SEEMOUS AND IMC PREPARATION, DAY 1, 2/12/2021
SEQUENCES, SERIES AND INEQUALITIES

1. Basic knowledge

1.1. **Sequences.** Sequences are one of the fundamental notions of Mathematical analysis. The most important notions about them are boundedness (bounded sequences), monotonicity (increasing and decreasing sequences) and convergence (the limit of a sequence).

We assume familiarity with the notions of a sequence, a subsequence and the limit of a sequence. The most important facts are the following ones:

1) Algebra of limits: if $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$, $x_n y_n \to xy$, etc.

2) Every bounded and monotone sequence converges to a finite limit.

3) Bolzano - Weierstrass: Every bounded sequence has a convergent subsequence.

4) Sandwich: If $a_n \leq b_n \leq c_n$ for every $n \geq n_0$ with n_0 some fixed index, then a_n, b_n, c_n converge, then

$$\lim a_n \leq \lim b_n \leq \lim c_n.$$

In particular, if a_n and c_n converge and $\lim a_n = \lim c_n$ then b_n converges and

$$\lim a_n = \lim b_n = \lim c_n.$$

1.2. **Series.** Series are one of the fundamental notions of Mathematical analysis. The most important notions about them are convergess and summation. Quite often, you must apply theory from inequalities to solve some exercises (Cauchy-Schwarz inequality, etc.)

Main criteria of convergence:

1) Well known series: geometric series, harmonic series, Taylor series of standard functions.

2) Comparison test.

3) Cauchy test: If a_n is a non-increasing sequence of non-negative real numbers, then

$$\sum_{n=1}^{\infty} a_n < \infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} 2^n a_{2^n} < \infty.$$

4) Leibniz test: If a_n decreasing and $a_n \to 0$, then

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

5) Cesáro - Stoltz theorem: Let $x_n, y_n \in \mathbb{R}$ be two sequences with y_n strictly positive, increasing and unbounded. If

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = a$$

then

$$\lim_{n\to\infty} \frac{x_n}{y_n} = a.$$

1.3. **Inequalities.** Proving inequalities often involves the (possibly repeated) application of just a few basic inequalities. The main examples are the following:

1. Squares are positive: for all $x \in \mathbb{R}$

$$x^2 \geq 0.$$

2. Rearrangement:

$$\sum_k x_k y_k$$

is maximised when $\{x_i\}$ and $\{y_i\}$ (positive) are sorted the same way.

3. Arithmetic mean-Geometric mean: for real nonnegative $x_1, x_2, ..., x_n$ we have

$$\frac{x_1 + x_2 + ... + x_n}{n} \geq \sqrt[n]{x_1 x_2 ... x_n}.$$

4. Cauchy-Schwarz:

$$\left(\sum_{k=1}^n x_k y_k\right)^2 \leq \left(\sum_{k=1}^n x_k^2\right)\left(\sum_{k=1}^n y_k^2\right)$$

5. Apply convexity:

   (1) A convex function of a compact domain attains its maximum on the boundary.
   (2) Jensen's inequality: $\frac{1}{n}\sum_i f(x_i) \geq f(\frac{1}{n}\sum_i x_i)$, for f convex.

6. Power-means:

$$\|x\|_p \geq \|x\|_q,$$

where $p \geq q$ and $\|x\|_k$ denotes the k-norm of the vector x;

7. Probabilistic inequalities: The maximum attained by a random variable is larger then its mean.

8. Triangle inequality(!).

On the other hand, the technique is often quite subtle, since it involves balancing simplifying an expression (e.g. by applying an inequality), with ensuring not too much is given up in the process. Note there are many other useful inequalities (Bernoulli's, Hölder's, Minkowski's, Chebyshev's etc.), although for problem solving purposes these give diminishing returns.

Hint: quite often, use induction!

2. EXERCISES

Warm-up problems for Sequences

1. Find the limit of the sequence

$$a_n = \sqrt{1+\sqrt{1+\sqrt{1+\ldots+\sqrt{1}}}}.$$

2. A sequence $S = \{x_n\}_{n\in\mathbb{N}}$ of real numbers satisfies $x_{n+1} = x_n \cos x_n$. Does it follow that S converges for all values of x_1?

3. What if $x_n \cos x_n$ in ex.1 is replaced with $x_n \sin x_n$?

4. Prove that the following sequence converges:

$$a_n = \sqrt{1+\sqrt{2+\sqrt{3+\ldots+\sqrt{n}}}}, \quad n \geq 1.$$

5. a) Let $a_1, a_2, ...$ be a sequence of real numbers such that $a_1 = 1$ and $a_{n+1} > \frac{3a_n}{2}$ for all n. Prove that the sequence

$$\frac{a_n}{\left(\frac{3}{2}\right)^{n-1}}$$

has a finite limit or tends to infinity.
b) Prove that for all $a > 1$ there exists a sequence $a_1, a_2, ...$ with the same properties such that

$$\lim_{n\to\infty} \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}} = a.$$

Warm-up problems for Series

6. Let $a_1, ...a_n, ...$ be a sequence of nonnegative numbers. Prove that if we have $\sum a_n < \infty$ then $\sum \sqrt{a_{n+1}a_n} < \infty$.

7. Show that if a_n is decreasing and $\sum a_n$ converges then $\lim na_n = 0$.

8. Find a sequence a_n of real numbers such that

$$\sum_{n=1}^{\infty} a_n$$

converges, but

$$\sum_{n=1}^{\infty} |a_n|^k$$

diverges for every $k \geq 1$.

9. Does there exist a positive sequence a_n such that $\sum a_n$ and $\sum 1/(n^2 a_n)$ are convergent?

10. Let $0 < a_n < 1$ for $n \in \mathbb{N}$. Assume that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{\log(1/a_n)}$$

converges. Prove that the series

$$\sum_{n=2}^{\infty} \frac{a_n}{\log n}$$

also converges.

11. Let a_n be a sequence of nonnegative real numbers such that

$$\frac{1}{n}\sum_{k=1}^{n} a_k \geq \sum_{k=n+1}^{2n} a_k$$

for every $n \geq 1$. Prove that

$$\sum_{n=1}^{\infty} a_n \leq 2ea_1.$$

More difficult problems

12. Define the sequence $\{x_i\}_{i\in\mathbb{N}}$ by $x_1 = \sqrt{5}$ and $x_{n+1} = x_n^2 - 2$. Compute

$$\lim_{n\to\infty} \frac{x_1 \cdot x_2 \cdot x_3 \cdots x_n}{x_{n+1}}.$$

13. Compute

$$\lim_{n\to\infty} \left|\sin\left(\pi\sqrt{n^2+n+1}\right)\right|.$$

14. Let $a_0 = \sqrt{2}, b_0 = 2$ and $a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}}$, $b_{n+1} = \frac{2b_n}{2+\sqrt{4+b_n^2}}$.

a) Prove that a_n and b_n are decreasing and converge to 0.

b) Prove that $2^n a_n$ is increasing, $2^n b_n$ is decreasing and they converge to the same limit.

c) Prove that there exist a constant C such that

$$0 < b_n - a_n < \frac{C}{8^n}.$$

15. Let $S = \{x_1, x_2, ..., x_n, ...\}$ be the set of all positive integers that do not contain the digit 9 in their decimal representation. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{x_n} < 80.$$

16. For which positive values of x does the series

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{x} - 1\right)$$

converge?

17. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)} < 2.$$

18. The sequences a_n and b_n of positive real numbers satisfy

$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \sum_{n=1}^{\infty} \frac{1}{b_n} = \infty.$$

Is it true that

$$\sum_{n=1}^{\infty} \frac{1}{a_n + b_n} = \infty?$$

19. Define the sequence $\{a_n\}_{n\in\mathbb{N}}$ by $a_0 = 1$, $a_1 = 1/2$ and

$$a_{n+1} = \frac{na_n^2}{1+(n+1)a_n} \,, \quad n \geq 1\,.$$

Does the series $\sum_{n\geq 0}(a_{n+1}/a_n)$ converge, and if so, to which value?

20. For positive integers n, let the sequence a_n be determined by the rules $a_1 = 1$, $a_{2n} = a_n$ and $a_{2n+1} = (-1)^n a_n$. Find the value of

$$\sum_{n=1}^{2016} a_n a_{n+2}.$$

21. Let $a_0, ..., a_n$ be real numbers in $(0, \pi/2)$ such that

$$\sum_{k=0}^{n} \tan\left(a_k - \frac{\pi}{4}\right) \geq n-1.$$

Prove that

$$\prod_{k=0}^{n} \tan(a_k) \geq n^{n+1}.$$

22. Let $(x_1, x_2, \dots)$ be a sequence of positive real numbers satisfying

$$\sum_{n=1}^{\infty} \frac{x_n}{2n-1} = 1.$$

Prove that

$$\sum_{k=1}^{\infty}\sum_{n=1}^{k} \frac{x_n}{k^2} \leq 2.$$

3. Solutions

1. Prove it is bounded by 2 and increasing. Recursive formula for the limit.

2. NO. For $x_1 = \pi$ we have $x_n = (-1)^{n-1}\pi$.

3. YES. Notice that $|y_n|$ is nonincreasing and hence converges to some number $a \geq 0$. If $a = 0$, then $\lim y_n = 0$ and we are done. If $a > 0$, then $a = \lim |y_n + 1| = \lim |y_n \sin y_n| = a|\sin a|$, so $\sin a = \pm 1$ and $a = (k + 1/2)\pi$ for some nonnegative integer k.

Notice that $t \cdot \sin t = (-t)\sin(-t) = |t| \sin |t|$ for all real t, hence $y_{n+1} = |y_n| \sin |y_n|$ for all $n \geq 2$. Since the function $t \to t \sin t$ is continuous, $y_{n+1} = |y_n| \sin |y_n| \to |a| \sin |a| = a$.

4. Since a_n is increasing, it suffices to prove that a_n is bounded. Notice that

$$\begin{aligned} a_n &= \sqrt{2}\sqrt{\frac{1}{2}+\sqrt{\frac{2}{4}+\sqrt{\frac{3}{8}+\ldots+\sqrt{\frac{n}{2^n}}}}} \\ &< \sqrt{2}\sqrt{1+\sqrt{1+\sqrt{1+\ldots+\sqrt{1}}}}. \end{aligned}$$

The last term is less than $2\sqrt{2}$.

5. (a) Let

$$b_n = \frac{a_n}{\left(\frac{3}{2}\right)^{n-1}}.$$

Then $b_{n+1} > b_n$, thus the sequence (b_n) is strictly increasing. Each increasing sequence has a finite limit or tends to infinity.
b) For all $a > 1$ there exists a sequence $1 = c_1 < c_2 < ...$ which converges to a. Choosing

$$a_n = c_n \left(\frac{3}{2}\right)^{n-1}.$$

we obtain the required sequence (a_n).

6. Cauchy-Schwarz for $x_n = \sqrt{a_n}, y_n = \sqrt{a_{n+1}}$. Or use the AM-GM inequality:

$$\sum_{n=1}^{\infty} \sqrt{a_{n+1}a_n} \leq \sum_{n=1}^{\infty} \frac{a_{n+1}+a_n}{2} = \sum_{n=1}^{\infty} \frac{a_{n+1}}{2} + \sum_{n=1}^{\infty} \frac{a_n}{2}.$$

7. Since the sequence of partial sums is Cauchy we have $a_{n+1} + ... + a_{2n} \to 0$. But a_n is decreasing i.e. $a_{n+1} + ... + a_{2n} \geq na_{2n}$. Hence $2na_{2n} \to 0$. For odd index follows by $a_{2n} \geq a_{2n+1}$.

8. Take $a_n = \frac{(-1)^{n+1}}{\log(n+1)}$.

9. No. By AM-GM we have

$$\sum \left(a_n + \frac{1}{n^2 a_n}\right) \geq \sum \frac{2}{n} = \infty.$$

10. Both series have nonnegative terms. If $\frac{\log(1/a_n)}{\log n} \leq 2$, then

$$\frac{a_n}{\log n} \leq \frac{2a_n}{\log(1/a_n)}.$$

If $\frac{\log(1/a_n)}{\log n} \geq 2$, then $a_n \leq 1/n^2$, hence

$$\frac{a_n}{\log n} \leq \frac{1}{n^2 \log n}.$$

Hence

$$\frac{a_n}{\log n} \leq \frac{2a_n}{\log(1/a_n)} + \frac{1}{n^2 \log n}.$$

11. For $M \in [2^L, 2^{L+1}]$, use $(1+x) \leq e^x$ to write

$$\sum_{n=1}^{M} a_n \leq (a_1 + ... + a_{2^L}) + (a_{2^L+1} + ... + a^{2^{L+1}})$$

$$\leq \left(1 + \frac{1}{2^L}\right)(a_1 + ... + a_{2^L}) \leq ... \leq \left(1 + \frac{1}{2^L}\right) ... \left(1 + \frac{1}{2}\right)(a_1 + a_2)$$

$$\leq 2a_1 e^{1/2+...+1/2^L} \leq 2ea_1.$$

12. Define $y_n = x_n^2$. Then $y_{n+1} = (y_n - 2)^2$ and $y_{n+1} - 4 = y_n(y_n - 4)$. We have $y_2 = 9 > 5$, hence inductively $y_n > 5$ for all $n \geq 2$. We get $y_{n+1} - y_n = y_n^2 - 5y_n + 4 > 4$ for all $n \geq 2$. Hence $y_n \to \infty$.

