsf{E}_k; \mathsf{W})$ is continuous if and only if it is bounded.*

Proof. ($\Leftarrow$) We shall simplify by letting $k = 2$. Let (a_1, a_2) and (v_1, v_2) be elements of $\mathsf{E}_1 \times \mathsf{E}_2$ and write

$$\begin{aligned} &\mu(v_1, v_2) - \mu(a_1, a_2) \\ &= \mu(v_1 - a_1, v_2) + \mu(a_1, v_2 - a_2). \end{aligned}$$

We then have

$$\begin{aligned} &\|\mu(v_1, v_2) - \mu(a_1, a_2)\| \\ &\le C\,\|v_1 - a_1\|\,\|v_2\| + C\,\|a_1\|\,\|v_2 - a_2\| \end{aligned}$$

and so if $\|(v_1, v_2) - (a_1, a_2)\| \to 0$ then $\|v_i - a_i\| \to 0$ and we see that

$$\|\mu(v_1, v_2) - \mu(a_1, a_2)\| \to 0.$$

(Recall that $\|(v_1, v_2)\| := \max\{\|v_1\|, \|v_2\|\}$).

($\Rightarrow$) Start out by assuming that μ is continuous at $(0,0)$. Then for $r > 0$ sufficiently small, $(v_1, v_2) \in B((0,0), r)$ implies that $\|\mu(v_1, v_2)\| \leq 1$ so if for $i = 1, 2$ we let

$$z_i := \frac{rv_i}{\|v_1\|_i + \epsilon} \text{ for some } \epsilon > 0$$

then $(z_1, z_2) \in B((0,0), r)$ and $\|\mu(z_1, z_2)\| \leq 1$. The case $(v_1, v_2) = (0,0)$ is trivial so assume $(v_1, v_2) \neq (0,0)$. Then we have

$$\begin{aligned}\mu(z_1, z_2) &= \mu(\frac{rv_1}{\|v_1\| + \epsilon}, \frac{rv_2}{\|v_2\| + \epsilon}) \\ &= \frac{r^2}{(\|v_1\| + \epsilon)(\|v_2\| + \epsilon)}\mu(v_1, v_2) \leq 1\end{aligned}$$

and so $\mu(v_1, v_2) \leq r^{-2}(\|v_1\| + \epsilon)(\|v_2\| + \epsilon)$. Now let $\epsilon \to 0$ to get the result. ■

We shall need to have several Banach spaces handy for examples. For the next example we need some standard notation.

Notation G.3 *In the context of $\mathbb{R}^n$, we often use the so called "multiindex notation". Let $\alpha = (\alpha_1, ..., \alpha_n)$ where the α_i are integers and $0 \leq \alpha_i \leq n$. Such an n-tuple is called a multiindex. Let $|\alpha| := \alpha_1 + ... + \alpha_n$*

$$\frac{\partial^\alpha f}{\partial x^\alpha} := \frac{\partial^{|\alpha|} f}{\partial(x^1)^{\alpha_1} \partial(x^1)^{\alpha_2} \cdots \partial(x^1)^{\alpha_n}}.$$

Example G.1 *Consider a bounded open subset Ω of $\mathbb{R}^n$. Let $L_k^p(\Omega)$ denote the Banach space obtained by taking the Banach space completion of the set $C^k(\Omega)$ of k-times continuously differentiable real valued functions on Ω with the norm given by*

$$\|f\|_{k,p} := \left(\int_\Omega \sum_{|\alpha| \leq k} \left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right|^p \right)^{1/p}$$

Note that in particular $L_0^p(\Omega) = L^p(\Omega)$ is the usual L^p-space from real analysis.

Exercise G.1 *Show that the map $C^k(\Omega) \to C^{k-1}(\Omega)$ given by $f \mapsto \frac{\partial f}{\partial x^i}$ is bounded if we use the norms $\|f\|_{2,p}$ and $\|f\|_{2-1,p}$. Show that we may extend this to a bounded map $L_2^p(\Omega) \to L_1^p(\Omega)$.*

Proposition G.2 *There is a natural linear isomorphism $L(\mathsf{V}, L(\mathsf{V}, \mathsf{W})) \cong L^2(\mathsf{V}, \mathsf{W})$ given by*

$$l(v_1)(v_2) \longleftrightarrow l(v_1, v_2)$$

and we identify the two spaces. In fact, $L(\mathsf{V}, L(\mathsf{V}, L(\mathsf{V}, \mathsf{W})) \cong L^3(\mathsf{V}; \mathsf{W})$ and in general $L(\mathsf{V}, L(\mathsf{V}, L(\mathsf{V}, ..., L(\mathsf{V}, \mathsf{W})) \cong L^k(\mathsf{V}; \mathsf{W})$ etc.

Proof. It is easily checked that if we just define $(\iota\ T)(v_1)(v_2) = T(v_1, v_2)$ then $\iota\ T \leftrightarrow T$ does the job for the $k = 2$ case. The $k > 2$ case can be done by an inductive construction and is left as an exercise. It is also not hard to show that the isomorphism is continuous and in fact, norm preserving. ■

Definition G.5 *A function $f : U \subset \mathsf{V} \to \mathsf{W}$ between Banach spaces and defined on an open set $U \subset \mathsf{V}$ is said to be **differentiable at** $p \in U$ if and only if there is a bounded linear map $A_p \in L(\mathsf{V}, \mathsf{W})$ such that*

$$\lim_{\|\mathsf{h}\| \to 0} \frac{f(p + \mathsf{h}) - f(p) - A_p \cdot \mathsf{h}}{\|\mathsf{h}\|} = 0$$

*In anticipation of the following proposition we write $A_p = Df(p)$. We will also use the notation $Df|_p$ or sometimes $f'(p)$. The linear map $Df(p)$ is called the **derivative** of f at p.*

We often write $Df|_p \cdot \mathsf{h}$. The dot in the notation just indicate a linear dependence and is not a literal "dot product". We could also write $Df(p)(\mathsf{h})$.

Exercise G.2 *Show that the map $F : L^2(\Omega) \to L^1(\Omega)$ given by $F(f) = f^2$ is differentiable at any $f_0 \in L^2(\Omega)$.*

Proposition G.3 *If A_p exists for a given function f then it is unique.*

Proof. Suppose that A_p and B_p both satisfy the requirements of the definition. That is the limit in question equals zero. For $p + \mathsf{h} \in U$ we have

$$\begin{aligned} A_p \cdot \mathsf{h} - B_p \cdot \mathsf{h} &= (f(p + \mathsf{h}) - f(p) - A_p \cdot \mathsf{h}) \\ &\quad - (f(p + \mathsf{h}) - f(p) - B_p \cdot \mathsf{h}). \end{aligned}$$

Dividing by $\|\mathsf{h}\|$ and taking the limit as $\|\mathsf{h}\| \to 0$ we get

$$\|A_p\mathsf{h} - B_p\mathsf{h}\| / \|\mathsf{h}\| \to 0$$

Now let $\mathsf{h} \neq 0$ be arbitrary and choose $\epsilon > 0$ small enough that $p + \epsilon\mathsf{h} \in U$. Then we have

$$\|A_p(\epsilon\mathsf{h}) - B_p(\epsilon\mathsf{h})\| / \|\epsilon\mathsf{h}\| \to 0.$$

But by linearity $\|A_p(\epsilon\mathsf{h}) - B_p(\epsilon\mathsf{h})\| / \|\epsilon\mathsf{h}\| = \|A_p\mathsf{h} - B_p\mathsf{h}\| / \|\mathsf{h}\|$ which doesn't even depend on ϵ so in fact $\|A_p\mathsf{h} - B_p\mathsf{h}\| = 0$. ■

If we are interested in differentiating "in one direction at a time" then we may use the natural notion of directional derivative. A map has a directional derivative $D_{\mathsf{h}}f$ at p in the direction h if the following limit exists:

$$(D_{\mathsf{h}}f)(p) := \lim_{\epsilon \to 0} \frac{f(p + \epsilon\mathsf{h}) - f(p)}{\epsilon}$$

In other words, $D_{\mathsf{h}}f(p) = \frac{d}{dt}\big|_{t=0} f(p+t\mathsf{h})$. But a function may have a directional derivative in every direction (at some fixed p), that is, for every $\mathsf{h} \in \mathsf{E}$ and yet still not be differentiable at p in the sense of definition **??**.

Notation G.4 *The directional derivative is written as $(D_{\mathsf{h}}f)(p)$ and in case f is actually differentiable at p equal to $Df|_p \mathsf{h} = Df(p) \cdot \mathsf{h}$ (the proof is easy). Look closely; $D_{\mathsf{h}}f$ should not be confused with $Df|_{\mathsf{h}}$.*

Definition G.6 *If it happens that a function f is differentiable for all p throughout some open set U then we say that f is differentiable on U. We then have a map $Df : U \subset \mathsf{V} \to L(\mathsf{V}, \mathsf{W})$ given by $p \mapsto Df(p)$. If this map is differentiable at some $p \in \mathsf{V}$ then its derivative at p is denoted $DDf(p) = D^2 f(p)$ or $D^2 f\big|_p$ and is an element of $L(\mathsf{V}, L(\mathsf{V}, \mathsf{W})) \cong L^2(\mathsf{V}; \mathsf{W})$. Similarly, we may inductively define $D^k f \in L^k(\mathsf{V}; \mathsf{W})$ whenever f is sufficiently nice that the process can continue.*

Definition G.7 *We say that a map $f : U \subset \mathsf{V} \to \mathsf{W}$ is C^r-differentiable on U if $D^r f|_p \in L^r(\mathsf{V}, \mathsf{W})$ exists for all $p \in U$ and if continuous $D^r f$ as map $U \to L^r(\mathsf{V}, \mathsf{W})$. If f is C^r-differentiable on U for all $r > 0$ then we say that f is C^∞ or **smooth** (on U).*

Exercise G.3 *Show directly that a bounded multilinear map is C^∞.*

Definition G.8 *A bijection f between open sets $U_\alpha \subset \mathsf{V}$ and $U_\beta \subset \mathsf{W}$ is called a C^r-**diffeomorphism** if and only if f and f^{-1} are both C^r-differentiable (on U_α and U_β respectively). If $r = \infty$ then we simply call f a diffeomorphism. Often, we will have $\mathsf{W} = \mathsf{V}$ in this situation.*

*Let U be open in V. A map $f : U \to \mathsf{W}$ is called a **local C^r diffeomorphism** if and only if for every $p \in U$ there is an open set $U_p \subset U$ with $p \in U_p$ such that $f|_{U_p} : U_p \to f(U_p)$ is a C^r-diffeomorphism.*

Remark G.1 *In the context of undergraduate calculus courses we are used to thinking of the derivative of a function at some $a \in \mathbb{R}$ as a number $f'(a)$ which is the slope of the tangent line on the graph at $(a, f(a))$. From the current point of view $Df(a) = Df|_a$ just gives the linear transformation $h \mapsto f'(a) \cdot h$ and the equation of the tangent line is given by $y = f(a) + f'(a)(x - a)$. This generalizes to an arbitrary differentiable map as $y = f(a) + Df(a) \cdot (x - a)$ giving a map which is the linear approximation of f at a.*

We will sometimes think of the derivative of a curve[1] $c : I \subset \mathbb{R} \to \mathsf{E}$ at $t_0 \in I$, written $\dot{c}(t_0)$, as a velocity vector and so we are identifying $\dot{c}(t_0) \in L(\mathbb{R}, \mathsf{E})$ with $Dc|_{t_0} \cdot 1 \in \mathsf{E}$. Here the number 1 is playing the role of the unit vector in $\mathbb{R}$.

Let $f : U \subset \mathsf{E} \to \mathsf{F}$ be a map and suppose that we have a splitting $\mathsf{E} = \mathsf{E}_1 \times \mathsf{E}_2$. We will write $f(x, y)$ for $(x, y) \in \mathsf{E}_1 \times \mathsf{E}_2$. Now for every $(a, b) \in \mathsf{E}_1 \times \mathsf{E}_2$ the partial map $f_{a,} : y \mapsto f(a, y)$ (resp. $f_{,b} : x \mapsto f(x, b)$) is defined in some neighborhood of b (resp. a). We define the partial derivatives when they exist by $D_2 f(a, b) = Df_{a,}(b)$ (resp. $D_1 f(a, b) = Df_{,b}(a)$). These are, of course, linear maps.

$$D_1 f(a, b) : \mathsf{E}_1 \to \mathsf{F}$$
$$D_2 f(a, b) : \mathsf{E}_2 \to \mathsf{F}$$

The partial derivative can exist even in cases where f might not be differentiable in the sense we have defined. The point is that f might be differentiable only in certain directions.

[1]We will often use the letter I to denote a generic (usually open) interval in the real line.

If f has continuous partial derivatives $D_i f(x, y) : \mathsf{E}_i \to \mathsf{F}$ near $(x, y) \in \mathsf{E}_1 \times \mathsf{E}_2$ then exists and is continuous for all directions v. In this case, we have for $\mathrm{v} = (\mathrm{v}_1, \mathrm{v}_2)$,

$$
\begin{aligned}
&Df(x, y) \cdot (\mathrm{v}_1, \mathrm{v}_2) \\
&= D_1 f(x, y) \cdot \mathrm{v}_1 + D_2 f(x, y) \cdot \mathrm{v}_2
\end{aligned}
$$

G.0.16 Chain Rule, Product rule and Taylor's Theorem

Theorem G.1 (Chain Rule) *Let U_1 and U_2 be open subsets of Banach spaces E_1 and E_2 respectively. Suppose we have continuous maps composing as*

$$U_1 \xrightarrow{f} U_2 \xrightarrow{g} \mathsf{E}_3$$

where E_3 is a third Banach space. If f is differentiable at p and g is differentiable at $f(p)$ then the composition is differentiable at p and $D(g \circ f) = Dg(f(p)) \circ Dg(p)$. In other words, if $v \in \mathsf{E}_1$ then

$$D(g \circ f)\big|_p \cdot v = Dg\big|_{f(p)} \cdot (Df\big|_p \cdot v).$$

Furthermore, if $f \in C^r(U_1)$ and $g \in C^r(U_2)$ then $g \circ f \in C^r(U_1)$.

Proof. Let us use the notation $O_1(v)$, $O_2(v)$ etc. to mean functions such that $O_i(v) \to 0$ as $\|v\| \to 0$. Let $y = f(p)$. Since f is differentiable at p we have $f(p + \mathsf{h}) = y + Df\big|_p \cdot \mathsf{h} + \|\mathsf{h}\| \, O_1(\mathsf{h}) := y + \Delta y$ and since g is differentiable at y we have $g(y + \Delta y) = Dg\big|_y \cdot (\Delta y) + \|\Delta y\| \, O_2(\Delta y)$. Now $\Delta y \to 0$ as $\mathsf{h} \to 0$ and in turn $O_2(\Delta y) \to 0$ hence

$$
\begin{aligned}
g \circ f(p + \mathsf{h}) &= g(y + \Delta y) \\
&= Dg\big|_y \cdot (\Delta y) + \|\Delta y\| \, O_2(\Delta y) \\
&= Dg\big|_y \cdot (Df\big|_p \cdot \mathsf{h} + \|\mathsf{h}\| \, O_1(\mathsf{h})) + \|\mathsf{h}\| \, O_3(\mathsf{h}) \\
&= Dg\big|_y \cdot Df\big|_p \cdot \mathsf{h} + \|\mathsf{h}\| \, Dg\big|_y \cdot O_1(\mathsf{h}) + \|\mathsf{h}\| \, O_3(\mathsf{h}) \\
&= Dg\big|_y \cdot Df\big|_p \cdot \mathsf{h} + \|\mathsf{h}\| \, O_4(\mathsf{h})
\end{aligned}
$$

which implies that $g \circ f$ is differentiable at p with the derivative given by the promised formula.

Now we wish to show that $f, g \in C^r$ $r \geq 1$ implies that $g \circ f \in C^r$ also. The bilinear map defined by composition $comp : L(\mathsf{E}_1, \mathsf{E}_2) \times L(\mathsf{E}_2, \mathsf{E}_3) \to L(\mathsf{E}_1, \mathsf{E}_3)$ is bounded . Define a map

$$m_{f,g} : p \mapsto (Dg(f(p), Df(p))$$

which is defined on U_1. Consider the composition $comp \circ m_{f,g}$. Since f and g are at least C^1 this composite map is clearly continuous. Now we may proceed inductively. Consider the $r - th$ statement:

composition of C^r maps are C^r

Suppose f and g are C^{r+1} then Df is C^r and $Dg \circ f$ is C^r by the inductive hypothesis so that $m_{f,g}$ is C^r. A bounded bilinear functional is C^∞. Thus *comp* is C^∞ and by examining $comp \circ m_{f,g}$ we see that the result follows. ∎

We will often use the following lemma without explicit mention when calculating:

Lemma G.1 *Let $f : U \subset \mathsf{V} \to \mathsf{W}$ be twice differentiable at $x_0 \in U \subset \mathsf{V}$ then the map $D_v f : x \mapsto Df(x) \cdot v$ is differentiable at x_0 and its derivative at x_0 is given by*

$$D(D_v f)|_{x_0} \cdot \mathsf{h} = D^2 f(x_0)(\mathsf{h}, v).$$

Proof. The map $D_v f : x \mapsto Df(x) \cdot v$ is decomposed as the composition

$$x \overset{Df}{\mapsto} Df|_x \overset{R^{\mathsf{v}}}{\mapsto} Df|_x \cdot v$$

where $R^{\mathsf{v}} : L(\mathsf{V}, \mathsf{W}) \mapsto \mathsf{W}$ is the map $(A, b) \mapsto A \cdot b$. The chain rule gives

$$\begin{aligned} D(D_{\mathsf{v}} f)(x_0) \cdot \mathsf{h} &= DR^{\mathsf{v}}(\,Df|_{x_0}) \cdot D(Df)|_{x_0} \cdot \mathsf{h}) \\ &= DR^{\mathsf{v}}(Df(x_0)) \cdot (D^2 f(x_0) \cdot \mathsf{h}). \end{aligned}$$

But R^{v} is linear and so $DR^{\mathsf{v}}(y) = R^v$ for all y. Thus

$$\begin{aligned} D(D_v f)|_{x_0} \cdot \mathsf{h} &= R^v(D^2 f(x_0) \cdot \mathsf{h}) \\ &= (D^2 f(x_0) \cdot \mathsf{h}) \cdot v = D^2 f(x_0)(\mathsf{h}, v). \end{aligned}$$

∎

Theorem G.2 *If $f : U \subset \mathsf{V} \to \mathsf{W}$ is twice differentiable on U such that $D^2 f$ is continuous, i.e. if $f \in C^2(U)$ then $D^2 f$ is symmetric:*

$$D^2 f(p)(w, v) = D^2 f(p)(v, w).$$

More generally, if $D^k f$ exists and is continuous then $D^k f\ (\mathbf{p}) \in \mathbf{L}^k_{sym}(\mathsf{V}; \mathsf{W})$.

Proof. Let $p \in U$ and define an affine map $A : \mathbb{R}^2 \to \mathsf{V}$ by $A(s, t) := p + sv + tw$. By the chain rule we have

$$\frac{\partial^2 (f \circ A)}{\partial s \partial t}(0) = D^2 (f \circ A)(0) \cdot (\mathbf{e}_1, \mathbf{e}_2) = D^2 f(p) \cdot (v, w)$$

where $\mathbf{e}_1, \mathbf{e}_2$ is the standard basis of $\mathbb{R}^2$. Thus it suffices to prove that

$$\frac{\partial^2 (f \circ A)}{\partial s \partial t}(0) = \frac{\partial^2 (f \circ A)}{\partial t \partial s}(0).$$

In fact, for any $\ell \in \mathsf{V}^*$ we have

$$\frac{\partial^2 (\ell \circ f \circ A)}{\partial s \partial t}(0) = \ell \left(\frac{\partial^2 (f \circ A)}{\partial s \partial t} \right)(0)$$

and so by the Hahn-Banach theorem it suffices to prove that $\frac{\partial^2 (\ell \circ f \circ A)}{\partial s \partial t}(0) = \frac{\partial^2 (\ell \circ f \circ A)}{\partial t \partial s}(0)$ which is the standard 1-variable version of the theorem which we assume known. The result for $D^k f$ is proven by induction. ∎

Theorem G.3 *Let $\varrho \in L(\mathsf{F}_1, \mathsf{F}_2; \mathsf{W})$ be a bilinear map and let $f_1 : U \subset \mathsf{E} \to \mathsf{F}_1$ and $f_2 : U \subset \mathsf{E} \to \mathsf{F}_2$ be differentiable (resp. $C^r, r \geq 1$) maps. Then the composition $\varrho(f_1, f_2)$ is differentiable (resp. $C^r, r \geq 1$) on U where $\varrho(f_1, f_2) : x \mapsto \varrho(f_1(x), f_2(x))$. Furthermore,*

$$D\varrho|_x (f_1, f_2) \cdot v = \varrho(Df_1|_x \cdot v, f_2(x)) + \varrho(f_1(x), Df_2|_x \cdot v).$$

In particular, if F is an algebra with differentiable product $\star$ and $f_1 : U \subset \mathsf{E} \to \mathsf{F}$ and $f_2 : U \subset \mathsf{E} \to \mathsf{F}$ then $f_1 \star f_2$ is defined as a function and

$$D(f_1 \star f_2) \cdot v = (Df_1 \cdot v) \star (f_2) + (Df_1 \cdot v) \star (Df_2 \cdot v).$$

Proof. This is completely similar to the usual proof of the product rule and is left as an exercise. ∎

The proof of this useful lemma is left as an easy exercise. It is actually quite often that this little lemma saves the day as it were.

It will be useful to define an integral for maps from an interval $[a, b]$ into a Banach space V. First we define the integral for step functions. A function f on an interval $[a, b]$ is a **step function** if there is a partition $a = t_0 < t_1 < \cdots < t_k = b$ such that f is constant, with value say f_i, on each subinterval $[t_i, t_{i+1})$. The set of step functions so defined is a vector space. We define the integral of a step function f over $[a, b]$ by

$$\int_{[a,b]} f := \sum_{i=0}^{k-1} (t_{i+1} - t_i) f_i = \sum_{i=0}^{k-1} (t_{i+1} - t_i) f(t_i).$$

One easily checks that the definition is independent of the partition chosen. Now the set of all step functions from $[a, b]$ into V is a linear subspace of the Banach space $\mathcal{B}(a, b, \mathsf{V})$ of all bounded functions of $[a, b]$ into V and the integral is a linear map on this space. Recall that the norm on $\mathcal{B}(a, b, \mathsf{V})$ is $\sup_{a \leq t < b}\{|f(t)|\}$. If we denote the closure of the space of step functions in this Banach space by $\bar{\mathcal{S}}(a, b, \mathsf{V})$ then we can extend the definition of the integral to $\bar{\mathcal{S}}(a, b, \mathsf{V})$ by continuity since on step functions we have

$$\left| \int_{[a,b]} f \right| \leq (b - a) \|f\|_\infty .$$

In the limit, this bound persists. This integral is called the **Cauchy-Bochner** integral and is a bounded linear map $\bar{\mathcal{S}}(a, b, \mathsf{V}) \to \mathsf{V}$. It is important to notice that $\bar{\mathcal{S}}(a, b, \mathsf{V})$ contains the continuous functions $C([a, b], \mathsf{V})$ because such may be uniformly approximated by elements of $\mathcal{S}(a, b, \mathsf{V})$ and so we can integrate these functions using the Cauchy-Bochner integral.

Lemma G.2 *If $\ell : \mathsf{V} \to \mathsf{W}$ is a bounded linear map of Banach spaces then for any $f \in \bar{\mathcal{S}}(a, b, \mathsf{V})$ we have*

$$\int_{[a,b]} \ell \circ f = \ell \circ \int_{[a,b]} f.$$

Proof. This is obvious for step functions. The general result follows by taking a limit for step functions converging in $\bar{\mathcal{S}}(a, b, \mathsf{V})$ to f. ∎

Some facts about maps on finite dimensional spaces.

Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$ be a map which is differentiable at $a = (a^1, ..., a^n) \in \mathbb{R}^n$. The map f is given by m functions $f^i : U \to \mathbb{R}^m$, $1 \leq i \leq m$. Now with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, the derivative is given by an $n \times m$ matrix called the **Jacobian** matrix:

$$J_a(f) := \begin{pmatrix} \frac{\partial f^1}{\partial x^1}(a) & \frac{\partial f^1}{\partial x^2}(a) & \cdots & \frac{\partial f^1}{\partial x^n}(a) \\ \frac{\partial f^2}{\partial x^1}(a) & & & \\ \vdots & & \ddots & \\ \frac{\partial f^m}{\partial x^1}(a) & & & \frac{\partial f^m}{\partial x^n}(a) \end{pmatrix}.$$

The rank of this matrix is called the rank of f at a. If $n = m$ so that $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ then the Jacobian is a square matrix and $\det(J_a(f))$ is called the **Jacobian determinant** at a. If f is differentiable near a then it follows from the inverse mapping theorem proved below that if $\det(J_a(f)) \neq 0$ then there is some open set containing a on which f has a differentiable inverse. The Jacobian of this inverse at $f(x)$ is the inverse of the Jacobian of f at x.

Notation G.5 *The Jacobian matrix was a bit tedious to write down. Of course we have the abbreviation but let us also use the suggestive notation*

$$\frac{\partial(f^1, .., f^m)}{\partial(x^1, .., x^n)}.$$

The following is the **mean value theorem**:

Theorem G.4 *Let V and W be Banach spaces. Let $c : [a, b] \to \mathsf{V}$ be a C^1-map with image contained in an open set $U \subset \mathsf{V}$. Also, let $f : U \to \mathsf{W}$ be a C^1 map. Then*

$$f(c(b)) - f(c(a)) = \int_0^1 Df(c(t)) \cdot c'(t)dt.$$

If $c(t) = (1 - t)x + ty$ then

$$f(y) - f(x) = \int_0^1 Df(c(t))dt \cdot (y - x)$$

Notice that $\int_0^1 Df(c(t))dt \in L(\mathsf{V}, \mathsf{W})$.

Proof. Use the chain rule and the 1-variable fundamental theorem of calculus for the first part. For the second use lemma **??**. ■

Corollary G.1 *Let U be a convex open set in a Banach space V and $f : U \to \mathsf{W}$ a C^1 map into another Banach space W. Then for any $x, y \in U$ we have*

$$\|f(y) - f(x)\| \leq C_{x,y} \|y - x\|$$

where $C_{x,y}$ is the supremum over all values taken by f along the line segment which is the image of the path $t \mapsto (1 - t)x + ty$.

Recall that for a fixed x higher derivatives $D^p f\big|_x$ are symmetric multilinear maps. For the following let $(y)^k$ denote $\underbrace{(y, y, ..., y)}_{k\text{-times}}$. With this notation we have the following version of Taylor's theorem.

Theorem G.5 (Taylor's theorem) *Given Banach spaces* V *and* W, *a* C^r *function* $f : U \to \mathsf{W}$ *and a line segment* $t \mapsto (1-t)x + ty$ *contained in* U, *we have that* $t \mapsto D^p f(x + ty) \cdot (y)^p$ *is defined and continuous for* $1 \le p \le k$ *and*

$$f(x+y) = f(x) + \frac{1}{1!}\, Df\big|_x \cdot y + \frac{1}{2!}\, D^2 f\big|_x \cdot (y)^2 + \cdots + \frac{1}{(k-1)!}\, D^{k-1} f\big|_x \cdot (y)^{k-1}$$
$$+ \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} D^k f(x+ty) \cdot (y)^k dt$$

Proof. The proof is by induction and follows the usual proof closely. The point is that we still have an integration by parts formula coming from the product rule and we still have the fundamental theorem of calculus. ∎

G.0.17 Local theory of maps

Inverse Mapping Theorem

The main reason for restricting our calculus to Banach spaces is that the inverse mapping theorem holds for Banach spaces and there is no simple and general inverse mapping theory on more general topological vector spaces. The so called hard inverse mapping theorems such as that of Nash and Moser require special estimates and are constructed to apply to a very limited situation. Recently, Michor and Kriegl et. al. have promoted an approach which defines differentiability in terms of mappings of $\mathbb{R}$ into the space that makes a lot of the formal parts of calculus valid under their definition of differentiability. However, the general (and easy) inverse and implicit mapping theorems still remain limited as before to Banach spaces and more general cases have to be handled case by case.

Definition G.9 *Let* E *and* F *be Banach spaces. A map will be called a* C^r ***diffeomorphism near*** p *if there is some open set* $U \subset \operatorname{dom}(f)$ *containing* p *such that* $f\big|_U : U \to f(U)$ *is a* C^r *diffeomorphism onto an open set* $f(U)$. *The set of all maps which are diffeomorphisms near* p *will be denoted* $\operatorname{Diff}^r_p(\mathsf{E}, \mathsf{F})$. *If* f *is a* C^r *diffeomorphism near* p *for all* $p \in U = \operatorname{dom}(f)$ *then we say that* f *is a **local*** C^r ***diffeomorphism**.*

Definition G.10 *Let* X, d_1 *and* Y, d_2 *be metric spaces. A map* $f : X \to Y$ *is said to be **Lipschitz continuous** (with constant* k*) if there is a* $k > 0$ *such that* $d(f(x_1), f(x_2)) \le kd(x_1, x_2)$ *for all* $x_1, x_2 \in X$. *If* $0 < k < 1$ *the map is called a **contraction mapping** (with constant* k*) or is said to be* k***-contractive**.*

The following technical result has numerous applications and uses the idea of iterating a map. **Warning**: For this theorem f^n will denote the $n-$fold composition $f \circ f \circ \cdots \circ f$ rather than a product.

Proposition G.4 (Contraction Mapping Principle) *Let F be a closed subset of a complete metric space (M, d). Let $f : F \to F$ be a k-contractive map such that*

$$d(f(x), f(y)) \leq kd(x, y)$$

for some fixed $0 \leq k < 1$. Then

1) there is exactly one $x_0 \in F$ such that $f(x_0) = x_0$. Thus x_0 is a fixed point for f. Furthermore,

2) for any $y \in F$ the sequence $y_n := f^n(y)$ converges to the fixed point x_0 with the error estimate $d(y_n, x_0) \leq \frac{k^n}{1-k} d(y_1, x_0)$.

Proof. Let $y \in F$. By iteration

$$d(f^n(y), f^{n-1}(y)) \leq kd(f^{n-1}(y), f^{n-2}(y)) \leq \cdots \leq k^{n-1}d(f(y), y)$$

as follows:

$$\begin{aligned} d(f^{n+j+1}(y), f^n(y)) &\leq d(f^{n+j+1}(y), f^{n+j}(y)) + \cdots + d(f^{n+1}(y), f^n(y)) \\ &\leq (k^{j+1} + \cdots + k)d(f^n(y), f^{n-1}(y)) \\ &\leq \frac{k}{1-k} d(f^n(y), f^{n-1}(y)) \\ &\frac{k^n}{1-k} d(f^1(y), y)) \end{aligned}$$

¿From this, and the fact that $0 \leq k < 1$, one can conclude that the sequence $f^n(y) = x_n$ is Cauchy. Thus $f^n(y) \to x_0$ for some x_0 which is in F since F is closed. On the other hand,

$$x_0 = \lim_{n \to 0} f^n(y) = \lim_{n \to 0} f(f^{n-1}(y)) = f(x_0)$$

by continuity of f. Thus x_0 is a fixed point. If u_0 where also a fixed point then

$$d(x_0, u_0) = d(f(x_0), f(u_0)) \leq kd(x_0, u_0)$$

which forces $x_0 = u_0$. The error estimate in (2) of the statement of the theorem is left as an easy exercise. ■

Remark G.2 *Note that a Lipschitz map f may not satisfy the hypotheses of the last theorem even if $k < 1$ since U is not a complete metric space unless $U = \mathsf{E}$.*

Definition G.11 *A continuous map $f : U \to \mathsf{E}$ such that $L_f := \mathrm{id}_U - f$ is injective has a not necessarily continuous inverse G_f and the invertible map $R_f := \mathrm{id}_{\mathsf{E}} - G_f$ will be called the **resolvent operator** for f.*

The resolvent is a term that is usually used in the context of linear maps and the definition in that context may vary slightly. Namely, what we have defined here would be the resolvent of $\pm L_f$. Be that as it may, we have the following useful result.

