Εσωτερικός διαγωνισμός επιλογής 2022
Τμήμα Μαθηματικών Πατρών

05/05/22

Διάρκεια εξέτασης: 3 ώρες.

Problem 1. For which positive real values of c does the series

$$\sum_{n=2}^{\infty}\left(1-\cos\left(\sqrt[3]{\frac{\pi}{n}}\right)\right)^{c}$$

converge?

Problem 2. Let A, B be 2×2 real matrices and set $C = A - B$. Assume that $C \neq O_2$ (the 2×2 zero matrix) and C is not a scalar matrix. If

$$A^2 + B^2 = AB + BA + \lambda C$$

with $\lambda \neq 0$ real, then prove that C is not invertible and calculate the trace of C.

Problem 3. Let $k \geq 1$ be a positive integer number and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be the function given by

$$f(x) = \sum_{m=1}^{k} x^{4m}.$$

Let $a_1, ..., a_n$ be positive real numbers and $b_1, ..., b_n$ be a rearrangement of $a_1, ..., a_n$. Prove that

$$\sum_{i=1}^{n} f\left(\frac{a_i}{b_i}\right) \geq nk.$$

Problem 4. Find all positive integers $n \geq 1$ with the following property: there exist 2×2 integer matrices A, B such that

$$(AB - BA)^n = nI_2,$$

where I_2 denotes the 2×2 identity matrix.

Εσωτερικός διαγωνισμός επιλογής για τον IMC 2022
Τμήμα Μαθηματικών Πατρών

09/06/22

Διάρκεια εξέτασης: 3 ώρες.

Problem 1. Assume that a_n is a sequence of nonnegative numbers such that the series

$$\sum_{n=1}^{\infty} a_n$$

converges.

(a) Prove that the series

$$\sum_{n=1}^{\infty} (a_n \log n)^c$$

also converges for any real $c > 1$.

(b) Does the statement hold if a_n are not necessarily nonnegative?

Problem 2. Estimate the sum

$$\sum_{n=1}^{\infty} \frac{1}{2^n(2n+1)(n+1)}.$$

Problem 3. Let A be a positive real $n \times n$ matrix. Prove that

$$trace(A) \geq n \det(A)^{\frac{k}{n}}.$$

Problem 4. Let $f(x), g(x)$ be two nonconstant polynomials in $\mathbb{Z}[x]$ such that $g(x)$ divides $f(x)$ in $\mathbb{Z}[x]$. Prove that if $f(x) - 226$ has at least 33 distinct integer roots, then the degree of $g(x)$ is at least 5.

Problem 5. For $n \geq 2$ define the sequence of positive integers a_n by the equality

$$(n-2)! \equiv a_n \mod n, \quad 0 \leq a_n < n.$$

Does the series

$$\sum_{n=2}^{\infty} \frac{a_n}{n}$$

converge?

Problem 6. Let p be a prime. Show that there are infinitely many positive integers n such that p divides $2^n - n$.