By $y_{n+1} - 4 = y_n(y_n - 4)$ we conclude

$$\begin{aligned}
\left(\frac{x_1 \cdot x_2 \cdot x_3 \cdots x_n}{x_{n+1}}\right)^2 &= \frac{y_1 \cdot y_2 \cdot y_3 \cdots y_n}{y_{n+1}} \\
&= \frac{y_1 \cdot y_2 \cdot y_3 \cdots y_n}{y_{n+1} - 4} \cdot \frac{y_{n+1} - 4}{y_{n+1}} = \frac{y_1 \cdot y_2 \cdot y_3 \cdots y_{n-1}}{y_n - 4} \cdot \frac{y_{n+1} - 4}{y_{n+1}} \\
&= ... = \frac{y_{n+1} - 4}{y_{n+1}} \cdot \frac{1}{y_1 - 4} \to 1.
\end{aligned}$$

13. The function $|\sin x|$ is periodic with period π. Hence

$$\lim_{n\to\infty} \left|\sin\left(\pi\sqrt{n^2+n+1}\right)\right| = \lim_{n\to\infty} \left|\sin\left(\pi(\sqrt{n^2+n+1} - n)\right)\right|.$$

But

$$\lim_{n\to\infty} \sqrt{n^2+n+1} - n) = \frac{1}{2}.$$

Hence

$$\lim_{n\to\infty} \left|\sin\left(\pi\sqrt{n^2+n+1}\right)\right| = |\sin(\pi/2)| = 1.$$

14. a) Obviously $a_2 < \sqrt{2} = a_1$. But the function $f(x) = \sqrt{2 - \sqrt{4 - x^2}}$ is increasing in $[0, 2]$. Hence a_n is decreasing. The limit satisfies the relation $f(x) = x$ hence the limit is equal to 0. Same for b_n.
b) The inequality $2f(x) > x$ for all $x \in (0, 2)$ implies $2^n a_n$ is strictly increasing. The inequality $2g(x) < x$ for all $x \in (0, 2)$ implies $2^n b_n$ is strictly decreasing. Use induction to prove that

$$a_n^2 = \frac{4b_n^2}{4 + b_n^2}.$$

This is decreasing.

The sequence $4^n(a_n)^2$ is increasing and

$$\lim 4^n(a_n)^2 = \lim \frac{44^n b_n^2}{4 + b_n^2} = \lim \frac{4}{4 + b_n^2} \lim 4^n(b_n)^2.$$

The last part follows from the equality

$$2^n(b_n - a_n) = \frac{(2^n b_n)^4}{4 + b_n^2} \cdot \frac{1}{4^n} \cdot \frac{1}{2^n(b_n + a_n)}.$$

15. Assume an n-digit such number. How many choices do we have? First digit cannot be 0 or 9

(hence 8 choices) and all the other digits cannot be 9 hence 9^{n-1} choices. Totally $8 \cdot 9^{n-1}$ choices. All are bounded by 10^n. We conclude

$$\sum_{x_k<10^n} \frac{1}{x_k} = \sum_{m=1}^{n} \sum_{x_k \in [10^{m-1}, 10^m)} \frac{1}{x_k} < \sum_{m=1}^{n} \frac{8 \cdot 9^{m-1}}{10^{m-1}} = 8 \cdot \sum_{m=1}^{n} \left(\frac{9}{10}\right)^{m-1} = 80$$

16. Clearly it converges for $x = 1$. For any other it diverges. To show this, assume $x \neq 1$. Then, applying the Mean Value theorem to $f_n(t) = t^{1/n}$ we find a c_n between 1 and x such that

$$\frac{\sqrt[n]{x} - 1}{x - 1} = \frac{1}{n} c_n^{1/n-1}.$$

Hence

$$\frac{\sqrt[n]{x} - 1}{x - 1} \geq \frac{1}{n}(\max(1, x))^{1/n-1} \geq \frac{1}{n}(\max(1, x))^{-1},$$

hence we compare it with the harmonic series.

17. Use calculations or AM-GM to prove that

$$\frac{1}{\sqrt{n}(n+1)} < \frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}},$$

hence, as a telescoping series, we get

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(n+1)} < \sum_{n=1}^{\infty} \left(\frac{2}{\sqrt{n}} - \frac{2}{\sqrt{n+1}}\right) < 2.$$

18. No. Assume $a_n = 1/n$ for n even and $1/n^2$ for n odd, and $b_n = 1/n$ for n odd and $1/n^2$ for n even.

19. Observe that

$$ka_k = \frac{(1 + (k+1)a_k)a_{k+1}}{a_k} = \frac{a_{k+1}}{a_k} + (k+1)a_{k+1}$$

for all $k \geq 1$. Hence, we have the telescopic sum

$$0 \leq \sum_{k=0}^{n} \frac{a_{k+1}}{a_k} = \frac{a_1}{a_0} + \sum_{k=1}^{n} (ka_k - (k+1)a_{k+1}) = \frac{1}{2} + a_1 - (n+1)a_{n+1} = 1 - (n+1)a_{n+1} \leq 1.$$

for all $n \geq 0$. Since all terms are positive, the sum is convergent. Hence $\frac{a_{n+1}}{a_n} \to 0$, i.e. there exists a n_0 such that for all $n \geq n_0$ we have $\frac{a_{n+1}}{a_n} \leq 1/2$. Induction proves there exists a constant $C > 0$ such that $0 \leq a_n \leq \frac{C}{2^n}$. This means $na_n \to 0$, hence

$$0 \leq \sum_{k=0}^{n} \frac{a_{k+1}}{a_k} = 1 - (n+1)a_{n+1} \to 1$$

as $n \to 0$.

20. Calculate many first terms and use induction.

21.

22. (IMC 2016, Problem 6) Interchange the sums to get

$$\sum_{k=1}^{\infty}\sum_{n=1}^{k}\frac{x_n}{k^2}=\sum_{n=1}^{\infty}\left(x_n\sum_{k=n}^{\infty}\frac{1}{k^2}\right).$$

We use telescopic series to get the upper bound

$$\sum_{k=n}^{\infty}\frac{1}{k^2}\leq\sum_{k=n}^{\infty}\frac{1}{k^2-\frac{1}{4}}\leq\sum_{k=n}^{\infty}\left(\frac{1}{k-\frac{1}{2}}-\frac{1}{k+\frac{1}{2}}\right)=\frac{2}{2n-1}$$

and we conclude the desired bound.

SEEMOUS AND IMC PREPARATION, DAY 2, 16/12/2021
SEQUENCES, SERIES AND INEQUALITIES

1. A bit more of theory

1.1. **Sequences.** A few more basic elements of the theory of real sequences.

1) Convergence of sequences can be deduced also by Cauchy criterion: a sequence is convergent if and only if it is Cauchy.

2) Search for patterns, especially if a_n is given by a recursive relation.

3) Root test, ratio test.

1.2. **Series.** More criteria of convergence:

1) Absolutely convergent series.

2) Integral criterion: Let f be a monotone decreasing function in the interval $[k, \infty)$. Then the series

$$\sum_{n=k}^{\infty} f(n)$$

converges if and only if

$$\int_k^{\infty} f(x)dx < \infty.$$

3) The Riemann zeta function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges for $s > 1$.

2. Exercises

Problems for Sequences

1. Let $k > 0$ be a real number. Compute the limit:

$$\lim_{n\to\infty} \frac{1^k + 2^k + 3^k + ... + n^k}{n^{k+1}}.$$

2. Let $x_0 \in (0, 1)$ and $x_{n+1} = x_n - x_n^2$ for $n \geq 0$. Compute $\lim_{n\to\infty} nx_n$.

3. Evaluate the infinite product

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right).$$

4. Evalueate the product

$$\prod_{n=3}^{\infty} \frac{(n^3+3n)^2}{n^6-64}.$$

Problems for Series

5. Prove that the series

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{e} - 1 \right)$$

diverges.