Theorem G.6 *Let E be a Banach space. If $f : \mathsf{E} \to \mathsf{E}$ is continuous map that is Lipschitz continuous with constant k where $0 \le k < 1$, then the resolvent R_f exists and is Lipschitz continuous with constant $\frac{k}{1-k}$.*

Proof. Consider the equation $x - f(x) = y$. We claim that for any $y \in \mathsf{E}$ this equation has a unique solution. This follows because the map $F : \mathsf{E} \to \mathsf{E}$ defined by $F(x) = f(x) + y$ is k-contractive on the complete normed space E as a result of the hypotheses. Thus by the contraction mapping principle there is a unique x fixed by F which means a unique x such that $f(x) + y = x$. Thus the inverse G_f exists and is defined on all of E. Let $R_f := \mathrm{id}_{\mathsf{E}} - G_f$ and choose $y_1, y_2 \in \mathsf{E}$ and corresponding unique x_i , $i = 1, 2$ with $x_i - f(x_i) = y_i$. We have

$$\begin{aligned} \|R_f(y_1) - R_f(y_2)\| &= \|f(x_1) - f(x_2)\| \\ &\le k \|x_1 - x_2\| \ \le \\ &\le k \|y_1 - R_f(y_1) - (y_2 - R_f(y_2))\| \le \\ &\le k \|y_1 - y_2\| + \|R_f(y_1) - R_f(y_2)\| . \end{aligned}$$

Solving this inequality we get

$$\|R_f(y_1) - R_f(y_2)\| \le \frac{k}{1-k} \|y_1 - y_2\| .$$

■

Lemma G.3 *The space $GL(\mathsf{E}, \mathsf{F})$ of continuous linear isomorphisms is an open subset of the Banach space $L(\mathsf{E}, \mathsf{F})$. In particular, if $\|\mathrm{id} - A\| < 1$ for some $A \in GL(\mathsf{E})$ then $A^{-1} = \lim_{N\to\infty} \sum_{n=0}^{N} (\mathrm{id} - A)^n$.*

Proof. Let $A_0 \in GL(\mathsf{E}, \mathsf{F})$. The map $A \mapsto A_0^{-1} \circ A$ is continuous and maps $GL(\mathsf{E}, \mathsf{F})$ onto $GL(\mathsf{E}, \mathsf{F})$. If follows that we may assume that $\mathsf{E} = \mathsf{F}$ and that $A_0 = \mathrm{id}_{\mathsf{E}}$. Our task is to show that elements of $\mathrm{L}(\mathsf{E}, \mathsf{E})$ close enough to id_{E} are in fact elements of $GL(\mathsf{E})$. For this we show that

$$\|\mathrm{id} - A\| < 1$$

implies that $A \in GL(\mathsf{E})$. We use the fact that the norm on $\mathrm{L}(\mathsf{E}, \mathsf{E})$ is an algebra norm. Thus $\|A_1 \circ A_2\| \le \|A_1\| \|A_2\|$ for all $A_1, A_2 \in \mathrm{L}(\mathsf{E}, \mathsf{E})$. We abbreviate id by "1" and denote $\mathrm{id} - A$ by Λ. Let $\Lambda^2 := \Lambda \circ \Lambda$, $\Lambda^3 := \Lambda \circ \Lambda \circ \Lambda$ and so forth. We now form a Neumann series :

$$\begin{aligned} \pi_0 &= 1 \\ \pi_1 &= 1 + \Lambda \\ \pi_2 &= 1 + \Lambda + \Lambda^2 \\ &\ \vdots \\ \pi_n &= 1 + \Lambda + \Lambda^2 + \cdots + \Lambda^n. \end{aligned}$$

By comparison with the Neumann series of real numbers formed in the same way using $\|A\|$ instead of A we see that $\{\pi_n\}$ is a Cauchy sequence since $\|\Lambda\| = \|\mathrm{id} - A\| < 1$. Thus $\{\pi_n\}$ is convergent to some element ρ. Now we have $(1-\Lambda)\pi_n = 1 - \Lambda^{n+1}$ and letting $n \to \infty$ we see that $(1-\Lambda)\rho = 1$ or in other words, $A\rho = 1$. ■

Lemma G.4 *The map* inv $:GL(\mathsf{E},\mathsf{F}) \to GL(\mathsf{E},\mathsf{F})$ *given by taking inverses is a* C^∞ *map and the derivative of* inv $:g \mapsto g^{-1}$ *at some* $g_0 \in GL(\mathsf{E},\mathsf{F})$ *is the linear map given by the formula:* $D\,\mathrm{inv}|_{g_0} : A \mapsto -g_0^{-1}Ag_0^{-1}$.

Proof. Suppose that we can show that the result is true for $g_0 = \mathrm{id}$. Then pick any $h_0 \in GL(\mathsf{E},\mathsf{F})$ and consider the isomorphisms $L_{h_0} : GL(\mathsf{E}) \to GL(\mathsf{E},\mathsf{F})$ and $R_{h_0^{-1}} : GL(\mathsf{E}) \to GL(\mathsf{E},\mathsf{F})$ given by $\phi \mapsto h_0\phi$ and $\phi \mapsto \phi h_0^{-1}$ respectively. The map $g \mapsto g^{-1}$ can be decomposed as

$$g \overset{L_{h_0^{-1}}}{\mapsto} h_0^{-1} \circ g \overset{\mathrm{inv}_{\mathsf{E}}}{\mapsto} (h_0^{-1} \circ g)^{-1} \overset{R_{h_0^{-1}}}{\mapsto} g^{-1}h_0h_0^{-1} = g^{-1}.$$

Now suppose that we have the result at $g_0 = \mathrm{id}$ in $GL(\mathsf{E})$. This means that $D\mathrm{inv}_{\mathsf{E}}|_{h_0} : A \mapsto -A$. Now by the chain rule we have

$$\begin{aligned}(D\,\mathrm{inv}|_{h_0}) \cdot \mathrm{A} &= D(R_{h_0^{-1}} \circ \mathrm{inv}_{\mathsf{E}} \circ L_{h_0^{-1}}) \cdot \mathrm{A} \\ &= \left(R_{h_0^{-1}} \circ D\,\mathrm{inv}_{\mathsf{E}}|_{\mathrm{id}} \circ L_{h_0^{-1}}\right) \cdot \mathrm{A} \\ &= R_{h_0^{-1}} \circ (-\mathrm{A}) \circ L_{h_0^{-1}} = -h_0^{-1}\mathrm{A}h_0^{-1}\end{aligned}$$

so the result is true for an arbitrary $h_0 \in GL(\mathsf{E},\mathsf{F})$. Thus we are reduced to showing that $D\mathrm{inv}_{\mathsf{E}}|_{\mathrm{id}} : A \mapsto -A$. The definition of derivative leads us to check that the following limit is zero.

$$\lim_{\|\mathrm{A}\| \to 0} \frac{\left\|(\mathrm{id}+\mathrm{A})^{-1} - (\mathrm{id})^{-1} - (-\mathrm{A})\right\|}{\|\mathrm{A}\|}.$$

Note that for small enough $\|\mathrm{A}\|$, the inverse $(\mathrm{id}+A)^{-1}$ exists and so the above limit makes sense. By our previous result (13) the above difference quotient becomes

$$\begin{aligned}&\lim_{\|\mathrm{A}\| \to 0} \frac{\left\|(\mathrm{id}+\mathrm{A})^{-1} - \mathrm{id} + \mathrm{A}\right\|}{\|\mathrm{A}\|} \\ &= \lim_{\|\mathrm{A}\| \to 0} \frac{\|\sum_{n=0}^\infty (\mathrm{id} - (\mathrm{id}+\mathrm{A}))^n - \mathrm{id} + \mathrm{A}\|}{\|\mathrm{A}\|} \\ &= \lim_{\|\mathrm{A}\| \to 0} \frac{\|\sum_{n=0}^\infty (-\mathrm{A})^n - \mathrm{id} + \mathrm{A}\|}{\|\mathrm{A}\|} \\ &= \lim_{\|\mathrm{A}\| \to 0} \frac{\|\sum_{n=2}^\infty (-\mathrm{A})^n\|}{\|\mathrm{A}\|} \le \lim_{\|\mathrm{A}\| \to 0} \frac{\sum_{n=2}^\infty \|\mathrm{A}\|^n}{\|\mathrm{A}\|} \\ &= \lim_{\|\mathrm{A}\| \to 0} \sum_{n=1}^\infty \|\mathrm{A}\|^n = \lim_{\|\mathrm{A}\| \to 0} \frac{\|\mathrm{A}\|}{1 - \|\mathrm{A}\|} = 0.\end{aligned}$$

■

Theorem G.7 (Inverse Mapping Theorem) *Let* E *and* F *be Banach spaces and* $f : U \to \mathsf{F}$ *be a* C^r *mapping defined an open set* $U \subset \mathsf{E}$*. Suppose that* $x_0 \in U$ *and that* $f'(x_0) = Df|_x : \mathsf{E} \to \mathsf{F}$ *is a continuous linear isomorphism. Then there exists an open set* $V \subset U$ *with* $x_0 \in V$ *such that* $f : V \to f(V) \subset \mathsf{F}$ *is a* C^r-*diffeomorphism. Furthermore the derivative of* f^{-1} *at* y *is given by* $Df^{-1}\big|_y = (Df|_{f^{-1}(y)})^{-1}$.

Proof. By considering $(Df|_x)^{-1} \circ f$ and by composing with translations we may as well just assume from the start that $f : \mathsf{E} \to \mathsf{E}$ with $x_0 = 0$, $f(0) = 0$ and $Df|_0 = \mathrm{id}_E$. Now if we let $g = x - f(x)$, then $Dg|_0 = 0$ and so if $r > 0$ is small enough then

$$\| Dg|_x \| < \frac{1}{2}$$

for $x \in B(0, 2r)$. The mean value theorem now tells us that $\|g(x_2) - g(x_1)\| \leq \frac{1}{2} \|x_2 - x_1\|$ for $x_2, x_1 \in \overline{B}(0,r)$ and that $g(\overline{B}(0,r)) \subset \overline{B}(0,r/2)$. Let $y_0 \in \overline{B}(0,r/2)$. It is not hard to show that the map $c : x \mapsto y_0 + x - f(x)$ is a contraction mapping $c : \overline{B}(0,r) \to \overline{B}(0,r)$ with constant $\frac{1}{2}$. The contraction mapping principle **??** says that c has a unique fixed point $x_0 \in \overline{B}(0,r)$. But $c(x_0) = x_0$ just translates to $y_0 + x_0 - f(x_0) = x_0$ and then $f(x_0) = y_0$. So x_0 is the unique element of $\overline{B}(0,r)$ satisfying this equation. But then since $y_0 \in \overline{B}(0,r/2)$ was an arbitrary element of $\overline{B}(0,r/2)$ it follows that the restriction $f : \overline{B}(0,r/2) \to f(\overline{B}(0,r/2))$ is invertible. But f^{-1} is also continuous since

$$\begin{aligned}
\left\|f^{-1}(y_2) - f^{-1}(y_1)\right\| &= \|x_2 - x_1\| \\
&\leq \|f(x_2) - f(x_1)\| + \|g(x_2) - g(x_1)\| \\
&\leq \|f(x_2) - f(x_1)\| + \frac{1}{2}\|x_2 - x_1\| \\
&= \|y_2 - y_1\| + \frac{1}{2}\left\|f^{-1}(y_2) - f^{-1}(y_1)\right\|
\end{aligned}$$

Thus $\left\|f^{-1}(y_2) - f^{-1}(y_1)\right\| \leq 2\|y_2 - y_1\|$ and so f^{-1} is continuous. In fact, f^{-1} is also differentiable on $B(0,r/2)$. To see this let $f(x_2) = y_2$ and $f(x_1) = y_1$ with $x_2, x_1 \in \overline{B}(0,r)$ and $y_2, y_1 \in \overline{B}(0,r/2)$. The norm of $Df(x_1))^{-1}$ is bounded (by continuity) on $\overline{B}(0,r)$ by some number B. Setting $x_2 - x_1 = \Delta x$ and $y_2 - y_1 = \Delta y$ and using $(Df(x_1))^{-1}Df(x_1) = \mathrm{id}$ we have

$$\begin{aligned}
&\left\|f^{-1}(y_2) - f^{-1}(y_1) - (Df(x_1))^{-1} \cdot \Delta y\right\| \\
&= \left\|\Delta x - (Df(x_1))^{-1}(f(x_2) - f(x_1))\right\| \\
&= \left\|\{(Df(x_1))^{-1}Df(x_1)\}\Delta x - \{(Df(x_1))^{-1}Df(x_1)\}(Df(x_1))^{-1}(f(x_2) - f(x_1))\right\| \\
&\leq B\|Df(x_1)\Delta x - (f(x_2) - f(x_1))\| \leq o(\Delta x) = o(\Delta y) \text{ (by continuity).}
\end{aligned}$$

Thus $Df^{-1}(y_1)$ exists and is equal to $(Df(x_1))^{-1} = (Df(f^{-1}(y_1)))^{-1}$. A simple argument using this last equation shows that $Df^{-1}(y_1)$ depends continuously on y_1 and so f^{-1} is C^1. The fact that f^{-1} is actually C^r follows from a simple induction argument that uses the fact that Df is C^{r-1} together with lemma **??**. This last step is left to the reader. ∎

Exercise G.4 *Complete the last part of the proof of theorem*

Corollary G.2 *Let $U \subset \mathsf{E}$ be an open set and $0 \in U$. Suppose that $f : U \to \mathsf{F}$ is differentiable with $Df(p) : \mathsf{E} \to \mathsf{F}$ a (bounded) linear isomorphism for each $p \in U$. Then f is a local diffeomorphism.*

Theorem G.8 (Implicit Mapping Theorem I) *Let $\mathsf{E}_1, \mathsf{E}_2$ and F be Banach spaces and $U \times V \subset \mathsf{E}_1 \times \mathsf{E}_2$ open. Let $f : U \times V \to \mathsf{F}$ be a C^r mapping such that $f(x_0, y_0) = 0$. If $D_2 f_{(x_0,y_0)} : \mathsf{E}_2 \to \mathsf{F}$ is a continuous linear isomorphism then there exists a (possibly smaller) open set $U_0 \subset U$ with $x_0 \in U_0$ and unique mapping $g : U_0 \to V$ with $g(x_0) = y_0$ and such that*

$$f(x, g(x)) = 0$$

for all $x \in U_0$.

Proof. Follows from the following theorem. ■

Theorem G.9 (Implicit Mapping Theorem II) *Let $\mathsf{E}_1, \mathsf{E}_2$ and F be Banach spaces and $U \times V \subset \mathsf{E}_1 \times \mathsf{E}_2$ open. Let $f : U \times V \to \mathsf{F}$ be a C^r mapping such that $f(x_0, y_0) = w_0$. If $D_2 f(x_0, y_0) : \mathsf{E}_2 \to \mathsf{F}$ is a continuous linear isomorphism then there exists (possibly smaller) open sets $U_0 \subset U$ and $W_0 \subset \mathsf{F}$ with $x_0 \in U_0$ and $w_0 \in W_0$ together with a unique mapping $g : U_0 \times W_0 \to V$ such that*

$$f(x, g(x, w)) = w$$

for all $x \in U_0$. Here unique means that any other such function h defined on a neighborhood $U_0' \times W_0'$ will equal g on some neighborhood of (x_0, w_0).

Proof. Sketch: Let $\Psi : U \times V \to \mathsf{E}_1 \times \mathsf{F}$ be defined by $\Psi(x, y) = (x, f(x, y))$. Then $D\Psi(x_0, y_0)$ has the operator matrix

$$\begin{bmatrix} \mathrm{id}_{E_1} & 0 \\ D_1 f(x_0, y_0) & D_2 f(x_0, y_0) \end{bmatrix}$$

which shows that $D\Psi(x_0, y_0)$ is an isomorphism. Thus Ψ has a unique local inverse Ψ^{-1} which we may take to be defined on a product set $U_0 \times W_0$. Now Ψ^{-1} must have the form $(x, y) \mapsto (x, g(x, y))$ which means that $(x, f(x, g(x, w))) = \Psi(x, g(x, w)) = (x, w)$. Thus $f(x, g(x, w)) = w$. The fact that g is unique follows from the local uniqueness of the inverse Ψ^{-1} and is left as an exercise. ■

In the case of a map $f : U \to V$ between open subsets of Euclidean spaces (say $\mathbb{R}^n$ and $\mathbb{R}^m$) we have the notion of **rank at** $p \in U$ which is just the rank of the linear map $D_p f : \mathbb{R}^n \to \mathbb{R}^m$.

Definition G.12 *Let X, Y be topological spaces. When we write $f :: \mathsf{X} \to \mathsf{Y}$ we imply only that f is defined on some open set in X. We shall employ a similar use of the symbol "$::$" when talking about continuous maps between (open subsets* *of) topological spaces in general. If we wish to indicate that f is defined near $p \in \mathsf{X}$ and that $f(p) = q$ we will used the pointed category notation together with the symbol "$::$":*

$$f :: (\mathsf{X}, p) \to (\mathsf{Y}, q)$$

We will refer to such maps as ***local maps at*** *p. Local maps may be composed with the understanding that the domain of the composite map may become smaller: If $f :: (\mathsf{X}, p) \to (\mathsf{Y}, q)$ and $g :: (\mathsf{Y}, q) \to (\mathsf{G}, z)$ then $g \circ f :: (\mathsf{X}, p) \to (\mathsf{G}, z)$ and the domain of $g \circ f$ will be a non-empty open set. Also, we will say that two such maps $f_1 : (\mathsf{X}, p) \to (\mathsf{Y}, q)$ and $f_2 : (\mathsf{X}, p) \to (\mathsf{Y}, q)$ are* ***equal near*** *p if there is an open set O with $p \in O \subset \operatorname{dom}(f_1) \cap \operatorname{dom}(f_2)$ such that the restrictions to O are equal:*

$$f_1|_O = f_2|_O$$

in this case will simply write "$f_1 = f_2$ (near p)".

Notation G.6 *Recall that for a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ which is injective with rank r there exist linear isomorphisms $C_1 : \mathbb{R}^m \to \mathbb{R}^m$ and $C_2 : \mathbb{R}^n \to \mathbb{R}^n$ such that $C_1 \circ A \circ C_2^{-1}$ is just a projection followed by an injection:*

$$\mathbb{R}^n = \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R}^r \times 0 \to \mathbb{R}^r \times \mathbb{R}^{m-r} = \mathbb{R}^m$$

We have obvious special cases when $r = n$ or $r = m$. This fact has a local version that applies to C^∞ nonlinear maps. In order to facilitate the presentation of the following theorems we will introduce the following terminology:

Linear case.

Definition G.13 *We say that a continuous linear map $A_1 : \mathsf{E}_1 \to \mathsf{F}_1$ is* ***equivalent*** *to a map $A_2 : \mathsf{E}_2 \to \mathsf{F}_2$ if there are continuous linear isomorphisms $\alpha : \mathsf{E}_1 \to \mathsf{E}_2$ and $\beta : \mathsf{F}_1 \to \mathsf{F}_2$ such that $A_2 = \beta \circ A_1 \circ \alpha^{-1}$.*

Definition G.14 *Let $A : \mathsf{E} \to \mathsf{F}$ be an injective continuous linear map. We say that A is a* ***splitting injection*** *if there are Banach spaces F_1 and F_2 with $\mathsf{F} \cong \mathsf{F}_1 \times \mathsf{F}_2$ and if A is equivalent to the injection $\operatorname{inj}_1 : \mathsf{F}_1 \to \mathsf{F}_1 \times \mathsf{F}_2$.*

Lemma G.5 *If $A : \mathsf{E} \to \mathsf{F}$ is a splitting injection as above then there exists a linear isomorphism $\delta : \mathsf{F} \to \mathsf{E} \times \mathsf{F}_2$ such that $\delta \circ A : \mathsf{E} \to \mathsf{E} \times \mathsf{F}_2$ is the injection $x \mapsto (x, 0)$.*

Proof. By definition there are isomorphisms $\alpha : \mathsf{E} \to \mathsf{F}_1$ and $\beta : \mathsf{F} \to \mathsf{F}_1 \times \mathsf{F}_2$ such that $\beta \circ A \circ \alpha^{-1}$ is the injection $\mathsf{F}_1 \to \mathsf{F}_1 \times \mathsf{F}_2$. Since α is an isomorphism we may compose as follows

$$\begin{aligned}
&(\alpha^{-1} \times \mathrm{id}_{\mathsf{E}}) \circ \beta \circ A \circ \alpha^{-1} \circ \alpha \\
&= (\alpha^{-1} \times \mathrm{id}_{\mathsf{E}}) \circ \beta \circ A \\
&= \delta \circ A
\end{aligned}$$

to get a map which is easily seen to have the correct form. ∎

If A is a splitting injection as above it easy to see that there are closed subspaces F_1 and F_2 of F such that $\mathsf{F} = \mathsf{F}_1 \oplus \mathsf{F}_2$ and such that A maps E isomorphically onto F_1 .

Definition G.15 *Let $A : \mathsf{E} \to \mathsf{F}$ be an surjective continuous linear map. We say that A is a **splitting surjection** if there are Banach spaces E_1 and E_2 with $\mathsf{E} \cong \mathsf{E}_1 \times \mathsf{E}_2$ and if A is equivalent to the projection $pr_1 : \mathsf{E}_1 \times \mathsf{E}_2 \to \mathsf{E}_1$.*

Lemma G.6 *If Let $A : \mathsf{E} \to \mathsf{F}$ is a splitting surjection then there is a linear isomorphism $\delta : \mathsf{F} \times \mathsf{E}_2 \to \mathsf{E}$ such that $A \circ \delta : \mathsf{F} \times \mathsf{E}_2 \to \mathsf{F}$ is the projection $(x, y) \mapsto x$.*

Proof. By definition there exist isomorphisms $\alpha : \mathsf{E} \to \mathsf{E}_1 \times \mathsf{E}_2$ and $\beta : \mathsf{F} \to \mathsf{E}_1$ such that $\beta \circ A \circ \alpha^{-1}$ is the projection $pr_1 : \mathsf{E}_1 \times \mathsf{E}_2 \to \mathsf{E}_1$. We form another map by composition by isomorphisms;

$$\begin{aligned} &\beta^{-1} \circ \beta \circ A \circ \alpha^{-1} \circ (\beta, \mathrm{id}_{\mathsf{E}_2}) \\ &= A \circ \alpha^{-1} \circ (\beta, \mathrm{id}_{\mathsf{E}_2}) := A \circ \delta \end{aligned}$$

and check that this does the job. ∎

If A is a splitting surjection as above it easy to see that there are closed subspaces E_1 and E_2 of E such that $\mathsf{E} = \mathsf{E}_1 \oplus \mathsf{E}_2$ and such that A maps E onto E_1 as a projection $x + y \mapsto x$.

Local (nonlinear) case.

Definition G.16 *Let $f_1 : (\mathsf{E}_1, p_1) \to (\mathsf{F}_1, q_1)$ be a local map. We say that f_1 is **locally equivalent** near p_1 to $f_2 : (\mathsf{E}_2, p_2) \to (\mathsf{F}_2, q_2)$ if there exist local diffeomorphisms $\alpha : \mathsf{E}_1 \to \mathsf{E}_2$ and $\beta : \mathsf{F}_1 \to \mathsf{F}_2$ such that $f_1 = \alpha \circ f_2 \circ \beta^{-1}$ (near p) or equivalently if $f_2 = \beta \circ f_1 \circ \alpha^{-1}$ (near p_2).*

Definition G.17 *Let $f :: \mathsf{E}, p \to \mathsf{F}, q$ be a local map. We say that f is a **locally splitting injection** or **local immersion** if there are Banach spaces F_1 and F_2 with $\mathsf{F} \cong \mathsf{F}_1 \times \mathsf{F}_2$ and if f is locally equivalent near p to the injection $\mathrm{inj}_1 :: (\mathsf{F}_1, 0) \to (\mathsf{F}_1 \times \mathsf{F}_2, 0)$.*

By restricting the maps to possibly smaller open sets we can arrange that $\beta \circ f \circ \alpha^{-1}$ is given by $U' \overset{\alpha^{-1}}{\to} U \overset{f}{\to} V \overset{\beta}{\to} U' \times V'$ which we will call a **nice** local injection.

Lemma G.7 *If f is a locally splitting injection as above there is an open set U_1 containing p and local diffeomorphism $\varphi : U_1 \subset \mathsf{F} \to U_2 \subset \mathsf{E} \times \mathsf{F}_2$ and such that $\varphi \circ f(x) = (x, 0)$ for all $x \in U_1$.*

Proof. This is done using the same idea as in the proof of lemma **??**.

$$\begin{array}{cccc} & (\mathsf{E},p) & \to & (\mathsf{F},q) \\ \alpha & \updownarrow & & \updownarrow \beta \\ & (\mathsf{F}_1,0) & \to & (\mathsf{F}_1\times\mathsf{F}_2,\quad (0,0)) \\ \alpha & \updownarrow & \mathrm{inj}_1 & \updownarrow \alpha^{-1}\times\mathrm{id} \\ & (\mathsf{E},p) & \to & (\mathsf{E}\times\mathsf{F}_2,\quad (p,0)) \end{array}$$

■

Definition G.18 *Let $f :: (\mathsf{E},p) \to (\mathsf{F},q)$ be a local map. We say that f is a **locally splitting surjection** or **local submersion** if there are Banach spaces E_1 and E_2 with $\mathsf{E} \cong \mathsf{E}_1 \times \mathsf{E}_2$ and if f is locally equivalent (at p) to the projection $pr_1 : \mathsf{E}_1 \times \mathsf{E}_2 \to \mathsf{E}_1$.*

Again, by restriction of the domains to smaller open sets we can arrange that projection $\beta \circ f \circ \alpha^{-1}$ is given by $U' \times V' \overset{\alpha^{-1}}{\to} U \overset{f}{\to} V \overset{\beta}{\to} U'$ which we will call a **nice** local projection.

Lemma G.8 *If f is a locally splitting surjection as above there are open sets $U_1 \times U_2 \subset \mathsf{F} \times \mathsf{E}_2$ and $V \subset \mathsf{F}$ together with a local diffeomorphism $\varphi : U_1 \times U_2 \subset \mathsf{F} \times \mathsf{E}_2 \to V \subset \mathsf{E}$ such that $f \circ \varphi(u,v) = u$ for all $(u,v) \in U_1 \times U_2$.*

Proof. *This is the local (nonlinear) version of lemma **??** and is proved just as easily. Examine the diagram for guidance if you get lost:*

$$\begin{array}{ccc} (\mathsf{E},p) & \to & (\mathsf{F},q) \\ \updownarrow & & \updownarrow \\ (\mathsf{E}_1\times\mathsf{E}_2,(0,0)) & \to & (\mathsf{E}_1,0) \\ \updownarrow & pr_1 & \updownarrow \\ (\mathsf{F}\times\mathsf{E}_2,(q,0)) & \to & (\mathsf{F},q) \end{array}$$

■

Theorem G.10 (local immersion) *Let $f :: (\mathsf{E},p)\to(\mathsf{F},q)$ be a local map. If $Df|_p : (\mathsf{E},p)\to(\mathsf{F},q)$ is a splitting injection then $f :: (\mathsf{E},p)\to(\mathsf{F},q)$ is a local immersion.*

Theorem G.11 (local immersion- finite dimensional case) *Let $f :: \mathbb{R}^n \to \mathbb{R}^n\times\mathbb{R}^k)$ be a map of constant rank n in some neighborhood of $0 \in \mathbb{R}^n$. Then there is $g_1 :: \mathbb{R}^n \to \mathbb{R}^n$ with $g_1(0) = 0$, and a $g_2 :: \mathbb{R}^n \to \mathbb{R}^n\times\mathbb{R}^k$ such that $g_2 \circ f \circ g_1^{-1}$ is just given by $x \mapsto (x,0) \in \mathbb{R}^n\times\mathbb{R}^k$.*

We have a similar but complementary theorem which we state in a slightly more informal manner.

Theorem G.12 (local submersion) *Let $f :: (\mathsf{E},p) \to (\mathsf{F},q)$ be a local map. If $Df|_p : (\mathsf{E},p) \to (\mathsf{F},q)$ is a splitting surjection then $f :: (\mathsf{E},p) \to (\mathsf{F},q)$ is a local submersion.*

Theorem G.13 (local submersion -finite dimensional case) *Let* $f :: (\mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R}^n, 0)$ *be a local map with constant rank* n *near* 0*. Then there are diffeomorphisms* $g_1 :: (\mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R}^n \times \mathbb{R}^k, 0)$ *and* $g_2 :: (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *such that near* 0 *the map*

$$g_2 \circ f \circ g_1^{-1} :: (\mathbb{R}^n \times \mathbb{R}^k, 0) \to (\mathbb{R}^n, 0)$$

is just the projection $(x, y) \mapsto x$*.*

If the reader thinks about what is meant by local immersion and local submersion he/she will realize that in each case the derivative map Df_p has full rank. That is, the rank of the Jacobian matrix in either case is a big as the dimensions of the spaces involved allow. Now rank is only a semicontinuous and this is what makes full rank extend from points out onto neighborhoods so to speak. On the other hand, we can get more general maps into the picture if we explicitly assume that the rank is locally constant. We will state and prove the following theorem only for the finite dimensional case. There is a Banach version of this theorem but

Theorem G.14 (The Constant Rank Theorem) *Let* $f : (\mathbb{R}^n, p) \to (\mathbb{R}^m, q)$ *be a local map such that* Df *has constant rank* r *in an open set containing* p*. Then there are local diffeomorphisms* $g_1 : (\mathbb{R}^n, p) \to (\mathbb{R}^n, 0)$ *and* $g_2 : (\mathbb{R}^m, q) \to (\mathbb{R}^m, 0)$ *such that* $g_2 \circ f \circ g_1^{-1}$ *is a local diffeomorphism near* 0 *with the form*

$$(x^1, ...x^n) \mapsto (x^1, ...x^r, 0, ..., 0).$$

Proof. Without loss of generality we may assume that $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ and that (reindexing) the $r \times r$ matrix

$$\left(\frac{\partial f^j}{\partial x^j} \right)_{1 \le i,j \le r}$$

is nonsingular in an open ball centered at the origin of $\mathbb{R}^n$. Now form a map $g_1(x^1, ...x^n) = (f^1(x), ..., f^r(x), x^{r+1}, ..., x^n)$. The Jacobian matrix of g_1 has the block matrix form

$$\begin{bmatrix} \left(\frac{\partial f^i}{\partial x^j} \right) & \\ 0 & I_{n-r} \end{bmatrix}$$

which has nonzero determinant at 0 and so by the inverse mapping theorem g_1 must be a local diffeomorphism near 0. Restrict the domain of g_1 to this possibly smaller open set. It is not hard to see that the map $f \circ g_1^{-1}$ is of the form $(z^1, ..., z^n) \mapsto (z^1, ..., z^r, \gamma^{r+1}(z), ..., \gamma^m(z))$ and so has Jacobian matrix of the form

$$\begin{bmatrix} I_r & 0 \\ * & \left(\frac{\partial \gamma^i}{\partial x^j} \right) \end{bmatrix}.$$

Now the rank of $\left(\frac{\partial \gamma^i}{\partial x^j} \right)_{r+1 \le i \le m,\ r+1 \le j \le n}$ must be zero near 0 since the $\text{rank}(f) = \text{rank}(f \circ h^{-1}) = r$ near 0. On the said (possibly smaller) neighborhood we now define the map $g_2 : (\mathbb{R}^m, q) \to (\mathbb{R}^m, 0)$ by

$$(y^1, ..., y^m) \mapsto (y^1, ..., y^r, y^{r+1} - \gamma^{r+1}(y_*, 0), ..., y^m - \gamma^m(y_*, 0))$$

where $(y_*, 0) = (y^1, ..., y^r, 0, ..., 0)$. The Jacobian matrix of g_2 has the form

$$\begin{bmatrix} I_r & 0 \\ * & I \end{bmatrix}$$

and so is invertible and the composition $g_2 \circ f \circ g_1^{-1}$ has the form

$$\begin{aligned} z &\overset{f\circ g_1^{-1}}{\mapsto} (z_*, \gamma_{r+1}(z), ..., \gamma_m(z)) \\ &\overset{g_2}{\mapsto} (z_*, \gamma_{r+1}(z) - \gamma_{r+1}(z_*, 0), ..., \gamma_m(z) - \gamma_m(z_*, 0)) \end{aligned}$$

where $(z_*, 0) = (z^1, ..., z^r, 0, ..., 0)$. It is not difficult to check that $g_2 \circ f \circ g_1^{-1}$ has the required form near 0. ■

Remark G.3 *In this section we have defined several things in terms of the canonical projections onto, or injections into, Cartesian products. The fact that we projected onto the first factor and injected into the first factor is irrelevant and it could have been either factor. Thus we will freely use the theorem as if we had made any other choice of factor. This is the case in our definitions of submanifolds, immersions, and submersions.*

G.0.18 The Tangent Bundle of an Open Subset of a Banach Space

Later on we will define the notion of a tangent space and tangent bundle for a differentiable manifold which locally looks like a Banach space. Here we give a definition that applies to the case of an open set U in a Banach space.

Definition G.19 *Let E be a Banach space and $U \subset \mathsf{E}$ an open subset. A **tangent vector** at $x \in U$ is a pair (x, v) where $v \in \mathsf{E}$. The **tangent space** at $x \in U$ is defined to be $T_xU := T_x\mathsf{E} := \{x\} \times \mathsf{E}$ and the tangent bundle TU over U is the union of the tangent spaces and so is just $TU = U \times \mathsf{E}$. Similarly the cotangent bundle over U is defined to be $T^*U = U \times \mathsf{E}^*$. A tangent space $T_x\mathsf{E}$ is also sometimes called the fiber at x.*

We give this definition in anticipation of our study of the tangent space at a point of a differentiable manifold. In this case however, it is often not necessary to distinguish between T_xU and E since we can often tell from context that an element $v \in \mathsf{E}$ is to be interpreted as based at some point $x \in U$. For instance a vector field in this setting is just a map $X : U \to \mathsf{E}$ but where $X(x)$ should be thought of as based at x.

