5 Tensor products

We have so far encountered vector fields and the derivatives of smooth functions as analytical objects on manifolds. These are examples of a general class of objects called *tensors* which we shall encounter in more generality. The starting point is pure linear algebra.

Let V, W be two finite-dimensional vector spaces over $\mathbf{R}$. We are going to define a new vector space $V \otimes W$ with two properties:

- if $v \in V$ and $w \in W$ then there is a product $v \otimes w \in V \otimes W$
- the product is bilinear:

$$\begin{aligned}(\lambda v_1 + \mu v_2) \otimes w &= \lambda v_1 \otimes w + \mu v_2 \otimes w \\ v \otimes (\lambda w_1 + \mu w_2) &= \lambda v \otimes w_1 + \mu v \otimes w_2\end{aligned}$$

In fact, it is the properties of the vector space $V \otimes W$ which are more important than what it is (and after all what is a real number? Do we always think of it as an equivalence class of Cauchy sequences of rationals?).

Proposition 5.1 *The tensor product $V \otimes W$ has the universal property that if $B : V \times W \to U$ is a bilinear map to a vector space U then there is a unique linear map*

$$\beta : V \otimes W \to U$$

such that $B(v, w) = \beta(v \otimes w)$.

There are various ways to define $V \otimes W$. In the finite-dimensional case we can say that $V \otimes W$ is *the dual space of the space of bilinear forms on $V \times W$*: i.e. maps $B : V \times W \to \mathbf{R}$ such that

$$\begin{aligned}B(\lambda v_1 + \mu v_2, w) &= \lambda B(v_1, w) + \mu B(v_2, w) \\ B(v, \lambda w_1 + \mu w_2) &= \lambda B(v, w_1) + \mu B(v, w_2)\end{aligned}$$

Given $v, w \in V, W$ we then define $v \otimes w \in V \otimes W$ as the map

$$(v \otimes w)(B) = B(v, w).$$

This satisfies the universal property because given $B : V \times W \to U$ and $\xi \in U^*$, $\xi \circ B$ is a bilinear form on $V \times W$ and defines a linear map from U^* to the space of bilinear forms. The dual map is the required homomorphism β from $V \otimes W$ to $(U^*)^* = U$.

A bilinear form B is uniquely determined by its values $B(v_i, w_j)$ on basis vectors $v_1, \ldots, v_m$ for V and $w_1, \ldots w_n$ for W which means the dimension of the vector space of bilinear forms is mn, as is its dual space $V \otimes W$. In fact, we can easily see that the mn vectors

$$v_i \otimes w_j$$

form a basis for $V \otimes W$. It is important to remember though that a typical element of $V \otimes W$ can only be written as a sum

$$\sum_{i,j} a_{ij} v_i \otimes w_j$$

and not as a pure product $v \otimes w$.

Taking $W = V$ we can form multiple tensor products

$$V \otimes V, \quad V \otimes V \otimes V = \otimes^3 V, \quad \ldots$$

We can think of $\otimes^p V$ as the dual space of the space of p-fold *multilinear forms* on V.

Mixing degrees we can even form the *tensor algebra*:

$$T(V) = \oplus_{k=0}^{\infty} (\otimes^k V).$$

An element of $T(V)$ is a finite sum

$$\lambda 1 + v_0 + \sum v_i \otimes v_j + \ldots + \sum v_{i_1} \otimes v_{i_2} \ldots \otimes v_{i_p}$$

of products of vectors $v_i \in V$. The obvious multiplication process is based on extending by linearity the product

$$(v_1 \otimes \ldots \otimes v_p)(u_1 \otimes \ldots \otimes u_q) = v_1 \otimes \ldots \otimes v_p \otimes u_1 \otimes \ldots \otimes u_q$$

It is associative, but noncommutative.

For the most part we shall be interested in only a quotient of this algebra, called the *exterior algebra*. A down-to-earth treatment of this is in the Section b3 Projective Geometry Notes on the Mathematical Institute website.

5.1 The exterior algebra

Let $T(V)$ be the tensor algebra of a real vector space V and let $I(V)$ be the *ideal* generated by elements of the form

$$v \otimes v$$

where $v \in V$. So $I(V)$ consists of all sums of multiples by $T(V)$ on the left and right of these generators.

Definition 19 *The exterior algebra of V is the quotient*

$$\Lambda^* V = T(V)/I(V).$$

If $\pi : T(V) \to \Lambda^* V$ is the quotient projection then we set

$$\Lambda^p V = \pi(\otimes^p V)$$

and call this the p-fold *exterior power* of V. We can think of this as the dual space of the space of multilinear forms $M(v_1, \dots, v_p)$ on V which vanish if any two arguments coincide – the so-called alternating multilinear forms. If $a \in \otimes^p V, b \in \otimes^q V$ then $a \otimes b \in \otimes^{p+q} V$ and taking the quotient we get a product called the exterior product:

Definition 20 *The exterior product of $\alpha = \pi(a) \in \Lambda^p V$ and $\beta = \pi(b) \in \Lambda^q V$ is*

$$\alpha \wedge \beta = \pi(a \otimes b).$$

Remark: As in the Projective Geometry Notes, if $v_1, \dots, v_p \in V$ then we define an element of the dual space of the space of alternating multilinear forms by

$$v_1 \wedge v_2 \wedge \dots \wedge v_p(M) = M(v_1, \dots, v_p).$$

The key properties of the exterior algebra follow:

Proposition 5.2 *If $\alpha \in \Lambda^p V, \beta \in \Lambda^q V$ then*

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha.$$

Proof: Because for $v \in V$, $v \otimes v \in I(V)$, it follows that $v \wedge v = 0$ and hence

$$0 = (v_1 + v_2) \wedge (v_1 + v_2) = 0 + v_1 \wedge v_2 + v_2 \wedge v_1 + 0.$$

So interchanging any two entries from V in an expression like

$$v_1 \wedge \dots \wedge v_k$$

changes the sign.

