IMC preparation seminar, Day 4

April 12, 2024

1 Theory/Background

Some necessary material from Linear Algebra.

Definition 1.1. (Characteristic polynomial) Let $A \in M_n(\mathbb{C})$, then the characteristic polynomial of A is defined as:

$$\chi_A(\lambda) = det(\lambda \mathbb{I}_n - A).$$

For $A \in M_2(\mathbb{C})$ we have

$$\chi_A(\lambda) = \lambda^2 - tr(A)\lambda + detA$$

The following theorem is crucial in the theory of finite matrices.

Theorem 1.2. *(Cayley-Hamilton) For $A \in M_n(\mathbb{C})$ we have $\chi_A(A) = O_n$. In particular, for $A \in M_2(\mathbb{C})$ we get*

$$A^2 - tr(A)A + detAI_2 = O_2.$$

Lemma 1.3. *Let $A, B \in M_2(\mathbb{C})$ and $z \in \mathbb{C}$, then*

$$det(A + zB) = detA + vz + detBz^2$$

If the entries of A, B are from a field $\mathbb{F}$, so is v.

We will need the basic parts of the theory of determinants. One particular example is the following famous determinant of Vandermonde:

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le j < i \le n} (x_i - x_j)$$

2 Problems

2.1 Warm up problems

Problem 1 Find a non-linear map $f : \mathbb{R}^n \to \mathbb{R}, n \in \mathbb{N}$, satisfying:

$$f(ax) = af(x)$$

for every $a \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Problem 2 Let $A, B \in M_n(\mathbb{R})$, such that $AB = BA$. Prove that

$$\det(A^2 + B^2) \ge 0.$$

When does equality hold? Does the inequality hold for non-commuting matrices?

Problem 3 Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$ such that $a^2+b^2+c^2+d^2 < \frac{1}{5}$. Show that $A+I$ is invertible.

Problem 4 Let $A \in M_2(\mathbb{Q})$ such that $\det(A^2 - 2I_2) = 0$. Prove that $A^2 = 2I_2$.

Problem 5 Let $A, B \in M_2(\mathbb{C})$, such that $\det(AB - BA) \leq 0$. Prove that

$$det(I_2 + AB) \leq \det\left(I_2 + \frac{1}{2}(AB + BA)\right).$$

2.2 Harder problems

Problem 6 Let $n \geq 2$ be a positive integer and $A, B \in M_2(\mathbb{C})$ two non-commuting matrices, such that $(AB)^n = (BA)^n$. Deduce that $(AB)^n = aI_2$, where $a \in \mathbb{C}$.

Problem 7 Prove that

$$\begin{vmatrix} (x^2+1)^2 & (xy+1)^2 & (xz+1)^2 \\ (xy+1)^2 & (y^2+1)^2 & (yz+1)^2 \\ (xz+1)^2 & (yz+1)^2 & (z^2+1)^2 \end{vmatrix} = 2(x-y)^2(y-z)^2(z-x)^2$$

Problem 8 Prove that for any matrix $A \in M_n(\mathbb{C}), n \in \mathbb{N}$ there exists a matrix $B \in M_n(\mathbb{C})$, such that

$$ABA = A.$$

Problem 9 Find all $A \in M_n(\mathbb{C})$ such that

$$A^{2023} = A^*A - AA^*.$$

Problem 10 Let $A \in M_{2020}(\mathbb{C})$, such $rank(A) = 2019$. Prove that

$$A^2 = A \Leftrightarrow A + adj(A) = I_{2020}.$$

Problem 11 (SEEMOUS 2024, P2) Let $A, B \in M_n(\mathbb{R})$ two real, symmetric matrices with nonnegative eigenvalues. Prove that $A^3 + B^3 = (A+B)^3$ if and only if $AB = O_n$.