Definition G.20 *If $f : U \to \mathsf{F}$ is a C^r map into a Banach space F then the **tangent map** $Tf : TU \to T\mathsf{F}$ is defined by*

$$Tf \cdot (x, v) = (f(x), Df(x) \cdot v).$$

The map takes the tangent space $T_xU = T_x\mathsf{E}$ linearly into the tangent space $T_{f(x)}\mathsf{F}$ for each $x \in U$. The projection onto the first factor is written τ_U :

$TU = U \times \mathsf{E} \to U$ and given by $\tau_U(x, v) = x$. We also have a projection $\pi_U : T^*U = U \times \mathsf{E}^* \to U$ defined similarly.

If $f : U \to V$ is a diffeomorphism of open sets U and V in E and F respectively then Tf is a diffeomorphism that is linear on the fibers and such that we have a commutative diagram:

$$\begin{array}{cccc} TU = & U \times \mathsf{E} & \overset{Tf}{\to} & V \times \mathsf{F} = TV \\ pr_1 & \downarrow & & \downarrow pr_1 \\ & U & \underset{f}{\to} & V \end{array}$$

The pair is an example of what is called a **local bundle map**. In this context we will denote the projection map $TU = U \times \mathsf{E} \to U$ by τ_U.

The chain rule looks much better if we use the tangent map:

Theorem G.15 *Let U_1 and U_2 be open subsets of Banach spaces E_1 and E_2 respectively. Suppose we have differentiable (resp. $C^r, r \geq 1$) maps composing as*

$$U_1 \xrightarrow{f} U_2 \xrightarrow{g} \mathsf{E}_3$$

where E_3 is a third Banach space. Then the composition is $g \circ f$ differentiable (resp. $C^r, r \geq 1$) and $T(g \circ f) = Tg \circ Tf$

$$\begin{array}{ccccc} TU_1 & \overset{Tf}{\to} & TU_2 & \overset{Tg}{\to} & T\mathsf{E}_3 \\ \tau_{U_1} \downarrow & & \tau_{U_2} \downarrow & & \downarrow \tau_{E_3} \\ U_1 & \overset{f}{\to} & U_2 & \overset{g}{\to} & \mathsf{E}_3 \end{array}$$

Notation G.7 *(and convention) There are three ways to express the "differential/derivative" of a differentiable map $f : U \subset \mathsf{E} \to \mathsf{F}$.*

1. *The first is just $Df : \mathsf{E} \to \mathsf{F}$ or more precisely $Df|_x : \mathsf{E} \to \mathsf{F}$ for any point $x \in U$.*

2. *This one is new for us. It is common but not completely standard :*

   $$dF : TU \to \mathsf{F}$$

   This is just the map $(x, v) \to Df|_x v$. We will use this notation also in the setting of maps from manifolds into vector spaces where there is a canonical trivialization of the tangent bundle of the target manifold (all of these terms will be defined). The most overused symbol for various "differentials" is d. We will use this in connection with Lie group also.

3. *Lastly the tangent map $Tf : TU \to T\mathsf{F}$ which we defined above. This is the one that generalizes to manifolds without problems.*

   In the local setting that we are studying now these three all contain essentially the same information so the choice to use one over the other is merely aesthetic.

In the local setting that we are studying now all three of these maps contain essentially the same information.

It should be noted that some authors use df to mean any of the above maps and their counterparts in the general manifold setting. This leads to less confusion than one might think since one always has context on one's side.

G.0.19 Extrema

A real valued function f on a topological space X is continuous at $x_0 \in X$ if for every $\epsilon > 0$ there is a open set U_ϵ containing x_0 such that

$$U_\epsilon \subset \{x : f(x) > f(x_0) - \epsilon\} \cap \{x : f(x) < f(x_0) + \epsilon\}.$$

Baire introduced the notion of semicontinuity by essentially using only one of the intersected sets above. Namely, f is lower semicontinuous at x_0 if for every $\epsilon > 0$ there is a open set U_ϵ containing x_0 such that

$$U_\epsilon \subset \{x : f(x) > f(x_0) - \epsilon\}.$$

Of course there is the symmetrical notion of upper semicontinuity. Lower semicontinuity is appropriately introduced in connection with the search for x such that $f(x)$ is a (local) minimum. Since replacing f by $-f$ interchanges upper and lower semicontinuity and also maxima and minima it we be sufficient to limit the discussion to minima. If the topological space is Hausdorff then we can have a simple and criterion for the existence of a minimum:

Theorem G.16 *Let M be Hausdorff and $f : M \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function such that there exists a number $c < \infty$ such that $M_{f,c} := \{x \in M : f(x) \leq c\}$ is nonempty and sequentially compact then there exists a minimizing $x_0 \in M$:*

$$f(x_0) = \inf_{x \in M} f(x).$$

This theorem is a sort of generalization of the theorem that a continuous function on a compact set achieve a minimum (and maximum). We need to include semicontinuous functions since even some of the most elementary examples for geometric minimization problems naturally involve functionals that are only semicontinuous for very natural topologies on the set M.

Now if M is a convex set in some vector space then if $f : M \to \mathbb{R}$ is strictly convex on M (meaning that $0 < t < 1 \implies f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2)$) then a simple argument shows that f can have at most one minimizer x_0 in M. We are thus led to a wish list in our pursuit of a minimum for a function. We hope to simultaneously have

1. M convex,
2. $f : M \to \mathbb{R}$ strictly convex,
3. f lower semicontinuous,

4. there exists $c < \infty$ so that $M_{f,c}$ is nonempty and sequentially compact.

Just as in elementary calculus, a differentiable function f has an extrema at a point x_0 only if x_0 is a boundary point of the domain of f or, if x_0 is an interior point, $Df(x_0) = 0$. In case $Df(x_0) = 0$ for some interior point x_0, we may not jump to the conclusion that f achieves an extrema at x_0. As expected, there is a second derivative test that may help in this case:

Theorem G.17 *Let $f : U \subset \mathsf{E} \to \mathsf{F}$ be twice differentiable at $x_0 \in U$ (assume U is open). If f has a local extrema at x_0 then $D^2 f(x_0)(v, v) \geq 0$ for all $v \in \mathsf{E}$. If $D^2 f(x_0)$ is a nondegenerate bilinear form then if $D^2 f(x_0)(v, v) > 0$ (resp. < 0) then f achieves a minimum (resp. maximum) at x_0.*

In practice we may have a choice of several possible sets M, topologies for M or the vector space contains M and so on. But then we must strike a balance since a topology with more open sets has more lower semicontinuous functions while one with less open sets means more sequentially compact sets.

G.1 Problem Set

1. Find the matrix that represents (with respects to standard bases) the derivative the map $f : \mathbb{R}^n \to \mathbb{R}^m$ given by

   a) $f(x) = Ax$ for an $m \times n$ matrix A.

   b) $f(x) = x^t Ax$ for an $n \times n$ matrix A (here $m = 1$).

   c) $f(x) = x^1 x^2 \cdots x^n$ (here $m = 1$).

2. Find the derivative of the map $F : L^2([0, 1]) \to L^2([0, 1])$ given by

$$F[f](x) = \int_0^1 k(x, y) \, [f(y)]^2 \, dy$$

   where $k(x, y)$ is a bounded continuous function on $[0, 1] \times [0, 1]$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and define $F : C[0, 1] \to C[0, 1]$ by

$$F(g) := f \circ g$$

   Show that F is differentiable and $DF|_g : C[0, 1] \to C[0, 1]$ is the linear map given by $\left(DF|_g \cdot u\right)(t) = f'(g(t)) \cdot u(t)$.

4. Let $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be C^∞ and define

$$S[c] = \int_0^1 L(c(t), c'(t), t) dt$$

which is defined on the Banach space B of all C^1 curves $c : [0,1] \to R^n$ with $c(0) = 0$ and $c(1) = 0$ and with the norm $\|c\| = \sup_{t\in[0,1]}\{|c(t)| + |c'(t)|\}$. Find a function $g_c : [0,1] \to R^n$ such that

$$DS|_c \cdot b = \int_0^1 \langle g_c(t), b(t)\rangle dt$$

or in other words,

$$DS|_c \cdot b = \int_0^1 \sum_{i=1}^n g_c^i(t) b^i(t) dt.$$

5. In the last problem, if we had not insisted that $c(0) = 0$ and $c(1) = 0$, but rather that $c(0) = x_0$ and $c(1) = x_1$, then the space wouldn't even have been a vector space let alone a Banach space. But this fixed endpoint family of curves is exactly what is usually considered for functionals of this type. Anyway, convince yourself that this is not a serious problem by using the notion of an affine space (like a vector space but no origin and only differences are defined.). Is the tangent space of the this space of fixed endpoint curves a Banach space?

   Hint: If we choose a fixed curve c_0 which is the point in the Banach space at which we wish to take the derivative then we can write $\mathcal{B}_{\vec{x}_0\vec{x}_1} = \mathcal{B} + c_0$ where

   $$\mathcal{B}_{\vec{x}_0\vec{x}_1} = \{c : c(0) = \vec{x}_0 \text{ and } c(1) = \vec{x}_1\}$$
   $$\mathcal{B} = \{c : c(0) = 0 \text{ and } c(1) = 0\}$$

   Then we have $T_{c_0}\mathcal{B}_{\vec{x}_0\vec{x}_1} \cong \mathcal{B}$. Thus we should consider $DS|_{c_0} : \mathcal{B} \to \mathcal{B}$.

6. Let $\mathrm{Fl}_t(.)$ be defined by $\mathrm{Fl}_t(x) = (t+1)x$ for $t \in (-1/2, 1/2)$ and $x \in \mathbb{R}^n$. Assume that the map is jointly C^1 in both variable. Find the derivative of

   $$f(t) = \int_{D(t)} (tx)^2 \, dx$$

   at $t = 0$, where $D(t) := \mathrm{Fl}_t(D)$ the image of the disk $D = \{|x| \le 1\}$.

   Hint: Find the Jacobian $J_t := \det[DFl_t(x)]$ and then convert the integral above to one just over $D(0) = D$.

7. Let $M_{n\times n}(\mathbb{R})$ be the vector space of $n \times n$ matrices with real entries and let $\det : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ be the determinant map. The derivative at the identity element I should be a linear map $D\det(I) : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$. Show that $D\det(I) \cdot B = Tr(B)$. More generally, show that $D\det(A) \cdot B = Tr((\mathrm{cof}\, A)^t B)$ where $\mathrm{cof}\, A$ is the matrix of cofactors of A.

   What is $\frac{\partial}{\partial x_{ij}} \det X$ where $X = (x_{ij})$?

8. Let $A : U \subset \mathsf{E} \to L(\mathsf{F}, \mathsf{F})$ be a C^r map and define $F : U \times \mathsf{F} \to \mathsf{F}$ by $F(u, f) := A(u)f$. Show that F is also C^r.

9. Show that if F is any closed subset of $\mathbb{R}^n$ there is a C^∞-function f whose zero set $\{x : f(x) = 0\}$ is exactly F.

10. Let U be an open set in $\mathbb{R}^n$. For $f \in C^k(U)$ and $S \subset U$ a compact set, let $\|f\|_k^S := \sum_{|\alpha| \le k} \sup_{x \in S} \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(x) \right|$. a) Show that (1) $\|rf\|_k^S = |r| \, \|f\|_k^S$ for any $r \in \mathbb{R}$, (2) $\|f_1 + f_2\|_k^S \le \|f_1\|_k^S + \|f_2\|_k^S$ for any $f_1, f_2 \in C^k(U)$, (3) $\|fg\|_k^S \le \|f\|_k^S \, \|g\|_k^S$ for $f, g \in C^k(U)$.
    b) Let $\{K_i\}$ be a compact subsets of U such that $U = \bigcup_i K_i$. Show that $d(f, g) := \sum_{i=0}^\infty \frac{1}{2^i} \frac{\|f-g\|_k^{K_i}}{1 + \|f-g\|_k^{K_i}}$ defines a complete metric space structure on $C^k(U)$.

11. Let E and F be real Banach spaces. A function $f : \mathsf{E} \to \mathsf{F}$ is said to be homogeneous of degree k if $f(rx) = rf(x)$ for all $r \in \mathbb{R}$ and $x \in \mathsf{E}$. Show that if f is homogeneous of degree k and is differentiable, then $Df(v) \cdot v = kf(v)$.

12. Show that the implicit mapping theorem implies the inverse mapping theorem. Hint: Consider $g(x, y) = f(x) - y$ for $f : U \to \mathsf{F}$.

G.2 Exterior Derivative

Let $\omega_U : U \to L^k_{alt}(\mathbb{R}^n; \mathbb{R}^n)$. In the following calculation we will identify $L^k_{alt}(\mathbb{R}^n; \mathbb{R}^n)$ with the $L(\wedge^k \mathbb{R}^n, \mathbb{R}^n)$. For $\xi_0, ..., \xi_k$ maps $\xi_i : U \to \mathbb{R}^n$ we have

$$
\begin{aligned}
&D\langle \omega_U, \xi_0, ..., \xi_k \rangle(x) \cdot \xi_i \\
&= \left.\frac{d}{dt}\right|_0 \langle \omega_U(x + t\xi_i), \xi_0(x + t\xi_i) \wedge ... \wedge \widehat{\xi_i} \wedge ... \wedge \xi_k(x + t\xi_i) \rangle \\
&= \langle \omega_U(x), \left.\frac{d}{dt}\right|_0 [\xi_0(x + t\xi_i) \wedge ... \wedge \widehat{\xi_i} \wedge ... \wedge \xi_k(x + t\xi_i)] \rangle \\
&+ \langle \left.\frac{d}{dt}\right|_0 \omega_U(x), \xi_0(x + t\xi_i) \wedge ... \wedge \widehat{\xi_i} \wedge ... \wedge \xi_k(x + t\xi_i) \rangle \\
&= \langle \omega_U(x), \sum_{j=0}^{i-1} (-1)^j \xi_j'(x)\xi_i \wedge [\xi_0(x) \wedge ... \wedge \widehat{\xi_j} \wedge ... \wedge \widehat{\xi_i} \wedge ... \wedge \xi_k(x)] \rangle \\
&+ \langle \omega_U(x), \sum_{j=i+1}^{k} (-1)^{j-1} \xi_j'(x)\xi_i \wedge [\xi_0(x) \wedge ... \wedge \widehat{\xi_i} \wedge ... \wedge \widehat{\xi_j} \wedge ... \wedge \xi_k(x)] \rangle \\
&+ \langle \omega_U'(x)\xi_i, \xi_0(x) \wedge ... \wedge \widehat{\xi_i} \wedge ... \wedge \xi_k(x) \rangle
\end{aligned}
$$

Theorem G.18 *There is a unique graded (sheaf) map $d : \Omega_M \to \Omega_M$, called the **exterior derivative**, such that*

1) $d \circ d = 0$

2) d is a graded derivation of degree one, that is

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta) \tag{G.1}$$

for $\alpha \in \Omega_M^k$.

Furthermore, if $\omega \in \Omega^k(U)$ and $X_0, X_1, ..., X_k \in \mathfrak{X}_M(U)$ then

$$\begin{aligned} d\omega = & \sum_{0 \le i \le k} (-1)^i X_i(\omega(X_0, ..., \widehat{X_i}, ..., X_k)) \\ & + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, ..., \widehat{X_i}, ..., \widehat{X_j}, ..., X_k). \end{aligned}$$

In particular, we have the following useful formula for $\omega \in \Omega_M^1$ and $X, Y \in \mathfrak{X}_M(U)$:

$$d\omega(X, Y) = X(\omega(Y)) - Y\omega(X) - \omega([X, Y])$$

Proof. First we give a local definition in terms of coordinates and then show that the global formula **??** agree with the local formula. Let U, ψ be a local chart on M. We will first define the exterior derivative on the open set $V = \psi(U) \subset \mathbb{R}^n$. Let $\xi_0, ..., \xi_k$ be local vector fields. The local representation of a form ω is a map $\omega_U : V \to L_{skew}^k(\mathbb{R}^n;\mathbb{R}^n)$ and so has at some $x \in V$ has a derivative $D\omega_U(x) \in L(\mathbb{R}^n, L_{skew}^k(\mathbb{R}^n; \mathbb{R}^n))$. We define

$$d\omega_U(x)(\xi_0, ..., \xi_k) := \sum_{i=0}^{k} (-1)^i (D\omega_U(x)\xi_i(x))(\xi_0(x), ..., \widehat{\xi_i}, ..., \xi_k(x))$$

where $D\omega_U(x)\xi_i(x) \in L_{skew}^k(\mathbb{R}^n; \mathbb{R}^n)$. This certainly defines a differential form in $\Omega^{k+1}(U)$. Let us call the right hand side of this local formula LOC. We wish to show that the global formula in local coordinates reduces to this local formula. Let us denote the first and second term of the global when expressed in local coordinates $L1$ and $L2$. Using, our calculation G.2 we have

$$L1 = \sum_{i=0}^{k} (-1)^i \xi_i(\omega(\xi_0, ..., \widehat{\xi_i}, ..., \xi_k)) = \sum_{i=0}^{k} (-1)^i D(\omega(\xi_0, ..., \widehat{\xi_i}, ..., \xi_k))(x)\xi_i(x)$$

$$
\begin{aligned}
=&\langle \omega_U(x), \sum_{i=0}^{k}(-1)^i \sum_{j=0}^{i-1}(-1)^j \xi_j^{'}(x)\xi_i \wedge [\xi_0(x) \wedge ... \wedge \widehat{\xi_j} \wedge ... \wedge \widehat{\xi_i} \wedge ... \wedge \xi_k(x)]\rangle \\
&\quad + \langle \omega_U(x), \sum_{i=0}^{k}(-1)^i \sum_{j=i+1}^{k}(-1)^{j-1}\xi_j^{'}(x)\xi_i \wedge [\xi_0(x) \wedge ... \wedge \widehat{\xi_i} \wedge ... \\
&\qquad\qquad ... \wedge \widehat{\xi_j} \wedge ... \wedge \xi_k(x)]\rangle \\
&\qquad\qquad + \sum_{i=0}^{k}(-1)^i \langle \omega_U'(x)\xi_i, \xi_0(x) \wedge ... \wedge \widehat{\xi_i} \wedge ... \wedge \xi_k(x)\rangle \\
&= \langle \omega_U(x), \sum_{i=0}^{k}\sum_{i<j}^{k}(-1)^{i+j}(\xi_j^{'}(x)\xi_i - \xi_i^{'}(x)\xi_j) \wedge \xi_0(x) \wedge ... \\
&\qquad\qquad .... \wedge \widehat{\xi_j} \wedge ... \wedge \widehat{\xi_i} \wedge ... \wedge \xi_k(x)\rangle \\
&= LOC + L2.
\end{aligned}
$$

So our global formula reduces to the local one when expressed in local coordinates. ■

Remark G.4 *d is a local operator and so commutes with restrictions to open sets. In other words, if U is an open subset of M and d_U denotes the analogous operator on the manifold U then $d_U\alpha\big|_U = (d\alpha)\big|_U$. This operator can thus be expressed locally. In order to save on notation we will use d to denote the exterior derivative on any manifold, forms of any degree and for the restrictions d_U for any open set. It is exactly because d is a natural operator that this will cause no harm.*

G.3 Topological Vector Spaces

Definition G.21 *A **topological vector space (TVS)** is a vector space V with a Hausdorff topology such that the addition and scalar multiplication operations are (jointly) continuous.*

Definition G.22 *A sequence (or net) x_n in a TVS is call a Cauchy sequence if and only if for every neighborhood U of 0 there is a number N_U such that $x_l - x_k \in U$ for all $k, l \geq N_U$. A TVS is called complete if every Cauchy sequence (or net) is convergent.*

A relatively nice situation is when V has a norm which induces the topology. Recall that a **norm** is a function $\|\| : v \mapsto \|v\| \in \mathbb{R}$ defined on V such that for all $v, w \in \mathsf{V}$ we have

1. $\|v\| \geq 0$ and $\|v\| = 0$ if and only if $v = 0$,
2. $\|v + w\| \leq \|v\| + \|w\|$,
3. $\|\alpha v\| = |\alpha| \, \|v\|$ for all $\alpha \in \mathbb{R}$.

In this case we have a metric on V given by $\operatorname{dist}(v, w) := \|v - w\|$.

Definition G.23 *A* ***seminorm*** *is a function* $\|.\| : v \mapsto \|v\| \in \mathbb{R}$ *such that 2) and 3) above hold but instead of 1) we require only that* $\|v\| \geq 0$.

Definition G.24 *A* ***locally convex topological vector space*** V *is a TVS such that it's topology is generated by a family of seminorms* $\{\|.\|_\alpha\}_\alpha$. *This means that we give* V *the weakest topology such that all* $\|.\|_\alpha$ *are continuous. Since we have taken a TVS to be Hausdorff we require that the family of seminorms is* ***sufficient*** *in the sense that for each* $x \in \mathsf{V}$ *we have* $\bigcap\{x : \|x\|_\alpha = 0\} = \{0\}$. *A locally convex topological vector space is sometimes called a locally convex space and so we abbreviate the latter to* ***LCS***.

Example G.2 *Let* Ω *be an open set in* $\mathbb{R}^n$ *or any manifold. For each* $x \in \Omega$ *define a seminorm* ρ_x *on* $C(\Omega)$ *by* $\rho_x(f) = f(x)$. *This family of seminorms makes* $C(\Omega)$ *a topological vector space. In this topology convergence is pointwise convergence. Also,* $C(\Omega)$ *is not complete with this TVS structure.*

Definition G.25 *An LCS which is complete (every Cauchy sequence converges) and metrizable is called a* ***Frechet space***.

Definition G.26 *A curve* $c : \mathbb{R} \to \mathsf{V}$ *where* V *is a LCS is differentiable if the limit*

$$\dot{c}(t) := \lim_{\epsilon \to 0} \frac{c(t+\epsilon) - c(t)}{\epsilon}$$

exists for all t *thus defining a new curve* $\frac{dc}{dt} : \mathbb{R} \to \mathsf{V}$. *This curve may also have a derivative and so on. It all the iterated derivatives* $\frac{d^k c}{dt^k}$ *exist then we say the curve is* ***smooth***.

Definition G.27 *Map* $f : \mathsf{V} \to \mathsf{W}$ *between locally convex spaces is called smooth if* $f \circ c$ *is smooth for every smooth curve* $c : \mathbb{R} \to \mathsf{V}$.

Notice that in this general setting we have so far only defined smoothness for curves and not for maps between open subsets of a LCS. We will however, make a new very flexible definition of smoothness that is appropriate for the locally convex spaces which are not Banach spaces.

Appendix H

Existence and uniqueness for differential equations

Theorem H.1 *Let E be a Banach space and let $X : U \subset E \to E$ be a smooth map. Given any $x_0 \in U$ there is a smooth curve $c : (-\epsilon, \epsilon) \to U$ with $c(0) = x_0$ such that $c'(t) = X(c(t))$ for all $t \in (-\epsilon, \epsilon)$. If $c_1 : (-\epsilon_1, \epsilon_1) \to U$ is another such curve with $c_1(0) = x_0$ and $c_1'(t) = X(c(t))$ for all $t \in (-\epsilon_1, \epsilon_1)$ then $c = c_1$ on the intersection $(-\epsilon_1, \epsilon_1) \cap (-\epsilon, \epsilon)$. Furthermore, there is an open set V with $x_0 \in V \subset U$ and a smooth map $\Phi : V \times (-a, a) \to U$ such that $t \mapsto c_x(t) := \Phi(x, t)$ is a curve satisfying $c'(t) = X(c(t))$ for all $t \in (-a, a)$.*

H.0.1 Differential equations depending on a parameter.

Theorem H.2 *Let J be an open interval on the real line containing 0 and suppose that for some Banach spaces E and F we have a smooth map $F : J \times U \times V \to \mathsf{F}$ where $U \subset \mathsf{E}$ and $V \subset \mathsf{F}$. Given any fixed point $(x_0, y_0) \in U \times V$ there exist a subinterval $J_0 \subset J$ containing 0 and open balls $B_1 \subset U$ and $B_2 \subset V$ with $(x_0, y_0) \in B_1 \times B_2$ and a unique smooth map*

$$\beta : J_0 \times B_1 \times B_2 \to V$$

such that

1) $\frac{d}{dt}\beta(t, x, y) = F(t, x, \beta(t, x, y))$ for all $(t, x, y) \in J_0 \times B_1 \times B_2$ and
2) $\beta(0, x, y) = y$.
Furthermore,
3) if we let $\beta(t, x) := \beta(t, x, y)$ for fixed y then

$$\begin{aligned} \frac{d}{dt} D_2\beta(t, x) \cdot v &= D_2F(t, x, \beta(t, x)) \cdot v \\ &\quad + D_3F(t, x, \beta(t, x)) \cdot D_2\beta(t, x) \cdot v \end{aligned}$$

for all $v \in \mathsf{E}$.

H.0.2 Smooth Banach Vector Bundles

The tangent bundle and cotangent bundle are examples of a general object called a (smooth) vector bundle which we have previously defined in the finite dimensional case. As a sort of review and also to introduce the ideas in the case of infinite dimensional manifolds we will define again the notion of a smooth vector bundle. For simplicity we will consider only smooth manifold and maps in this section. Let E be a Banach space. The most important case is when E is a finite dimensional vector space and in that case we might as well take $\mathsf{E} = \mathbb{R}^n$. It will be convenient to introduce the concept of a general fiber bundle and then specialize to vector bundles. The following definition is not the most efficient logically since there is some redundancy built in but is presented in this form for pedagogical reasons.

Definition H.1 *Let F be a smooth manifold modeled on F. A smooth **fiber bundle** $\xi = (E, \pi_E, M, F)$ with typical fiber F consists of*

*1) smooth manifolds E and M referred to as the **total space** and the **base space** respectively and modeled on Banach spaces $\mathsf{M} \times \mathsf{F}$ and M respectively;*

*2) a smooth surjection $\pi_E : E \to M$ such that each **fiber** $E_x = \pi^{-1}\{x\}$ is diffeomorphic to F;*

*3) a cover of the **base space** M by domains of maps $\phi_\alpha : E_{U_\alpha} := \pi_E^{-1}(U_\alpha) \to U_\alpha \times F$, called **bundle charts,** which are such that the following diagram commutes:*

$$\begin{tikzcd} E_{U_\alpha} \arrow[rr, "(\pi_{U_\alpha}{,}\Phi_\alpha)"] \arrow[dr, "\pi_E"'] & & U_\alpha \times F \arrow[dl] \\ & U_\alpha & \end{tikzcd} .$$

Thus each ϕ_α is of the form $(\pi_{U_\alpha}, \Phi_\alpha)$ where $\pi_{U_\alpha} := \pi_E | U_\alpha$ and $\Phi_\alpha : E_{U_\alpha} \to F$ is a smooth submersion.

Definition H.2 *The family of bundle charts whose domains cover the base space of a fiber bundle as in the above definition is called a **bundle atlas**.*

For all $x \in U_\alpha$, each restriction $\Phi_{\alpha,x} := \Phi_\alpha|_{E_x}$ is a diffeomorphism onto $\mathbf{F}$. Whenever we have two bundle charts $\phi_\alpha = (\pi_E, \Phi_\alpha)$ and $\phi_\beta = (\pi_E, \Phi_\beta)$ such that $U_\alpha \cap U_\beta \neq \emptyset$ then for every $x \in U_\alpha \cap U_\beta$ we have the diffeomorphism $\Phi_{\alpha\beta,x} = \Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1} : F \to F$. Thus we have map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$ given by $g_{\alpha\beta}(x) := \Phi_{\alpha\beta,x}$. Notice that $g_{\beta\alpha} \circ g_{\alpha\beta}^{-1} = \mathrm{id}$. The maps so formed satisfy the following cocycle conditions:

$$g_{\gamma\beta} \circ g_{\alpha\gamma} = g_{\alpha\beta} \text{ whenever } U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset.$$

Let ξ be as above and let U be open in M. Suppose we have a smooth map $\phi : E_U \to U \times F$ such that the diagram

$$\begin{tikzcd} E_U \arrow[rr, "\phi"] \arrow[dr, "\pi_U"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

where $E_U := \pi_E^{-1}(U)$ as before. We call ϕ a trivialization and even if ϕ was not one of the bundle charts of a given bundle atlas, it must have the form (π_E, Φ) and we may enlarge the atlas by including this map. We do not wish to consider the new atlas as determining a new bundle so instead we say that the new atlas is equivalent. There is a unique maximal atlas for the bundle which just contains every possible trivialization.

Now given any open set U in the base space M of a fiber bundle $\xi = (E, \pi_E, M, F)$ we may form the restriction $\xi| U$ which is the fiber bundle $(\pi_E^{-1}(U), \pi_E| U, U, F)$. To simplify notation we write $E_U := \pi_E^{-1}(U)$ and $\pi_E| U := \pi_U$. This is a special case of the notion of a pull-back bundle.

One way in which vector bundles differ from general fiber bundles is with regard to the existence on global sections. A vector bundle always has at least one global section. Namely, the **zero section** $0_E : M \to E$ which is given by $x \mapsto 0_x \in E_x$. Our main interest at this point is the notion of a vector bundle. Before we proceed with our study of vector bundles we include one example of fiber bundle that is not a vector bundle.

Example H.1 *Let M and F be smooth manifolds and consider the projection map $pr_1 : M \times F \to M$. This is a smooth fiber bundle with typical fiber F and is called a* ***product bundle*** *or a* ***trivial bundle****.*

Example H.2 *Consider the tangent bundle $\pi : TM \to M$ of a smooth manifold modeled on $\mathbb{R}^n$. This is certainly a fiber bundle (in fact, a vector bundle) with typical fiber $\mathbb{R}^n$ but we also have the* ***bundle of nonzero vectors*** *$\pi^\times : TM^\times \to M$ defined by letting $TM^\times := \{v \in TM : v \neq 0\}$ and $\pi^\times := \pi|_{TM^\times}$. This bundle may have no global sections.*

Remark H.1 *A "structure" on a fiber bundle is determined by requiring that the atlas be paired down so that the transition maps all have values in some subset G (usually a subgroup) of $\mathrm{Diff}(F)$. Thus we speak of a $G-$***atlas** *for $\xi = (E, \pi_E, M, F)$. In this case, a trivialization $(\pi_E, \Phi) : E_U \to U \times F$ is compatible with a given $G-$atlas $\mathcal{A}(G)$ if $\Phi_{\alpha,x} \circ \Phi_x^{-1} \in G$ and $\Phi_x \circ \Phi_{\alpha,x}^{-1} \in G$ for all $(\pi_E, \Phi_\alpha) \in \mathcal{A}(G)$. The set of all trivializations (bundle charts) compatible which a given $G-$atlas is a maximal $G-$atlas and is called a G***-structure.*** *Clearly, any $G-$atlas determines a G-structure. A fiber bundle $\xi = (E, \pi_E, M, F)$ with a G-atlas is called a G-bundle and any two G-atlases contained in the same maximal G-atlas are considered equivalent and determine the same G-bundle. We will study this idea in detail after we have introduced the notion of a Lie group.*

We now introduce our current object of interest.

Definition H.3 *A (real)* ***vector bundle*** *is a fiber bundle $\xi = (E, \pi_E, M, \mathsf{E})$ with typical fiber a (real) Banach space E such that for each pair of bundle chart domains U_α and U_β with nonempty intersection, the map*

$$g_{\alpha\beta} : x \mapsto \Phi_{\alpha\beta,x} := \Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1}$$

is a C^∞ morphism $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathsf{E})$. If E is finite dimensional, say $\mathsf{E} = \mathbb{R}^n$, then we say that $\xi = (E, \pi_E, M, \mathbb{R}^n)$ has rank n.

So if $v_x \in \pi_E^{-1}(x) \subset E_{U_\alpha}$ then $\phi_\alpha(v_x) = (x, \Phi_{\alpha,x}(v_x))$ for $\Phi_{\alpha,x} : E_x \to \mathsf{E}$ a diffeomorphism. Thus we can transfer the vector space structure of V to each fiber E_x in a well defined way since $\Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1} \in GL(\mathsf{E})$ for any x in the intersection of two VB-chart domains U_α and U_β. Notice that we have $\phi_\alpha \circ \phi_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x) \cdot v)$ where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathsf{E})$ is differentiable and is given by $g_{\alpha\beta}(x) = \Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1}$. Notice that $g_{\alpha\beta}(x) \in GL(\mathsf{E})$.

A **complex vector bundle** is defined in an analogous way. For a complex vector bundle the typical fiber is a complex vector space (Banach space) and the transition maps have values in $GL(\mathsf{E}; \mathbb{C})$.

The set of all sections of real (resp. complex) vector bundle is a vector space over $\mathbb{R}$ (resp. $\mathbb{C}$) and a module over the ring of smooth real valued (resp. complex valued) functions.