Write α as a linear combination of terms $v_1 \wedge \dots \wedge v_p$ and β as a linear combination of $w_1 \wedge \dots \wedge w_q$ and then, applying this rule to bring w_1 to the front we see that

$$(v_1 \wedge \dots \wedge v_p) \wedge (w_1 \wedge \dots \wedge w_q) = (-1)^p w_1 \wedge v_1 \wedge \dots v_p \wedge w_2 \wedge \dots \wedge w_q.$$

For each of the q w_i's we get another factor $(-1)^p$ so that in the end

$$(w_1 \wedge \dots \wedge w_q)(v_1 \wedge \dots \wedge v_p) = (-1)^{pq} (v_1 \wedge \dots \wedge v_p)(w_1 \wedge \dots \wedge w_q).$$

□

Proposition 5.3 *If* $\dim V = n$ *then* $\dim \Lambda^n V = 1$.

Proof: Let $w_1, \ldots, w_n$ be n vectors on V and relative to some basis let M be the square matrix whose columns are $w_1, \ldots, w_n$. then

$$B(w_1, \ldots, w_n) = \det M$$

is a non-zero n-fold multilinear form on V. Moreover, if any two of the w_i coincide, the determinant is zero, so this is a non-zero alternating n-linear form – an element in the dual space of $\Lambda^n V$.

On the other hand, choose a basis $v_1, \ldots, v_n$ for V, then anything in $\otimes^n V$ is a linear combination of terms like $v_{i_1} \otimes \ldots \otimes v_{i_n}$ and so anything in $\Lambda^n V$ is, after using Proposition 5.2 a linear combination of $v_1 \wedge \ldots \wedge v_n$.

Thus $\Lambda^n V$ is non-zero and at most one-dimensional hence is one-dimensional. □

Proposition 5.4 *let* $v_1, \ldots, v_n$ *be a basis for* V*, then the* $\binom{n}{p}$ *elements* $v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_p}$ *for* $i_1 < i_2 < \ldots < i_p$ *form a basis for* $\Lambda^p V$.

Proof: By reordering and changing the sign we can get any exterior product of the v_i's so these elements clearly span $\Lambda^p V$. Suppose then that

$$\sum a_{i_1 \ldots i_p} v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_p} = 0.$$

Because $i_1 < i_2 < \ldots < i_p$, each term is uniquely indexed by the subset $\{i_1, i_2, \ldots, i_p\} = I \subseteq \{1, 2, \ldots, n\}$, and we can write

$$\sum_I a_I v_I = 0 \tag{8}$$

If I and J have a number in common, then $v_I \wedge v_J = 0$, so if J has $n - p$ elements, $v_I \wedge v_J = 0$ unless J is the complementary subset I' in which case the product is a multiple of $v_1 \wedge v_2 \ldots \wedge v_n$ and by Proposition 5.3 this is non-zero. Thus, multiplying (8) by each term $v_{I'}$ we deduce that each coefficient $a_I = 0$ and so we have linear independence. □

Proposition 5.5 *The vector* v *is linearly dependent on the vectors* $v_1, \ldots, v_p$ *if and only if* $v_1 \wedge v_2 \wedge \ldots \wedge v_p \wedge v = 0$.

Proof: If v is linearly dependent on $v_1, \ldots, v_p$ then $v = \sum a_i v_i$ and expanding

$$v_1 \wedge v_2 \wedge \ldots \wedge v_p \wedge v = v_1 \wedge v_2 \wedge \ldots \wedge v_p \wedge (\sum_1^p a_i v_i)$$

gives terms with repeated v_i, which therefore vanish. If not, then $v_1, v_2 \ldots, v_p, v$ can be extended to a basis and Proposition 5.4 tells us that the product is non-zero. □

Proposition 5.6 *If $A : V \to W$ is a linear transformation, then there is an induced linear transformation*

$$\Lambda^p A : \Lambda^p V \to \Lambda^p W$$

such that

$$\Lambda^p A(v_1 \wedge \ldots \wedge v_p) = Av_1 \wedge Av_2 \wedge \ldots \wedge Av_p.$$

Proof: From Proposition 5.4 the formula

$$\Lambda^p A(v_1 \wedge \ldots \wedge v_p) = Av_1 \wedge Av_2 \wedge \ldots \wedge Av_p$$

actually defines what $\Lambda^p A$ is on basis vectors but doesn't prove it is independent of the choice of basis. But the universal property of tensor products gives us

$$\otimes^p A : \otimes^p V \to \otimes^p W$$

and $\otimes^p A$ maps the ideal $I(V)$ to $I(W)$ so defines $\Lambda^p A$ invariantly. □

Proposition 5.7 *If* $\dim V = n$*, then the linear transformation $\Lambda^n A : \Lambda^n V \to \Lambda^n V$ is given by* $\det A$.