6. Prove that

$$\log(n+1) < \sum_{k=1}^{n} \frac{1}{k}.$$

7. Prove the inequalities

$$\sum_{n=1}^{\infty} \log\left(\frac{n^k+1}{n^k}\right) < \zeta(k)$$

for $k > 1$ and, for $a \in (0,1)$,

$$\sum_{n=1}^{\infty} \log\left(1+a^n\right) < \frac{a}{1-a}.$$

8. Prove that the limit

$$\gamma := \lim_{n\to\infty} \left(1 + \frac{1}{2} + ... + \frac{1}{n} - \log n\right)$$

exists.

9. Evaluate the sum

$$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)}.$$

10. Use an elementary method to prove that

$$\sum_{primes} \frac{1}{p}$$

diverges.

3. Solutions

1. Apply Cesáro-Stoltz theorem for $a_n = 1^k + 2^k + 3^k + ... + n^k$ and $b_n = n^{k+1}$, then the desired limit exists and is equal to

$$\lim_n \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_n \frac{(n+1)^k}{(n+1)^{k+1}-n^{k+1}} = \lim_n \frac{n^k + O(n^{k-1})}{(k+1)n^k + O(n^{k-1})} = \frac{1}{k+1}.$$

2. You can easily see by induction that $0 < x_n < 1$ for every n. Hence x_n is decreasing. Bounded and monotonic, x_n must converge to some limit x. Then $x = x - x^2$ gives $x = 0$. The Cesàro-Stolz theorem gives

$$\lim n x_n = \lim \frac{n+1-n}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} = \lim \frac{x_n - x_n^2}{x_n} = \lim 1 - x_n = 1.$$

3. Use induction to show that the product up to $n \leq N$ is equal to $\frac{N+1}{2N}$.

4. (IMC 2019, Problem 1) Notice that

$$\frac{(n^3+3n)^2}{n^6-64} = \frac{n}{n-2} \cdot \frac{n}{n+2} \cdot \frac{n^2+3}{(n-1)^2+3} \cdot \frac{n^2+3}{(n+1)^2+3}.$$

Thus we get

$$\prod_{n=3}^{N} \frac{(n^3+3n)^2}{n^6-64} = \frac{N(N-1)}{1 \cdot 2} \frac{3 \cdot 4}{(N+1)(N+2)} \frac{N^2+3}{2^2+3} \frac{3^3+3}{(N+1)^2+3} \to \frac{12^2}{2 \cdot 7} = \frac{72}{7}.$$

5. We know that

$$\left(1+\frac{1}{n}\right)^n \to e$$

from below, hence

$$\sqrt[n]{e} - 1 > \frac{1}{n}.$$

6. Using the elementary inequality $1 + x \leq e^x$ for $x \geq 0$, we get

$$\prod_{n=1}^{N} (1+a_n) < \exp\left(\sum_{n=1}^{N} a_n\right).$$

For $a_n = 1/n$ we telescopically get

$$N+1 < \exp\left(\sum_{n=1}^{N} \frac{1}{n}\right),$$

hence the result.

7. Work as in the previous exercise.

8. Use the graph of $\log x$ to deduce $1/2 + ... + 1/(n+1) < \log n < 1 + ... + 1/n$, i.e. $a_{n+1} - 1 < \log n < a_n$. Hence

$$\log n < a_n < a_{n+1} < \log n + 1.$$

The sequence $c_n = a_n - \log n$ is increasing iff

$$\frac{1}{n+1} + \log n - \log(n+1).$$

Study f' of the above function to deduce monotonicity.

9. Consider the function

$$F(x) = \sum_{k=0}^{\infty} \frac{x^{4k+4}}{(4k+1)(4k+2)(4k+3)(4k+4)}.$$

The series converges for $|x| \leq 1$ and we want to evaluate $F(1)$. Differentiating 4 times we get

$$F^{(4)}(x) = \sum_{k=0}^{\infty} x^{4k} = \frac{1}{1-x^4}.$$

We have $F(0) = F^{'}(0) = F^{''}(0) = F^{'''}(0) = 0$. Thus

$$F(1) = \int_{t=0}^{1} \int_{z=0}^{t} \int_{y=0}^{z} \int_{x=0}^{y} \frac{1}{1-x^4} dxdydzdt = ... = \frac{\log 2}{4} - \frac{\pi}{24}.$$

10. We have seen that $\sum 1/n^2 < 2$. We prove now that

$$\sum_{k=1}^{n} \frac{1}{k} \leq \prod_{p \leq n} \left(1 + \frac{1}{p}\right) \sum_{m=1}^{n} \frac{1}{m^2}.$$

To see this:

$$\sum_{k=1}^{n} \frac{1}{k} = \sum_{m=1}^{n} \frac{1}{m^2} + \sum_{m=1, p \leq n}^{n} \frac{1}{pm^2} + \sum_{m=1, p_1, p_2 \leq n}^{n} \frac{1}{p_1 p_2 m^2} + ...$$

and this last expression is bounded by

$$\prod_{p \leq n} \left(1 + \frac{1}{p}\right) \sum_{m=1}^{n} \frac{1}{m^2} < 2 \prod_{p \leq n} \left(1 + \frac{1}{p}\right) \leq 2 \prod_{p \leq n} e^{\frac{1}{p}} = 2 \exp\left(\sum_{p \leq n} \frac{1}{p}\right).$$

Then

$$\log\log(n+1) < \log\left(\sum_{k=1}^{n} \frac{1}{k}\right) \leq \log 2 + \sum_{p \leq n} \frac{1}{p}.$$

Hence

$$\sum_{p \leq n} \frac{1}{p} > \log\log(n+1) - 1.$$

Theory

Weierstrass Polynomial Approximation Theorem. Let $f(x)$ be a real-valued (or complex-valued) continuous function on $[0,1]$. There exists a sequence of polynomials $p_n(x)$, $n \in \mathbb{N}$, which converges, as $n \to \infty$, to $f(x)$ uniformly on $[0,1]$. For instance, we may take

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} f(k/n) x^k (1-x)^{n-k}.$$

Proof. (Taken from *Functional Analysis* - K. Yosida) By applying $x\partial_x$ and $x^2\partial_x^x$ to $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$ we get $nx(x+y)^n = \sum_{k=0}^n k\binom{n}{k} x^k y^{n-k}$ and $n(n-1)x^2(x+y)^n = \sum_{k=0}^n k(k-1)\binom{n}{k} x^k y^{n-k}$, respectively. Letting $r_k(x) = \binom{n}{k} f(k/n) x^k (1-x)^{n-k}$, we have

$$\sum_{k=0}^{n} r_k(x) = 1, \quad \sum_{k=0}^{n} k r_k(x) = nx \quad \text{and} \quad \sum_{k=0}^{n} k(k-1) r_k(x) = n(n-1)x^2. \tag{1}$$