Remark H.2 *If E is of finite dimension then the smoothness of the maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathsf{E})$ is automatic.*

Definition H.4 *The maps $g_{\alpha\beta}(x) := \Phi_{\alpha,x} \circ \Phi_{\beta,x}^{-1}$ are called **transition maps**.*

The transition maps always satisfy the following **cocycle condition**:

$$g_{\gamma\beta}(x) \circ g_{\beta\alpha}(x) = g_{\gamma\alpha}(x)$$

In fact, these maps encode the entire vector bundle up to isomorphism:

Remark H.3 *The following definition assumes the reader knows the definition of a Lie group and has a basic familiarity with Lie groups and Lie group homomorphisms. We shall study Lie groups in Chapter* **??**. *The reader may skip this definition.*

Definition H.5 *Let G be a Lie subgroup of $GL(\mathsf{E})$. We say that $\pi_E : E \to M$ has a **structure group** G if there is a cover by trivializations (vector bundle charts) $\phi_\alpha : E_{U_\alpha} \to U_\alpha \times \mathsf{E}$ such that for every non-empty intersection $U_\alpha \cap U_\beta$, the transition maps $g_{\alpha\beta}$ take values in G.*

Remark H.4 *Sometimes it is convenient to define the notion of vector bundle chart in a slightly different way. Notice that U_α is an open set in M and so ϕ_α is not quite a chart for the total space manifold E. But by choosing a possibly smaller open set inside U_α we may assume that U_α is the domain of an admissible chart U_α, ψ_α for M. Then we can compose to get a map $\widetilde{\phi}_\alpha : E_{U_\alpha} \to \psi_\alpha(U_\alpha) \times \mathsf{E}$. The maps now can serve as admissible charts for the differentiable manifold E. This leads to an alternative definition of VB-chart which fits better with what we did for the tangent bundle and cotangent bundle:*

Definition H.6 (Type II vector bundle charts) *A* ***(type II) vector bundle chart*** *on an open set $V \subset E$ is a fiber preserving diffeomorphism $\phi : V \to O \times \mathsf{E}$ which covers a diffeomorphism $\underline{\phi}$:$\pi_E(V) \to O$ in the sense that the following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\pi_E"'] & O \times \mathsf{E} \arrow[d, "pr_1"] \\ \pi_E(V) \arrow[r, "\underline{\phi}"'] & O \end{tikzcd}$$

and which is a linear isomorphism on each fiber.

Example H.3 *The maps $T\psi_\alpha : TU_\alpha \to \psi_\alpha(U_\alpha) \times \mathsf{E}$ are (type II) VB-charts and so not only give TM a differentiable structure but also provide TM with a vector bundle structure. Similar remarks apply for T^*M.*

Example H.4 *Let E be a vector space and let $E = M \times \mathsf{E}$. The using the projection $pr_1 : M \times \mathsf{E} \to M$ we obtain a vector bundle. A vector bundle of this simple form is called a trivial vector bundle.*

Define the sum of two section s_1 and s_2 by $(s_1 + s_2)(p) := s_1(p) + s_2(p)$. For any $f \in C^\infty(U)$ and $s \in \Gamma(U, E)$ define a section fs by $(fs)(p) = f(p)s(p)$. Under these obvious definitions $\Gamma(U, E)$ becomes a $C^\infty(U)$-module.

The the appropriate morphism in our current context is the vector bundle morphism:

Definition H.7 ***Definition H.8*** *Let (E, π_E, M) and $(F, \pi_F, N\ )$ be vector bundles. A* ***vector bundle morphism*** *$(E, \pi_E, M) \to (F, \pi_F, N)$ is a pair of maps $f : E \to F$ and $f_0 : M \to N$ such that*

1. ***Definition H.9*** *1) The following diagram commutes:*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "f_0"'] & N \end{tikzcd}$$

and $f|_{E_p}$ is a continuous linear map from E_p into $F_{f_0(p)}$ for each $p \in M$.

2) For each $x_0 \in M$ there exist VB-charts $(\pi_E, \Phi) : E_U \to U \times \mathsf{E}$ and $(\pi_E, \Phi'_\alpha) : F_{U'} \to U' \times \mathsf{E}'$ with $x_0 \in U$ and $f_0(U) \subset V$ such that

$$x \mapsto \Phi'|_{F_{f(x)}} \circ f_0 \circ \Phi|_{E_x}$$

is a smooth map from U into $GL(\mathsf{E}, \mathsf{E}')$.

Notation H.1 *Each of the following is a valid way to refer to a vector bundle morphism:*

1) $(f, f_0) : (E, \pi_E, M, \mathsf{E}) \to (F, \pi_F, N, \mathsf{F})$

2) $f : (E, \pi_E, M) \to (F, \pi_F, N)$ *(the map* f_0 *is induced and hence understood)*

3) $f : \xi_1 \to \xi_2$ *(this one is concise and fairly exact once it is set down that* $\xi_1 = (E_1, \pi_1, M)$ *and* $\xi_2 = (E_2, \pi_2, M)$

4) $f : \pi_E \to \pi_F$

5) $E \xrightarrow{f} F$

Remark H.5 *There are many variations of these notations in use and the reader would do well to get used to this kind of variety. Actually, there will be much more serious notational difficulties in store for the novice. It has been said that notation is one of the most difficult aspects of differential geometry. On the other hand, once the underlying geometric picture has been properly understood , one may "see through" the notation. Drawing diagrams while interpreting equations is often a good idea.*

Definition H.10 ***Definition H.11*** *If* f *is an (linear) isomorphism on each fiber* E_p *then we say that* f *is a **vector bundle isomorphism** and the two bundles are considered equivalent.*

Notation H.2 *If* $\widetilde{f}$ *is a vector bundle morphism from a vector bundle* $\pi_E : E \to M$ *to a vector bundle* $\pi_F : F \to M$ *we will sometimes write this as* $\widetilde{f} : \pi_E \to \pi_F$ *or* $\pi_E \xrightarrow{\widetilde{f}} \pi_F$.

Definition H.12 *A vector bundle is called **trivial** if there is a there is a vector bundle isomorphism onto a trivial bundle:*

$$\begin{array}{ccc} & \cong & \\ E & \to & M \times \mathsf{E} \\ \pi_E \searrow & & \swarrow pr_1 \\ & M & \end{array}$$

Now we make the observation that a section of a trivial bundle is in a sense, nothing more than a vector-valued function since all sections $s \in \Gamma(M, M \times \mathsf{E})$ are of the form $p \to (p, f(p))$ for a unique function $f \in C^\infty(M, \mathsf{E})$. It is common to identify the function with the section.

Now there is an important observation to be made here; a trivial bundle always has a section which never takes on the value zero. There reason is that we may always take a trivialization $\phi : E \to M \times \mathsf{E}$ and then transfer the obvious nowhere-zero section $p \mapsto (p, 1)$ over to E. In other words, we let $s_1 : M \to E$ be defined by $s_1(p) = \phi^{-1}(p, 1)$. We now use this to exhibit a very simple example of a non-trivial vector bundle:

Example H.5 (Möbius bundle) *Let E be the quotient space of $[0,1]\times\mathbb{R}$ under the identification of $(0,t)$ with $(1,-t)$. The projection $[0,1]\times\mathbb{R}\to[0,1]$ becomes a map $E\to S^1$ after composition with the quotient map:*

$$\begin{array}{ccc} [0,1]\times\mathbb{R} & \to & [0,1] \\ \downarrow & & \downarrow \\ E & \to & S^1 \end{array}$$

Here the circle arises as $[0,1]/\sim$ where we have the induced equivalence relation given by taking $0\sim 1$ in $[0,1]$. The familiar Mobius band has an interior which is diffeomorphic to the Mobius bundle.

Now we ask if it is possible to have a nowhere vanishing section of E. It is easy to see that sections of E correspond to continuous functions $f:[0,1]\to\mathbb{R}$ such that $f(0)=-f(1)$. But then continuity forces such a function to take on the value zero which means that the corresponding section of E must vanish somewhere on $S^1=[0,1]/\sim$. Of course, examining a model of a Mobius band is even more convincing; any nonzero section of E could be, if such existed, normalized to give a map from S^1 to the boundary of a Möbius band which only went around once, so to speak, and inspection of a model would convince the reader that this is impossible.

Let $\xi_1=(E_1,\pi_1,M,\mathsf{E}_1)$ and $\xi_2=(E_2,\pi_2,M,\mathsf{E}_2)$ be vector bundles locally isomorphic to $\mathsf{M}\times\ \mathsf{E}_1$ and $\mathsf{M}\times\ \mathsf{E}_2$ respectively. We say that the sequence of vector bundle morphisms

$$0\to\xi_1\xrightarrow{f}\xi_2$$

is **exact** if the following conditions hold:

1. There is an open covering of M by open sets U_α together with trivializations $\phi_{1,\alpha}:\pi_1^{-1}(U_\alpha)\to U_\alpha\times\ \mathsf{E}_1$ and $\phi_{2,\alpha}:\pi_2^{-1}(U_\alpha)\to U_\alpha\times\ \mathsf{E}_2$ such that $\mathsf{E}_2=\mathsf{E}_1\times\mathsf{F}$ for some Banach space F;

2. the diagram below commutes for all α :

$$\begin{array}{ccccc} & \pi_1^{-1}(U_\alpha) & \to & \pi_2^{-1}(U_\alpha) & \\ \phi_{1,\alpha} & \downarrow & & \downarrow & \phi_{2,\alpha} \\ & U_\alpha\times\mathsf{E}_1 & \to & U_\alpha\times\mathsf{E}_1\times\mathsf{F} & \end{array}$$

Definition H.13 *A **subbundle** of a vector bundle $\xi=(E,\pi,M)$ is a vector bundle of the form $\xi=(L,\pi|_L,M)$ where $\pi|_L$ is the restriction to $L\subset E$, and where $L\subset E$ is a submanifold such that*

$$0\to\xi|_L\to\xi$$

is exact. Here, $\xi|_L\to\xi$ is the bundle map given by inclusion: $L\hookrightarrow E$.

*Equivalently, $\pi|_L:L\to M$ is a **subbundle** if $L\subset E$ is a submanifold and there is a splitting $\mathsf{E}=\mathsf{E}_1\times\mathsf{F}$ such that for each $p\in M$ there is a bundle chart $\phi:\pi^{-1}U\to U\times\mathsf{E}$ with $p\in U$ and $\phi((\pi^{-1}U)\cap L)=U\times\mathsf{E}_1\times\{0\}$.*

Definition H.14 *The chart ϕ from the last definition is said to be adapted to the subbundle.*

Notice that if $L \subset E$ is as in the previous definition then $\pi|_L : L \to M$ is a vector bundle with VB-atlas given by the various VB-charts U, ϕ restricted to $(\pi^{-1}U) \cap S$ and composed with projection $U \times \mathsf{E}_1 \times \{0\} \to U \times \mathsf{E}_1$ so $\pi|_L$ is a bundle locally isomorphic to $M \times \mathsf{E}_1$. The fiber of $\pi|_L$ at $p \in L$ is $L_p = E_p \cap L$. Once again we remind the reader of the somewhat unfortunate fact that although the bundle includes and is indeed determined by the map $\pi|_L : L \to M$ we often refer to L itself as the subbundle. In order to help the reader see what is going on here lets us look at how the definition of subbundle looks if we are in the finite dimensional case. We take $\mathsf{M} = \mathbb{R}^n$, $\mathsf{E} = \mathbb{R}^m$ and $\mathsf{E}_1 \times \mathsf{F}$ is the decomposition $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{m-k}$. Thus the bundle $\pi : E \to M$ has rank m (i.e. the typical fiber is $\mathbb{R}^m$) while the subbundle $\pi|_L : L \to M$ has rank k. The condition described in the definition of subbundle translates into there being a VB-chart $\phi : \pi^{-1}U \to U \times \mathbb{R}^k \times \mathbb{R}^{m-k}$ with $\phi((\pi^{-1}U) \cap L) = U \times \mathbb{R}^k \times \{0\}$. What if our original bundle was the trivial bundle $pr_1 : U \times \mathbb{R}^m \to U$? Then the our adapted chart must be a map $U \times \mathbb{R}^m \to U \times \mathbb{R}^k \times \mathbb{R}^{m-k}$ which must have the form $(x, v) \mapsto (x, f(x)v, 0)$ where for each x the $f(x)$ is a linear map $\mathbb{R}^m \to \mathbb{R}^k$.

H.0.3 Formulary

We now define the pseudogroup(s) relevant to the study of foliations. Let $\mathsf{M} = \mathsf{E} \times \mathsf{F}$ be a (split) Banach space. Define $\mathcal{G}_{\mathsf{M},\mathsf{F}}$ to be the set of all diffeomorphisms from open subsets of $\mathsf{E} \times \mathsf{F}$ to open subsets of $\mathsf{E} \times \mathsf{F}$ of the form

$$\Phi(x, y) = (f(x, y), g(y)).$$

In case M is n dimensional and $\mathsf{M} = \mathbb{R}^n$ is decomposed as $\mathbb{R}^k \times \mathbb{R}^q$ we write $\mathcal{G}_{\mathsf{M},\mathsf{F}} = \mathcal{G}_{n,q}$. We can then the following definition:

Definition H.15 *A $\mathcal{G}_{\mathsf{M},\mathsf{F}}$ structure on a manifold M modeled on $\mathsf{M} = \mathsf{E} \times \mathsf{F}$ is a maximal atlas of charts satisfying the condition that the overlap maps are all members of $\mathcal{G}_{\mathsf{M},\mathsf{F}}$.*

1) $\iota_{[X,Y]} = \mathcal{L}_X \circ \iota_Y + \iota_Y \circ \mathcal{L}_X$
2) $\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge \iota_X\omega$ for all $\omega \in \Omega(M)$
3) $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$
4) $\frac{d}{dt}\varphi_t^{X*}Y = \varphi_t^{X*}(L_X Y)$
5) $[X, Y] = \sum_{i,j=1}^m \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial}{\partial x^i}$

Example H.6 (Frame bundle) *Let M be a smooth manifold of dimension n. Let $F_x(M)$ denote the set of all bases (frames) for the vector space T_xM. Now let $F(M) := \bigcup_{x \in M} F_x(M)$. Define the natural projection $\pi : F(M) \to M$ by $\pi(\mathbf{f}) = x$ for all frames $\mathbf{f} = (f_i)$ for the space T_xM. It can be shown that $F(M)$ has a natural smooth structure. It is also a $GL(n, \mathbb{R})$-bundle whose typical fiber* *is also $GL(n,\mathbb{R})$. The bundle charts are built using the charts for M in the following way: Let U_α, ψ_α be a chart for M. Any frame $\mathbf{f} = (f_i)$ at some point $x \in U_\alpha$ may be written as*

$$f_i = \sum c_i^j \left.\frac{\partial}{\partial x^j}\right|_x .$$

We then map $\mathbf{f}$ to $(x, (c_i^j)) \in U_\alpha \times GL(n,\mathbb{R})$. This recipe gives a map $\pi^{-1}(U_\alpha) \to U_\alpha \times GL(n,\mathbb{R})$ which is a bundle chart.

Definition H.16 *A bundle morphism $(f, f_0) : \xi_1 \to \xi_2$ from one fiber bundle $\xi_1 = (E_1, \pi_{E_1}, M_1, F_1)$ to another $\xi_2 = (E_2, \pi_{E_2}, M_2, F_2)$ is a pair of maps (f, f_0) such that the following diagram commutates*

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \pi_{E_1} \downarrow & & \pi_{E_2} \downarrow \\ M_1 & \xrightarrow{f_0} & M_2 \end{array}$$

*In case $M_1 = \quad M_2$ and $f_0 = \mathrm{id}_M$ we call f a strong bundle morphism. In the latter case if $f : E_1 \to E_2$ is also a diffeomorphism then we call it a **bundle isomorphism.***

Definition H.17 *Let ξ_1 and ξ_2 be fiber bundles with the same base space M. If there exists a bundle isomorphism $(f, \mathrm{id}_M) : \xi_1 \to \xi_2$ we say that ξ_1 and ξ_2 are **isomorphic as fiber bundles over** M and write $\xi_1 \overset{fib}{\cong} \xi_2$.*

Now given any open set U in the base space M of a fiber bundle $\xi = (E, \pi_E, M, F)$ we may form the restriction $\xi|\,U$ which is the fiber bundle $(\pi_E^{-1}(U),\, \pi_E|\,U, U, F)$. To simplify notation we write $E_U := \pi_E^{-1}(U)$ and $\pi_E|\,U := \pi_U$.

Example H.7 *Let M and F be smooth manifolds and consider the projection map $pr_1 : M \times F \to M$. This is a smooth fiber bundle with typical fiber F and is called a **product bundle** or a **trivial bundle**.*

A fiber bundle which is isomorphic to a product bundle is also called a trivial bundle. The definition of a fiber bundle ξ with typical fiber F includes the existence of a cover of the base space by a family of open sets $\{U_\alpha\}_{\alpha \in A}$ such that $\xi|\,U_\alpha \overset{fib}{\cong} U \times F$ for all $\alpha \in A$. Thus, fiber bundles as we have defined them, are all locally trivial.

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1−form $\theta = \sum e_j \theta^i$ which takes any vector to itself:

$$\begin{aligned} \theta(v_p) &= \sum e_j(p)\theta^i(v_p) \\ &= \sum v^i e_j(p) = v_p \end{aligned}$$

Let us write $d^{\nabla}\theta = \frac{1}{2}\sum e_k \otimes T^k_{ij}\theta^i \wedge \theta^j = \frac{1}{2}\sum e_k \otimes \tau^k$. If ∇ is the Levi-Civita connection on M then consider the projection $P^\wedge : E \otimes TM \otimes T^*M$ given by $P^\wedge T(\xi, v) = T(\xi, v) - T(v, \xi)$. We have

$$\begin{aligned}\nabla e_j &= \omega^k_j e_k = e\omega \\ \nabla\theta^j &= -\omega^j_k\theta^k\end{aligned}$$

$\nabla_\xi(e_j \otimes \theta^j)$
$P^\wedge(\nabla_\xi\theta^j)(v) = -\omega^j_k(\xi)\theta^k(v) + \omega^j_k(v)\theta^k(\xi) = -\omega^j_k \wedge \theta^k$
Let $T(\xi, v) = \nabla_\xi(e_i \otimes \theta^j)(v)$
$= (\nabla_\xi e_i) \otimes \theta^j(v) + e_i \otimes (\nabla_\xi\theta^j)(v) = \omega^k_i(\xi)e_k \otimes \theta^j(v) + e_i \otimes (-\omega^j_k(\xi)\theta^k(v))$
$= \omega^k_i(\xi)e_k \otimes \theta^j(v) + e_i \otimes (-\omega^k_j(\xi)\theta^j(v)) = e_k \otimes (\omega^k_i(\xi) - \omega^k_j(\xi))\theta^j(v)$
Then

$$\begin{aligned}(P^\wedge T)(\xi, v) &= T(\xi, v) - T(v, \xi) \\ &= (\nabla e_j) \wedge \theta^j + e_j \otimes d\theta^j \\ &= d^{\nabla}(e_j \otimes \theta^j)\end{aligned}$$

$$\begin{aligned}d^{\nabla}\theta &= d^{\nabla}\sum e_j\theta^j \\ &= \sum(\nabla e_j) \wedge \theta^j + \sum e_j \otimes d\theta^j \\ &= \sum(\sum_k e_k \otimes \omega^k_j) \wedge \theta^j + \sum e_k \otimes d\theta^k \\ &= \sum_k e_k \otimes (\sum_j \omega^k_j \wedge \theta^j + d\theta^k)\end{aligned} \tag{H.1}$$

So that $\sum_j \omega^k_j \wedge \theta^j + d\theta^k = \frac{1}{2}\tau^k$. Now let $\sigma = \sum f^j e_j$ be a vector field

$$\begin{aligned}d^{\nabla}d^{\nabla}\sigma = d^{\nabla}\left(d^{\nabla}\sum e_j f^j\right) &= d^{\nabla}\left(\sum(\nabla e_j)f^j + \sum e_j \otimes df^j\right) \\ \left(\sum(\nabla e_j)df^j + \sum(d^{\nabla}\nabla e_j)f^j\right. &\left.+ \sum\nabla e_j df^j + \sum e_j \otimes ddf^j\right)\end{aligned}$$

$$\sum f^j(d^{\nabla}\nabla e_j) = \sum f^j$$

So we seem to have a map $f^j e_j \mapsto \Omega^k_j f^j e_k$.

$$\begin{aligned}e_r\Omega^r_j = d^{\nabla}\nabla e_j &= d^{\nabla}(e_k\omega^k_j) \\ &= \nabla e_k \wedge \omega^k_j + e_k d\omega^k_j \\ &= e_r\omega^r_k \wedge \omega^k_j + e_k d\omega^k_j \\ &= e_r\omega^r_k \wedge \omega^k_j + e_r d\omega^r_j \\ &= e_r(d\omega^r_j + \omega^r_k \wedge \omega^k_j)\end{aligned}$$

$$d^{\nabla}\nabla e = d^{\nabla}(e\omega) = \nabla e \wedge \omega + ed\omega$$

¿From this we get $0 = d(A^{-1}A)A^{-1} = (dA^{-1})AA^{-1} + A^{-1}dAA^{-1}$
$dA^{-1} = A^{-1}dAA^{-1}$

$$\begin{aligned}
\Omega^r_j &= d\omega^r_j + \omega^r_k \wedge \omega^k_j \\
\Omega &= d\omega + \omega \wedge \omega \\
\Omega' &= d\omega' + \omega' \wedge \omega' \\
\Omega' &= d\left(A^{-1}\omega A + A^{-1}dA\right) + \left(A^{-1}\omega A + A^{-1}dA\right) \wedge \left(A^{-1}\omega A + A^{-1}dA\right) \\
&= d\left(A^{-1}\omega A\right) + d\left(A^{-1}dA\right) + A^{-1}\omega \wedge \omega A + A^{-1}\omega \wedge dA \\
&+ A^{-1}dAA^{-1} \wedge \omega A + A^{-1}dA \wedge A^{-1}dA \\
&= d\left(A^{-1}\omega A\right) + dA^{-1} \wedge dA + A^{-1}\omega \wedge \omega A + A^{-1}\omega \wedge dA \\
&+ A^{-1}dAA^{-1}\omega A + A^{-1}dA \wedge A^{-1}dA \\
&= dA^{-1}\omega A + A^{-1}d\omega A - A^{-1}\omega dA + dA^{-1} \wedge dA + A^{-1}\omega \wedge \omega A + A^{-1}\omega \wedge dA \\
&+ A^{-1}dAA^{-1} \wedge \omega A + A^{-1}dA \wedge A^{-1}dA \\
&= A^{-1}d\omega A + A^{-1}\omega \wedge \omega A \\
\Omega' &= A^{-1}\Omega A
\end{aligned}$$

$\omega' = A^{-1}\omega A + A^{-1}dA$

These are interesting equations let us approach things from a more familiar setting so as to interpret what we have.

H.1 Curvature

An important fact about covariant derivatives is that they don't need to commute. If $\sigma : M \to E$ is a section and $X \in \mathfrak{X}(M)$ then $\nabla_X \sigma$ is a section also and so we may take it's covariant derivative $\nabla_Y \nabla_X \sigma$ with respect to some $Y \in \mathfrak{X}(M)$. In general, $\nabla_Y \nabla_X \sigma \neq \nabla_X \nabla_Y \sigma$ and this fact has an underlying geometric interpretation which we will explore later. A measure of this lack of commutativity is the **curvature operator** which is defined for a pair $X, Y \in \mathfrak{X}(M)$ to be the map $F(X,Y) : \Gamma(E) \to \Gamma(E)$ defined by

$$F(X,Y)\sigma := \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma.$$

or

$$[\nabla_X, \nabla_Y]\sigma - \nabla_{[X,Y]}\sigma$$

Appendix I

Notation and font usage guide

Category	Space or object	Typical elements	Typical morphisms
Vector Spaces	V , W , $\mathbb{R}^n$	v,w, x, y	A, B, K, λ, L
Banach Spaces	E,F,M,N, V,W, $\mathbb{R}^n$	v, w, x, y etc.	A, B, K, λ, L
Open sets in vector spaces	U, V, O, U_α	p, q, x, y, v, w	f, g, φ, ψ
Differentiable manifolds	M, N, P, Q	p, q, x, y	f, g, φ, ψ
Open sets in manifolds	U, V, O, U_α	p, q, x, y	$f, \varphi, \psi, \mathtt{x}_\alpha, \mathtt{y}_\alpha$
Bundles	$E \to M$	v, w, ξ, p, q, x	$(\bar{f}, f), (g, \mathrm{id}), h$
Sections of bundles	$\Gamma(M, E)$	$s, s_1, \sigma, ...$	f^*
Sections over open sets	$\Gamma(U, E) = \mathcal{S}_M^E(U)$	$s, s_1, \sigma, ...$	f^*
Lie Groups	G, H, K	g, h, x, y	h, f, g
Lie Algebras	$\mathfrak{g}, \mathfrak{h}, \mathfrak{k}, \mathfrak{a}, \mathfrak{b}$	v, x, y, z, ξ	h, g, df, dh
Fields	$\mathbb{F}, \mathbb{R}, \mathbb{C}, \mathbb{K}$	t, s, x, y, z, r	f, g, h
Vector Fields	$\mathfrak{X}_M(U)$,$\mathfrak{X}(M)$	X, Y, Z	f^*, f_*

Also we have the following notations

$C^\infty(U)$ or $\mathcal{F}(U)$	Smooth functions on U
$C_c^\infty(U)$ or $\mathcal{D}(U)$	"...." with compact support in U
T_pM	Tangent space at p
TM, with $\pi : TM \to M$	Tangent bundle of M
$T_pf : T_pM \to T_{f(p)}N$	Tangent map of $f : M \to N$ at p
$Tf : TM \to TN$	Tangent map of $f : M \to N$
T_p^*M	Cotangent space at p
T^*M, with $\pi_M : T^*M \to M$	Cotangent bundle of M
$J_x(M, N)_y$	k-jets of maps $f :: M, x \to N, y$
$\mathfrak{X}(U), \mathfrak{X}_M(U)$ (or $\mathfrak{X}(M)$)	Vector field over U (or over M)
$(U, \mathtt{x}), (U_\alpha, \mathtt{x}_\alpha), (U_\beta, \psi_\beta), (\varphi, U)$	Typical charts
$T_s^r(\mathrm{V})$ r-contravariant s-covariant	Tensors on V
$\mathfrak{T}_s^r(M)$ r-contravariant s-covariant	Tensor fields on M

d	exterior derivative, differential
∇	covariant derivative
M, g	Riemannian manifold with metric tensor g
M, ω	Symplectic manifold with symplectic form ω
$L(\mathrm{V}, \mathrm{W})$	Linear maps from V to W. Assumed bounded if V, W are Banach
$L^r_s(\mathrm{V}, \mathrm{W})$	r-contravariant, s-covariant multilinear maps $\mathrm{V}^{*r} \times \mathrm{V}^s \to \mathrm{W}$

In keeping with this we will later think of the space $L(\mathrm{V}, \mathrm{W})$ of linear maps from V to W as being identified with $\mathrm{W} \otimes \mathrm{V}^*$ rather than $\mathrm{V}^* \otimes \mathrm{W}$ (this notation will be explained in detail). Thus $(w \otimes \alpha)(v) = w\alpha(v) = \alpha(v)w$. This works nicely since if $w = (w^1, ..., w^m)^t$ is a column and $\alpha = (a_1, ..., a_n)$ then the linear transformation $w \otimes \alpha$ defined above has as matrix

$$w\alpha = \begin{bmatrix} w^1\alpha_1 & w^1\alpha_2 & \cdots & w^1\alpha_n \\ w^2\alpha_1 & w^2\alpha_2 & \cdots & w^2\alpha_n \\ \vdots & \vdots & & \vdots \\ w^m\alpha_1 & w^m\alpha_2 & \cdots & w^m\alpha_n \end{bmatrix}$$

while if $\mathrm{V} = \mathrm{W}$ the number $\alpha(w)$ is just the 1×1 matrix αw. To be consistent with this and with tensor notation we will write a matrix which is to be thought of as a linear map with one index down and one up. For instance, if $A = (a^i_j)$ then the i-th entry of $w = Av$ is

$$w^i = \sum_j a^i_j v^j.$$

A basis for a vector space is usually thought of as an ordered set of vectors (which are linearly independent and span) but it is not always convenient to index the basis elements by a set of integers such as $I = \{1, 2, ..., n\}$. For example, a basis for the space $\mathbb{R}^n_n$ of all $n \times n$ vectors (also denoted $\mathbb{M}_{n\times n}$) is more conveniently indexed by $I = \{1, 2, ..., n\} \times \{1, 2, ..., n\}$. We write elements of this set as ij rather than (i, j) so that a basis would be written $\{E_{ij}\}_{(ij)\in I^2}$. Thus a matrix which represents a linear transformation from $\mathbb{R}^n_n$ to $\mathbb{R}^n_n$ would be written something like $A = (A^{ij}_{kl})$ and given another such matrix say$B = (B^{ij}_{kl})$, the matrix multiplication $C = AB$ is given by

$$C^{ij}_{kl} = \sum_{a,b} A^{ij}_{ab} B^{ab}_{kl}.$$

The point is that it is convenient to define a basis for a (real) vector space to be an indexed set of vectors $\{v_\alpha\}_{\alpha\in\mathcal{I}}$ for some general index set $\mathcal{I}$ such that every vector $v \in \mathrm{V}$ has a unique representation as a sum

$$v = \sum c^\alpha v_\alpha$$

where all but a finite number of the coefficients $c^\alpha \in \mathbb{R}$ are zero. Since vector spaces use scalars which commute there is no harm in write $\sum v_\alpha c^\alpha$ instead of $\sum c^\alpha v_\alpha$. But notice that only the first (rather strange) expression is consistent with our matrix conventions. Thus $\sum v_\alpha c^\alpha = \sum c^\alpha v_\alpha = cv$ (not vc). In noncommutative geometry, these seemingly trivial issues acquire a much more serious character.

Remark I.1 *The use of superscripts is both useful and annoying. Useful because of the automatic bookkeeping features of expressions (like the above) but annoying because of potential confusion with powers. For example, the sphere would have to be denoted as the set of all* (x^1, x^2) *such that* $(x^1)^2 + (x^2)^2 = 1$. *This expression looks rather unfortunate to the author. In cases like this we might just use subscripts instead and hope the reader realizes from context that we have not employed any index lowering transformation such as* $x^i g_{ij} = x_i$ *but rather simply mean that* $x_i = x^i$. *Context must be the guide. Thus we write the equation for an n-sphere more comfortably as* $\sum_{i=1}^n x_i^2 = 1$.

I.1 Positional index notation and consolidation

With regard to the tensor algebras, recall our convention of covariant variables to the left and all the contravariant variables to the right. Also recall that when we formed the tensor product of, for example, $\tau \in \mathfrak{T}^1{}_1$ with $\sigma \in \mathfrak{T}^2{}_0$ we did not get a map $\mathfrak{X}^* \times \mathfrak{X} \times \mathfrak{X}^* \times \mathfrak{X}^* \to C^\infty$ which would be from a space which we could denote by $\mathfrak{T}^1{}_1{}^2{}_0$. Instead, we were formed what we once called the consolidated tensor product which put all the covariant variables together on the left. Thus the result was in $\mathfrak{T}^3{}_1$.

Putting all of the covariant variables on the left is just a convention and we could have done it the other way and used the space $\mathfrak{T}_s{}^r$ instead of $\mathfrak{T}^r{}_s$. But have we lost anything by this consolidation? In accordance with the above convention we have sometimes written the components of a tensor $\tau \in \mathfrak{T}^r{}_s$ as $\tau^{i_1 \dots i_r}{}_{j_i \dots j_s}$ instead of $\tau^{i_1 \dots i_r}_{j_i \dots j_s}$. But one sometimes sees components written such as, say, $\tau^i{}_j{}^k$ which is interpreted by many authors to refer to a multilinear map $\mathfrak{X}^* \times \mathfrak{X} \times \mathfrak{X}^* \to C^\infty$. This is indeed one possible meaning but we shall employ a different interpretation on most occasions. For example, if one defines a tensor whose components are at first written $\tau^i{}_{j\,k}$ and then later we see $\tau^i{}_j{}^k$, this may simply mean that an index operation has taken place. Index raising (and lowering) is defined and studied later on but basically the idea is that if there is a special 2−covariant tensor g_{ij} (metric or symplectic) defined which is a nondegenerate bilinear form on each tangent space then we can form the tensor g_{ij} defined by $\mathrm{g}_{ir}\mathrm{g}^{rj} = \delta^i_j$. Then from $\tau^i{}_{j\,k}$ we form the tensor whose components are $\tau^i{}_j{}^k := \tau^i{}_{j\,r}\mathrm{g}^{rk}$. Now some authors insist that this must mean that the new tensor is a multilinear map $\mathfrak{X}^* \times \mathfrak{X} \times \mathfrak{X}^* \to C^\infty$ but there is another approach which says that the index being in a new "unnatural" position is just a reminder that we have defined a new tensor from a previously defined tensor in certain way but that the result is still map $\mathfrak{X}^* \times \mathfrak{X}^* \times \mathfrak{X} \to C^\infty$. We have thus consolidated, putting covariant variables on the left. So $\tau^i{}_j{}^k$ actually refers to a new tensor $\widetilde{\tau} \in \mathfrak{T}^2{}_1$ whose components are $\widetilde{\tau}^{ik}{}_j := \tau^i{}_j{}^k := \tau^i{}_{j\,r}\mathrm{g}^{rk}$. We might also use the index position to indicate how the tensor was constructed. Many schemes are possible.