Proof: From Proposition 5.3, $\Lambda^n V$ is one-dimensional and so $\Lambda^n A$ is multiplication by a real number $\lambda(A)$. So with a basis $v_1, \ldots, v_n$,

$$\Lambda^n A(v_1 \wedge \ldots \wedge v_n) = Av_1 \wedge Av_2 \wedge \ldots Av_n = \lambda(A) v_1 \wedge \ldots \wedge v_n.$$

But

$$Av_i = \sum_j A_{ji} v_j$$

and so

$$\begin{aligned} Av_1 \wedge Av_2 \wedge \ldots \wedge Av_n &= \sum A_{j_1,1} v_{j_1} \wedge A_{j_2,2} v_{j_2} \wedge \ldots \wedge A_{j_n,n} v_{j_n} \\ &= \sum_{\sigma \in S_n} A_{\sigma 1,1} v_{\sigma 1} \wedge A_{\sigma 2,2} v_{\sigma 2} \wedge \ldots \wedge A_{\sigma n,n} v_{\sigma n} \end{aligned}$$

where the sum runs over all permutations σ. But if σ is a transposition then the term $v_{\sigma 1} \wedge v_{\sigma 2} \ldots \wedge v_{\sigma n}$ changes sign, so

$$Av_1 \wedge Av_2 \wedge \ldots \wedge Av_n = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma A_{\sigma 1,1} A_{\sigma 2,2} \ldots A_{\sigma n,n} v_1 \wedge \ldots \wedge v_n$$

which is the definition of $(\det A) v_1 \wedge \ldots \wedge v_n$. □

6 Differential forms

6.1 The bundle of p-forms

Now let M be an n-dimensional manifold and T_x^* the cotangent space at x. We form the p-fold exterior power

$$\Lambda^p T_x^*$$

and, just as we did for the tangent bundle and cotangent bundle, we shall make

$$\Lambda^p T^* M = \bigcup_{x \in M} \Lambda^p T_x^*$$

into a vector bundle and hence a manifold.

If $x_1, \ldots, x_n$ are coordinates for a chart (U, φ_U) then for $x \in U$, the elements

$$dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}$$

for $i_1 < i_2 < \ldots < i_p$ form a basis for $\Lambda^p T_x^*$. The $\binom{n}{p}$ coefficients of $\alpha \in \Lambda^p T_x^*$ then give a coordinate chart Ψ_U mapping to the open set

$$\varphi_U(U) \times \Lambda^p \mathbf{R}^n \subseteq \mathbf{R}^n \times \mathbf{R}^{\binom{n}{p}}.$$

When $p = 1$ this is just the coordinate chart we used for the cotangent bundle:

$$\Phi_U(x, \sum y_i dx_i) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$$

and on two overlapping coordinate charts we there had

$$\Phi_\beta \Phi_\alpha^{-1}(x_1, \ldots, x_n, y_1 \ldots, y_n) = (\tilde{x}_1, \ldots, \tilde{x}_n, \sum_j \frac{\partial \tilde{x}_i}{\partial x_1} y_i, \ldots, \sum_i \frac{\partial \tilde{x}_i}{\partial x_n} y_i).$$

For the p-th exterior power we need to replace the Jacobian matrix

$$J = \frac{\partial \tilde{x}_i}{\partial x_j}$$

by its induced linear map

$$\Lambda^p J : \Lambda^p \mathbf{R}^n \to \Lambda^p \mathbf{R}^n.$$

It's a long and complicated expression if we write it down in a basis but it is invertible and each entry is a polynomial in C^∞ functions and hence gives a smooth map with smooth inverse. In other words,

$$\Psi_\beta \Psi_\alpha^{-1}$$

satisfies the conditions for a manifold of dimension $n + \binom{n}{p}$.

Definition 21 *The bundle of p-forms of a manifold M is the differentiable structure on $\Lambda^p T^* M$ defined by the above atlas. There is natural projection $p : \Lambda^p T^* M \to M$ and a section is called a differential p-form*

Examples:

1. A zero-form is a section of $\Lambda^0 T^*$ which by convention is just a smooth function f.

2. A 1-form is a section of the cotangent bundle T^*. From our definition of the derivative of a function, it is clear that df is an example of a 1-form. We can write in a coordinate system

$$df = \sum_j \frac{\partial f}{\partial x_j} dx_j.$$

By using a bump function we can extend a locally-defined p-form like $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_p$ to the whole of M, so sections always exist. In fact, it will be convenient at various points to show that any function, form, or vector field can be written as a sum of these local ones. This involves the concept of *partition of unity*.

6.2 Partitions of unity

Definition 22 *A partition of unity on M is a collection $\{\varphi_i\}_{i \in I}$ of smooth functions such that*

- $\varphi_i \geq 0$

- $\{\operatorname{supp}\varphi_i : i \in I\}$ *is locally finite*
- $\sum_i \varphi_i = 1$

Here *locally finite* means that for each $x \in M$ there is a neighbourhood U which intersects only finitely many supports $\operatorname{supp}\varphi_i$.

In the appendix, the following general theorem is proved:

Theorem 6.1 *Given any open covering $\{V_\alpha\}$ of a manifold M there exists a partition of unity $\{\varphi_i\}$ on M such that* $\operatorname{supp}\varphi_i \subset V_{\alpha(i)}$ *for some $\alpha(i)$.*

We say that such a partition of unity is *subordinate* to the given covering.

Here let us just note that in the case when M is compact, life is much easier: for each point $x \in \{V_\alpha\}$ we take a coordinate neighbourhood $U_x \subset \{V_\alpha\}$ and a bump function which is 1 on a neighbourhood V_x of x and whose support lies in U_x. Compactness says we can extract a finite subcovering of the $\{V_x\}_{x\in X}$ and so we get smooth functions $\psi_i \geq 0$ for $i = 1, \ldots, N$ and equal to 1 on V_{x_i}. In particular the sum is positive, and defining

$$\varphi_i = \frac{\psi_i}{\sum_1^N \psi_i}$$

gives the partition of unity.

