PROBLEM SOLVING SEMINAR 19/4/2024
NUMBER THEORY DAY 5

1. Basic knowledge

In Number Theory we deal with problems related with the natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$ (or the integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$) and functions $f : \mathbb{N} \to \mathbb{N}$ or $f : \mathbb{Z} \to \mathbb{Z}$. The prerequisites we want are little, but problems can be extremely difficult. Basic theory: Induction and descent method, Divisibility and prime numbers, Mod arithmetic, Arithmetic functions, Pell equations, Sums of squares.

2. Warm-up problems

Use the above theory and techniques to solve the following warm-up problems.

1) Prove that for every natural number $n \geq 3$, at least one of $2^n - 1$ and $2^n + 1$ is composite.

2) If p is a prime then $p|(p-1)! + 1$ (Wilson's theorem).

3) Prove that if $p \leq n$ then $p \nmid (n! + 1)$. Deduce there are infinitely many primes.

4) Can we find a polynomial $f(x)$ with integer coefficients such $f(231) = 554$ and $f(161) = 496$?

5) If $k \geq 1$ then $k(k+1)$ is not a power > 1.

6) Let r be a real number such that $r + r^{-1} \in \mathbb{N}$. Prove that for every $n \in \mathbb{Z}$, $r^n + r^{-n} \in \mathbb{N}$.

7) If n is a sum of two squares, then also $2n$ is.

8) Find all $n \in \mathbb{N}$ such that $[\sqrt{n}] \mid n$.

3. Challenging problems

More difficulet problems.

9) Prove that for any natural number $n \neq 2, 6$ we have
$$\phi(n) \geq \sqrt{n}.$$

10) Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers.

11) Prove that for no integer $n > 1$ does n divide $2^n - 1$.

12) Show that the equation
$$x^2 + 10y^2 = 3z^2$$
has no solution in the positive integers.

13) Let $m, n \in \mathbb{N}$ such that mn divides $m^2 + n^2 + m$. Then m is a square number.

14) Find all n such that $d(n) = n$ or $d(n)^2 = n$.

15) Prove that the product of four consecutive natural numbers cannot be the square of an integer.

16) Let $f(x)$ be a polynomial with all coefficients being natural numbers. Can we find $f(x)$ by determining only two values of $f(n), f(m)$ for two integers n, m?

17) If n is an integer, then
$$1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\notin\mathbb{Z}.$$

18) Prove that there exists no n such that $\phi(n)=14$. Are there infinitely many m such that $\phi(n)=m$ has no solution?
19) We denote by $[x]$ the integral part of x. Prove that if $n\in\mathbb{N}$ and $a\geq 0$ real then
$$\sum_{k=0}^{n-1}\left[a+\frac{k}{n}\right]=[na].$$

20) Prove that if $n\in\mathbb{N}$ then
$$\sum_{k=0}^{\infty}\left[\frac{n+2^k}{2^{k+1}}\right]=n.$$

21) Prove that for every $n\in\mathbb{N}$
$$\frac{1}{\zeta(2)}<\frac{\sigma(n)\phi(n)}{n^2}\leq 1.$$

22) (IMC 2012) For every $n>1$ let $p(n)$ denote the number of partitions of n, i.e. the number of ways we can write n as a sum of natural numbers. For instance $p(3)=3, p(4)=5, ...$ Prove that $p(n)-p(n-1)$ is the number of ways to express n as a sum of integers each of which is strictly greater than 1.

23) Let $f:\mathbb{N}\to\mathbb{N}$ such that $f(2)=2$, $f(mn)=f(m)f(n)$ and $f(m)>f(n)$ if $m>n$. Prove that $f(n)=n$.

24) (IMC 2014) Let $n>6$ be a perfect number, and let $n=p^{a_1}...p^{a_k}$ its prime factorization, with $p_1<...<p_k$. Prove that a_1 is an even number.

25) Let A be the set of natural numbers representable in the form a^2+2b^2 for some integers a and b with $b\neq 0$. Show that if $p^2\in A$ for a prime p, then $p\in A$.

26) A prime number p cannot be written as a sum of two squares in two different ways.

27) Let $f(x)$ be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer k. Prove that all coefficients of f are divisible by 5.

28) If $29\mid(x^4+y^4+z^4)$ then $29^4\mid(x^4+y^4+z^4)$.

29) If a,b,c are natural numbers satisfying $a^2+b^2+1=c^2$ then the quantity
$$\left[\frac{a}{2}\right]+\left[\frac{c}{2}\right]$$
is even.

30) If $n>1$ then n^4+4^n is not a prime.

31) Show that if n has $p-1$ digits all equal to 1, where $p>7$ is a prime, then n is divisible by p.

32) Assume $n\geq 2$. If $f(n)=2^n+n^2$ is prime, then $3|f(f(n))$.

33) Determine all functions $f:\mathbb{N}\to\mathbb{N}$ such that
$$(f(m)+n)(m+f(n))$$
is a perfect square for all $m,n\in\mathbb{N}$.

34) (Putnam 2018, B3) Find all positive integers $n<10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides 2^n-1, and $n-2$ divides 2^n-2.