Why is it that in practice the interpretation of index position and the choice to consolidate or not seems to make little difference. The answer can best be understood by noticing that when one is doing a component calculation the indices take care of themselves. For example, in the expression $\tau^a{}_{bc}\,\theta_a X^b Y^c$ all one needs to know is how the components $\tau^a{}_{bc}$ are defined (e.g. $\tau_{rbc} g^{ar}\,\tau^a{}_{bc}$) and that X^b matches up with the b in $\tau^a{}_{bc}$ and so on. We have $\tau^a{}_{bc}\,\theta_a Y^c X^b = \tau^a{}_{bc}\,\theta_a X^b \theta_a Y^c$ and it matters not which variable comes first. One just has to be consistent. As another example, just notice how nicely it works out to say that the tensor product of $\tau^a{}_{bc}$ and $\eta^a{}_{bc}$ is the tensor $\tau^a{}_{bc}\,\eta^d{}_{ef}$. Does $\tau^a{}_{bc}\,\eta^d{}_{ef}$ refer to a multilinear map $\mathfrak{X}^* \times \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X}^* \times \mathfrak{X} \times \mathfrak{X} \to C^\infty$ or to a multilinear map $\mathfrak{X}^* \times \mathfrak{X}^* \times \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to C^\infty$? Does it matter?

Appendix J

Review of Curves and Surfaces

First of all a C^k map σ from an open set $U \subset \mathbb{R}^m$ into another Euclidean space $\mathbb{R}^n$ is given by n functions (say $\sigma_1, ..., \sigma_n$) of m variables (say $u^1, ..., u^m$) such that the functions and all possible partial derivatives of order $\leq k$ exist and are continuous throughout U. The total derivative at $p \in U$ of such a map is a linear map represented by the matrix of first partials:

$$\begin{bmatrix} \frac{\partial \sigma_1}{\partial u^1}(p) & \cdots & \frac{\partial \sigma_1}{\partial u^m}(p) \\ \vdots & & \vdots \\ \frac{\partial \sigma_n}{\partial u^1}(p) & \cdots & \frac{\partial \sigma_n}{\partial u^m}(p) \end{bmatrix}$$

and if this map is full rank for each $p \in U$ then we say that the

map is an C^k **immersion**. A C^k map $\phi : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ which is a homeomorphism and such that ϕ^{-1} is also a C^k map is called a **diffeomorphism**. By convention, if $k = 0$ then a diffeomorphism is really just a homeomorphism between open set of Euclidean space. (Here we have given rough and ready definitions that will refined in the next section on calculus on general Banach spaces).

J.0.1 Curves

Let O be an open set in $\mathbb{R}$. A continuous map $c : O \to \mathbb{R}^n$ is C^k if the $k-$th derivative $c^{(k)}$ exist on all of O and is a continuous. If I is a subset of $\mathbb{R}$ then a map $c : I \to \mathbb{R}^n$ is said to be C^k if there exists a C^k extension $\widetilde{c} : O \to \mathbb{R}^n$ for some open set O containing I. We are particularly interested in the case where I is an interval. This interval may be open, closed, half open etc. We also allow intervals for which one of the "end points" a or b is $\pm\infty$.

Definition J.1 *Let $O \subset \mathbb{R}$ be open. A continuous map $\mathbf{c} : O \to \mathbb{R}^n$ is said to be piecewise C^k if there exists a discrete sequence of points $\{t_n\} \subset O$ with*

$t_i < t_{i+1}$ such that $\mathbf{c}$ *restricted to each* $(t_i, t_{i+1}) \cap O$ *is* C^k. *If* I *is subset of* $\mathbb{R}$, *then a map* $\mathbf{c} : I \to \mathbb{R}^n$ *is said to be piecewise* C^k *if there exists a piecewise extension of* $\mathbf{c}$ *to some open set containing* I.

Definition J.2 *A* ***parametric curve*** *in* $\mathbb{R}^n$ *is a piecewise differentiable map* $\mathbf{c} : I \to \mathbb{R}^n$ *where* I *is either an interval or finite union of intervals. If* $\mathbf{c}$ *is an interval then we say that* $\mathbf{c}$ *is a* ***connected parametric curve***.

Definition J.3 *If* $\mathbf{c} : I \to \mathbb{R}^n$ *is a* C^1-*parametric curve then* $\mathbf{c}'$ *is well defined on* I *except, possibly, at a discrete set of points in* I *and is called the velocity of* $\mathbf{c}$. *If* $\|\mathbf{c}'\| = 1$ *where defined then we say that* $\mathbf{c}$ *is a* ***unit speed*** *curve. If* $\mathbf{c} : I \to \mathbb{R}^n$ *is a* C^2-*parametric curve then* $\mathbf{c}''$ *is a piecewise continuous map and is referred to as the acceleration of* c. *We call the vector* $c'(t)$ *the velocity at time* t *and call* $c''(t)$ *the acceleration at time* t.

Definition J.4 *A parametric curve is called* ***regular*** *if* c' *is defined and nonzero on all of* I.

We shall now restrict our attention to curves which are piecewise C^∞.

Definition J.5 *A elementary parametric curve is a regular curve* $c : I \to \mathbb{R}^n$ *such that* I *is an open connected interval.*

When should two curves be considered geometrically equivalent? Part of the answer comes from considering reparametrization:

Definition J.6 *If* $\mathbf{c} : I_1 \to \mathbb{R}^n$ *and* $b : I_2 \to \mathbb{R}^n$ *are curves then we say that* b *is a positive (resp. negative) reparametrization of* $\mathbf{c}$ *if there exists a bijection* $h : I_2 \to I_1$ *with* $\mathbf{c} \circ h = b$ *such that* h *is smooth and* $h'(t) > 0$ *(resp.* $h'(t) > 0$*) for all* $t \in I_2$.

Now we can think of two parametric curves as being in some sense the same if one is a C^k reparametrization of the other. On the other hand this form of congruence is appropriate to the topological and differentiable structure of $\mathbb{R}^n$.

Definition J.7 *Two parametric curves,* $\mathbf{c} : I_1 \to \mathbb{R}^n$ *and* $b : I_2 \to \mathbb{R}^n$, *are said to be congruent if there is a Euclidean motion* $T : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $T \circ \mathbf{c}$ *is a reparametrization of* b.

We distinguish between a C^1 map $c : I \to \mathbb{R}^n$ and its image (or trace) $c(I)$ as a subset of $\mathbb{R}^n$. The geometric curve is the set $c(I)$ itself while the parameterized curve is the map in question. Of course, two sets, S_1 and S_2, are congruent (equivalent) if there exists a Euclidean motion $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(S_1) = S_2$. The question now arises as to what our point of view should be. Are we intending to study certain subsets of $\mathbb{R}^n$ or are we ultimately interested in maps into $\mathbb{R}^n$. Sets or Maps? In the study of curves and surfaces both points of view are valuable. Unfortunately the two approaches are not always clearly distinguished in the literature. If subset S is the trace $\mathbf{c}(I)$ of some parametric curve then we say that $\mathbf{c}$ parameterizes the set S. Then one way to study this set geometrically is to study those aspects of the set which may be described in terms of a parameterization $\mathbf{c}$ but which would remain the same in some sense if we where to choose a different parameterization. If the parameterization we use are bijections onto the image S then this approach works fairly well. On the other hand if we are interested in including self intersections then we have to be a bit careful. We can consider two maps $\mathbf{c} : I_1 \to \mathbb{R}^n$ and $b : I_2 \to \mathbb{R}^n$ to be congruent (in a generalized sense) if there exists a Euclidean motion $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T \circ \mathbf{c} = b$ is a reparametrization of $\mathbf{c}$. As we know from elementary calculus, every C^1 parameterized curve has a parameterization by arc length. In this case we often use the letter s to denote the independent variable. Now if two curves $\mathbf{c}_1$ and $\mathbf{c}_2$ are both unit speed reparametrizations of the same parameterized curve $\mathbf{c}$ then there is a transformation $s' = s + s_0$ so that $\mathbf{c}_2(s') = \mathbf{c}_1(s + s_0)$ where s_0 is a constant. Whenever we have a unit speed curve we say that the curve (or more precisely, the geometric curve $S = \mathbf{c}(I)$) is parameterized by arc length. Thus arc length parameterization is unique up to this kind of simple change of variables: $s = s' + s_0$

Given a parameterized curve $\mathbf{c} : I \to \mathbb{R}^n$ one may, in principal, immediately define the arc length of the curve from a reference point $p_0 = \mathbf{c}(t_0)$ to another point $p = \mathbf{c}(t)$. The formula familiar from elementary calculus is

$$l_{\mathbf{c},t_0}(t) = \int_{t_0}^{t} |\dot{\mathbf{c}}(\tau)|\, d\tau.$$

We will assume for simplicity that $|\dot{\mathbf{c}}(t)| > 0$ all $t \in I$. We will sometimes abbreviate $l_{\mathbf{c},t_0}$ to l whenever there is no danger of confusion. Since, $\frac{ds}{dt} = |\dot{\mathbf{c}}(t)| > 0$ we may invert and obtain $t = l^{-1}(s)$ with $t(t_0) = 0$ and $\frac{dt}{ds}(s) = 1/\frac{ds}{dt}(l^{-1}(s))$. Having done this we reparametrize our curve using l^{-1} and obtain $\widetilde{\mathbf{c}}(s) := \mathbf{c} \circ l^{-1}(s)$. We will abuse notation by using the same symbol $\mathbf{c}$ for this new parameterized curve. Now the unit tangent to the curve is defined by $\frac{d\mathbf{c}}{ds}(s) := \mathbf{T}(s)$. Notice that

$$\begin{aligned}
\sqrt{\mathbf{T}(s) \cdot \mathbf{T}(s)} &= |\mathbf{T}(s)| \\
&= \left|\frac{d\mathbf{c}}{ds}(s)\right| = \left|\frac{d\mathbf{c}}{dt}\frac{dt}{ds}\right| \\
&= \left|\frac{d\mathbf{c}}{dt}\right| \frac{dt}{ds} = |\dot{\mathbf{c}}(t)|\, |\dot{\mathbf{c}}(t)|^{-1} = 1
\end{aligned}$$

and so $\mathbf{T}$ is a unit vector. If $\frac{d\mathbf{T}}{ds}(s)$ is identically zero over some finite interval then $\mathbf{c}$ is easily seen to be a straight line. Because of this we will may as well assume that $\left|\frac{d\mathbf{T}}{ds}(s)\right| > 0$ for all $t \in I$. Now we have define $\mathbf{N}(s)$ as $\frac{d\mathbf{T}}{ds}(s)/\left|\frac{d\mathbf{T}}{ds}(s)\right|$ so we automatically get that $\mathbf{N}(s) = |\kappa|\,\frac{d\mathbf{T}}{ds}(s)$ where $|\kappa| := \left|\frac{d\mathbf{T}}{ds}(s)\right|$. The function $|\kappa|$ is often denoted simply as κ and is called the unsigned curvature. Observe that since $1 = |\mathbf{T}(s)|^2 = \mathbf{T}(s) \cdot \mathbf{T}(s)$ we get

$$0 = 2\frac{d\mathbf{T}}{ds}(s) \cdot \mathbf{T}(s)$$

and then $\mathbf{N}(s) \cdot \mathbf{T}(s) = 0$ so $\mathbf{N}(s)$ is a unit vector normal to the tangent $\mathbf{T}(s)$. The next logical step is to consider $\frac{d\mathbf{N}}{ds}$. Once again we separate out the case where $\frac{d\mathbf{N}}{ds}$ is identically zero in some interval. In this case we see that $\mathbf{N}$ is constant and it is not hard to see that the curve must remain in the fixed plane determined by $\mathbf{N}$. This plane is also oriented by the $\mathbf{T}$ and $\mathbf{N}$. The case of a curve in an oriented plane is equivalent to a curve in $\mathbb{R}^2$ and will be studied separately below. For now we assume that $\left|\frac{d\mathbf{N}}{ds}\right| > 0$ on I.

J.0.2 Curves in $\mathbb{R}^3$

In this case we may define the unit binormal vector to be the vector at $c(s)$ such that $\mathbf{T}, \mathbf{N}, \mathbf{B}$ is a positively oriented triple of orthonormal unit vectors. By positively oriented we mean that

$$\det[\mathbf{T}, \mathbf{N}, \mathbf{B}] = 1.$$

We now show that $\frac{d\mathbf{N}}{ds}$ is parallel to $\mathbf{B}$. For this it suffices to show that $\frac{d\mathbf{N}}{ds}$ is normal to both $\mathbf{T}$ and $\mathbf{N}$. First, we have $\mathbf{N}(s) \cdot \mathbf{T}(s) = 0$. If this equation is differentiated we obtain $2\frac{d\mathbf{N}}{ds} \cdot \mathbf{T}(s) = 0$. On the other hand we also have $1 = \mathbf{N}(s) \cdot \mathbf{N}(s)$ which differentiates to give $2\frac{d\mathbf{N}}{ds} \cdot \mathbf{N}(s) = 0$. From this we see that there must be a function $\tau = \tau(s)$ such that $\frac{d\mathbf{N}}{ds} := \tau\mathbf{B}$. This is a function of arc length but should really be thought of as a function on the curve. In a sense that we shall eventually make precise, τ is an invariant of the geometric curve. This invariant is called the **torsion**.

Theorem J.1 *If on some interval I we have $|\dot{\mathbf{c}}(s)| > \mathbf{0}$, $\left|\frac{d\mathbf{T}}{ds}\right| > 0$, and $\left|\frac{d\mathbf{N}}{ds}\right| > 0$*

with $\mathbf{T}$ *and* $\mathbf{N}$ *defined as above then*

$$\frac{d}{ds}[\mathbf{T},\mathbf{N},\mathbf{B}] = [\mathbf{T},\mathbf{N},\mathbf{B}]\begin{bmatrix} 0 & |\kappa| & 0 \\ -|\kappa| & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix}$$

or in other words

$$\begin{aligned} \tfrac{d\mathbf{T}}{ds} &= & |\kappa|\,\mathbf{N} & \\ \tfrac{d\mathbf{N}}{ds} &= -|\kappa|\,\mathbf{T} & & \tau\mathbf{B} \\ \tfrac{d\mathbf{B}}{ds} &= & \tau\mathbf{N} & \end{aligned}$$

Proof. Since $F = [\mathbf{T},\mathbf{N},\mathbf{B}]$ is by definition an orthogonal matrix we have $F(s)F^t(s) = I$. It is also clear that there is some matrix function $A(s)$ such that $F' = F(s)A(s)$. Also, Differentiating we have $\frac{dF}{ds}(s)F^t(s) + F(s)\frac{dF}{ds}^t(s) = 0$ and so

$$\begin{aligned} FAF^t + FA^tF^t &= 0 \\ A + A^t &= 0 \end{aligned}$$

since F is invertible. Thus $A(s)$ is antisymmetric. But we already have established that $\frac{d\mathbf{T}}{ds} = |\kappa|\,\mathbf{N}$ and $\frac{d\mathbf{B}}{ds} = \tau\mathbf{N}$ and so the result follows. ∎

As indicated above, it can be shown that the functions $|\kappa|$ and τ completely determine a sufficiently regular curve up to reparameterization and rigid motions of space. This is not hard to establish but we will save the proof for a later time. The three vectors form a vector field along the curve c. At each point $p = \mathbf{c}(s)$ along the curve c the provide and oriented orthonormal basis (or frame) for vectors based at p. This basis is called the Frenet frame for the curve. Also, $|\kappa|(s)$ and $\tau(s)$ are called the (unsigned) curvature and torsion of the curve at $\mathbf{c}(s)$. While, $|\kappa|$ is never negative by definition we may well have that $\tau(s)$ is negative. The curvature is, roughly speaking, the reciprocal of the radius of the circle which is tangent to $\mathbf{c}$ at $\mathbf{c}(s)$ and best approximates the curve at that point. On the other hand, τ measures the twisting of the plane spanned by $\mathbf{T}$ and $\mathbf{N}$ as we move along the curve. If $\gamma : I \to \mathbb{R}^3$ is an arbitrary speed curve then we define $|\kappa|_\gamma(t) := |\kappa| \circ h^{-1}$ where $h : I' \to I$ gives a unit speed reparameterization $\mathbf{c} = \gamma \circ h : I' \to \mathbb{R}^n$. Define the torsion function τ_γ for γ by $\tau \circ h^{-1}$. Similarly we have

$$\begin{aligned} \mathbf{T}_\gamma(t) &:= \mathbf{T} \circ h^{-1}(t) \\ \mathbf{N}_\gamma(t) &:= \mathbf{N} \circ h^{-1}(t) \\ \mathbf{B}_\gamma(t) &:= B \circ h^{-1}(t) \end{aligned}$$

Exercise J.1 *If* $\mathbf{c} : I \to \mathbb{R}^3$ *is a unit speed reparameterization of* $\gamma : I \to \mathbb{R}^3$ *according to* $\gamma(t) = \mathbf{c} \circ h$ *then show that*

1. $\mathbf{T}_\gamma(t) = \gamma' / \|\gamma'\|$

2. $\mathbf{N}_\gamma(t) = \mathbf{B}_\gamma(t) \times \mathbf{T}_\gamma(t)$
3. $\mathbf{B}_\gamma(t) = \frac{\gamma' \times \gamma''}{\|\gamma' \times \gamma''\|}$
4. $|\kappa|_\gamma = \frac{\|\gamma' \times \gamma''\|}{\|\gamma''\|^3}$
5. $\tau_\gamma = \frac{(\gamma' \times \gamma'') \cdot \gamma'''}{\|\gamma' \times \gamma''\|^2}$

Exercise J.2 *Show that* $\gamma'' = \frac{dv}{dt}\mathbf{T}_\gamma + v^2 |\kappa|_\gamma \mathbf{N}_\gamma$ *where* $v = \|\gamma'\|$.

For a curve confined to a plane we haven't got the opportunity to define $\mathbf{B}$ or τ. However, we can obtain a more refined notion of curvature.

We now consider the special case of curves in $\mathbb{R}^2$. Here it is possible to define a signed curvature which will be positive when the curve is turning counterclockwise. Let $J : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $J(a, b) := (-b, a)$. The signed curvature $\kappa_{2,\gamma}$ of γ is given by

$$\kappa_\gamma(t) := \frac{\gamma''(t) \cdot J\gamma'(t)}{\|\gamma'(t)\|^3}.$$

Exercise J.3 *If* γ *is a parameterized curve in* $\mathbb{R}^2$ *then* $\kappa_\gamma \equiv 0$ *then* γ *(parameterizes) a straight line. If* $\kappa_\gamma \equiv k_0 > 0$ *(a constant) then* γ *parameterizes a portion of a circle of radius* $1/k_0$.

The unit tangent is $\mathbf{T} = \frac{\gamma'}{\|\gamma'\|}$. We shall redefine the normal $\mathbf{N}$ to a curve to be such that $\mathbf{T}, \mathbf{N}$ is consistent with the orientation given by the standard basis of $\mathbb{R}^2$. In fact, we have $\mathbf{N} = J\mathbf{c}'(s) = J\mathbf{T}$.

Exercise J.4 *If* $\mathbf{c} : I \to \mathbb{R}^2$ *is a unit speed curve then*

1. $\frac{d\mathbf{T}}{ds}(s) = \kappa_{\mathbf{c}}(s)\mathbf{N}(s)$
2. $\mathbf{c}''(s) = \kappa_{\mathbf{c}}(s)\,(J\mathbf{T}(s))$

J.1 Frenet Frames along curves in $\mathbb{R}^n$

We have already discussed the Frenet frames for curves in $\mathbb{R}^3$. We generalize this now for $n > 3$. We assume that $\gamma : I \to \mathbb{R}^n$ is a regular curve so that $\|\gamma'\| > 0$. For convenience we will assume that γ is infinitely differentiable (smooth). An **adapted moving orthonormal frame along** γ is a orthonormal set $\mathbf{E}_1(t), ..., \mathbf{E}_n(t)$ of smooth vector fields along γ such $\mathbf{E}_1(t) = \gamma'/\|\gamma'\|$. The moving frame will be called positively oriented if there is be a smooth matrix function $Q(t)$ of $n \times n$ orthogonal matrices of determinant 1 such that

$$[\mathbf{e}_1, ..., \mathbf{e}_n]Q(t) = [\mathbf{E}_1(t), ..., \mathbf{E}_n(t)]$$

where $\mathbf{e}_1, ..., \mathbf{e}_n$ is the standard basis of $\mathbb{R}^n$. Let us refer to such a moving frame simply as an orthonormal frame along the given curve.

We shall call a curve γ in R^n **fully regular** if $\gamma'(t), \gamma''(t), ..., \gamma^{(n)}(t)$ is a linearly independent set for each t in the domain of the curve. Now for a fully regular curve the existence of a moving orthonormal frame an easily proved. One essentially applies a Gram-Schmidt process: If $\mathbf{E}_1(t), ..., \mathbf{E}_k(t)$ are already defined then

$$\mathbf{E}_{k+1}(t) := \gamma^{(k+1)}(t) - \sum_{j=1}^{k} \gamma^{(j)}(t) \cdot \mathbf{E}_j(t).$$

This at least gives us an orthonormal moving frame and we leave it as an exercise to show that the $\mathbf{E}_k(t)$ are all smooth. Furthermore by making one possible adjustment of sign on $\mathbf{E}_n(t)$ we may guarantee that the moving frame is a positively oriented moving frame. So far we have a moving frame along our fully regular curve which is clearly quite nicely related to the curve itself. By our construction we have a nice list of properties:

1. For each k, $1 \leq k \leq n$ the vectors $\mathbf{E}_1(t), ..., \mathbf{E}_k(t)$ are in the linear span of $\gamma'(t), ..., \gamma^{(k)}(t)$ so that there is a upper triangular matrix function $L(t)$ such that
$$[\gamma'(t), ..., \gamma^{(n)}(t)]L(t) = [\mathbf{E}_1(t), ..., \mathbf{E}_n(t)]$$
2. For each k, $1 \leq k \leq n-1$ the vectors $\mathbf{E}_1(t), ..., \mathbf{E}_k(t)$ have the same orientation as $\gamma'(t), ..., \gamma^{(k)}(t)$. Thus the matrix $L(t)$ has its first $n-1$ diagonal elements all positive.
3. $[\mathbf{E}_1(t), ..., \mathbf{E}_n(t)]$ is positively oriented as a basis of $\mathbb{R}^n$.

Exercise J.5 *Convince yourself that the moving frame we have constructed is unique one with these properties.*

We call this moving frame the **Frenet frame** along γ. However, we can deduce more nice properties. The derivative of each $\mathbf{E}_i(t)$ is certainly expressible as a linear combination of the basis $\mathbf{E}_1(t), ..., \mathbf{E}_n(t)$ and so we may write

$$\frac{d}{dt}\mathbf{E}_i(t) = \sum_{j=1}^{n} \omega_{ij}\mathbf{E}_j(t).$$

Of course, $\omega_{ij} = \frac{d}{dt}\mathbf{E}_i(t) \cdot \mathbf{E}_j(t)$ but since $\mathbf{E}_i(t) \cdot \mathbf{E}_j(t) = \delta_{ij}$ we conclude that $\omega_{ij} = -\omega_{ji}$ so the matrix $\omega = (\omega_{ij})$ is antisymmetric. Furthermore, for $1 \leq j < n$ we have $\mathbf{E}_j(t) = \sum_{k=1}^{j} L_{kj}\gamma^{(k)}(t)$ where $L = (L_{kj})$ is the upper triangular matrix mentioned above. using the fact that L, $\frac{d}{dt}L$ and L^{-1} are all upper triangular we have

$$\frac{d}{dt}\mathbf{E}_j(t) = \sum_{k=1}^{j}\left(\frac{d}{dt}L_{kj}\right)\gamma^{(k)}(t) + \sum_{k=1}^{j} L_{kj}\gamma^{(k+1)}(t)$$

but $\gamma^{(k+1)}(t) = \sum_{r=1}^{k+1}\left(L^{-1}\right)_{(k+1)r}\mathbf{E}_r(t)$ and $\gamma^{(k)}(t) = \sum_{r=1}^{k}\left(L^{-1}\right)_{kr}\mathbf{E}_r(t)$ so that

$$
\begin{aligned}
\frac{d}{dt}\mathbf{E}_j(t) &= \sum_{k=1}^{j}\left(\frac{d}{dt}L_{kj}\right)\gamma^{(k)}(t) + \sum_{k=1}^{j} L_{kj}\gamma^{(k+1)}(t) \\
&= \sum_{k=1}^{j}\left(\frac{d}{dt}L_{kj}\right)\sum_{r=1}^{k}\left(L^{-1}\right)_{kr}\mathbf{E}_r(t) \\
&+ \sum_{k=1}^{j} L_{kj}\sum_{r=1}^{k+1}\left(L^{-1}\right)_{(k+1)r}\mathbf{E}_r(t)
\end{aligned}
$$

From this we see that $\frac{d}{dt}\mathbf{E}_j(t)$ is in the span of $\{\mathbf{E}_r(t)\}_{1\le r\le j+1}$. So $\omega = (\omega_{ij})$ can have no nonzero entries below the subdiagonal. But ω is antisymmetric and so we have no choice but to conclude that ω has the form

$$
\begin{bmatrix}
0 & \omega_{12} & 0 & \cdots & 0 \\
-\omega_{12} & 0 & \omega_{23} & & \\
 & -\omega_{23} & 0 & & \\
 & & & \ddots & \omega_{n-1,n} \\
 & & & -\omega_{n-1,n} & 0
\end{bmatrix}
$$

J.2 Surfaces

J.2.1 C^k Singular elementary surfaces.

This approach is seldom examined explicitly but is a basic conceptual building block for all versions of the word surface. At first we define a C^k singular surface to simply be a C^k map $\mathbf{x} : U \to \mathbb{R}^n$ where U is an open set in the plane $\mathbb{R}^2$. We are actually studying the maps themselves even though we often picture its image $\mathbf{x}(U)$ and retain a secondary interest in this image. The image may be a very irregular and complicated set and is called the trace of the singular elementary surface. A morphism[1] $\mu : \mathbf{x}_1 \to \mathbf{x}_2$ between two such elementary surfaces $\mathbf{x}_1 : U_1 \to \mathbb{R}^n$ and $\mathbf{x}_2 : U_2 \to \mathbb{R}^m$ is a pair of C^k maps $h : U_1 \to U_2$ and $\bar{h} : \mathbb{R}^n \to \mathbb{R}^m$ such that the following diagram commutes

$$
\begin{array}{ccc}
\mathbb{R}^n & \overset{\bar{h}}{\to} & \mathbb{R}^m \\
\mathbf{x}_1 \uparrow & & \mathbf{x}_2 \uparrow \\
U_1 & \overset{h}{\to} & U_2
\end{array} .
$$

If $m = n$ and both h and $\bar{h}$ are C^k diffeomorphisms then we say that the morphism is an isomorphism and that the elementary surfaces $\mathbf{x}_1 : U_1 \to \mathbb{R}^n$ and $\mathbf{x}_2 : U_2 \to \mathbb{R}^n$ are **equivalent** or **isomorphic**.

[1] Don't let the use of the word "morphism" here cause to much worry. So far the use of this word doesn't represent anything deep.

J.2.2 Immersed elementary surfaces.

Here we change things just a bit. An **immersed C^k elementary surface** (also called a **regular elementary surface**) is just a map as above but with the restriction that such $k > 0$ and that the derivative has full rank on all of U.

Remark J.1 *We may also need to deal with elementary surfaces which are regular on part of U and so we say that $\mathbf{x}$ is regular at p if the Jacobian matrix has rank 2 at p and also we say that $\mathbf{x}$ is regular on a set $A \subset U$ if it is regular at each $p \in A$. Often, A will be of the form $U \backslash N$ where N is a set consisting of a finite number of points or maybe some small set in some other set. For example N might be the image (trace) of a regular curve.*

What should be the appropriate notion of morphism between two such maps $\mathbf{x}_1 : U_1 \to \mathbb{R}^n$ and $\mathbf{x}_2 : U_2 \to \mathbb{R}^n$? Clearly, it is once again a pair of maps $h : U_1 \to U_2$ and $\bar{h} : \mathbb{R}^n \to \mathbb{R}^n$ but we must not end up creating maps which do not satisfy our new defining property. There is more than one approach that might work (or at least be self consistent) but in view of our ultimate goals we take the appropriate choice to be a pair of C^k diffeomorphisms $h : U_1 \to U_2$ and $\bar{h} : \mathbb{R}^n \to \mathbb{R}^n$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^n & \stackrel{\bar{h}}{\to} & \mathbb{R}^n \\ \mathbf{x}_1 \uparrow & & \mathbf{x}_2 \uparrow \\ U_1 & \stackrel{h}{\to} & U_2 \end{array} .$$

J.2.3 Embedded elementary surface

This time we add the requirement that the maps $\mathbf{x} : U \to \mathbb{R}^n$ actually be regular elementary surfaces which are injective and in fact homeomorphism onto their images $\mathbf{x}(U)$.

Such a map $\mathbf{x}_1 : U_1 \to \mathbb{R}^n$ is called an **embedded elementary surface**. A morphism between two such embeddings is a pair of C^k diffeomorphisms with the obvious commutative diagram as before. All such a morphisms are in fact isomorphisms. (In some categories all morphisms are isomorphisms.)

J.2.4 Surfaces as Subspaces

The common notion of a surfaces is not a map but rather a set. It is true that in many if not most cases the set is the image of one or more maps. One approach is to consider a surface as a 2-dimensional smooth submanifold (defined shortly) of some $\mathbb{R}^n$. Of course for the surfaces that we see around us and are most compatible with intuition sit in $\mathbb{R}^3$ (or more precisely $\mathbf{E}^3$)

J.2.5 Geometric Surfaces

Here we mean the word geometric in the somewhat narrow sense of keeping track of such things as curvatures, volumes, lengths etc. There is a sort of division into extrinsic geometry and intrinsic geometry which is a theme which survives in more general settings such as Riemannian geometry.

Each of the ideas of the previous section has an obvious extension that starts with maps from open sets in $\mathbb{R}^m$ into $\mathbb{R}^n$. The case $m = 1$ is the study of curves. In each case we can take a topological and then differentiable view point. We can also consider the maps as the main object to be studied under the appropriate equivalences or we can think of sets a that which we study. The theory of differentiable manifolds which is one of the main topics of this book provides the tools to study our subject from each of these related views. Now when we actually start to get to the extra structures found in Riemannian, semi-Riemannian and symplectic geometry we still have the basic choice to center attention on maps (immersions, embedding, etc.) or sets but now the morphisms and therefore the notion of equivalence is decidedly more intricate. For example, if we are considering curves or surfaces as maps into some $\mathbb{R}^n$ then the notion of equivalence takes advantage of the metric structure and the structure provided by the natural inner product. Basically, is we have a map $\mathbf{x} : X \to \mathbb{R}^n$ from some open set X in $\mathbb{R}^m$, or even more generally, some abstract smooth surface or differentiable manifold. We might well decide to consider two such maps $\mathbf{x}_1 : X_1 \to \mathbb{R}^n$ and $\mathbf{x}_2 : X_2 \to \mathbb{R}^n$ to be equivalent if there is a diffeomorphism $h : X_1 \to X_2$ (the reparameterization) and a Euclidean motion $A : \mathbb{R}^n \to \mathbb{R}^n$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \\ \mathbf{x}_1 \uparrow & & \mathbf{x}_2 \uparrow \\ X_1 & \xrightarrow{h} & X_2 \end{array} .$$

In the next section we study some of the elementary ideas from the classical theory of curves and surfaces. Because we have not yet studied the notion of a differentiable manifold in detail or even given more than a rough definition of a smooth surface we will mainly treat surfaces which are merely elementary surfaces. These are also called parameterized surfaces and are actually maps rather than sets. The read will notice however that our constructions and notions of equivalence has one eye as it were on the idea of a curve or surface as a special kind of set with a definite shape. In fact, "shape" is the basic motivating idea in much of differential geometry.

J.3 Surface Theory

We have studied elementary surfaces theory earlier in the book. We shall now redo some of that theory but now that we are clear on what a differentiable manifold is we shall be able to treat surfaces in a decidedly more efficient way. In particular, surfaces now are first conceived of as submanifolds. Thus they are now sets rather than maps. Of course we can also study immersions which possible have self crossings with the new advantage that the domain space can be an abstract (or concrete) 2-dimensional manifold and so may have rather complicated topology. When we first studied surfaces as maps the domains were open sets in R^2. We called these elementary parameterized surfaces. Let S be a submanifold of $\mathbb{R}^3$. The inverse of a coordinate map $\psi : V \to U \subset \mathbb{R}^2$ is a parameterization $\mathbf{x} : U \to V \subset S$ of a portion V of our surface. Let (u_1, u_2) the coordinates of points in V. there will be several such parameterization that cover the surface. We reiterate, what we are now calling a parameterization of a surface is not necessarily an onto map and is just the inverse of a coordinate map. For example, we have the usual parameterization of the sphere

$$\mathbf{x}(\varphi, \theta) = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi).$$

This parameterization is deficient at the north and south poles but covers all but a set of measure zero. For some purposes this is good enough and we can just use this one parameterization.