Therefore,

$$\begin{aligned} \sum_{k=0}^{n} (k-nx)^2 r_k(x) &= n^2x^2 \sum_{k=0}^{n} r_k(x) - 2nx \sum_{k=0}^{n} k r_k(x) + \sum_{k=0}^{n} k^2 r_k(x) \\ &= n^2x^2 - 2nx \cdot nx + (nx + n(n-1)x^2) \quad = \quad nx(1-x). \end{aligned} \tag{2}$$

For any $\varepsilon > 0$, due to uniform continuity, there exists a $\delta > 0$ such that $|f(x) - f(\widetilde{x})| < \varepsilon$ whenever $|x - \widetilde{x}| < \delta$. Hence, if we choose $M > 0$ such that $|f(x)| \leq M$, $x \in [0,1]$, by (1) and (2) we deduce

$$\begin{aligned} |f(x) - \sum_{k=0}^{n} f(k/n) r_k(x)| &= |\sum_{k=0}^{n} (f(x) - f(k/n)) r_k(x)| \\ &\leq |\sum_{|k-nx| \leq \delta n} (f(x) - f(k/n)) r_k(x)| + |\sum_{|k-nx| > \delta n} (f(x) - f(k/n)) r_k(x)| \\ &\leq \varepsilon \sum_{|k-nx| \leq \delta n} r_k(x) + 2M \sum_{|k-nx| > \delta n} r_k(x) \\ &\leq \varepsilon \sum_{k=0}^{n} r_k(x) + \frac{2M}{n^2\delta^2} \sum_{k=0}^{n} (k-nx)^2 r_k(x) = \varepsilon + \frac{2Mx(1-x)}{n\delta^2} \leq \varepsilon + \frac{M}{2n\delta^2}. \end{aligned}$$

☐

Bernoulli inequality. For any $x \geq -1$ and $n \in \mathbb{N}$, we have $(1+x)^n \geq 1 + nx$.
Proof. Follows by induction. ☐

Young inequality. Let $p, q \in (1, \infty)$ satisying $p^{-1} + q^{-1} = 1$. Then, for any $a, b > 0$ we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Let $\rho \in (0,1)$ and $y > 0$. Let

$$g(x) = \rho \ln(x) + (1-\rho)\ln(y) - \ln(\rho x + (1-\rho)y), \quad x \geq y.$$

We have

$$g(y) = 0 \quad \text{and} \quad \partial_x g(x) = \frac{\rho}{x} - \frac{\rho}{\rho x + (1-\rho)y} = \rho \frac{(\rho - 1)(x-y)}{x(\rho x + (1-\rho)y)} \leq 0.$$

Therefore $g(x) \leq 0$, i.e.

$$\rho \ln(x) + (1-\rho)\ln(y) \leq \ln(\rho x + (1-\rho)y) \quad \text{when} \quad x \geq y.$$

Without loss of generality, setting $r = p^{-1}$, $x = a^p$ and $y = b^q$, we get

$$\frac{\ln(a^p)}{p} + \frac{\ln(b^p)}{q} \leq \ln\Big(\frac{a^p}{p} + \frac{b^q}{q}\Big).$$

The right hand side of the above inequality equals $\ln(ab)$, so that the result follows by the fact that the function $\ln(\cdot)$ is monotone increasing. ☐

Hölder inequality. Let $p, q \in (1, \infty)$ satisfying $p^{-1} + q^{-1} = 1$ and let $n \in \mathbb{N}$. For any $x, y \in \mathbb{C}^n$, $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, we have

$$\sum_{j=1}^{n} |x_j y_j| \leq \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{\frac{1}{p}} \Big(\sum_{j=1}^{n} |y_j|^p\Big)^{\frac{1}{q}}.$$

Proof. Setting

$$a = \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{\frac{1}{p}} \quad \text{and} \quad b = \Big(\sum_{j=1}^{n} |y_j|^p\Big)^{\frac{1}{q}},$$

from Young inequality we get

$$\sum_{j=1}^{n} \frac{|x_j|}{a}\frac{|y_j|}{b} \leq \frac{1}{p}\sum_{j=1}^{n}\Big(\frac{|x_j|}{a}\Big)^p + \frac{1}{q}\sum_{j=1}^{n}\Big(\frac{|y_j|}{b}\Big)^q \leq \frac{1}{p} + \frac{1}{q} = 1,$$

where we have assumed $a, b \neq 0$, since, otherwise, the result is trivial. ☐

Hölder integral inequality. Let Ω be a measurable subset in $\mathbb{R}^n$ and f, g be measurable real (or complex)-valued functions on Ω. Then

$$\int_\Omega |f(x)g(x)|dx \leq \Big(\int_\Omega |f(x)|^p dx\Big)^{1/p}\Big(\int_\Omega |g(x)|^q dx\Big)^{1/q},$$

where $p^{-1} + q^{-1} = 1$, in the sense that finiteness of the right side implies that fg is summable.

Minkowski inequality. Let $p \in (1, \infty)$ and $n \in \mathbb{N}$. For any $x, y \in \mathbb{C}^n$, $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, we have

$$\Big(\sum_{j=1}^{n} |x_j + y_j|^p\Big)^{\frac{1}{p}} \leq \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{\frac{1}{p}} + \Big(\sum_{j=1}^{n} |y_j|^p\Big)^{\frac{1}{p}}.$$

Proof. For any $j \in \{1, ..., n\}$ we have

$$|x_j + y_j|^p = |x_j + y_j||x_j + y_j|^{p-1} \leq |x_j||x_j + y_j|^{p-1} + |y_j||x_j + y_j|^{p-1},$$

where, by summing up from $j = 1$ to $j = n$ and then using Hölder's inequality, we obtan

$$\sum_{j=1}^{n} |x_j + y_j|^p \leq \Big(\Big(\sum_{j=1}^{n} |x_j|^p\Big)^{\frac{1}{p}} + \Big(\sum_{j=1}^{n} |y_j|^p\Big)^{\frac{1}{p}}\Big)\Big(\sum_{j=1}^{n} |x_j + y_j|^{q(p-1)}\Big)^{\frac{1}{q}},$$

where $p^{-1} + q^{-1} = 1$. The result follows since $q(p-1) = p$. ☐

Minkowski integral inequality. Let Ω be a measurable subset in $\mathbb{R}^n$, f, g be measurable real (or complex)-valued functions on Ω and $p \in [1, \infty)$. Then

$$\Big(\int_\Omega |f(x) + g(x)|^p dx\Big)^{1/p} \leq \Big(\int_\Omega |f(x)|^p dx\Big)^{1/p} + \Big(\int_\Omega |g(x)|^p dx\Big)^{1/p},$$

in the sense that finiteness of the right side implies that $f + g$ is summable.

Problems

(1) Show that there exists no continuous and onto map $f : [0,1] \to (0,1)$.

Solution. Assume that there exists such a map. Let $x_n \in [0,1]$ such that $0 < f(x_n) < 1/n$, $n \in \mathbb{N}$. By the Bolzano-Weierstrass theorem, let $\{x_{n_j}\}_{j\in\mathbb{N}}$ be a subsequence of $\{x_n\}_{n\in\mathbb{N}}$ that converges to some $x \in [0,1]$. By continuity we get that $f(x) = 0$, which yields a contradiction. ☐

(2) Let $a < b$, $a, b \in \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ be a continuous function such that $[a,b] \subseteq f([a,b])$. Show that the equation $f(x) = x$ has at least one solution in $[a,b]$.