Now, not only can we create global p-forms by taking local ones, multiplying by φ_i and extending by zero, but conversely if α is *any* p-form, we can write it as

$$\alpha = (\sum_i \varphi_i)\alpha = \sum_i (\varphi_i \alpha)$$

which is a sum of extensions of locally defined ones.

At this point, it may not be clear why we insist on introducing these complicated exterior algebra objects, but there are two motivations. One is that the algebraic theory of determinants is, as we have seen, part of exterior algebra, and multiple integrals involve determinants. We shall later be able to integrate p-forms over p-dimensional manifolds.

The other is the appearance of the skew-symmetric cross product in ordinary three-dimensional calculus, giving rise to the curl differential operator taking vector fields to vector fields. As we shall see, to do this in a coordinate-free way, and in all dimensions, we have to dispense with vector fields and work with differential forms instead.

6.3 Working with differential forms

We defined a differential form in Definition 21 as a section of a vector bundle. In a local coordinate system it looks like this:

$$\alpha = \sum_{i_1 < i_2 < \ldots < i_p} a_{i_1 i_2 \ldots i_p}(x) dx_{i_1} \wedge dx_{i_2} \ldots \wedge dx_{i_p} \tag{9}$$

where the coefficients are smooth functions. If $x(y)$ is a different coordinate system, then we write the derivatives

$$dx_{i_k} = \sum_j \frac{\partial x_{i_k}}{\partial y_j} dy_j$$

and substitute in (9) to get

$$\alpha = \sum_{j_1 < j_2 < \ldots < j_p} \tilde{a}_{j_1 j_2 \ldots j_p}(y) dy_{j_1} \wedge dy_{j_2} \ldots \wedge dy_{j_p}.$$

Example: Let $M = \mathbf{R}^2$ and consider the 2-form $\omega = dx_1 \wedge dx_2$. Now change to polar coordinates on the open set $(x_1, x_2) \neq (0, 0)$:

$$x_1 = r \cos\theta, \quad x_2 = r \sin\theta.$$

We have

$$\begin{aligned} dx_1 &= \cos\theta dr - r \sin\theta d\theta \\ dx_2 &= \sin\theta dr + r \cos\theta d\theta \end{aligned}$$

so that

$$\omega = (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) = r dr \wedge d\theta.$$

We shall often write

$$\Omega^p(M)$$

as the infinite-dimensional vector space of all p-forms on M.

Although we first introduced vector fields as analytical objects on manifolds, in many ways differential forms are better behaved. For example, suppose we have a smooth map

$$F : M \to N.$$

The derivative of this gives at each point $x \in M$ a linear map

$$DF_x : T_x M \to T_{F(x)} N$$

but if we have a *section* of the tangent bundle TM – a vector field X – then $DF_x(X_x)$ doesn't in general define a vector field on N – it doesn't tell us what to choose in $T_a N$ if $a \in N$ is not in the image of F.

On the other hand suppose α is a section of $\Lambda^p T^* N$ – a p-form on N. Then the dual map

$$DF'_x : T^*_{F(x)} N \to T^*_x M$$

defines

$$\Lambda^p(DF'_x) : \Lambda^p T^*_{F(x)} N \to \Lambda^p T^*_x M$$

and then

$$\Lambda^p(DF'_x)(\alpha_{F(x)})$$

is defined for all x and is a section of $\Lambda^p T^* M$ – a p-form on M.

Definition 23 *The pull-back of a p-form $\alpha \in \Omega^p(N)$ by a smooth map $F : M \to N$ is the p-form $F^*\alpha \in \Omega^p(M)$ defined by*

$$(F^*\alpha)_x = \Lambda^p(DF'_x)(\alpha_{F(x)}).$$

Examples:

1. The pull-back of a 0-form $f \in C^\infty(N)$ is just the composition $f \circ F$.

2. Let $F : \mathbf{R}^3 \to \mathbf{R}^2$ be given by

$$F(x_1, x_2, x_3) = (x_1 x_2, x_2 + x_3) = (x, y)$$

and take

$$\alpha = x dx \wedge dy.$$

Then

$$\begin{aligned} F^*\alpha &= (x \circ F) d(x \circ F) \wedge d(y \circ F) \\ &= x_1 x_2 d(x_1 x_2) \wedge d(x_2 + x_3) \\ &= x_1 x_2 (x_1 dx_2 + x_2 dx_1) \wedge d(x_2 + x_3) \\ &= x_1^2 x_2 dx_2 \wedge dx_3 + x_1 x_2^2 dx_1 \wedge dx_2 + x_1 x_2^2 dx_1 \wedge dx_3 \end{aligned}$$

From the algebraic properties of the maps

$$\Lambda^p A : \Lambda^p V \to \Lambda^p V$$

we have the following straightforward properties of the pull-back:

- $(F \circ G)^* \alpha = G^*(F^* \alpha)$
- $F^*(\alpha + \beta) = F^* \alpha + F^* \beta$
- $F^*(\alpha \wedge \beta) = F^* \alpha \wedge F^* \beta$

6.4 The exterior derivative

We now come to the construction of the basic differential operator on forms – the exterior derivative which generalizes the grads, divs and curls of three-dimensional calculus. The key feature it has is that it is defined naturally by the manifold structure without any further assumptions.