A curve on a surface S may be given by first letting $t \mapsto (u_1(t), u_2(t))$ be a smooth curve into U and then composing with $\mathbf{x} : U \to S$. For concreteness let the domain of the curve be the interval $[a, b]$. By the ordinary chain rule

$$\dot{\mathbf{x}} = \dot{u}_1 \partial_1 \mathbf{x} + \dot{u}_2 \partial_2 \mathbf{x}$$

and so the length of such a curve is

$$\begin{aligned} L &= \int_a^b |\dot{\mathbf{x}}(t)|\, dt = \int_a^b |\dot{u}_1 \partial_1 \mathbf{x} + \dot{u}_2 \partial_2 \mathbf{x}|\, dt \\ &= \int_a^b (g_{ij} \dot{u}_1 \dot{u}_2)^{1/2} dt \end{aligned}$$

where $g_{ij} = \partial_i \mathbf{x} \cdot \partial_j \mathbf{x}$. Let $p = \mathbf{x}(u_1, u_2)$ be arbitrary in $V \subset S$. The bilinear form g_p given on each $T_p S \subset T\mathbb{R}^3$ where $p = \mathbf{x}(u_1, u_2)$ given by

$$g_p(v, w) = g_{ij} v^i w^j$$

for $v_p = v^1 \partial_1 \mathbf{x} + v^2 \partial_2 \mathbf{x}$ gives a tensor g is called the first fundamental form or metric tensor. The classical notation is $ds^2 = \sum g_{ij} du_j du_j$ which does, whatever it's shortcomings, succinctly encode the first fundamental form. For example, if we parameterize the sphere $S^2 \subset \mathbb{R}^3$ using the usual spherical coordinates φ, θ we have

$$ds^2 = d\varphi^2 + \sin^2(\varphi) d\theta^2$$

from which the length of a curve $c(t) = \mathbf{x}(\varphi(t), \theta(t))$ is given by

$$L(c) = \int_{t_0}^{t} \sqrt{(\frac{d\varphi}{dt})^2 + \sin^2 \varphi(t) (\frac{d\theta}{dt})^2} dt.$$

Now it may seem that we have something valid only in a single parameterization. Indeed the formulas are given using a single chart and so for instance the curve should not stray from the chart domain V. On the other hand, the expression $g_p(v, w) = g_{ij} v^i w^j$ is an invariant since it is just the length of the vector v as it sits in $\mathbb{R}^3$. So, as the reader has no doubt anticipated, $g_{ij}v^iw^j$ would give the same answer now matter what chart we used. By breaking up a curve into segments each of which lies in some chart domain we may compute it's length using a sequence of integrals of the form $\int (g_{ij}\dot{u}_1\dot{u}_2)^{1/2}dt$. It is a simple consequence of the chain rule that the result is independent of parameter changes. We also have a well defined notion of surface area of on S. This is given by

$$Area(S) := \int_S dS$$

and where dS is given locally by $\sqrt{g(u^1,u^2)}du^1du^2$ where $g := \det(g_{ij})$.

We will need to be able to produce normal fields on S. In a coordinate patch we may define

$$\begin{aligned} N &= \partial_1\mathbf{x}(u_1,u_2) \times \partial_2\mathbf{x}(u_1,u_2) \\ &= \det\begin{bmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \mathbf{i} \\ \frac{\partial x^2}{\partial u^1} & \frac{\partial x^2}{\partial u^2} & \mathbf{j} \\ \frac{\partial x^3}{\partial u^1} & \frac{\partial x^3}{\partial u^2} & \mathbf{k} \end{bmatrix}. \end{aligned}$$

The unit normal field is then $\mathbf{n} = N/|N|$. Of course, $\mathbf{n}$ is defined independent of coordinates up to sign because there are only two possibilities for a normal direction on a surface in $\mathbb{R}^3$. The reader can easily prove that if the surface is orientable then we may choose a global normal field. If the surface is a closed submanifold (no boundary) then the two choices are characterized as inward and outward.

We have two vector bundles associated with S that are of immediate interest. The first one is just the tangent bundle of S which is in this setting embedded into the tangent bundle of $\mathbb{R}^3$. The other is the normal bundle NS which has as its fiber at $p \in S$ the span of either normal vector $\pm\mathbf{n}$ at p. The fiber is denoted N_pS. Our plan now is to take the obvious connection on $T\mathbb{R}^3$, restrict it to S and then decompose into tangent and normal parts. Restricting to the tangent and normal bundles appropriately,what we end up with is three connections. The obvious connection on $\mathbb{R}^3$ is simply $D_\xi(\sum_{i=1}^3 Y^i\frac{\partial}{\partial x^i}) := dY^i(\xi)\frac{\partial}{\partial x^i}$ which exist simply because we have a global distinguished coordinate frame $\{\frac{\partial}{\partial x^i}\}$. The fact that this standard frame is orthonormal with respect to the dot product on $\mathbb{R}^3$ is of significance here. We have both of the following:

1. $D_\xi(X\cdot Y) = D_\xi X\cdot Y + X\cdot D_\xi Y$ for any vector fields X and Y on $\mathbb{R}^3$ and any tangent vector ξ.

2. $D_\xi \circ D_v = D_v \circ D_\xi$ (This means the connection has no "torsion" as we define the term later).

Now the connection on the tangent bundle of the surface is defined by projection. Let ξ be tangent to the surface at p and Y a tangent vector field on the surface. Then by definition

$$\nabla_\xi Y = (D_\xi Y)^{\tan}$$

where $(D_\xi Y)^{\tan}(p)$ is the projection of $D_\xi Y$ onto the tangent planes to the surface at p. This gives us a map $\nabla : TS \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ which is easily seen to be a connection. Now there is the left over part $(D_\xi Y)^\perp$ but as a map $(\xi, Y) \mapsto (D_\xi Y)^\perp$ this does *not* give a connection. On the other hand, if η is a normal field, that is, a section of the normal bundle NS we define $\nabla_\xi^\perp \eta := (D_\xi \eta)^\perp$. The resulting map $\nabla^\perp : TS \times \Gamma(S, NS) \to \Gamma(S, NS)$ given by $(\xi, \eta) \mapsto \nabla_\xi^\perp \eta$ is indeed a connection on the normal bundle. Here again there is a left over part $(D_\xi \eta)^{\tan}$. What about these two other left over pieces $(D_\xi \eta)^{\tan}$ and $(D_\xi Y)^\perp$? These pieces measure the way the surface bends in $\mathbb{R}^3$. We define the shape operator at a point $p \in S$ with respect to a unit normal direction in the following way. First choose the unit normal field $\mathbf{n}$ in the chosen direction as we did above. Now define the shape operator $S(p) : T_pS \to T_pS$ by

$$S(p)\xi = -\nabla_\xi \mathbf{n}.$$

To see that the result is really tangent to the sphere just notice that $\mathbf{n} \cdot \mathbf{n} = 1$ and so $\nabla_\xi^\perp \mathbf{n}$

$$\begin{aligned} 0 &= \xi 1 = \xi(\mathbf{n} \cdot \mathbf{n}) \\ &= 2 D_\xi \mathbf{n} \cdot \mathbf{n} \end{aligned}$$

which means that $D_\xi \mathbf{n} \in T_pS$. Thus the fact, that $\mathbf{n}$ had constant length gave us $\overline{\nabla}_\xi \mathbf{n} = (D_\xi \mathbf{n})^{\tan}$ and we have made contact with one of the two extra pieces. For a general normal section η we write $\eta = f\mathbf{n}$ for some smooth function on the surface and then

$$\begin{aligned} (D_\xi \eta)^{\tan} &= (D_\xi f\mathbf{n})^{\tan} \\ &= (df(\xi)\mathbf{n} + f D_\xi \mathbf{n})^{\tan} \\ &= -f S(p)\xi. \end{aligned}$$

so we obtain

Lemma J.1 $S(p)\xi = f^{-1}(D_\xi f\mathbf{n})^{\tan}$

The next result tell us that $S(p) : T_pS \to T_pS$ is symmetric with respect to the first fundamental form.

Lemma J.2 *Let* $v, w \in T_pS$. *Then we have* $g_p(S(p)v, w) = g_p(v, S(p)w)$.

Proof. The way we have stated the result hide something simple. Namely, tangent vector to the surface are also vectors in $\mathbb{R}^3$ under the usual identification of $T\mathbb{R}^3$ with $\mathbb{R}^3$. With this in mind the result is just $S(p)v \cdot w = v \cdot S(p)w$. Now this is easy to prove. Note that $\mathbf{n} \cdot w = 0$ and so $0 = v(\mathbf{n} \cdot w) = \overline{\nabla}_v \mathbf{n} \cdot w + \mathbf{n} \cdot \overline{\nabla}_v w$.

But the same equation holds with v and w interchanged. Subtracting the two expressions gives

$$\begin{aligned}
0 &= \overline{\nabla}_v \mathbf{n} \cdot w + \mathbf{n} \cdot \overline{\nabla}_v w \\
&\quad - \left(\overline{\nabla}_w \mathbf{n} \cdot v + \mathbf{n} \cdot \overline{\nabla}_w v\right) \\
&= \overline{\nabla}_v \mathbf{n} \cdot w - \overline{\nabla}_w \mathbf{n} \cdot v + \mathbf{n} \cdot (\overline{\nabla}_v w - \overline{\nabla}_w v) \\
&= \overline{\nabla}_v \mathbf{n} \cdot w - \overline{\nabla}_w \mathbf{n} \cdot v
\end{aligned}$$

from which the result follows. ∎

Since $S(p)$ is symmetric with respect to the dot product there are eigenvalues κ_1, κ_2 and eigenvectors $v_{\kappa_1}, v_{\kappa_2}$ such that $v_{\kappa_i} \cdot S(p) v_{\kappa_j} = \delta_{ij}\kappa_i$. Let us calculate in a special coordinate system containing our point p obtained by projecting onto the tangent plane there. Equivalently, we rigidly move the surface until p is at the origin of $\mathbb{R}^3$ and is tangent to the x, y plane. Then the surface is parameterized near p by $(u^1, u^2) \mapsto (u^1, u^2, f(u^1, u^2))$ for some smooth function f with $\frac{\partial f}{\partial u^1}(0) = \frac{\partial f}{\partial u^2}(0) = 0$. At the point p, which is now the origin, we have $g_{ij}(0) = \delta_{ij}$. Since S is now the graph of the function f the tangent space T_pS is identified with the x, y plane. A normal field is given by $\operatorname{grad} F = \operatorname{grad}(f(x,y) - z) = (\frac{\partial f}{\partial u^1}, \frac{\partial f}{\partial u^2}, -1)$ and the unit normal is

$$\mathbf{n}(u^1, u^2) = \frac{1}{\sqrt{(\frac{\partial f}{\partial u^1})^2 + (\frac{\partial f}{\partial u^2})^2 + 1}} (\frac{\partial f}{\partial u^1}, \frac{\partial f}{\partial u^2}, -1)$$

Letting $r(u^1, u^2) := ((\frac{\partial f}{\partial u^1})^2 + (\frac{\partial f}{\partial u^2})^2 + 1)^{1/2}$ and using lemma J.1 we have $S(p)\xi = r^{-1}(\overline{\nabla}_\xi r\mathbf{n})^{\tan} = r^{-1}(\overline{\nabla}_\xi N)^{\tan}$ where $N := (\frac{\partial f}{\partial u^1}, \frac{\partial f}{\partial u^2}, -1)$. Now at the origin $r = 1$ and so $\xi \cdot S(p)\xi = \overline{\nabla}_\xi N \cdot \xi = \frac{\partial}{\partial u^k} \frac{\partial F}{\partial u^i} \xi^k \xi^i = \frac{\partial^2 F}{\partial u^k \partial u^i} \xi^k \xi^i$ from which we get the following:

$$\xi \cdot S(p)v = \sum_{ij} \xi^i v^j \frac{\partial f}{\partial u^i \partial u^j}(0)$$

valid for these special type of coordinates and only at the central point p. Notice that this means that once we have the surface positioned as a graph over the x, y-plane and parameterized as above then

$$\xi \cdot S(p)v = D^2 f(\xi, v) \text{ at } 0.$$

Here we must interpret ξ and v on the right hand side to be (ξ^1, ξ^2) and (v^1, v^2) where as on the left hand side $\xi = \xi^1 \frac{\partial \mathbf{x}}{\partial u^1} + \xi^2 \frac{\partial \mathbf{x}}{\partial u^2}$, $v = v^1 \frac{\partial \mathbf{x}}{\partial u^1} + v^2 \frac{\partial \mathbf{x}}{\partial u^2}$.

Exercise J.6 *Position S to be tangent to the x, y plane as above. Let the x, z plane intersect S in a curve c_1 and the y, z plane intersect S in a curve c_2. Show that by rotating we can make the coordinate vectors $\frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^2}$ be eigenvectors for $S(p)$ and that the curvatures of the two curves at the origin are κ_1 and κ_2.*

We have two important invariants at any point p. The first is the Gauss curvature $K := \det(S) = \kappa_1\kappa_2$ and the second is the mean curvature $H = \frac{1}{2}trace(S) = \frac{1}{2}(\kappa_1 + \kappa_2)$.

The sign of H depends on which of the two normal directions we have chosen while the sign of κ does not. In fact, the Gauss curvature turns out to be "intrinsic" to the surface in the sense that it remains constant under any deformation of the surface the preserves lengths of curves. More on this below but first let us establish a geometric meaning for H. First of all, we may vary the point p and then S becomes a function of p and the same for H (and K).

Theorem J.2 *Let S_t be a family of surfaces given as the image of maps $h_t : S \to \mathbb{R}^3$ and given by $p \mapsto p + t\mathbf{v}$ where $\mathbf{v}$ is a section of $T\mathbb{R}^3\big|_S$ with $\mathbf{v}(0) = 1$ and compact support. Then*

$$\left.\frac{d}{dt}\right|_{t=0} area(S_t) = -\int_S (\mathbf{v} \cdot H\mathbf{n})dS$$

More generally, the formula is true if $h : (-\epsilon, \epsilon) \times S \to \mathbb{R}^3$ is a smooth map and $\mathbf{v}(p) := \left.\frac{d}{dt}\right|_{t=0} h(t,p)$.

Exercise J.7 *Prove the above theorem by first assuming that $\mathbf{v}$ has support inside a chart domain and then use a partition of unity argument to get the general case.*

Surface S is called a **minimal surface** if $H \equiv 0$ on S. It follows from theorem J.2 that if S_t is a family of surfaces given as in the theorem that if S_0 is a minimal surface then 0 is a critical point of the function $a(t) := area(S_t)$. Conversely, if 0 is a critical point for all such variations of S then S is a minimal surface.

Exercise J.8 *(doody this one in Maple) Show that Sherk's surface, which is given by $e^z \cos(y) = \cos x$, is a minimal surface. If you haven't seen this surface do a plot of it using Maple, Mathematica or some other graphing software. Do the same for the helicoid $y \tan z = x$.*

Now we move on to the Gauss curvature K. Here the most important fact is that K may be written in terms of the first fundamental form. The significance of this is that if S_1 and S_2 are two surfaces and if there is a map $\phi : S_1 \to S_2$ that preserves the length of curves, then κ^{S_1} and κ^{S_2} are the same in the sense that $K^{S_1} = K^{S_2} \circ \phi$. In the following theorem, "$g_{ij} = \delta_{ij}$ to first order" means that $g_{ij}(0) = \delta_{ij}$ and $\frac{\partial g_{ij}}{\partial u}(0) = \frac{\partial g_{ij}}{\partial v}(0) = 0$.

Theorem J.3 (Gauss's Theorema Egregium) *Let $p \in S$. There always exist coordinates u, v centered at p (so $u(p) = 0, v(p) = 0$) such that $g_{ij} = \delta_{ij}$ to first order at 0 and for which we have*

$$K(p) = \frac{\partial^2 g_{12}}{\partial u \partial v}(0) - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u^2}(0) - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial v^2}(0).$$

Proof. In the coordinates described above which give the parameterization $(u,v) \mapsto (u,v,f(u,v))$ where p is the origin of $\mathbb{R}^3$ we have

$$\begin{bmatrix} g_{11}(u,v) & g_{12}(u,v) \\ g_{21}(u,v) & g_{22}(u,v) \end{bmatrix} = \begin{bmatrix} 1+(\frac{\partial f}{\partial x})^2 & \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} & 1+(\frac{\partial f}{\partial y})^2 \end{bmatrix}$$

from which we find after a bit of straightforward calculation

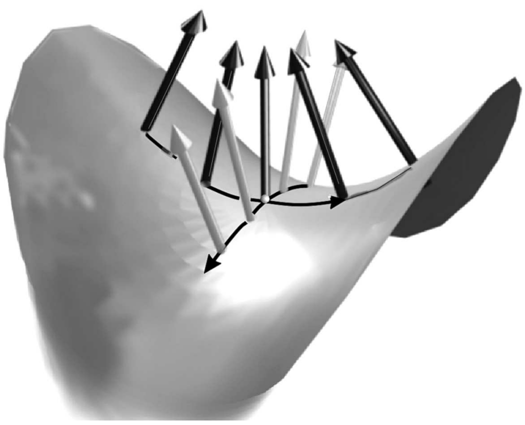
$$\begin{aligned} &\frac{\partial^2 g_{12}}{\partial u \partial v}(0) - \frac{1}{2}\frac{\partial^2 g_{22}}{\partial u^2}(0) - \frac{1}{2}\frac{\partial^2 g_{11}}{\partial v^2}(0) \\ &= \frac{\partial^2 f}{\partial u^2}\frac{\partial^2 f}{\partial v^2} - \frac{\partial^2 f}{\partial u \partial v} = \det D^2 f(0) \\ &= \det S(p) = K(p). \end{aligned}$$

■

Note that if we have any other coordinate system s,t centered at p then writing $(u,v) = (x^1, x^2)$ and $(s,t) = (\bar{x}^1, \bar{x}^2)$ we have the transformation law

$$\bar{g}_{ij} = g_{kl}\frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}$$

which means that if we know the metric components in any coordinate system then we can get them, and hence $K(p)$, at any point in any coordinate system. The conclusion is the that the metric determines the Gauss curvature. We say that K is an **intrinsic invariant**.



Negative Gauss Curvature

Now every field $\bar{X} \in \mathfrak{X}(\overline{M})\big|_M$ is uniquely written as $X^{\tan} + X^{\perp}$ where $X^{\tan} \in \mathfrak{X}(M)$ and $X^{\perp} \in \mathfrak{X}(M)^{\perp}$. Now for any $\bar{Y} \in \mathfrak{X}(\overline{M})\big|_M$ and $X \in \mathfrak{X}(M)$ we have the decomposition

$$\bar{\nabla}_X \bar{Y} = \left(\bar{\nabla}_X \bar{Y}\right)^{\tan} + \left(\bar{\nabla}_X \bar{Y}\right)^{\perp}$$

and writing $\bar{Y} = Y^{\tan} + Y^{\perp}$ we have

$$\begin{aligned}&\left(\bar{\nabla}_X\left(Y^{\tan}+Y^{\perp}\right)\right)^{\tan}+\left(\bar{\nabla}_X\left(Y^{\tan}+Y^{\perp}\right)\right)^{\perp}\\&=\left(\bar{\nabla}_X Y^{\tan}\right)^{\tan}+\left(\bar{\nabla}_X Y^{\perp}\right)^{\tan}\\&+\left(\bar{\nabla}_X Y^{\tan}\right)^{\perp}+\left(\bar{\nabla}_X Y^{\perp}\right)^{\perp}\end{aligned}$$

Now we will show that $\langle\left(\bar{\nabla}_X Y^{\tan}\right)^{\perp}, Y^{\perp}\rangle = -\langle Y^{\tan}, \left(\bar{\nabla}_X Y^{\tan}\right)^{\tan}\rangle$. Indeed, $\langle Y^{\tan}, Y^{\perp}\rangle = 0$ and so

$$\begin{aligned}0=\bar{\nabla}_X\langle Y^{\tan},Y^{\perp}\rangle&=\langle\bar{\nabla}_X Y^{\tan},Y^{\perp}\rangle+\langle Y^{\tan},\bar{\nabla}_X Y^{\perp}\rangle\\&=\langle\left(\bar{\nabla}_X Y^{\tan}\right)^{\perp},Y^{\perp}\rangle+\langle Y^{\tan},\left(\bar{\nabla}_X Y^{\tan}\right)^{\tan}\rangle\end{aligned}$$

Notice that $Y^{\tan}$ and $Y^{\perp}$ may vary independently. We now want to interpret what we have so far.

Definition J.8 *For $X_1, X_2 \in \mathfrak{X}(M)$ we define $\nabla_{X_1}X_2 := \left(\bar{\nabla}_{X_1}X_2\right)^{\tan}$.*

$(X_1, X_2) \to \nabla_{X_1}X_2$ defines a connection on M (i.e. a connection on the bundle TM). Thus $(X_1, X_2) \to \nabla_{X_1}X_2$ is $C^{\infty}(M)$-linear in X_1. It follows, as usual, that for any $v \in T_pM$ the map $v \mapsto \nabla_v X$ is well defined. The connection is actually the Levi-Civita connection for M. To verify this we just need to check that $\nabla_{X_1}X_2 - \nabla_{X_2}X_1 = [X_1, X_2]$ for all $X_1, X_2 \in \mathfrak{X}(M)$ and that

$$\nabla_X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

for all $X, Y, Z \in \mathfrak{X}(M)$. But both of these follow directly from the corresponding facts for $\bar{\nabla}$.

Definition J.9 *For any $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(M)^{\perp}$ we define $\nabla^{\perp}_X Y := \left(\bar{\nabla}_X T\right)^{\perp}$*

$(X, Y) \to \nabla^{\perp}_X Y$ defines a connection on the vector bundle $TM^{\perp} \to M$ which is again something easily deduced.

Exercise J.9 *Check the details, show that $\bar{\nabla}$ restricted to the bundle $T\bar{M}\big|_M$ is a connection and that $\nabla^{\perp}$ is a connection on $TM^{\perp}$. Show also that not only does $\nabla_{X_1}X_2 := \left(\bar{\nabla}_{X_1}X_2\right)^{\tan}$ define a connection of TM but that this connection is the Levi-Civita connection for M were M has the semi-Riemannian metric given by restriction of the metric on $\bar{M}$.*

Definition J.10 *Let $II : \mathfrak{X}(M)\times\mathfrak{X}(M) \to \mathfrak{X}(M)^{\perp}$ be defined by $II(X_1, X_2) := \left(\bar{\nabla}_{X_1}X_2\right)^{\perp}$. This $\mathfrak{X}(M)^{\perp}$-valued bilinear map called the second fundamental tensor. The set of all elements $u \in TM^{\perp}$ of unit length is called the unit normal bundle of M in $\bar{M}$. It is denoted by $TM^{\perp}_1$.*

We now have the decomposition

$$\bar{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + II(X_1, X_2)$$

for any $X_1, X_2 \in \mathfrak{X}(M)$.

Lemma J.3 ***Proposition J.1*** *II is $C^\infty(M)$ bilinear and symmetric.*

Proof. The linearity in the first variable is more or less obvious. We shall be content to show that

$$\begin{aligned} II(X_1, fX_2) &= \left(\bar\nabla_{X_1} fX_2\right)^\perp \\ &= \left((X_1 f)\, X_2 + f\bar\nabla_{X_1} X_2\right)^\perp \\ &= f\left(\bar\nabla_{X_1} X_2\right)^\perp = fII(X_1, X_2). \end{aligned}$$

Symmetry: $II(X_1, X_2) - II(X_2, X_1) = \left(\bar\nabla_{X_1} X_2 - \bar\nabla_{X_2} X_1\right)^\perp$
$= ([X_1, X_2])^\perp = 0$ ■

Definition J.11 *Let $p \in M$. For each unit vector u normal to M at p we have a map called the* ***shape operator*** *S_u defined by $S_u(v) := \left(\bar\nabla_v U\right)^{\tan}$ where U is any unit normal field defined near p such that $U(p) = u$.*

Now for any $Z \in \mathfrak{X}(M)^\perp$ and $X \in \mathfrak{X}(M)$ we consider $\bar\nabla_X Z$. We decompose this as $\bar\nabla_X Z = \left(\bar\nabla_X Z\right)^{\tan} + \left(\bar\nabla_X Z\right)^\perp = S_Z(X) + \nabla_X^\perp Z$

Proof. e job using the fact that $x \mapsto |x|$ is assumed to be smooth (resp. C^r). ■

Corollary J.1 *If a manifold M is modeled on a smooth (resp. C^r) Banach space M (for example, if M is a finite dimensional smooth manifold) then for every $\alpha_p \in T^*M$, there is a (global) smooth (resp. C^r) function f such that $Df|_p = \alpha_p$.*

Proof. Let $x_0 = \psi(p) \in \mathsf{M}$ for some chart ψ, U. Then the local representative $\bar\alpha_{x_0} = (\psi^{-1})^* \alpha_p$ can be considered a linear function on M since we have the canonical identification $\mathsf{M} \cong \{x_0\} \times \mathsf{M} = \mathsf{T}_{x_0}\mathsf{M}$. Thus we can define

$$\varphi(x) = \begin{cases} \beta(x)\bar\alpha_{x_0}(x) & \text{for} \quad x \in B_R(x_0) \\ 0 & \text{otherwise} \end{cases}$$

and now making sure that R is small enough that $B_R(x_0) \subset \psi(U)$ we can transfer this function back to M via ψ^{-1} and extend to zero outside of U get f. Now the differential of φ at x_0 is $\bar\alpha_{x_0}$ and so we have for $v \in T_pM$

$$\begin{aligned} df(p) \cdot v &= d(\psi^*\varphi)(p) \cdot v \\ &= (\psi^* d\varphi)(p)v \\ & d\varphi(T_p\psi \cdot v) \\ &= \bar\alpha_{x_0}(T_p\psi \cdot v) = (\psi^{-1})^*\alpha_p(T_p\psi \cdot v) \\ &= \alpha_p(T\psi^{-1}T_p\psi \cdot v) = \alpha_p(v) \end{aligned}$$

so $df(p) = \alpha_p$ ■

It is usually taken for granted that derivations on smooth functions are vector fields and that all C^∞ vector fields arise in this way. In fact, this not true in general. It is true however, for finite dimensional manifold. More generally, we have the following result:

Proposition J.2 *The map from $\mathfrak{X}(M)$ to the vector space of derivations* $\mathsf{Der}(M)$ *given by* $X \mapsto \mathcal{L}_X$ *is a linear monomorphism if M is modeled on a C^∞ Banach space.*

Proof. The fact that the map is linear is straightforward. We just need to get the injectivity. For that, suppose $\mathcal{L}_X f = 0$ for all $f \in \mathcal{C}^\infty(M)$. Then $Df|_p X_p = 0$ for all $p \in M$. Thus by corollary J.1 $\alpha_p(X_p) = 0$ for all $\alpha_p \in T_p^* M$. By the Hahn-Banach theorem this means that $X_p = 0$. Since p was arbitrary we concluded that $X = 0$. ∎

Another very useful result is the following:

Theorem J.4 *Let $L : \mathfrak{X}(M) \to \mathcal{C}^\infty(M)$ be a $\mathcal{C}^\infty(M)-$linear function on vector fields. If M admits (spherical?)cut off functions then $L(X)(p)$ depends only on the germ of X at p.*
If M is finite dimensional then $L(X)(p)$ depends only on the value of X at p.

Proof. Suppose $X = 0$ in a neighborhood U and let $p \in U$ be arbitrary. Let O be a smaller open set containing p and with closure inside U. Then letting β be a function that is identically 1 on a neighborhood of p contained in O and identically zero outside of O then $(1-\beta)X = X$. Thus we have

$$\begin{aligned} L(X)(p) &= L((1-\beta)X)(p) \\ &= (1-\beta(p))L(X)(p) = 0 \times L(X)(p) \\ &= 0. \end{aligned}$$

Applying this to $X - Y$ we see that if two fields X and Y agree in an open set then $L(X) = L(Y)$ on the same open set. The result follows from this.

Now suppose that M is finite dimensional and suppose that $X(p) = 0$. Write $X = X^i \frac{\partial}{\partial x^i}$ in some chart domain containing p with smooth function X^i satisfying $X^i(p) = 0$. Letting β be as above we have

$$\beta^2 L(X) = \beta X^i L(\beta \frac{\partial}{\partial x^i})$$

which evaluated at p gives

$$L(X)(p) = 0$$

since $\beta(p) = 1$. Applying this to $X - Y$ we see that if two fields X and Y agree at p then $L(X)(p) = L(Y)(p)$. ∎

Corollary J.2 *If M is finite dimensional and $L : \mathfrak{X}(M) \to \mathcal{C}^\infty(M)$ is a $\mathcal{C}^\infty(M)-$linear function on vector fields then there exists an element $\alpha \in \mathfrak{X}^*(M)$ such that $\alpha(X) = L(X)$ for all $X \in \mathfrak{X}(M)$.*

Definition J.12 *The support of a smooth function is the closure of the set in its domain where it takes on nonzero values. The support of a function f is denoted* $\mathrm{supp}(f)$.

For finite dimensional manifolds we have the following stronger result.ALLREADYDIDTHIS?

Lemma J.4 (Existence of cut-off functions) *Let K be a compact subset of $\mathbb{R}^n$ and U an open set containing K. There exists a smooth function β on $\mathbb{R}^n$ that is identically equal to 1 on K, has compact support in U and $0 \leq \beta \leq 1$.*

Proof. Special case: Assume that $U = B(0, R)$ and $K = \overline{B}(0, r)$. In this case we may take

$$\phi(x) = \frac{\int_{|x|}^{R} g(t)dt}{\int_{r}^{R} g(t)dt}$$

where

$$g(t) = \begin{cases} e^{-(t-r)^{-1}} e^{-(t-R)^{-1}} & \text{if } 0 < t < R \\ 0 & \text{otherwise.} \end{cases}$$

This is the circular cut-off that always exists for smooth Banach spaces.

General case: Let $K \subset U$ be as in the hypotheses. Let $K_i \subset U_i$ be concentric balls as in the special case above but now with various choices of radii and such that $K \subset \cup K_i$. The U_i are chosen small enough that $U_i \subset U$. Let ϕ_i be the corresponding functions provided in the proof of the special case. By compactness there are only a finite number of pairs $K_i \subset U_i$ needed so assume that this reduction to a finite cover has been made. Examination of the following function will convince the reader that it is well defined and provides the needed cut-off function;

$$\beta(x) = 1 - \prod_i (1 - \phi_i(x)).$$

■

Definition J.13 *A topological space is called **locally convex** if every point has a neighborhood with compact closure.*

Note that a finite dimensional smooth manifold is always locally compact and we have agreed that a finite dimensional manifold should be assumed to be Hausdorff unless otherwise stated. The following lemma is sometimes helpful. It shows that we can arrange to have the open sets of a cover and a locally refinement of the cover to be indexed by the same set in a consistent way:

Lemma J.5 *If X is a paracompact space and $\{U_i\}_{i \in I}$ is an open cover, then there exists a locally finite refinement $\{O_i\}_{i \in I}$ of $\{U_i\}_{i \in I}$ with $O_i \subset U_i$.*

Proof. Let $\{V_k\}_{i \in K}$ be a locally finite refinement of $\{U_i\}_{i \in I}$ with the index map $k \mapsto i(k)$. Let O_i be the union of all V_k such that $i(k) = k$. Notice that if an open set U intersects an infinite number of the O_i then it will meet an infinite number of the V_k. It follows that $\{O_i\}_{i \in I}$ is locally finite. ■

Theorem J.5 *A second countable, locally compact Hausdorff space X is paracompact.*

Sketch of proof. If follows from the hypotheses that there exists a sequence of open sets $U_1, U_2, ...$that cover X and such that each U_i has compact closure $\overline{U_i}$. We start an inductive construction: Set $V_n = U_1 \cup U_2 \cup ... \cup U_n$ for each positive integer n. Notice that $\{V_n\}$ is a new cover of X and each V_n has compact closure. Now let $O_1 = V_1$. Since $\{V_n\}$ is an open cover and $\overline{O_1}$ is compact we have

$$\overline{O_1} \subset V_{i_1} \cup V_{i_2} \cup ... \cup V_{i_k}.$$

Next put $O_2 = V_{i_1} \cup V_{i_2} \cup ... \cup V_{i_k}$ and continue the process. Now we have that X is the countable union of these open sets $\{O_i\}$ and each O_{i-1} has compact closure in O_i. Now we define a sequence of compact sets; $K_i = \overline{O_i} \setminus O_{i-1}$.
Now if $\{W_\beta\}_{\beta \in B}$ is *any* open cover of X we can use those W_β that meet K_i to cover K_i and then reduce to a finite subcover since K_i is compact. We can arrange that this cover of K_i consists only of sets each of which is contained in one of the sets $W_\beta \cap O_{i+1}$ and disjoint from O_{i-1}. Do this for all K_i and collect all the resulting open sets into a countable cover for X. This is the desired locally finite refinement. ■

Definition J.14 *A C^r partition of unity on a C^r manifold M is a collection $\{V_i, \rho_i\}$ where*
(i) $\{V_i\}$ is a locally finite cover of M;
(ii) each ρ_i is a C^r function with $\rho_i \geq 0$ and compact support contained in V_i;
(iii) for each $x \in M$ we have $\sum \rho_i(x) = 1$ (This sum is finite since $\{V_i\}$ is locally finite).
If the cover of M by chart domains $\{U_\alpha\}$ of some atlas $\mathcal{A} = \{(U_\alpha, \mathtt{x}_\alpha)\}$ for M has a partition of unity $\{V_i, \rho_i\}$ such that each V_i is contained in one of the chart domains $U_{\alpha(i)}$ (locally finite refinement), then we say that $\{V_i, \rho_i\}$ is ***subordinate to** $\mathcal{A}$. We will say that a manifold* ***admits a smooth partition of unity*** *if every atlas has a subordinate smooth partition of unity.*

Smooth (C^r, $r > 0$) partitions of unity do not necessarily exist on a Banach space and less so for manifolds modeled on such Banach spaces. On the other hand, some Banach spaces do admit partitions of unity. It is a fact that all separable Hilbert spaces admit partitions of unity. For more information see [A,B,R]. We will content ourselves with showing that all finite dimensional manifolds admit smooth partitions of unity.