Solution. Let $f([a,b]) = [m,M]$, for some $m, M \in \mathbb{R}$, $m \leq M$, and $x_m, x_M \in [a,b]$ such that $f(x_m) = m$ and $f(x_M) = M$. For the continuous function $g(x) = f(x) - x$ defined on $[a,b]$, we have $g(x_m) \leq 0$ and $g(x_M) \geq 0$. Hence, the result follows by the intermediate value theorem. ☐

(3) Let $f : [-1,1] \to \mathbb{R}$ be continuous satisfying $f(2x^2 - 1) = 2xf(x)$, for all $x \in [-1,1]$. Show that f equals to zero identically on $[-1,1]$.

Solution. Define $g(t) = f(\cos(t))/\sin(t)$, $t \in \mathbb{R}\backslash\{k\pi \mid k \in \mathbb{Z}\}$. We have $g(t+\pi) = g(t)$ and, moreover,

$$g(2t) = \frac{f(2\cos^2(t) - 1)}{\sin(2t)} = \frac{2\cos(t)f(\cos(t))}{\sin(2t)} = g(t).$$

In particular,

$$g(1 + \frac{k\pi}{2^m}) = g(2^m + k\pi) = g(2^m) = g(1), \quad \text{for all} \quad k, m \in \mathbb{Z}.$$

Therefore, g is constant on $S = \{1 + \frac{k\pi}{2^m} \mid k, m \in \mathbb{Z}\}$. Noting that S is dense in $\mathbb{R}$, we get that g is constant on $\mathbb{R}$. Since g is odd, we conclude that $g = 0$ on $\mathbb{R}$. The result now follows by noting that $f(0) = f(1) = 0$, due to the functional equation of f. ☐

(4) Show that for all $p \geq 1$ and $a, b \geq 0$ we have

$$a^p + b^p \leq (a+b)^p \leq 2^{p-1}(a^p + b^p).$$

Solution. Let $f : [0,1] \to \mathbb{R}$ defined by $f(x) = (1+x)^p/(1+x^p)$. We have $f'(x) \geq 0$, for all $x \in [0,1]$. This implies $1 = f(0) \leq f(x) \leq f(1) = 2^{p-1}$, $x \in [0,1]$. Without los of generality, we may assume that $b \neq 0$ and $a \leq b$. Then, the result follows by setting $x = a/b$ to the above inequality. ☐

(5) Let $a \in \mathbb{R}$ and $f : (a,\infty) \to \mathbb{R}$ be a nonconstant bounded twice differentiable function such that $f' \geq 0$ and $f'' \geq 0$ on (a,∞). Show that $\lim_{x\to\infty} f(x) = 0$.

Solution. We have that f and f' are nondecreasing on (a,∞). Let $\ell = \lim_{x\to\infty} f(x)$. Suppose that $\ell \neq 0$. By l' Hospital rule we have

$$0 = \lim_{a\to\infty} \frac{f(x)}{x} = \lim_{a\to\infty} f'(x).$$

Since $f' \geq 0$ is nondecreasing, it follows that $f' = 0$ on (a,∞), i.e. f is constant map, which is a contradiction. ☐

(6) (IMC 2002) Does there exist a continuously differentiable function $f : \mathbb{R} \to (0,+\infty)$ satisfying $f' = f \circ f$?

Solution. Suppose that such a function exists. Then, it has to be increasing. Therefore, $f(x) > 0$ implies $f(f(x)) > f(0)$ for all $x \in \mathbb{R}$. That is $f(0)$ is a lower bound for f'. By mean value theorem, for $x < 0$, we have

$$\frac{f(x)-f(0)}{x-0} = f'(\xi) > f(0)$$

for certain $\xi \in (x,0)$, which implies $f(x) < (x+1)f(0)$. By choosing $x < -1$ we get a contradiction. ☐

(7) (Math. Magazine, Problem 1005) Find all differentiable functions f and g defined on $(0,+\infty)$ that satisfy

$$f'(x) = -\frac{g(x)}{x} \quad \text{and} \quad g'(x) = -\frac{f(x)}{x} \quad \text{for all} \quad x > 0.$$

Solution. We have

$$(x(f(x)+g(x)))' = xf'(x) + xg'(x) + f(x) + g(x) = 0.$$

Hence, there exists a $c_0 \in \mathbb{R}$ such that $f(x) + g(x) = 2c_1/x$, for all $x > 0$. Furthermore,

$$\Big(\frac{f(x)-g(x)}{x}\Big)' = \frac{xf'(x) - xg'(x) - f(x) + g(x)}{x^2} = 0.$$

Thus, there exists some $c_1 \in \mathbb{R}$ such that $f(x) - g(x) = 2c_1 x$. Summarising we obtain

$$f(x) = c_0 x^{-1} + c_1 x \quad \text{and} \quad g(x) = c_0 x^{-1} - c_1 x \quad \text{for all} \quad x > 0.$$

☐

(8) Find all integers a and b sucht hat $0 < a < b$ and $a^b = b^a$.

Solution. Let $f(x) = \ln(x)/x$, $x > 0$. Then, $a^b = b^a$ if and only if $f(a) = f(b)$. We have $f'(x) = (1-\ln(x))/x^2$, so that f increases on $(0,e)$ and decreases on (e,∞). So, for $f(a) = f(b)$ to be true, we must have $a \in (0,e)$ (i.e. $a \in \{1,2\}$) and $b > e$. The case of $a = 1$ gives no solution. On the other hand, for $a = 2$ we get a solution with $b = 4$. Due to monotonicity of f, this is the unique solution. ☐

(9) (Harvard-MIT Mathematics Tournament, 2002) Suppose that f is a differentiable real function satisfying $f(0) = 0$ and $f(x) + f'(x) \leq 1$ for all $x \in \mathbb{R}$. What is the largest possible value of $f(1)$?

Solution. Consider the function $g(x) = e^x(f(x)-1)$, $x \in \mathbb{R}$. We have $g'(x) = e^x(f(x)+f'(x)-1) \leq 0$ for all $x \in \mathbb{R}$. In particular,

$$e(f(1)-1) = g(1) \leq g(0) = -1, \tag{3}$$

i.e. $f(1) \leq 1 - e^{-1}$. On the other hand, if f is a function for which $f(1)$ achieves its largest possible value, then by (3) and the monotonicity of g, we have that g must be constant on $[0,1]$, i.e. $g(x) = g(1)$ for $x = [0,1]$. This implies that $f(x) = 1 - e^{-x}$, $x \in \mathbb{R}$. ☐

(10) Let $f : [1,\infty) \to \mathbb{R}$ be such that $f(1) = 1$ and

$$f'(x) = \frac{1}{x^2 + f^2(x)}, \qquad x \geq 1.$$

Show that the limit $\lim_{x\to\infty} f(x)$ exists and is $\leq 1 + \frac{\pi}{4}$.