Theorem 6.2 *On any manifold M there is a natural linear map*

$$d : \Omega^p(M) \to \Omega^{p+1}(M)$$

called the exterior derivative such that

1. *if $f \in \Omega^0(M)$, then $df \in \Omega^1(M)$ is the derivative of f*
2. *$d^2 = 0$*
3. *$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ if $\alpha \in \Omega^p(M)$*

Examples: Before proving the theorem, let's look at $M = \mathbf{R}^3$, following the rules of the theorem, to see d in all cases $p = 0, 1, 2$.

$\mathbf{p} = 0$: by definition

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3$$

which we normally would write as grad f.

$\mathbf{p} = 1$: take a 1-form

$$\alpha = a_1 dx_1 + a_2 dx_2 + a_3 dx_3$$

then applying the rules we have

$$\begin{aligned} d(a_1 dx_1 + a_2 dx_2 + a_3 dx_3) &= da_1 \wedge dx_1 + da_2 \wedge dx_2 + da_3 \wedge dx_3 \\ &= \left(\frac{\partial a_1}{\partial x_1} dx_1 + \frac{\partial a_1}{\partial x_2} dx_2 + \frac{\partial a_1}{\partial x_3} dx_3\right) \wedge dx_1 + \ldots \end{aligned}$$

$$= \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right) dx_3 \wedge dx_1 + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) dx_1 \wedge dx_2 + \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) dx_2 \wedge dx_3.$$

The coefficients of this define what we would call the curl of the vector field **a** but **a** has now become a 1-form α and not a vector field and $d\alpha$ is a 2-form, not a vector field. The geometrical interpretation has changed. Note nevertheless that the invariant statement $d^2 = 0$ is equivalent to $\operatorname{curl} \operatorname{grad} f = 0$.

$\mathbf{p} = 2$: now we have a 2-form

$$\beta = b_1 dx_2 \wedge dx_3 + b_2 dx_3 \wedge dx_1 + b_3 dx_1 \wedge dx_2$$

and

$$\begin{aligned} d\beta &= \frac{\partial b_1}{\partial x_1} dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial b_2}{\partial x_2} dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial b_3}{\partial x_3} dx_1 \wedge dx_2 \wedge dx_3 \\ &= \left(\frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3 \end{aligned}$$

which would be the divergence of a vector field **b** but in our case is applied to a 2-form β. Again $d^2 = 0$ is equivalent to $\operatorname{div} \operatorname{curl} \mathbf{b} = 0$.

Here we see familiar formulas, but acting on unfamiliar objects. The fact that we can pull differential forms around by smooth maps will give us a lot more power, even in three dimensions, than if we always considered these things as vector fields.

Let us return to the Theorem 6.2 now and give its proof.

Proof: We shall define $d\alpha$ by first breaking up α as a sum of terms with support in a local coordinate system (using a partition of unity), define a local d operator using a coordinate system, and then show that the result is independent of the choice.

So to begin with write a p-form locally as

$$\alpha = \sum_{i_1 < i_2 < \ldots < i_p} a_{i_1 i_2 \ldots i_p}(x) dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}$$

and define

$$d\alpha = \sum_{i_1<i_2<\ldots<i_p} da_{i_1 i_2\ldots i_p} \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}.$$

When $p = 0$, this is just the derivative, so the first property of the theorem holds.

For the second part, we expand

$$d\alpha = \sum_{j,i_1<i_2<\ldots<i_p} \frac{\partial a_{i_1 i_2\ldots i_p}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}$$

and then calculate

$$d^2\alpha = \sum_{j,k,i_1<i_2<\ldots<i_p} \frac{\partial^2 a_{i_1 i_2\ldots i_p}}{\partial x_j \partial x_k} dx_k \wedge dx_j \wedge dx_{i_1} \wedge dx_{i_2} \ldots \wedge dx_{i_p}.$$

The term

$$\frac{\partial^2 a_{i_1 i_2\ldots i_p}}{\partial x_j \partial x_k}$$

is symmetric in j, k but it multiplies $dx_k \wedge dx_j$ in the formula which is skew-symmetric in j and k, so the expression vanishes identically and $d^2\alpha = 0$ as required.

For the third part, we check on decomposable forms

$$\begin{aligned} \alpha &= f dx_{i_1} \wedge \ldots \wedge dx_{i_p} = f dx_I \\ \beta &= g dx_{j_1} \wedge \ldots \wedge dx_{j_q} = g dx_J \end{aligned}$$

and extend by linearity. So

$$\begin{aligned} d(\alpha \wedge \beta) &= d(fg dx_I \wedge dx_J) \\ &= d(fg) \wedge dx_I \wedge dx_J \\ &= (f dg + g df) \wedge dx_I \wedge dx_J \\ &= (-1)^p f dx_I \wedge dg \wedge dx_J + df \wedge dx_I \wedge g dx_J \\ &= (-1)^p \alpha \wedge d\beta + d\alpha \wedge \beta \end{aligned}$$

So, using one coordinate system we have defined an operation d which satisfies the three conditions of the theorem. Now represent α in coordinates $y_1, \ldots, y_n$:

$$\alpha = \sum_{i_1<i_2<\ldots<i_p} b_{i_1 i_2\ldots i_p} dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p}$$

and define in the same way

$$d'\alpha = \sum_{i_1<i_2<\ldots<i_p} db_{i_1i_2\ldots i_p} \wedge dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p}.$$

We shall show that $d = d'$ by using the three conditions.