Notice that in theorem J.5 we have proven a bit more than is part of the definition of paracompactness. Namely, the open sets of the refinement $V_i \subset U_{\beta(i)}$ have compact closure in $U_{\beta(i)}$. This is actually true for any paracompact space but we will not prove it here. Now for the existence of a smooth partition of unity we basically need the paracompactness but since we haven't proved the above statement about compact closures (shrink?ing lemma) we state the theorem in terms of second countability:

Theorem J.6 *Every second countable finite dimensional C^r manifold admits a C^r−partition of unity.*

Let M be the manifold in question. We have seen that the hypotheses imply paracompactness and that we may choose our locally finite refinements to have the compact closure property mentioned above. Let $\mathcal{A} = \{U_i, \mathtt{x}_i\}$ be an atlas for M and let $\{W_i\}$ be a locally finite refinement of the cover $\{U_i\}$ with $\overline{W}_i \subset U_i$. By lemma J.4 above there is a smooth cut-off function β_i with $\operatorname{supp}(\beta_i) = \overline{W}_i$. For any $x \in M$ the following sum is finite and defines a smooth function:

$$\beta(x) = \sum_i \beta_i(x).$$

Now we normalize to get the functions that form the partition of unity:

$$\rho_i = \frac{\beta_i}{\beta}.$$

It is easy to see that $\rho_i \geq 0$, and $\sum \rho_i = 1$.

J.3.1 Notation

Differential geometry is one of the subjects where notation is a continual problem. Notation that is highly precise from the vantage point of set theory and logic tends to be fairly opaque. On the other hand, notation that is true to intuition is difficult to make precise. Notation that is uncluttered and handy for calculations tends to suffer from ambiguities when looked at closely. It is perhaps worth pointing out that the kind of ambiguities we are talking about include some that are of the same sort as are accepted by every calculus student without much thought. For instance, we find (x, y, z) being used to refer variously to "indeterminates", "a triple of numbers", or a triple of functions of some variable as for example when we write

$$\vec{x}(t) = (x(t), y(t), z(t)).$$

Also, we often write $y = f(x)$ and then, even write $y = y(x)$ and $y'(x)$ or dy/dx instead of $f'(x)$ or df/dx. Polar coordinates are generally thought of as living on the $xy-$plane even though it could be argued that $r\theta-$space is really a (subset of a) different copy of $\mathbb{R}^2$. The ambiguities of this type of notation are as apparent. This does not mean that this notation is bad. In fact, it can be quite useful to use slightly ambiguous notation. Human beings are generally very good at handling ambiguity. In fact, if a self conscious desire to avoid logical inconsistency in notation is given priority over everything else we quickly begin to become immobilized. The reader should be warned that while we will develop fairly pedantic notation, perhaps too pedantic at times, we shall also not hesitate to resort to abbreviation and notational shortcuts as the need arises. This will be done with increasing frequency in later chapters.

J.4 Naive Functional Calculus.

We have recalled the basic definitions of the directional derivative of a map such as $f : \mathbb{R}^n \to \mathbb{R}^m$. This is a good starting point for making the generalizations to come but let us think about a bit more about our "directions" h and "points" p. In both cases these refer to $n-$tuples in $\mathbb{R}^n$. The values taken by the function are also tuples ($m-$tuples in this instance). From one point of view a $n-$tuple is just a function whose domain is the finite set $\{1, 2, ..., n\}$. For instance, the n-tuple $h = (h^1, ..., h^n)$ is just the function $i \mapsto h^i$ which may as well have been written $i \mapsto h(i)$. This suggests that we generalize to functions whose domain is an infinite set. A sequence of real numbers is just such an example but so is any real (or complex) valued function. This brings us to the notion of a function space. An example of a function space is $C([0, 1])$, the space of continuous functions on the unit interval $[0, 1]$. So, whereas an element of $\mathbb{R}^3$, say $(1, \pi, 0)$ has 3 components or entries, an element of $C([0, 1])$, say $(t \mapsto \sin(2\pi t))$ has a continuum of "entries". For example, the 1/2 entry of the latter element is $\sin(2\pi(1/2)) = 0$. So one approach to generalizing the usual setting of calculus might be to consider replacing the space of $n-$tuples $\mathbb{R}^n$ by a space of functions. Now we are interested in differentiating functions whose arguments are themselves functions. This type of function is sometimes called a functional. We shall sometimes follow the tradition of writing $F[f]$ instead of $F(f)$. Some books even write $F[f(x)]$. Notice that this is *not* a composition of functions. A simple example of a functional on $C([0, 1])$ is

$$F[f] = \int_0^1 f^2(x)dx.$$

We may then easily define a formal notion of directional derivative:

$$(D_h F)[f] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (F[f + \epsilon h] - F[f])$$

where h is some function which is the "direction vector". This also allows us to define the differential δF which is a linear map on the functions space given at f by $\delta F|_f h = (D_h F)[f]$. We use a δ instead of a d to avoid confusion between dx^i and δx^i which comes about when x^i is simultaneously used to denote a number and also a function of , say, t.

It will become apparent that choosing the right function space for a particular problem is highly nontrivial and in each case the function space must be given an appropriate topology. In the following few paragraphs our discussion will be informal and we shall be rather cavalier with regard to the issues just mentioned. After this informal presentation we will develop a more systematic approach (Calculus on Banach spaces).

The following is another typical example of a functional defined on the space $C^1([0, 1])$ of continuously differentiable functions defined on the interval $[0, 1]$:

$$S[c] := \int_0^1 \sqrt{1 + (dc/dt)^2}dt$$

The reader may recognize this example as the arc length functional. The derivative *at* the function c in the *direction of* a function $h \in C^1([0, 1])$ would be given by

$$\left.\delta S\right|_c(h) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(S[c+\varepsilon h] - S[c]\right).$$

It is well known that if $\left.\delta S\right|_c(h) = 0$ for every h then c is a linear function; $c(t) = at + b$. The condition $\left.\delta S\right|_c(h) = 0 = 0$ (for all h) is often simply written as $\delta S = 0$. We shall have a bit more to say about this notation shortly. For examples like this one, the analogy with multi-variable calculus is summarized as

$$\begin{aligned} \text{The index or argument becomes continuous:} &\quad i \rightsquigarrow t \\ d\text{-tuples become functions:} &\quad x^i \rightsquigarrow c(t) \\ \text{Functions of a vector variable become functionals of functions:} &\quad f(\vec{x}) \rightsquigarrow S[c] \end{aligned}$$

Here we move from $d-$tuples (which are really functions with finite domain) to functions with a continuous domain. The function f of x becomes a functional S of functions c.

We now exhibit a common example from the mechanics which comes from considering a bead sliding along a wire. We are supposed to be given a so called "Lagrangian function" $L : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ which will be the basic ingredient in building an associated functional. A typical example is of the form $L(x, v) = \frac{1}{2}mv^2 - V(x)$. Define the action functional S by using L as follows: For a given function $t \longmapsto q(t)$ defined on $[a, b]$ let

$$S[q] := \int_a^b L(q(t), \dot{q}(t))dt.$$

We have used x and v to denote variables of L but since we are eventually to plug in $q(t), \dot{q}(t)$ we could also follow the common tradition of denoting these variables by q and $\dot{q}$ but then it must be remembered that we are using these symbols in two ways. In this context, one sometimes sees something like following expression for the so-called variation

$$\delta S = \int \frac{\delta S}{\delta q(t)}\delta q(t)dt \tag{J.1}$$

Depending on one's training and temperament, the meaning of the notation may be a bit hard to pin down. First, what is the meaning of δq as opposed to, say, the differential dq? Second, what is the mysterious $\frac{\delta S}{\delta q(t)}$? A good start might be to go back and settle on what we mean by the differential in ordinary multivariable calculus. For a differentiable function $f : \mathbb{R}^d \to \mathbb{R}$ we take df to just mean the map

$$df : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

given by $df(p, h) = f'(p)h$. We may also fix p and write $\left.df\right|_p$ or $df(p)$ for the linear map $h \mapsto df(p, h)$. With this convention we note that $\left.dx^i\right|_p(h) = h^i$ where $h = (h^1, ..., h^d)$. Thus applying both sides of the equation

$$\left.df\right|_p = \sum \frac{\partial f}{\partial x^i}(p)\left.dx^i\right|_p \tag{J.2}$$

to some vector h we get

$$f'(p)h = \sum \frac{\partial f}{\partial x^i}(p)h^i. \tag{J.3}$$

In other words, $df|_p = D_h f(p) = \nabla f \cdot h = f'(p)$. Too many notations for the same concept. Equation J.2 is clearly very similar to $\delta S = \int \frac{\delta S}{\delta q(t)} \delta q(t) dt$ and so we expect that δS is a linear map and that $t \mapsto \frac{\delta S}{\delta q(t)}$ is to δS as $\frac{\partial f}{\partial x^i}$ is to df:

$$\begin{aligned} df &\rightsquigarrow \delta S \\ \frac{\partial f}{\partial x^i} &\rightsquigarrow \frac{\delta S}{\delta q(t)}. \end{aligned}$$

Roughly, $\frac{\delta S}{\delta q(t)}$ is taken to be whatever function (or distribution) makes the equation J.1 true. We often see the following type of calculation

$$\begin{aligned} \delta S &= \delta \int L dt \\ &= \int \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q}\right) dt \\ &= \int \left\{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right\} \delta q dt \end{aligned} \tag{J.4}$$

from which we are to conclude that

$$\frac{\delta S}{\delta q(t)} = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$$

Actually, there is a subtle point here in that we must restrict δS to variations for which the integration by parts is justified. We can make much better sense of things if we have some notion of derivative for functionals defined on some function space. There is also the problem of choosing an appropriate function space. On the one hand, we want to be able to take (ordinary) derivatives of these functions since they may appear in the very definition of S. On the other hand, we must make sense out of limits so we must pick a space of functions with a tractable and appropriate topology. We will see below that it is very desirable to end up with what is called a Banach space. Often one is forced to deal with more general topological vector spaces. Let us ignore all of these worries for a bit longer and proceed formally. If δS is somehow the variation due to a variation $h(t)$ of $q(t)$ then it depends on both the starting position in function space (namely, the function $q(.)$) and also the direction in function space that we move (which is the function $h(.)$). Thus we interpret $\delta q = h$ as some appropriate function and then interpret δS as short hand for

$$\begin{aligned} \delta S|_{q(.)}\, h &:= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(S[q + \varepsilon h] - S[q]) \\ &= \int \left(\frac{\partial L}{\partial q} h + \frac{\partial L}{\partial \dot{q}} \dot{h}\right) dt \end{aligned}$$

Note: Here and throughout the book the symbol ":= " is sued to indicate equality by definition.

If we had been less conventional and more cautious about notation we would have used c for the function which we have been denoting by $q : t \mapsto q(t)$. Then we could just write $\delta S|_c$ instead of $\delta S|_{q(.)}$. The point is that the notation $\delta S|_q$ might leave one thinking that $q \in \mathbb{R}$ (which it is under one interpretation!) but then $\delta S|_q$ would make no sense. It is arguably better to avoid letting q refer both to a number and to a function even though this is quite common. At any rate, from here we restrict attention to "directions" h for which $h(a) = h(b) = 0$ and use integration by parts to obtain

$$\delta S|_{q(.)}\, h = \int \{ \frac{\partial L}{\partial x^i}(q(t), \dot{q}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) \} h^i(t) dt.$$

So it seems that the function $E(t) := \frac{\partial L}{\partial q}(q(t), \dot{q}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}(q(t), \dot{q}(t))$ is the right candidate for the $\frac{\delta S}{\delta q(t)}$. However, once again, we must restrict to h which vanish at the boundary of the interval of integration. On the other hand, this family is large enough to force the desired conclusion. Despite this restriction the function $E(t)$ is clearly important. For instance, if $\delta S|_{q(.)} = 0$ (or even $\delta S|_{q(.)}\, h = 0$ for all functions that vanish at the end points) then we may conclude easily that $E(t) \equiv 0$. This gives an equation (or system of equations) known as the Euler-Lagrange equation for the function $q(t)$ corresponding to the action functional S :

$$\frac{\partial L}{\partial q}(q(t), \dot{q}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t)) = 0$$

Exercise J.10 *Replace $S[c] = \int L(c(t), \dot{c}(t))dt$ by the similar function of several variables $S(c_1, ...c_N) = \sum L(c_i, \triangle c_i)$. Here $\triangle c_i := c_i - c_{i-1}$ (taking $c_0 = c_N$) and L is a differentiable map $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$. What assumptions on $c = (c_1, ...c_N)$ and $h = (h_1, ...h_N)$ justify the following calculation?*

$$\begin{aligned} dS|_{(c_1,...c_N)}\, h &= \sum \frac{\partial L}{\partial c_i} h^i + \frac{\partial L}{\partial \triangle c_i} \triangle h^i \\ &= \sum \frac{\partial L}{\partial c_i} h^i + \sum \frac{\partial L}{\partial \triangle c_i} h^i - \sum \frac{\partial L}{\partial \triangle c_i} h^{i-1} \\ &= \sum \frac{\partial L}{\partial c_i} h^i + \sum \frac{\partial L}{\partial \triangle c_i} h^i - \sum \frac{\partial L}{\partial \triangle c_{i+1}} h^i \\ &= \sum \frac{\partial L}{\partial c_i} h^i - \sum (\frac{\partial L}{\partial \triangle c_{i+1}} - \frac{\partial L}{\partial \triangle c_i}) h^i \\ &= \sum \{ \frac{\partial L}{\partial c_i} h^i - (\triangle \frac{\partial L}{\partial \triangle c_i}) \} h^i \\ &= \sum \frac{\partial S}{\partial c_i} h^i. \end{aligned}$$

The upshot of our discussion is that the δ notation is just an alternative notation to refer to the differential or derivative. Note that q^i might refer to a coordinate or to a function $t \mapsto q^i(t)$ and so dq^i is the usual differential and maps $\mathbb{R}^d$ to $\mathbb{R}$ whereas $\delta x^i(t)$ is either taken as a variation function $h^i(t)$ as above or as the map $h \mapsto \delta q^i(t)(h) = h^i(t)$. In the first interpretation $\delta S = \int \frac{\delta S}{\delta q^i(t)} \delta q^i(t) dt$ is an abbreviation for $\delta S(h) = \int \frac{\delta S}{\delta q^i(t)} h^i(t) dt$ and in the second interpretation it is the map $\int \frac{\delta S}{\delta q^i(t)} \delta q^i(t) dt : h \mapsto \int \frac{\delta S}{\delta q^i(t)} (\delta q^i(t)(h)) dt = \int \frac{\delta S}{\delta q^i(t)} h^i(t) dt$. The various formulas make sense in either case and both interpretations are ultimately equivalent. This much the same as taking the dx^i in $df = \frac{\partial f}{\partial x i} dx^i$ to be components of an arbitrary vector $(dx^1, ..., dx^d)$ or we may take the more modern view that dx^i is a linear map given by $dx^i : h \mapsto h^i$. If this seems strange recall that x^i itself is also interpreted both as a number and as a coordinate function.

Example J.1 *Let $F[c] := \int_{[0,1]} c^2(t) dt$ as above and let $c(t) = t^3$ and $h(t) = \sin(t^4)$. Then*

$$\begin{aligned}
\delta F|_c (h) &= D_h F(c) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(F[c + \varepsilon h] - F[c] \right) \\
&= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} F[c + \varepsilon h] \\
&= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_{[0,1]} (c(t)) + \varepsilon h(t))^2 dt \\
&= 2 \int_{[0,1]} c(t) h(t) dt = 2 \int_0^1 t^3 \sin(\pi t^4) dx \\
&= \frac{1}{\pi}
\end{aligned}$$

Note well that h and c are functions but here they are, more importantly, "points" in a function space! What we are differentiating is F. Again, $F[c]$ is *not* a composition of functions; the function c itself is the dependent variable here.

Exercise J.11 *Notice that for a smooth function $s : \mathbb{R} \to \mathbb{R}$ we may write*

$$\frac{\partial s}{\partial x^i}(x_0) = \lim_{h \to 0} \frac{s(x_0 + h e_i) - s(x_0)}{h}$$
$$\text{where } e_i = (0, ..., 1, ...0)$$

Consider the following similar statement which occurs in the physics literature quite often.

$$\frac{\delta S}{\delta c(t)} = \lim_{h \to 0} \frac{S[c + h\delta_t] - S[c]}{h}$$

Here δ_t is the Dirac delta function (distribution) with the defining property $\int \delta_t \phi = \phi(t)$ for all continuous ϕ. To what extent is this rigorous? Try a formal calculation using this limit to determine $\frac{\delta S}{\delta c(t)}$ in the case that

$$S(c) := \int_0^1 c^3(t) dt.$$

Geometry has its philosophical aspects. Differential geometry makes explicit, and abstracts from, our intuitions about what the world itself is as revealed by, as the German philosopher Kant would have it, the action of understanding on the manifold presentations of the senses. Kant held that space and time were "categories of the understanding". Kant thought that the truths of Euclidean geometry where a prior truths; what he called a priori synthetic truths. This often taken to imply that Kant believed that non-Euclidean geometries were impossible (at least as far as physical space is concerned) but perhaps this doesn't give enough credit to Kant. Another interpretation of Kant's arguments simply concludes that human intuition about the geometry of space must necessarily be Euclidean; a statement that is still questionable but certainly not the same as the claim that the real geometry of space must be of logical necessity Euclidean in nature. Since the advent of Einstein's theory of gravitation it is assumed that space may in fact be non-Euclidean. Despite this it does seem to be a fact that our innate conceptualization of space is Euclidean. Contrary to Kant's assertions, this seems to be the result of evolution and the approximately Euclidean nature of space at human scales. Einstein's theory of gravitation is an extension of special relativity which can be seen as unifying space and time. From that point of view, the theory is more a theory about spacetime rather than space. For philosophers and scientist alike, *space* and *time* are words that refer to basic aspects of the world and/or, depending on one philosophical persuasion, basic aspects of the human experience of the world. An implicit understanding of space and time is part and parcel of what if means to be conscious. As the continental philosophers might describe it, space and time are what stand between us and our goals. Heidegger maintained that there was in human beings an understanding of a deeper kind of time that he called "temporality[2]" that was lost to the explicitly self conscious scientific form of understanding. I believe that while this might be true in some sense, Heidegger and many of his modern admirers underestimate the extent to which mathematics has succeeded in capturing and making explicit our implicit existential understanding of spatiality and temporality. Again, evolutionary theory suggests a different analysis of human temporality. In any case, geometry as a branch of mathematics can be seen as related to humankind's effort to come to an exact and explicit understanding of these ubiquitous aspects of experienced reality. In this form geometry is abstracted and idealized. Heidegger would say that much is lost once the abstraction is effected, even so, what is gained is immense. Geometry is the first exact science.

At root, what is the actual nature of reality which makes us experience it in this largely geometric way? As hinted at above, this is the topic of much discussion among philosophers and scientists. That the abstractions of geometry are not mere abstractions is proved by the almost unbelievable usefulness of geometric thinking in physics. Differential geometry is especially useful for classical physics-including and especially Einstein's theory of gravitation (general relativity). On the other hand, there is also the presumably more fundamental

[2]Strangely, Heidegger did not see the need to form a similar concept of primordial spatiality.

quantum nature of physical reality. One has to face the possibility that the quantum nature of the physical world is not a drama played out on a preexisting stage of a classically conceived space (or spacetime) but rather it may be the case that, like temperature, space and time are emergent "macroscopic properties" of nature. In fact, it is popular among physicists to look for so called "background free" theories where the most basic entities that make up (a model of) reality conspire to create what we perceive as space and time. On the mathematical side, this idea is connected with the emerging fields of discrete differential geometry and noncommutative geometry or quantum geometry[3]. Whatever the outcome of this drive toward background free theory, geometry in a broad sense can be expected to remain important for physics. After all, whatever nature is in her own secret nature, it is evident that her choreography adds up to something highly geometric when viewed at a large scale. If physics wants to replace geometry as we know it today by something else, then it is left to explain why geometry emerges in the macroscopic world. Thus even the most ambitious background free theory must be shown to contain the seeds of geometry and the mechanism by which macroscopic geometry emerges must be explained. It would be a mistake to underestimate this task.

J.4.1 Lagrange Multipliers and Ljusternik's Theorem

`Note: This section is under construction. It is still uncertain if this material will be included at all.`

The next example show how to use Lagrange multipliers to handle constraints.

Example J.2 *Let* E *and* F *and* F_0 *be as in the previous example. We define two functionals*

$$\mathcal{F}[f] := \int_D \nabla f \cdot \nabla f dx$$
$$\mathcal{C}[f] = \int_D f^2 dx$$

We want a necessary condition on f *such that* f *extremizes* $\mathcal{D}$ *subject to the constraint* $\mathcal{C}[f] = 1$. *The method of Lagrange multipliers applies here and so we have the equation* $D\mathcal{F}|_f = \lambda \; D\mathcal{C}|_f$ *which means that*

$$\langle \frac{\delta \mathcal{F}}{\delta f}, h \rangle = \lambda \langle \frac{\delta \mathcal{C}}{\delta f}, h \rangle \text{ for all } h \in C_c^2(D)$$

or

$$\frac{\delta \mathcal{F}}{\delta f} = \lambda \frac{\delta \mathcal{C}}{\delta f}$$

[3] Quantum geometry is often taken to by synonymous with noncommutative geometry but there are also such things as Ashtekar's "quantum Riemannian geometry" which is doesn't quite fit so neatly into the field of noncommutative geometry as it is usually conceived.

After determining the functional derivatives we obtain

$$-\nabla^2 f = \lambda f$$

This is not a very strong result since it is only a necessary condition and only hints at the rich spectral theory for the operator ∇^2.

Theorem J.7 *Let* E *and* F *be Banach spaces and* $U \subset \mathsf{E}$ *open with a differentiable map* $f : U \to \mathsf{F}$. *If for* $x_0 \in U$ *with* $y_0 = f(x_0)$ *we have that* $Df|_{x_0}$ *is onto and* $\ker Df|_{x_0}$ *is complemented in* E *then the set* $x_0 + \ker Df|_{x_0}$ *is tangent to the level set* $f^{-1}(y_0)$ *in the following sense: There exists a neighborhood* $U' \subset U$ *of* x_0 *and a homeomorphism* $\phi : U' \to V$ *where* V *is another neighborhood of* x_0 *and where* $\phi(x_0 + h) = x_0 + h + \varepsilon(h)$ *for some continuous function* ε *with the property that*

$$\lim_{h \to 0} \frac{\|\varepsilon(h)\|}{\|h\|} = 0.$$

Proof. $Df|_{x_0}$ is surjective. Let $K := \ker Df|_{x_0}$ and let L be the complement of K in E. This means that there are projections $p : \mathsf{E} \to K$ and $q : \mathsf{E} \to L$

$$p^2 = p \text{ and } q^2 = q$$
$$p + q = id$$

Let $r > 0$ be chosen small enough that $x_0 + B_r(0) + B_r(0) \subset U$. Define a map

$$\psi : K \cap B_r(0) \times L \cap B_r(0) \to \mathsf{F}$$

by $\psi(h_1, h_2) := f(x_0 + h_1 + h_2)$ for $h_1 \in K \cap B_r(0)$ and $h_2 \in L \cap B_r(0)$. We have $\psi(0,0) = f(x_0) = y_0$ and also one may verify that ψ is C^1 with $\partial_1 \psi = Df(x_0)|K = 0$ and $\partial_2 \psi = Df(x_0)|L$. Thus $\partial_2 \psi : L \to \mathsf{F}$ is a continuous isomorphism (use the open mapping theorem) and so we have a continuous linear inverse $(\partial_2 \psi)^{-1} : \mathsf{F} \to L$. We may now apply the implicit function theorem to the equation $\psi(h_1, h_2) = y_0$ to conclude that there is a locally unique function $\varepsilon : K \cap B_\delta(0) \to L$ for small $\delta > 0$ (less than r) such that

$$\psi(h, \varepsilon(h)) = y_0 \text{ for all } h \in K \cap B_\delta(0)$$
$$\varepsilon(0) = 0$$
$$D\varepsilon(0) = -(\partial_2 \psi)^{-1} \circ \partial_1 \psi|_{(0,0)}$$

But since $\partial_1 \psi = Df(x_0)|K = 0$ this last expression means that $D\varepsilon(0) = 0$ and so

$$\lim_{h \to 0} \frac{\|\varepsilon(h)\|}{\|h\|} = 0$$

Clearly the map $\phi : (x_0 + K \cap B_\delta(0)) \to \mathsf{F}$ defined by $\phi(x_0 + h) := x_0 + h + \varepsilon(h)$ is continuous and also since by construction $y_0 = \psi(h, \varepsilon(h)) = \phi(x_0 + h + \varepsilon(h))$ we have that ϕ has its image in $f^{-1}(y_0)$. Let the same symbol ϕ denote the map $\phi : (x_0 + K \cap B_\delta(0)) \to f^{-1}(y_0)$ which only differs in its codomain. Now h and $\varepsilon(h)$ are in complementary subspaces and so ϕ must be injective. Thus its restriction to the set $V := \{x_0 + h + \varepsilon(h) : h \in K \cap B_\delta(0)$ is invertible and in fact we have $\phi^{-1}(x_0+h+\varepsilon(h)) = x_0+h$. That V is open follows from the way we have used the implicit function theorem. Now recall the projection p. Since the range of p is K and its kernel is L we have that $\phi^{-1}(x_0 + h + \varepsilon(h)) = x_0 + p(h + \varepsilon(h))$ and we see that ϕ^{-1} is continuous on V. Thus ϕ (suitably restricted) is a homeomorphism of $U' := x_0 + K \cap B_\delta(0)$ onto $V \subset f^{-1}(y_0)$. We leave it to the reader to provide the easy verification that ϕ has the properties claimed by statement of the theorem. ■

A transitive $G-$space, M, is essentially a group theoretic object in a sense that we now describe. The reader will understand the situation much better if s/he does the following exercises before proceeding to the abstract situation.

Exercise J.12 *Show that the action of $Aff^+(\mathbf{A}^2)$ on $\mathbf{A}^2$ is transitive and effective but not free.*

Exercise J.13 *Fix a point x_0 (say $(0,0)$) in $\mathbf{A}^2$ and show that $H := \{g \in Aff^+(\mathbf{A}^2) : gx_0 = x_0\}$ is a closed subgroup of $Aff^+(\mathbf{A}^2)$ isomorphic to $Sl(2)$.*

Exercise J.14 *Let $H \cong Sl(2)$ be as in the previous exercise. Show that there is a natural 1-1 correspondence between the cosets of $Aff^+(\mathbf{A}^2)/H$ and the points of $\mathbf{A}^2$.*

Exercise J.15 *Show that the bijection of the previous example is a homeomorphism if we give $Aff^+(\mathbf{A}^2)/H$ its quotient topology.*

Exercise J.16 *Let S^2 be the unit sphere considered a subset of $\mathbb{R}^3$ in the usual way. Let the group $SO(3)$ act on S^2 by rotation. This action is clearly continuous and transitive. Let $n = (0,0,1)$ be the "north pole". Show that if H is the (closed) subgroup of $SO(3)$ consisting of all $g \in SO(3)$ such that $g \cdot n = n$ then $x = g \cdot n \mapsto g$ gives a well defined bijection $S^2 \cong SO(3)/H$. Note that $H \cong \mathrm{SO}(2)$ so we may write $S^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2)$.*

Exercise J.17 *Show that the bijection of the previous example is a homeomorphism if we give $\mathrm{SO}(3)/H$ its quotient topology.*

Definition J.15 *Let M be a transitive G-space. Fix $x_0 \in M$. The isotropy subgroup for the point x_0 is*

$$G_{x_0} := \{g \in G : gx_0 = x_0\}$$

Theorem J.8 *Let M be a transitive (left) G-space and fix $x_0 \in M$. Let G_{x_0} be the corresponding **isotropy subgroup** Then we have a natural bijection*

$$G \cdot x_0 \cong G/G_{x_0}$$

given by $g \cdot x_0 \mapsto gG_{x_0}$. In particular, if the action is transitive then $G/G_{x_0} \cong M$ and x_0 maps to H.

For the following discussion we let M be a transitive (left) G-space and fix $x_0 \in M$. The action of G on M may be transferred to an equivalent action on G/G_{x_0} via the bijection of the above theorem. To be precise, if we let Φ denote the bijection of the theorem $G/G_{x_0} \cong M$ and let λ denote the actions of G on M and G respectively, then λ' is defined so that the following diagram commutes:

$$\begin{array}{ccccc} & G \times G/G_{x_0} & \xrightarrow{\lambda'} & G/G_{x_0} \\ id_G \times \Phi & \downarrow & & \downarrow \Phi \\ & G \times M & \xrightarrow{\lambda} & M \end{array}$$

This action turns out to be none other than the most natural action of G on G/G_{x_0}. Namely,

$$(g, xG_{x_0}) \mapsto gxG_{x_0}.$$

We have left out consideration of the topology. That is, what topology should we take on G/G_{x_0} so that all of the maps of the above diagram are continuous? If we just use the bijection to transfer the topology of M over to G/G_{x_0}. In this case, all of the actions are continuous. On the other hand, G/G_{x_0} has the quotient topology and we must face the possibility that the two topologies do not coincide! In order to not interrupt discussion we will defer dealing with this question until chapter 13. For now we just comment that for each of the examples discussed in this chapter the two topologies are easily seen to coincide.

Exercise J.18 *Prove theorem J.8 and the above statement about the equivalent action of G on G/G_{x_0}.*

Now any closed subgroup $H \subset G$ is the isotropy subgroup of the coset H for the natural action $G \times G/H \to G/H$ (where $(g, xH) \mapsto gxH$). The upshot of all this is that we may choose to study coset spaces G/H as $G-$spaces since every transitive $G-$spaces is equivalent to one of this form.

J.4.2 Euclidean space

As we have mentioned, if we have an affine space A modeled on an n-dimensional vector space V then if we are also given an inner product $\langle , \rangle$ on V, we may at once introduce several new structures. For one thing, we have a notion of length of vectors and this can be used to define a distance function on A itself. From the definition of affine space we know that for any two elements $p, q \in A$, there is a unique translation vector v such that $p + v = q$. If we denote this v suggestively by $q - p$ then the distance is $\text{dist}(p, q) := \|q - p\|$ where $\|.\|$ is the norm defined by the inner product $\|v\| = \langle v, v\rangle^{1/2}$. There is also a special family of coordinates that are constructed by choosing an orthonormal basis $e_1, ..., e_n$ for V. Having chosen the orthonormal basis, we get a map $V \to \mathbb{R}^n$

$$v = v^1 e_1 + \cdots + v^n e_n \mapsto (v^1, ..., v^n).$$

Now picking one of the bijections $A \to V$ centered at some point and composing this with the map $V \to \mathbb{R}^n$ that we got from the basis we get the desired coordinate map $A \to \mathbb{R}^n$. All coordinate systems constructed in this way are related by affine transformations of the form $L : x \mapsto Lx_0 + Q(x - x_0)$ for some $L \in O(V)$. The set of all such transformations is a group which is called the **Euclidean motion** group $\mathrm{Euc}(A, V, \langle,\rangle)$. This group acts transitively on A. If we require L be orientation preserving then we have the proper **Euclidean motion** group $\mathrm{Euc}^+(A, V, \langle,\rangle)$ which also acts transitively on A.

Exercise J.19 *Show that for any point $x_0 \in A$, the isotropy subgroup $G_{x_0} \subset \mathrm{Euc}^+(A, V, \langle,\rangle)$ is isomorphic to $\mathrm{SO}(V)$.*

Exercise J.20 *Show that action of $G = \mathrm{Euc}(A, V, \langle,\rangle)$ on A is transitive and continuous. So that if H is the isotropy subgroup corresponding to some point $x_0 \in A$ then we have the ($G-$space) isomorphism $A \cong G/H$. Exhibit this isomorphism in the concrete case of $A = \mathbf{E}^2$ and $G = Euc(2)$.*

In the finite dimensional situation we will usually stick to the concrete case of $\mathrm{Euc}(n)$ (resp. $\mathrm{Euc}^+(n)$) acting on $\mathbf{E}^n$ and take $x_0 = 0 \in \mathbf{E}^n$. Using the matrix representation described above, the isotropy subgroup of $x_0 = 0$ consists of all matrices of the form $\begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$ for $Q \in \mathrm{O}(n)$ (resp. $S\mathrm{O}(n)$). Identifying this isotropy subgroup with $\mathrm{O}(n)$ (resp. $S\mathrm{O}(n)$) we have

$$\mathbf{E}^n \cong \mathrm{Euc}(n)/\mathrm{O}(n) \cong \mathrm{Euc}^+(n)/S\mathrm{O}(n).$$

We also have the homogeneous space presentation

$$M^{1+3} \cong P/O(1,3)$$

where we have identified the isotropy subgroup of a point in M^{1+3} with $O(1,3)$
.