Solution. We have that f is increasing on $[1,\infty)$, so that $f(t) \geq f(1) = 1$, $t \geq 1$. Hence,

$$f'(t) \leq \frac{1}{1+t^2}, \qquad t \geq 1.$$

Therefore, for each $x \geq 1$ we have

$$f(x) = f(1) + \int_1^x f'(t)dt \leq 1 + \int_1^x \frac{1}{1+t^2}dt \leq 1 + \int_1^\infty \frac{1}{1+t^2}dt = 1 + \frac{\pi}{4}.$$

Thus, $\lim_{x\to\infty} f(x)$ exists and is $\leq 1 + \frac{\pi}{4}$. □

(11) Let $f : [0, +\infty)$ be a continuous function. Show that

$$\int_0^x \Big(\int_0^y f(t)dt \Big) dy = \int_0^x (x-y) f(y) dy, \quad \text{for all} \quad x \geq 0.$$

Solution. For any $x \geq 0$, by integrating by parts, we get

$$\int_0^x \Big(\int_0^y f(t)dt \Big) dy = \int_0^x (y)' \Big(\int_0^y f(t)dt \Big) dy = x \int_0^x f(y)dy - \int_0^x y f(y) dy.$$

□

(12) (IMC, 1998) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function satisfying $xf(y) + yf(x) \leq 1$ for all $x, y \in [0, 1]$. Show that

$$\int_0^1 f(x)dx \leq \pi/4.$$

In addition, show that the above bound is optimal.

Solution. We have

$$J = \int_0^1 f(x)dx = \int_0^{\frac{\pi}{2}} f(\cos(t)) \sin(t) dt = \int_0^{\frac{\pi}{2}} f(\sin(t)) \cos(t) dt.$$

By summing up, we deduce

$$2J \leq \int_0^{\frac{\pi}{2}} 1 dt = \pi/2.$$

Equality holds with $f(x) = \sqrt{1 - x^2}$. □

(13) (Math. Magazine, Problem Q887) Show that

$$\sin(x) \ln \Big(\frac{1 + \sin(x)}{1 - \sin(x)} \Big) \geq 2x^2, \quad x \in (-\pi/2, \pi/2).$$

Solution. Since both sides of the inequality are even functions, it suffices to consider the case $x \in [0, \pi/2)$. We have

$$\int_0^x \cos(t) dt = \sin(x) \quad \text{and} \quad \int_0^x \frac{1}{\cos(t)} dt = \frac{1}{2} \ln \Big(\frac{1 + \sin(x)}{1 - \sin(x)} \Big).$$

Then, by Cauchy-Schwarz inequality (i.e. Hölder for $p = 2$), namely

$$\Big(\int_0^x f(t)g(t)dt \Big)^2 \leq \Big(\int_0^x f^2(t)dt \Big) \Big(\int_0^x f^2(t)dt \Big),$$

with $f(t) = \sqrt{\cos(t)}$ and $g(t) = 1/\sqrt{\cos(t)}$, we obtain the result. □

SEEMOUS AND IMC PREPARATION, DAY 4, 20/01/2022
BASIC LINEAR ALGEBRA

1. BASIC KNOWLEDGE

Vector spaces, matrices, diagonalization, eigenvalues.

2. EXERCISES

Warm-up problems

1. Determine

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}^n .$$

2. Determine

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^n .$$

3. Let A, B and C be real square matrices of the same size, and suppose A is invertible. Prove that if $(A-B)C = BA^{-1}$, then $C(A-B) = A^{-1}B$.

4. Let A and B be real symmetric matrices with all eigenvalues strictly greater than 1. Let λ be a real eigenvalue of matrix AB. Prove that $|\lambda| > 1$.

5. Let $A = (a_{ij})_{n\times n}$ be the $n \times n$ matrix with $a_{ij} = i + j$ for all $i, j = 1, 2, ..., n$. What is the rank of A?

More difficult problems

6. In the linear space of all real $n \times n$ matrices, find the maximum possible dimension of a linear subspace V such that

$$\forall X, Y \in V \Rightarrow tr(XY) = 0.$$

7. Let n be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the main diagonal and strictly positive real numbers off the main diagonal.

8. Call a polynomial $P(x_1, x_2, ..., x_k)$ good if there exist 2×2 matrices $A_1, ..., A_k$ such that

$$P(x_1, x_2, ..., x_k) = \det\left(\sum_{i=1}^{k} x_i A_i\right).$$

Find all values of k for which all homogeneous polynomials with k variables of degree 2 are good.

Solutions

1. Write $A = \lambda I + B$. Deduce

$$A^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}.$$

2. Write $A = J - I$ where J is the matrix having 1 in every entry. Prove $J^m = 3^{m-1}J$ for every $m \geq 1$. Use binomial theorem to deduce

$$A^n = (-1)^n I + \frac{1}{3} J \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} 3^k$$

$$= (-1)^n I + \frac{2^n + (-1)^{n-1}}{3} J.$$

3.The relation $(A - B)C = BA^{-1}$ is equivalent with $AC - BC - BA^{-1} + AA^{-1} = I$. This is equivalent with $(A - B)(C + A^{-1}) = I$. Hence $(C + A^{-1})(A - B) = I$ also holds. Expand this to deduce the desired relation.

4. The transforms given by A and B strictly increase the length of every nonzero vector, this can be seen in a basis where the matrix is diagonal with entries greater than 1 in the diagonal. Hence their product AB also strictly increases the length of any nonzero vector, and therefore its real eigenvalues are all greater than 1 or less than -1

5. For $n = 1$ the rank is 1. Now assume $n \geq 2$. Since $A = (i)_{i,j}^n + (j)_{i,j}^n$ is the sum of two matrices of rank 1, we conclude $rank(A)$ is at most 2. The rank of the top-let 2×2 minor is -1, so the rank is exactly 2.

6. If A is a nonzero symmetric matrix, then $tr(A^2) = tr(A^T A)$ is the sum of the squared entries of A which is positive. Hence, V does not contain any symmetric matrix except 0.

Denote by S the linear space of all symmetric matrices. Then $\dim S = \frac{n(n+1)}{2}$. Since $V \cup S = \{0\}$, we have $\dim V + \dim S \leq n^2$ hence $\dim V \leq n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

The space of strictly upper triangular matrices has dimension $\frac{n(n-1)}{2}$ and satisfies the condition of the problem. Therefore the maximum dimension of V is $\frac{n(n-1)}{2}$.

7. For $n = 1$ we have $rank = 0$. For $n = 2$ the determinant of such a matrix is negative, so the rank is 2. We prove that for $n \geq 3$ the minimal rank is 3. Notice that the first three rows are linearly independent. This can been proved as follows. Assume a linear combination of them with coefficients c_1, c_2, c_3. From the first column we have c_2, c_3 have opposite signs or are both zero. Same applies to (c_1, c_2) and (c_1, c_2). Hence they are all zero.

To prove the rank is at most 3, we consider the matrix

$$A = ((i - j))_{i,j=1}^n,$$

which has rank at most 3.