From (1) and (3),

$$d\alpha = d(\sum b_{i_1i_2\ldots i_p} dy_{i_1} \wedge dy_{i_2} \ldots \wedge dy_{i_p}) =$$

$$\sum db_{i_1i_2\ldots i_p} \wedge dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p} + \sum b_{i_1i_2\ldots i_p} d(dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p})$$

and from (3)

$$d(dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p}) = d(dy_{i_1}) \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p} - dy_{i_1} \wedge d(dy_{i_2} \wedge \ldots \wedge dy_{i_p}).$$

From (1) and (2) $d^2 y_{i_1} = 0$ and continuing similarly with the right hand term, we get zero in all terms.

Thus on each coordinate neighbourhood U $d\alpha = \sum_{i_1<i_2<\ldots<i_p} db_{i_1i_2\ldots i_p} \wedge dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_p} = d'\alpha$ and $d\alpha$ is thus globally well-defined. □

One important property of the exterior derivative is the following:

Proposition 6.3 *Let $F : M \to N$ be a smooth map and $\alpha \in \Omega^p(N)$. then*

$$d(F^*\alpha) = F^*(d\alpha).$$

Proof: Recall that the derivative $DF_x : T_xM \to T_{F(x)}N$ was defined in (11) by

$$DF_x(X_x)(f) = X_x(f \circ F)$$

so that the dual map $DF'_x : T^*_{F(x)}N \to T^*_xM$ satisfies

$$DF'_x(df)_{F(x)} = d(f \circ F)_x.$$

From the definition of pull-back this means that

$$F^*(df) = d(f \circ F) = d(F^*f) \tag{10}$$

Now if

$$\alpha = \sum_{i_1<i_2<\ldots<i_p} a_{i_1 i_2 \ldots i_p}(x) dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p},$$

$$F^*\alpha = \sum_{i_1<i_2<\ldots<i_p} a_{i_1 i_2 \ldots i_p}(F(x)) F^*dx_{i_1} \wedge F^*dx_{i_2} \wedge \ldots \wedge F^*dx_{i_p}$$

by the multiplicative property of pull-back and then using the properties of d and (10)

$$\begin{aligned} d(F^*\alpha) &= \sum_{i_1<i_2<\ldots<i_p} d(a_{i_1 i_2 \ldots i_p}(F(x))) \wedge F^*dx_{i_1} \wedge F^*dx_{i_2} \wedge \ldots \wedge F^*dx_{i_p} \\ &= \sum_{i_1<i_2<\ldots<i_p} F^*da_{i_1 i_2 \ldots i_p} \wedge F^*dx_{i_1} \wedge F^*dx_{i_2} \wedge \ldots \wedge F^*dx_{i_p} \\ &= F^*(d\alpha). \end{aligned}$$

□

6.5 The Lie derivative of a differential form

Suppose φ_t is the one-parameter (locally defined) group of diffeomorphisms defined by a vector field X. Then there is a naturally defined *Lie derivative*

$$\mathcal{L}_X \alpha = \frac{\partial}{\partial t} \varphi_t^* \alpha \bigg|_{t=0}$$

of a p-form α by X. It is again a p-form. We shall give a useful formula for this involving the exterior derivative.

Proposition 6.4 *Given a vector field X on a manifold M, there is a linear map*

$$i(X) : \Omega^p(M) \rightarrow \Omega^{p-1}(M)$$

(called the interior product) such that

- $i(X)df = X(f)$
- $i(X)(\alpha \wedge \beta) = i(X)\alpha \wedge \beta + (-1)^p \alpha \wedge i(X)\beta$ *if* $\alpha \in \Omega^p(M)$

The proposition tells us exactly how to work out an interior product: if

$$X = \sum_i a_i \frac{\partial}{\partial x_i},$$

and $\alpha = dx_1 \wedge dx_2 \wedge \ldots \wedge dx_p$ is a basic p-form then

$$i(X)\alpha = a_1 dx_2 \wedge \ldots \wedge dx_p - a_2 dx_1 \wedge dx_3 \wedge \ldots \wedge dx_p + \ldots \tag{11}$$

In particular

$$i(X)(i(X)\alpha) = a_1 a_2 dx_3 \wedge \ldots \wedge dx_p - a_2 a_1 dx_3 \wedge \ldots \wedge dx_p + \ldots = 0.$$

Example: Suppose

$$\alpha = dx \wedge dy, \quad X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

then

$$i(X)\alpha = xdy - ydx.$$

The interior product is just a linear algebra construction. Above we have seen how to work it out when we write down a form as a sum of basis vectors. We just need to prove that it is well-defined and independent of the way we do that, which motivates the following more abstract proof:

Proof: In Remark 5.1 we defined $\Lambda^p V$ as the dual space of the space of alternating p-multilinear forms on V. If M is an alternating $(p-1)$-multilinear form on V and ξ a linear form on V then

$$(\xi M)(v_1, \ldots, v_p) = \xi(v_1)M(v_2, \ldots, v_p) - \xi(v_2)M(v_1, v_3, \ldots, v_p) + \ldots \tag{12}$$

is an alternating p-multilinear form. So if $\alpha \in \Lambda^p V$ we can define $i(\xi)\alpha \in \Lambda^{p-1} V$ by

$$(i(\xi)\alpha)(M) = \alpha(\xi M).$$

Taking $V = T^*$ and $\xi = X \in V^* = (T^*)^* = T$ gives the interior product. Equation (12) gives us the rule (11) for working out interior products. □

Here then is the formula for the Lie derivative:

