IMC preparation seminar, Day 1
Calculus I

April 19, 2024

1 Theory/Background

Calculus I and Calculus II: Sequences, Series, Sums, Real functions, Limits, Continuity, Differentiation, Riemann Integration.

Reminder: x is a fixed point of $f : X \to X \iff f(x) = x$.

2 Problems

2.1 Warm up problems:

Problem 1 Calculate the sum

$$\sum_{n=0}^{\infty} \arctan\left(\frac{1}{n^2+n+1}\right).$$

Problem 2 Calculate the integral

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx.$$

Problem 3 Calculate the following limit:

$$\lim_{n\to\infty} \sqrt{\frac{1}{n} + \sqrt{\frac{1}{n} + \sqrt{\frac{1}{n} + \dots}}}$$

Problem 4 Is it always true that $\lim\limits_{n\to\infty} \lim\limits_{m\to\infty} a_{n,m} = \lim\limits_{m\to\infty} \lim\limits_{n\to\infty} a_{n,m}$?

Problem 5 Let $f \in C^2_{\mathbb{R}}(\mathbb{R})$ which is also bounded with bounded second derivative. Let

$$A = \sup_{x\in\mathbb{R}} |f(x)|$$

and

$$B = \sup_{x\in\mathbb{R}} |f''(x)|.$$

Prove that the derivative of f is bounded by $2\sqrt{AB}$, that is

$$\sup_{x\in\mathbb{R}} |f'(x)| \leq 2\sqrt{AB}.$$

Problem 6 Show that the integer nearest to $\dfrac{n!}{e}$ is divisible by $n-1$ but not by n.

2.2 Harder problems:

Problem 7 Let $h : [0,1) \to \mathbb{R}$. Prove that if h is uniformly continuous then there exists a continuous $g : [0,1] \to \mathbb{R}$, such that

$$g\Big|_{[0,1)} = h,$$

that is $g(x) = h(x),\ \forall\, x \in [0,1)$.

Problem 8 Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. Prove that for every n there exists an ascending finite sequence $q_1 < q_2 < \cdots < q_n$ in (a,b) such that

$$\frac{f(b)-f(a)}{b-a} = \frac{f'(q_1)+f'(q_2)+\cdots+f'(q_n)}{n}.$$

Problem 9 Let $f_0 : [0,1] \to \mathbb{R}$ a continuous function. We define a sequence of functions $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \int_0^x f_{n-1}(t)\,dt.$$

Prove that if there is an integer $m \in \mathbb{Z}_{\geq 0}$ such that

$$\int_0^1 f_m(t)\,dt = \frac{1}{(m+1)!}$$

then the function f_0 must have a fixed point.

Problem 9 Let $f : [a,b] \to [a,b]$ a continuous function. Let $x_1 \in [a,b]$ and define $x_{n+1} = f(x_n)$. Show that the sequence $(x_n)_{n\geq 1}$ converges to a fixed point of f if and only if $\lim\limits_{n\to\infty} x_{n+1} - x_n = 0$.

Problem 10 (IMC 2023, Problem 2, Day 2) We define as V the set of all continuous functions $f : [0,1] \to \mathbb{R}$ differentiable on $(0,1)$ with the property $f(0) = 0$ and $f(1) = 1$. Determine all $a \in \mathbb{R}$ such that for every $f \in V$, there exists some $z \in (0,1)$ with the property $f(z) + a = f'(z)$.

Problem 11 Let $a > 0$ and $\mathcal{A}$ the class of all sequences of positive numbers $(a_n)_{n\geq 1}$ such that

$$\sum_{n=1}^{\infty} a_n = a.$$

Find the set

$$\left\{ \sum_{n=1}^{\infty} a_n^2 : (a_n)_{n\geq 1} \in \mathcal{A} \right\}.$$