Appendix K

Interlude: A Bit of Physics

K.1 The Basic Ideas of Relativity

We will draw an analogy with the geometry of the Euclidean plane. Recall that the abstract plane $\mathcal{P}$ is *not* the same thing as coordinate space $\mathbb{R}^2$ but rather there are many "good" bijections $\Psi : \mathcal{P} \to \mathbb{R}^2$ called coordinatizations such that points $p \in \mathcal{P}$ corresponding under Ψ to coordinates $(x(p), y(p))$ are temporarily identified with the pair $(x(p), y(p))$ is such a way that the distance between points is given by $dist(p, q) = \sqrt{(x(p) - x(q))^2 + (y(p) - y(q))^2}$ or

$$dist(p, q)^2 = \Delta x^2 + \Delta y^2$$

for short. Now the numbers Δx and Δy separately have no absolute meaning since a different good-coordinatization $\Phi : \mathcal{P} \to \mathbb{R}^2$ would give something like $(X(p), Y(p))$ and then for the same two points p, q we expect that in general $\Delta x \neq \Delta X$ and $\Delta y \neq \Delta Y$. On the other hand, the notion of distance is a geometric reality that should be independent of coordinate choices and so we always have $\Delta x^2 + \Delta y^2 = \Delta X^2 + \Delta Y^2$. But what is a "good coordinatization"? Well, one thing can be said for sure and that is if x, y are good then X, Y will be good also if

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & \pm\sin\theta \\ \mp\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

for some θ and numbers T_1, T_2. The set of all such transformations form a group under composition is called the Euclidean motion group. Now the idea of points on an abstract plane is easy to imagine but it is really just a "set" of objects with some logical relations; an idealization of certain aspects of our experience. Similarly, we now encourage the reader to make the following idealization. Imagine the set of all ideal local events or possible events as a sort of 4-dimensional plane. Imagine that when we speak of an event happening at location (x, y, z) in rectangular coordinates and at time t it is only because we have imposed some sort of coordinate system on the set of events that is implicit in our norms regarding measuring procedures etc. What if some other system were used to describe the same set of events, say, two explosions $e1$ and $e2$. You would not be surprised to find out that the spatial separations for the two events

$$\Delta X, \Delta Y, \Delta Z$$

would not be absolute and would not individually equal the numbers

$$\Delta x, \Delta y, \Delta z.$$

But how about ΔT and Δt. Is the time separation, in fixed units of seconds say, a real thing?

The answer is actually *no* according to the special theory of relativity. In fact, not even the quantities $\Delta X^2+\Delta Y^2+\Delta Z^2$ will agree with $\Delta x^2+\Delta y^2+\Delta y^2$ under certain circumstances! Namely, if two observers are moving relative to each other at constant speed, there will be objectively irresolvable disagreements. Who is right? There simply is no fact of the matter. The objective or absolute quantity is rather

$$-\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta y^2$$

which always equals $-\Delta T^2 + \Delta X^2 + \Delta Y^2 + \Delta Y^2$ for good coordinates systems. But what is a good coordinate system? It is one in which the equations of physics take on their simplest form. Find one, and then all others are related by the Poincaré group of linear transformations given by

$$\begin{pmatrix} X \\ Y \\ Z \\ T \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ t_0 \end{pmatrix}$$

where the matrix A is a member of the Lorentz group. The Lorentz group is characterized as that set $O(1,3)$ of matrices A such that

$$A^T \eta A = \eta \quad \text{where}$$

$$\eta := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is exactly what makes the following true.

Fact If $(t, \tilde{\mathbf{x}})$ and $(T, \tilde{\mathbf{X}})$ are related by $(t, \tilde{\mathbf{x}})^t = A(T, \tilde{\mathbf{X}})^t + (t_0, \tilde{\mathbf{x}}_0)^t$ for $A \in O(1,3)$ then $-t^2 + |\tilde{\mathbf{x}}|^2 = T^2 + \left|\tilde{\mathbf{X}}\right|^2$.

A 4−vector is described relative to any inertial coordinates $(t, \vec{x})$ by a 4-tuple $v = (v^0, v^1, v^2, v^3)$ such that its description relative to $(T, \tilde{\mathbf{X}})$ as above is given by

$$\mathsf{V}^t = A\mathsf{v}^t \text{ (contravariant)}.$$

Notice that we are using superscripts to index the components of a vector (and are not to be confused with exponents). This due to the following convention: vectors written with components up are called contravariant vectors while those with indices down are called covariant. Contravariant and covariant vectors transform differently and in such a way that the contraction of a contravariant with a covariant vector produces a quantity that is the same in any inertial coordinate system. To change a contravariant vector to its associated covariant form one uses the matrix η introduced above which is called the Lorentz metric tensor. Thus $(v^0, v^1, v^2, v^3)\eta = (-v^0, v^1, v^2, v^3) := (v_0, v_1, v_2, v_3)$ and thus the pseudo-length $v_i v^i = -(v^0)^2 + (v^1)^2 + (v^2)^2 + (v^3)^2$ is an invariant with respect to coordinate changes via the Lorentz group or even the Poincaré group. Notice also that $v_i v^i$ actually means $\sum_{i=0}^{3} v_i v^i$ which is in turn the same thing as

$$\sum \eta_{ij} v^i v^j$$

The so called **Einstein summation convention** say that when an index is repeated once up and once down as in $v_i v^i$, then the summation is implied.

Minkowski Space

One can see from the above that lurking in the background is an inner product space structure: If we fix the origin of space time then we have a vector space with scalar product $\langle v, v\rangle = v_i v^i = -(v^0)^2 + (v^1)^2 + (v^2)^2 + (v^3)^2$. This scalar product space (indefinite inner product!) is called Minkowski space. The scalar product just defined is the called the **Minkowski metric** or **the Lorentz metric.**.

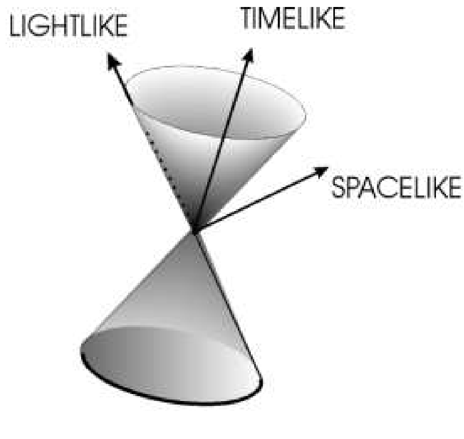
Definition K.1 *A 4-vector v is called **space-like** if and only if $\langle v, v\rangle > 0$, **time-like** if and only if $\langle v, v\rangle < 0$ and **light-like** if and only if $\langle v, v\rangle = 0$. The set of all light-like vectors at a point in Minkowski space form a double cone in $\mathbb{R}^4$ referred to as the **light cone**.*

Remark K.1 (Warning) *Sometimes the definition of the Lorentz metric given is opposite in sign from the one we use here. Both choices of sign are popular. One consequence of the other choice is that time-like vectors become those for which $\langle v, v\rangle > 0$.*

Definition K.2 *At each point of $x \in \mathbb{R}^4$ there is a set of vectors parallel to the 4-axes of $\mathbb{R}^4$. We will denote these by $\partial_0, \partial_1, \partial_2$,and ∂_3 (suppressing the point at which they are based).*

Definition K.3 *A vector v based at a point in $\mathbb{R}^4$ such that $\langle \partial_0, v\rangle < 0$ will be called **future pointing** and the set of all such forms the interior of the "future" light-cone.*

One example of a 4-vector is the momentum 4-vector written $\mathsf{p} = (E, \overrightarrow{\mathbf{p}})$ which we will define below. We describe the motion of a particle by referring



to its career in space-time. This is called its world-line and if we write c for the speed of light and use coordinates $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$ then a world line is a curve $\gamma(s) = (x^0(s), x^1(s), x^2(s), x^3(s))$ for some parameter. The momentum 4-vector is then $\mathsf{p} = mc\mathsf{u}$ where u is the unite vector in the direction of the 4-velocity $\gamma'(s)$. The action functional for a free particle in Relativistic mechanics is invariant with respect to Lorentz transformations described above. In the case of a free particle of mass m it is

$$A_L = -\int_{s_1}^{s_2} mc\,|\langle\dot\gamma(s), \dot\gamma(s)\rangle|^{1/2}\,ds.$$

The quantity c is a constant equal to the speed of light in any inertial coordinate system. Here we see the need the absolute value since for a timelike path to assume that $\langle\dot\gamma(s), \dot\gamma(s)\rangle \leq 0$ by our convention. Define

$$\tau(s) = \int_{s_1}^{s_2} |\langle\dot\gamma(s), \dot\gamma(s)\rangle|^{1/2}\,ds$$

Then $\frac{d}{ds}\tau(s) = cm\,|\langle\dot\gamma(s), \dot\gamma(s)\rangle|^{1/2} \geq 0$ so we can reparameterize by τ:

$$\gamma(\tau) = (x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau)).$$

The parameter τ is called the proper time of the particle in question. A stationary curve will be a geodesic or straight line in $\mathbb{R}^4$.

Let us return to the Lorentz group. The group of rotations generated by rotations around the spatial x, y, and z-axes are a copy of $SO(3)$ sitting inside $SO(1,3)$ and consists of matrices of the form

$$\begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$$

where $R \in SO(3)$. Now a rotation of the x, y-plane about the z, t-plane[1] for example has the form

$$\begin{pmatrix} cT \\ X \\ Y \\ Z \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

where as a Lorentz "rotation" of the $t, x-$plane about the $y, z-$plane has the form

$$\begin{pmatrix} cT \\ X \\ Y \\ Z \end{pmatrix} = \left[\begin{matrix} \cosh(\beta) & \sinh(\beta) & 0 & 0 \\ \sinh(\beta) & \cosh(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \right]$$

$= [cT, X, Y, Z] = [(\cosh \beta)\, ct + (\sinh \beta)\, x, (\sinh \beta)\, ct + (\cosh \beta)\, x, y, z]$ Here the parameter β is usually taken to be the real number given by

$$\tanh \beta = v/c$$

where v is a velocity indicating that in the new coordinates the observer in travelling at a velocity of magnitude v in the x direction as compared to an observer in the original before the transformation. Indeed we have for an observer motionless at the spatial origin the T, X, Y, Z system the observers path is given in T, X, Y, Z coordinates as $T \mapsto (T, 0, 0, 0)$

$$\begin{aligned} \frac{dX}{dt} &= c \sinh \beta \\ \frac{dX}{dx} &= \cosh \beta \\ v = \frac{dx}{dt} &= \frac{\sinh \beta}{\cosh \beta} = c \tanh \beta. \end{aligned}$$

Calculating similarly, and using $\frac{dY}{dt} = \frac{dZ}{dt} = 0$ we are lead to $\frac{dy}{dt} = \frac{dz}{dt} = 0$. So the t, x, y, z observer sees the other observer (and hence his frame) as moving in the x direction at speed $v = c \tanh \beta$. The transformation above is called a **Lorentz boost** in the x-direction.

Now as a curve parameterized by the parameter τ (the proper time) the 4-**momentum** is the vector

$$\mathsf{p} = mc \frac{d}{d\tau} \mathsf{x}(\tau).$$

In a specific inertial (Lorentz) frame

$$\mathsf{p}(t) = \frac{dt}{d\tau} \frac{d}{dt} mc\mathsf{x}(t) = \left(\frac{mc^2}{c\sqrt{1 - (v/c)^2}}, \frac{mc\dot{x}}{\sqrt{1 - (v/c)^2}}, \frac{mc\dot{y}}{\sqrt{1 - (v/c)^2}}, \frac{mc\dot{z}}{\sqrt{1 - (v/c)^2}} \right)$$

[1] It is the z,t-plane rather than just the z-axis since we are in four dimensions and both the z-axis and the t-axis would remain fixed.

which we abbreviate to $\mathsf{p}(t) = (E/c, \overrightarrow{\mathbf{p}})$ where $\overrightarrow{\mathbf{p}}$ is the 3-vector given by the last there components above. Notice that

$$m^2c^2 = \langle \mathsf{p}(t), \mathsf{p}(t) \rangle = E^2/c^2 + \left|\overrightarrow{\mathbf{p}}\right|^2$$

is an invariant quantity but the pieces E^2/c^2 and $\left|\overrightarrow{\mathbf{p}}\right|^2$ are dependent on the choice of inertial frame.

What is the energy of a moving particle (or tiny observer?) in this theory? We claim it is the quantity E just introduced. Well, if the Lagrangian is any guide we should have from the point of view of inertial coordinates and for a particle moving at speed v in the positive $x-$direction

$$\begin{aligned} E &= v\frac{\partial L}{\partial v} = v\frac{\partial}{\partial v}\left(-mc^2\sqrt{1-(v/c)^2}\right) - (-mc^2\sqrt{1-(v/c)^2}) \\ &= m\frac{c^3}{\sqrt{(c^2-v^2)}} = m\frac{c^2}{\sqrt{(1-(v/c)^2)}}. \end{aligned}$$

Expanding in powers of the dimensionless quantity v/c we have $E = mc^2 + \frac{1}{2}mv^2 + O\left((v/c)^4\right)$. Now the term $\frac{1}{2}mv^2$ is just the nonrelativistic expression for kinetic energy. What about the mc^2? If we take the Lagrangian approach seriously, this must be included as some sort of energy. Now if v had been zero then we would still have a "rest energy" of mc^2! This is interpreted as the energy possessed by the particle by virtue of its mass. A sort of energy of being as it were. Thus we have $v = 0$ here and the famous equation for the equivalence of mass and energy follows:

$$E = mc^2.$$

If the particle is moving then mc^2 is only part of the energy but we can define $E_0 = mc^2$ as the "rest energy". Notice however, that although the length of the momentum 4-vector $m\dot{\mathsf{x}}$=p is always m;

$$|\langle \mathsf{p}, \mathsf{p} \rangle|^{1/2} = m\left|-(dt/d\tau)^2 + (dx/d\tau)^2 + (dy/d\tau)^2 + (dz/d\tau)^2\right|^{1/2} = m$$

and is therefore conserved in the sense of being constant one must be sure to remember that the mass of a body consisting of many particles is *not* the sum of the individual particle masses.

K.2 Electricity and Magnetism

Up until now it has been mysterious how any object of matter could influence any other. It turns out that most of the forces we experience as middle sized objects pushing and pulling on each other is due to a single electromagnetic force. Without the help of special relativity there appears to be two forces; electric and magnetic. Elementary particles that carry electric charges such as electrons or protons, exert forces on each other by means of a field. In a particular Lorentz frame, the electromagnetic field is described by a skew-symmetric matrix of functions called the electromagnetic field tensor:

$$(F_{\mu\nu}) = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}.$$

Where we also have the forms $F_\mu^\nu = \eta^{s\nu} F_{\mu s}$ and $F^{\mu\nu} = \eta^{s\mu} F_s^\nu$. This tensor can be derived from a potential $\mathsf{A} = (A_0, A_1, A_2, A_3)$ by $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$. The contravariant form of the potential is $(A_0, -A_1, -A_2, -A_3)$ is a four vector often written as

$$\mathsf{A} = (\phi, \overrightarrow{\mathbf{A}}).$$

The action for a charged particle in an electromagnetic field is written in terms of A in a manifestly invariant way as

$$\int_a^b -mcd\tau - \frac{e}{c} A_\mu dx^\mu$$

so writing $\mathsf{A} = (\phi, \overrightarrow{\mathbf{A}})$ we have

$$S(\tilde{\mathbf{x}}) = \int_a^b (-mc\frac{d\tau}{dt} - e\phi(t) + \overrightarrow{\mathbf{A}} \cdot \frac{d\tilde{\mathbf{x}}}{dt})dt$$

so in a given frame the Lagrangian is

$$L(\tilde{\mathbf{x}}, \frac{d\tilde{\mathbf{x}}}{dt}, t) = -mc^2\sqrt{1-(v/c)^2} - e\phi(t) + \overrightarrow{\mathbf{A}} \cdot \frac{d\tilde{\mathbf{x}}}{dt}.$$

Remark K.2 *The system under study is that of a particle in a field and does not describe the dynamics of the field itself. For that we would need more terms in the Lagrangian.*

This is a time dependent Lagrangian because of the $\phi(t)$ term but it turns out that one can re-choose A so that the new $\phi(t)$ is zero and yet still have $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$. This is called change of gauge. Unfortunately, if we wish to express things in such a way that a constant field is given by a constant potential then we cannot make this choice. In any case, we have

$$L(\overrightarrow{\mathbf{x}}, \frac{d\overrightarrow{\mathbf{x}}}{dt}, t) = -mc^2\sqrt{1-(v/c)^2} - e\phi + \overrightarrow{\mathbf{A}} \cdot \frac{d\overrightarrow{\mathbf{x}}}{dt}$$

and setting $\overrightarrow{\mathbf{v}} = \frac{d\tilde{\mathbf{x}}}{dt}$ and $\left|\overrightarrow{\mathbf{v}}\right| = v$ we get the follow form for energy

$$\overrightarrow{\mathbf{v}} \cdot \frac{\partial}{\partial \overrightarrow{\mathbf{v}}} L(\tilde{\mathbf{x}}, \overrightarrow{\mathbf{v}}, t) - L(\tilde{\mathbf{x}}, \overrightarrow{\mathbf{v}}, t) = \frac{mc^2}{\sqrt{1-(v/c)^2}} + e\phi.$$

Now this is not constant with respect to time because $\frac{\partial L}{\partial t}$ is not identically zero. On the other hand, this make sense from another point of view; the particle is interacting with the field and may be picking up energy from the field.

The Euler-Lagrange equations of motion turn out to be

$$\frac{d\tilde{\mathbf{p}}}{dt} = e\tilde{\mathbf{E}} + \frac{e}{c}\overrightarrow{\mathbf{v}} \times \tilde{\mathbf{B}}$$

where $\tilde{\mathbf{E}} = -\frac{1}{c}\frac{\partial \tilde{\mathbf{A}}}{\partial t} - \operatorname{grad}\phi$ and $\tilde{\mathbf{B}} = \operatorname{curl}\tilde{\mathbf{A}}$ are the electric and magnetic parts of the field respectively. This decomposition into electric and magnetic parts is an artifact of the choice of inertial frame and may be different in a different frame. Now the momentum $\tilde{\mathbf{p}}$ is $\frac{m\overrightarrow{\mathbf{v}}}{\sqrt{1-(v/c)^2}}$ but a speeds $v << c$ this becomes nearly equal to $m\mathbf{v}$ so the equations of motion of a charged particle reduce to

$$m\frac{d\overrightarrow{\mathbf{v}}}{dt} = e\tilde{\mathbf{E}} + \frac{e}{c}\overrightarrow{\mathbf{v}} \times \tilde{\mathbf{B}}.$$

Notice that is the particle is not moving, or if it is moving parallel the magnetic field $\tilde{\mathbf{B}}$ then the second term on the right vanishes.

The electromagnetic field equations.

We have defined the 3-vectors $\tilde{\mathbf{E}} = -\frac{1}{c}\frac{\partial \tilde{\mathbf{A}}}{\partial t} - \operatorname{grad}\phi$ and $\tilde{\mathbf{B}} = \operatorname{curl}\tilde{\mathbf{A}}$ but since the curl of a gradient we see that $\operatorname{curl}\tilde{\mathbf{E}} = -\frac{1}{c}\frac{\partial \tilde{\mathbf{B}}}{\partial t}$. Also, from $\tilde{\mathbf{B}} = \operatorname{curl}\tilde{\mathbf{A}}$ we get $\operatorname{div}\tilde{\mathbf{B}} = \mathbf{0}$. This easily derived pair of equations is the first two of the four famous Maxwell's equations. Later we will see that the electromagnetic field tensor is really a differential 2-form F and these two equations reduce to the statement that the (exterior) derivative of F is zero:

$$dF = 0$$

Exercise K.1 *Apply Gauss' theorem and Stokes' theorem to the first two Maxwell's equations to get the integral forms of the equations. What do these equations say physically?*

One thing to notice is that these two equations do not determine $\frac{\partial}{\partial t}\tilde{\mathbf{E}}$.

Now we have not really written down a action or Lagrangian that includes terms that represent the field itself. When that part of the action is added in we get

$$S = \int_a^b (-mc - \frac{e}{c}A_\mu \frac{dx^\mu}{d\tau})d\tau + a\int_V F^{\nu\mu}F_{\nu\mu}dx^4$$

where in so called Gaussian system of units the constant a turns out to be $\frac{-1}{16\pi c}$. Now in a particular Lorentz frame and recalling 16.5 we get $= a\int_V F^{\nu\mu}F_{\nu\mu}dx^4 = \frac{1}{8\pi}\int_V \left|\tilde{\mathbf{E}}\right|^2 - \left|\tilde{\mathbf{B}}\right|^2 dtdxdydz$.

In order to get a better picture in mind let us now assume that there is a continuum of charged particle moving through space and that volume density of charge at any given moment in space-time is ρ so that if $dxdydz = dV$ then ρdV is the charge in the volume dV. Now we introduce the four vector $\rho \mathsf{u} = \rho(d\mathsf{x}/d\tau)$ where u is the velocity 4-vector of the charge at (t, x, y, z). Now recall that $\rho d\mathsf{x}/d\tau = \frac{d\tau}{dt}(\rho, \rho\overrightarrow{\mathbf{v}}) = \frac{d\tau}{dt}(\rho, \tilde{\mathbf{j}}) = \mathsf{j}$. Here $\tilde{\mathbf{j}} = \rho\overrightarrow{\mathbf{v}}$ is the charge current density as viewed in the given frame a vector field varying smoothly from point to point. Write $\mathsf{j} = (j^0, j^1, j^2, j^3)$.

Assuming now that the particle motion is determined and replacing the discrete charge e be the density we have applying the variational principle with the region $U = [a, b] \times V$ says

$$\begin{aligned} 0 &= -\delta \left(\int_V \int_a^b \frac{\rho dV}{c} dV A_\mu \frac{dx^\mu}{d\tau} d\tau + a \int_U F^{\nu\mu} F_{\nu\mu} dx^4 \right) \\ &= -\delta \left(\frac{1}{c} \int_U j^\mu A_\mu + a F^{\nu\mu} F_{\nu\mu} dx^4 \right) \end{aligned}$$

Now the Euler-Lagrange equations become

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0$$

where $\mathcal{L}(A_\mu, \partial_\mu A_\eta) = \frac{\rho}{c} A_\mu \frac{dx^\mu}{dt} + a F^{\nu\mu} F_{\nu\mu}$ and $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$. If one is careful to remember that $\partial_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu}$ is to be treated as an independent variable one can arrive at some complicated looking equations and then looking at the matrix 16.5 we can convert the equations into statements about the fields $\tilde{\mathbf{E}}$, $\tilde{\mathbf{B}}$,and $(\rho, \tilde{\mathbf{j}})$. We will not carry this out since we instead discover a much more efficient formalism for dealing with the electromagnetic field. Namely, we will use differential forms and the Hodge star operator. At any rate the last two of Maxwell's equations read

$$\begin{aligned} \operatorname{curl} \tilde{\mathbf{B}} &= 0 \\ \operatorname{div} \tilde{\mathbf{E}} &= 4\pi\rho. \end{aligned}$$

Accordingly, if M only one isomorphism of Banach spaces is implicated by an atlas.

For completeness we add one more technical definoition. If a C^r differentiable manifold is specified by an atlas with values in a fixed Banach space E we say that it is **modeled** on E or that it is an E-manifold (of class C^r).

Finite dimensional manifolds are far more commonly studied than those modeled on more general Banach spaces and in the literature the notion of a manifold is usually defined in such a way as to be automatically finite dimensional. Furthermore manifolds are usual required to satisfy further topological requirements. This is discussed in the following.

Remark K.3 *In the literature the definition of a finite dimensional differentiable manifold usually includes the requirement that the topology is Hausdorff and second countable. Recall that our definition of paracompact requires the* *space to be Hausdorff. The main reason that second countability is assumed because second countability implies paracompactness and paracompactness allows the construction of the so called smooth partitions of unity (discussed later). For finite dimensional manifolds, paracompact is equivalent to each connected component being second countable. A finite dimensional paracompact manifold with a finite or countably infinite number of components would also be second countable. Some authors take the alternative route of defining a finite dimensional differentiable manifold to be paracompact. However, this leave open the possibility of an uncountable number of connected components and this would create problems for the celebrated theorem of Sard.*

On the other hand, an infinite dimensional Banach space need not be paracompact itself and even when it is we do not seem to automatically get the existence of smooth (C^r with $r > 1$) partitions of unity. It seems then that as long as differentiable manifolds modeled on general Banach spaces are under consideration the motivation for putting extra topological conditions into the very definition of a differentiable manifold is undercut.

K.2.1 Maxwell's equations.

$\mathbb{R}^{3,1}$ is just $\mathbb{R}^4$ but with the action of the symmetry group $O(1,3)$.

Recall the electromagnetic field tensor

$$(F_{\mu\nu}) = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix}.$$

Let us work in units where $c = 1$. Since this matrix is skew symmetric we can form a 2-form called the electromagnetic field 2-form:

$$F = \frac{1}{2}\sum_{\mu,\nu} F_{\mu\nu} dx^\mu \wedge dx^\nu = \sum_{\mu<\nu} F_{\mu\nu} dx^\mu \wedge dx^\nu.$$

Let write $E = E_x dx + E_y dy + E_z dz$ and $B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$. One can check that we now have

$$F = B + E \wedge dt.$$

Now we know that F comes from a potential $A = A_\nu dx^\nu$. In fact, we have

$$\begin{aligned} dA = d(A_\nu dx^\nu) &= \sum_{\mu<\nu} \left(\frac{\partial}{\partial x^\mu} A_\nu - \frac{\partial}{\partial x^\nu} A_\mu\right) dx^\mu \wedge dx^\nu \\ &= \sum_{\mu<\nu} F_{\mu\nu} dx^\mu \wedge dx^\nu = F. \end{aligned}$$

Thus we automatically have $dF = ddA = 0$. Now what does $dF = 0$ translate into in terms of the E and B? We compute:

$$
\begin{aligned}
0 &= dF = d(B + E \wedge dt) = dB + dE \wedge dt \\
&= (\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}) + d\,(E_x dx + E_y dy + E_z dz) \wedge dt \\
&= (\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}) dx \wedge dy \wedge dz + \frac{\partial B}{\partial t} \wedge dt + \\
&+ \left[(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) dy \wedge dz + (\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}) dz \wedge dx + (\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) dy \wedge dx \right] \wedge dt
\end{aligned}
$$

From this we conclude that

$$
\begin{aligned}
\operatorname{div}(\tilde{\mathbf{B}}) &= 0 \\
\operatorname{curl}(\tilde{\mathbf{E}}) + \frac{\partial \tilde{\mathbf{B}}}{\partial t} &= 0
\end{aligned}
$$

which is Maxwell's first two equations. Thus Maxwell's first two equations end up being equivalent to just the single equation

$$dF = 0$$

which was true just from the fact that $dd = 0$ since we assuming that there is a potential A! This equation does not involve the scalar product structure encoded by the matrix η.

As for the second pair of Maxwell's equations, they too combine to give a single equation. The appropriate star operator is given by

Definition K.4 *Define $\epsilon(\mu)$ to be entries of the diagonal matrix $\eta = diag(-1, 1, 1, 1)$. Let $*$ be defined on $\Omega^k(\mathbb{R}^4)$ by letting $*(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = \pm\epsilon(j_1)\epsilon(j_2)\cdots\epsilon(j_k) dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}}$ where $dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_{n-k}} = \pm dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. (Choose the sign to that makes the last equation true and then the first is true by definition). Extend $*$ linearly to a map $\Omega^k(\mathbb{R}^4) \to \Omega^{4-k}(\mathbb{R}^4)$. More simply and explicitly*

Exercise K.2 *Show that $* \circ *$ acts on $\Omega^k(\mathbb{R}^4)$ by $(-1)^{k(4-k)+1}$.*

Exercise K.3 *Show that if F is the electromagnetic field tensor defined above then*

$$*F = (*F)_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

where $(*F)_{\mu\nu}$ are the components of the matrix

$$
(*F)_{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{bmatrix}
$$

Now let J be the differential 1-form constructed from the 4-current $\mathrm{j} = (\rho, \tilde{\mathbf{j}})$ introduced in section F.0.13 by letting $(j_0, j_1, j_2, j_3) = (-\rho, \tilde{\mathbf{j}})$ and then setting $J = j_\mu dx^\mu$.

Now we add the second equation

$$*d * F = J.$$

Exercise K.4 *Show that the single differential form equation $*d * F = J$ is equivalent to Maxwell's second two equations*

$$\begin{aligned}\operatorname{curl}(\tilde{\mathbf{B}}) &= \frac{\partial \tilde{\mathbf{E}}}{\partial t} + \tilde{\mathbf{j}} \\ \operatorname{div}(\tilde{\mathbf{E}}) &= \rho.\end{aligned}$$

In summary, we have that Maxwell's 4 equations (in free space) in the formalism of differential forms and the Hodge star operator are simply the pair

$$\begin{aligned}dF &= 0 \\ *d * F &= J.\end{aligned}$$

The first equation is equivalent to Maxwell's first two equations and interestingly does not involve the metric structure of space $\mathbb{R}^3$ or the metric structure of spacetime $\mathbb{R}^{1,3}$. The second equation above is equivalent to Maxwell's second two equations an through the star operator essentiality involves the Metric structure of $\mathbb{R}^{1,3}$.

Now an interesting thing happens if the Lagrangian density is invariant under some set of transformations. Suppose that $\delta\phi$ is an infinitesimal "internal" symmetry of the Lagrangian density so that $\delta S(\delta\phi) = 0$ even though $\delta\phi$ does not vanish on the boundary. Then if ϕ is already a solution of the field equations then

$$0 = \delta S = \int_U \partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) d^4x$$

for all regions U. This means that $\partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$ so if we define $j^\mu = \delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$ we get

$$\partial_\mu j^\mu = 0$$

or

$$\frac{\partial}{\partial t} j^0 = -\nabla \cdot \overrightarrow{\mathbf{j}}$$

where $\overrightarrow{\mathbf{j}} = (j^1, j^2, j^3)$ and $\nabla \cdot \overrightarrow{\mathbf{j}} = \operatorname{div}(\overrightarrow{\mathbf{j}})$ is the spatial divergence. This looks like some sort of conservation. Indeed, if we define the total charge at any time t by

$$Q(t) = \int j^0 d^3x$$

the assuming $\overrightarrow{\mathbf{j}}$ shrinks to zero at infinity then the divergence theorem gives

$$\begin{aligned}\frac{d}{dt}Q(t) &= \int \frac{\partial}{\partial t} j^0 d^3x \\ &= -\int \nabla \cdot \overrightarrow{\mathbf{j}} d^3x = 0\end{aligned}$$

so the charge $Q(t)$ is a conserved quantity. Let $Q(U,t)$ denote the total charge inside a region U. The charge inside any region U can only change via a flux through the boundary:

$$\begin{aligned}\frac{d}{dt}Q(U,t) &= \int_U \frac{\partial}{\partial t} j^0 d^3x \\ &= \int_{\partial U} \overrightarrow{\mathbf{j}} \cdot \mathbf{n} dS\end{aligned}$$

which is a kind of "local conservation law". To be honest the above discussion only takes into account so called internal symmetries. An example of an internal symmetry is given by considering a curve of linear transformations of $\mathbb{R}^k$ given as matrices $C(s)$ with $C(0) = I$. Then we vary ϕ by $C(s)\phi$ so that $\delta\phi = \frac{d}{ds}\big|_0 C(s)\phi = C'(0)\phi$. Another possibility is to vary the underlying space so that $C(s,.)$ is now a curve of transformations of $\mathbb{R}^4$ so that if $\phi_s(x) = \phi(C(s,x))$ is a variation of fields then we must take into account the fact that the domain of integration is also varying:

$$L(\phi_s, \partial\phi_s) = \int_{U_s \subset \mathbb{R}^4} \mathcal{L}(\phi_s, \partial\phi_s) d^4x$$

We will make sense of this later.

	Global	local	Sheaf notation
functions on M	$C^\infty(M)$	$C^\infty(U)$	$C^\infty M$
Vector fields on M	$\mathfrak{X}(M)$	$\mathfrak{X}(U)$	$\mathfrak{X}_M$
Sections of E	$\Gamma(E)$	$\Gamma(U,E)$	$-_E$
Forms on M	$\Omega(M)$	$\Omega(U)$	Ω_M
Tensor fields on M	$\mathfrak{T}^k_l(M)$	$\mathfrak{T}^k_l(U)$	$\mathfrak{T}^k_l M$

Example K.1 *For matrix Lie groups there is a simple way to compute the Maurer-Cartan form. Let g denote the identity map $G \to G$ so that $dg : TG \to TG$ is also the identity map. Then*

$$\omega_G = g^{-1}dg.$$

For example, let $G = \mathrm{Euc}(n)$ presented as matrices of the form

$$\begin{bmatrix} 1 & 0 \\ x & Q \end{bmatrix} \text{ for } Q \in \mathrm{O}(n) \text{ and } x \in \mathbb{R}^n.$$

Then $g = \begin{bmatrix} 1 & 0 \\ x & Q \end{bmatrix}$ *and* $dg = d\begin{bmatrix} 1 & 0 \\ x & Q \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ dx & dQ \end{bmatrix}$.

$$\begin{aligned} g^{-1}dg &= \begin{bmatrix} 1 & 0 \\ -x & Q^t \end{bmatrix}\begin{bmatrix} 0 & 0 \\ dx & dQ \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ Q^t dx & dQ \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ (Q^t)_r dx^r & (Q^t)^{ik} dQ_{kj} \end{bmatrix} \end{aligned}$$

Index

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