Proposition 6.5 *The Lie derivative $\mathcal{L}_X \alpha$ of a p-form α is given by*

$$\mathcal{L}_X \alpha = d(i(X)\alpha) + i(X)d\alpha.$$

Proof: Consider the right hand side

$$R_X(\alpha) = d(i(X)\alpha) + i(X)d\alpha.$$

Now $i(X)$ reduces the degree p by 1 but d increases it by 1, so R_X maps p-forms to p-forms. Also,

$$d(d(i(X)\alpha) + i(X)d\alpha) = di(X)d\alpha = (di(X) + i(X)d)d\alpha$$

because $d^2 = 0$, so R_X commutes with d. Finally, because

$$\begin{aligned} i(X)(\alpha \wedge \beta) &= i(X)\alpha \wedge \beta + (-1)^p \alpha \wedge i(X)\beta \\ d(\alpha \wedge \beta) &= d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \end{aligned}$$

we have

$$R_X(\alpha \wedge \beta) = (R_X\alpha) \wedge \beta + \alpha \wedge R_X(\beta).$$

On the other hand

$$\varphi_t^*(d\alpha) = d(\varphi_t^*\alpha)$$

so differentiating at $t = 0$, we get

$$\mathcal{L}_X d\alpha = d(\mathcal{L}_X \alpha)$$

and

$$\varphi_t^*(\alpha \wedge \beta) = \varphi_t^*\alpha \wedge \varphi_t^*\beta$$

and differentiating this, we have

$$\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X\beta.$$

Thus both $\mathcal{L}_X$ and R_X preserve degree, commute with d and satisfy the same Leibnitz identity. Hence, if we write a p-form as

$$\alpha = \sum_{i_1 < i_2 < \ldots < i_p} a_{i_1 i_2 \ldots i_p}(x) dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_p}$$

$\mathcal{L}_X$ and R_X will agree so long as they agree on functions. But

$$R_X f = i(X)df = X(f) = \frac{\partial}{\partial t} f(\varphi_t)\bigg|_{t=0} = \mathcal{L}_X f$$

so they do agree. □

6.6 de Rham cohomology

In textbooks on vector calculus, you may read not only that $\operatorname{curl}\operatorname{grad} f = 0$, but also that if a vector field $\mathbf{a}$ satisfies $\operatorname{curl}\mathbf{a} = 0$, then it can be written as $\mathbf{a} = \operatorname{grad} f$ for some function f. Sometimes the statement is given with the proviso that the open set of $\mathbf{R}^3$ on which $\mathbf{a}$ is defined satisfies the topological condition that it is simply connected (any closed path can be contracted to a point).

In the language of differential forms on a manifold, the analogue of the above statement would say that if a 1-form α satisfies $d\alpha = 0$, and M is simply-connected, there is a function f such that $df = \alpha$.

While this is true, the criterion of simply connectedness is far too strong. We want to know when the kernel of

$$d : \Omega^1(M) \to \Omega^2(M)$$

is equal to the image of

$$d : \Omega^0(M) \to \Omega^1(M).$$

Since $d^2 f = 0$, the second vector space is contained in the first and what we shall do is simply to study the quotient, which becomes a topological object in its own right, with an algebraic structure which can be used to say many things about the global topology of a manifold.

Definition 24 *The p-th de Rham cohomology group of a manifold M is the quotient vector space:*

$$H^p(M) = \frac{\operatorname{Ker} d : \Omega^p(M) \to \Omega^{p+1}(M)}{\operatorname{Im} d : \Omega^{p-1}(M) \to \Omega^p(M)}$$

Remark:

1. Although we call it the cohomology *group*, it is simply a real vector space. There are analogous structures in algebraic topology where the additive group structure is more interesting.

2. Since there are no forms of degree -1, the group $H^0(M)$ is the space of functions f such that $df = 0$. Now each connected component M_i of M is an open set of M and hence a manifold. The mean value theorem tells us that on any open ball in a coordinate neighbourhood of M_i, $df = 0$ implies that f is equal to a constant c, and the subset of M_i on which $f = c$ is open and closed and hence equal to M_i.

Thus if M is connected, the de Rham cohomology group $H^0(M)$ is naturally isomorphic to $\mathbf{R}$: the constant value c of the function f. In general $H^0(M)$ is the vector space of real valued functions on the set of components. Our assumption that M

has a countable basis of open sets means that there are at most countably many components. When M is compact, there are only finitely many, since components provide an open covering. The cohomology groups for all p of a compact manifold are finite-dimensional vector spaces, though we shall not prove that here.

It is convenient in discussing the exterior derivative to introduce the following terminology:

Definition 25 *A form $\alpha \in \Omega^p(M)$ is closed if $d\alpha = 0$.*

Definition 26 *A form $\alpha \in \Omega^p(M)$ is exact if $\alpha = d\beta$ for some $\beta \in \Omega^{p-1}(M)$.*

The de Rham cohomology group $H^p(M)$ is by definition the quotient of the space of closed p-forms by the subspace of exact p-forms. Under the quotient map, a closed p-form α defines a cohomology class $[\alpha] \in H^p(M)$, and $[\alpha'] = [\alpha]$ if and only if $\alpha' - \alpha = d\beta$ for some β.

Here are some basic features of the de Rham cohomology groups:

Proposition 6.6 *The de Rham cohomology groups of a manifold M of dimension n have the following properties:*

- *$H^p(M) = 0$ if $p > n$*
- *for $a \in H^p(M), b \in H^q(M)$ there is a bilinear product $ab \in H^{p+q}(M)$ which satisfies*
$$ab = (-1)^{pq} ba$$
- *if $F : M \to N$ is a smooth map, it defines a natural linear map*
$$F^* : H^p(N) \to H^p(M)$$
which commutes with the product.

Proof: The first part is clear since $\Lambda^p T^* = 0$ for $p > n$.

For the product, this comes directly from the exterior product of forms. If $a = [\alpha], b = [\beta]$ we define
$$ab = [\alpha \wedge \beta]$$

but we need to check that this really does define a cohomology class. Firstly, since α, β are closed,

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta = 0$$

so there is a class defined by α and β. Suppose we now choose a different representative $\alpha' = \alpha + d\gamma$ for a. Then

$$\alpha' \wedge \beta = (\alpha + d\gamma) \wedge \beta = \alpha \wedge \beta + d(\gamma \wedge \beta)$$

since $d\beta = 0$, so $d(\gamma \wedge \beta) = d\gamma \wedge \beta$. Thus $\alpha' \wedge \beta$ and $\alpha \wedge \beta$ differ by an exact form and define the same cohomology class. Changing β gives the same result.

The last part is just the pull-back operation on forms. Since

$$dF^*\alpha = F^* d\alpha$$

F^* defines a map of cohomology groups. And since

$$F^*(\alpha \wedge \beta) = F^*\alpha \wedge F^*\beta$$

it respects the product. □

Perhaps the most important property of the de Rham cohomology, certainly the one that links it to algebraic topology, is the deformation invariance of the induced maps F. We show that if F_t is a smooth family of smooth maps, then the effect on cohomology is independent of t. As a matter of terminology (because we have only defined smooth maps of manifolds) we shall say that a map

$$F : M \times [a, b] \to N$$

is smooth if it is the restriction of a smooth map on the product with some slightly bigger open interval $M \times (a - \epsilon, b + \epsilon)$.

Theorem 6.7 *Let $F : M \times [0, 1] \to N$ be a smooth map. Set $F_t(x) = F(x, t)$ and consider the induced map on de Rham cohomology $F_t^* : H^p(N) \to H^p(M)$. Then*

$$F_1^* = F_0^*.$$

Proof: Represent $a \in H^p(N)$ by a closed p-form α and consider the pull-back form $F^*\alpha$ on $M \times [0, 1]$. We can decompose this uniquely in the form

$$F^*\alpha = \beta + dt \wedge \gamma \tag{13}$$

where β is a p-form on M (also depending on t) and γ is a $(p-1)$-form on M, depending on t. In a coordinate system it is clear how to do this, but more invariantly, the form β is just $F_t^*\alpha$. To get γ in an invariant manner, we can think of

$$(x, s) \mapsto (x, s + t)$$

as a local one-parameter group of diffeomorphisms of $M \times (a, b)$ which generates a vector field $X = \partial/\partial t$. Then

$$\gamma = i(X)F^*\alpha.$$

Now α is closed, so from (13),

$$0 = d_M\beta + dt \wedge \frac{\partial \beta}{\partial t} - dt \wedge d_M\gamma$$

where d_M is the exterior derivative in the variables of M. It follows that

$$\frac{\partial}{\partial t} F_t^*\alpha = \frac{\partial \beta}{\partial t} = d_M\gamma$$

so that, integrating with respect to the parameter t,

$$F_1^*\alpha - F_0^*\alpha = \int_0^1 \frac{\partial}{\partial t} F_t^*\alpha \ dt = d \int_0^1 \gamma \ dt.$$

So the closed forms $F_1^*\alpha$ and $F_0^*\alpha$ differ by an exact form and

$$F_1^*(a) = F_0^*(a).$$

□

Here is an immediate corollary:

Proposition 6.8 *The de Rham cohomology groups of* $M = \mathbf{R}^n$ *are zero for* $p > 0$.

Proof: Define $F : \mathbf{R}^n \times [0, 1] \to \mathbf{R}^n$ by

$$F(x, t) = tx.$$

Then $F_1(x) = x$ which is the identity map, and so

$$F_1^* : H^p(\mathbf{R}^n) \to H^p(\mathbf{R}^n)$$

is the identity.

But $F_0(x) = 0$ which is a constant map. In particular the derivative vanishes, so the pull-back of any p-form of degree greater than zero is the zero map. So for $p > 0$

$$F_0^* : H^p(\mathbf{R}^n) \to H^p(\mathbf{R}^n)$$

vanishes.

From Theorem 6.7 $F_0^* = F_1^*$ and we deduce that $H^p(\mathbf{R}^n)$ vanishes for $p > 0$. Of course $\mathbf{R}^n$ is connected so $H^0(\mathbf{R}^n) \cong \mathbf{R}$. □

Example: Show that the previous proposition holds for a *star shaped region* in $\mathbf{R}^n$: an open set U with a point $a \in U$ such that for each $x \in U$ the straight-line segment $\overline{ax} \subset U$. This is usually called the *Poincaré lemma*.

We are in no position yet to calculate many other de Rham cohomology groups, but here is one non-trivial example. Consider the case of $\mathbf{R}/\mathbf{Z}$, diffeomorphic to the circle. In the atlas given earlier, we had $\varphi_1\varphi_0^{-1}(x) = x$ or $\varphi_1\varphi_0^{-1}(x) = x - 1$ so the 1-form $dx = d(x-1)$ is well-defined, and nowhere zero. It is not the derivative of a function, however, since $\mathbf{R}/Z$ is compact and any function must have a minimum where $df = 0$. We deduce that

$$H^1(\mathbf{R}/\mathbf{Z}) \neq 0.$$

To get more information we need to study the other aspect of differential forms: integration.