8. The possible values for k are 1 and 2.

If $k = 1$ then $P(x) = ax^2$ and we can choose

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}.$$

If $k = 2$, then $P(x, y) = ax^2 + by^2 + cxy$ and we can choose

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & b \\ -1 & c \end{pmatrix}.$$

If $k \geq 3$, then $P(x_1, x_2, ..., x_k) = \sum_{i=0}^{k} x_i^2$ is not good. Suppose that

$$P(x_1, x_2, ..., x_k) = \det\left(\sum_{i=1}^{k} x_i A_i\right).$$

Since the first columns of $A_1, ..., A_k$ are linearly independent, the first column of some non-trivial linear combination $y_1A_1 + ... + y_kA_k$ is zero. Then $\det(y_1A_1 + ... + y_kA_k) = 0$, but $P(y_1, ..., y_k) \neq 0$, a contradiction.

PROBLEM SOLVING SEMINAR 2/12/2021
NUMBER THEORY DAY 1

1. Basic knowledge

In Number Theory we deal with problems related with the natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$ (or the integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$) and functions $f : \mathbb{N} \to \mathbb{N}$ or $f : \mathbb{Z} \to \mathbb{Z}$. The prerequisites we want are little, but problems can be extremely difficult.

1.1. **Divisibility.**

One of the most basic notions in number theory is that of divisibility.

Definition 1.1. Let a, b be integer numbers. We say that a divides b (and we write $a|b$) if there exists a number $c \in \mathbb{Z}$ such that $b = ac$.

If an integer number n is divided by an integer number $a > 1$, then we call it composite. Otherwise, we call it prime.

Proposition 1.2. *In what follows, we denote a, b, c denote integer numbers and p denotes a prime number. Then, we have the following properties:*

- *If $a|b$ and $b|c$ then $a|c$.*
- *If $a|b$ and $a|c$ then $a|bx + cy$ for every $x, y \in \mathbb{N}$.*
- *If $p|ab$ then $p|a$ or $p|b$.*

Theorem 1.3 (Division with remainder)**.** *Assume a and b are natural numbers. Then there exists unique natural numbers q, r with $0 \leq r < b$ such that*

$$a = bq + r.$$

There are infinitely many prime numbers (if you do not know a proof of this theorem, see exercise 4). We denote them by $p_1 = 2, p_2 = 3, ...$ and by $\mathcal{P} = \{2, 3, ..\}$ the set of the prime numbers. Further, we have the following main theorem.

Theorem 1.4 (Unique factorization theorem)**.** *Every natural number n has a unique factorization of the form*

$$n = p_1^{a_1} p_2^{a_2} ... p_k^{a_k}$$

with $a_i \geq 1$.

We denote by $\gcd(a, b)$ the geatest common divisor of a and b and by $\mathrm{lcm}(a, b)$ their least common multiple. We call them relatively prime or coprime if $\gcd(a, b) = 1$. Note that $\gcd(a, b) = \gcd(a - b, b)$. Hence, if $a = bq + r$ then $\gcd(a, b) = \gcd(r, b)$. Use the Unique factorization theorem to prove the following proposition.

Proposition 1.5. *For every a, b we have*

$$\gcd(a, b) \cdot \mathrm{lcm}(a, b) = a \cdot b.$$

Moreover, we have the following property of the geatest common divisor.

Proposition 1.6. *Fix two natural numbers a, b. Then, there exists integers x, y such that*

$$ax + by = \gcd(a, b).$$

1.2. **Mod arithmetic and properties.**

Definition 1.7. If $a|b-c$ we write $b \equiv c \pmod a$ and we say that b is congruent to c mod a.

Notice that, if we write $b = aq_1 + r_1$ and $c = aq_2 + r_2$ with $0 \leq r_1, r_2 < a$, then $b \equiv c \pmod a$ iff $r_1 = r_2$. Congruences can be added, subtracted and multiplied. We have the following proposition.

Proposition 1.8 (Properties of congruences)**.** *Let* a, b, c, d, k, m *be integer numbers such that* $a \equiv b \pmod m$ *and* $c \equiv d \pmod m$*. Then:*

- $a \pm c \equiv b \pm d \pmod m$.
- $ka \equiv kb \pmod m$.
- $a^k \equiv b^k \pmod m$.
- $f(a) \equiv f(b) \pmod m$ *for every polynomial* $f(x)$ *with integer coefficients.*

Further, if $\gcd(n, m) = 1$ *then* $na \equiv nb \pmod m$ *implies* $a \equiv b \pmod m$.

1.3. **Induction.**

Our intuition about the natural numbers is fundamentally based on the following principle.

Theorem 1.9. *Assume a property* $\mathcal{P}(n)$ *defined for natural numbers* n *such that:*

- $\mathcal{P}(1)$ *is true.*
- *If* $\mathcal{P}(n-1)$ *is true then* $\mathcal{P}(n)$ *is true.*

Then $\mathcal{P}(n)$ *is true for every* $n \in \mathbb{N}$.

This theorem can be proved using the axiom that every non-empty subset of $\mathbb{N}$ has a minimum element.

1.4. **Arithmetic functions.**

There is a whole zoo of arithmetic functions (i.e. functions $f : \mathbb{N} \to \mathbb{N}$ or $f : \mathbb{Z} \to \mathbb{Z}$) with nice properties. An arithmetic function $f : \mathbb{N} \to \mathbb{N}$ is called multiplicative if

$$\gcd(m, n) = 1 \Longrightarrow f(mn) = f(m)f(n).$$

An arithmetic function f is called totally multiplicative if for every m, n we have

$$f(mn) = f(m)f(n).$$

Definition 1.10. The most important arithmetic functions are the following ones:

1) The divisor function $d(n)$ defined as:

$$\sum_{d|n} 1.$$

2) The sum of divisors $\sigma(n)$ defined as:

$$\sum_{d|n} d.$$

A number n is called perfect iff $\sigma(n) = 2n$.

3) The Euler function $\phi(n)$ defined as the number of integers $0 \leq k \leq n$ such that $\gcd(k, n) = 1$.

4) The Möbius function $\mu(n)$ is defined as $\mu(1) = 1$, $\mu(n) = (-1)^k$ if $n = p_1 ... p_k > 1$ is squarefree and $\mu(n) = 0$ otherwise.

5) The function $r_k(n)$ counts the number of ways n can be written as a sum of k squares.

Theorem 1.11 (Fermat-Euler)**.** *If* $\gcd(a, m) = 1$ *then*

$$a^{\phi(m)} \equiv 1 \pmod{m}.$$

In particular, if p is a prime not dividing a then

$$a^{p-1} \equiv 1 \pmod{p}.$$

1.5. **Formulas for arithmetic functions.** Some basic formulas.

The function $\phi(n)$ can be calculated using the following formula: if $n = p_1^{a_1}...p_k^{a_k}$, then

$$\phi(n) = n \prod_{p_i} \left(1 - \frac{1}{p_i}\right).$$

We also have

$$d(n) = \prod_i (a_i + 1)$$

and

$$\sigma(n) = \prod_{p_i} \left(\frac{p_k^{a_k+1} - 1}{p_i - 1} \right).$$

Finally, we denote by $[x]$ the integral part of x and by $\{x\}$ the fractional part of x.

1.6. **Sums of two squares.** There are many well-known facts about $r_k(n)$, and many things that we don't know yet. Lagrange's 4-squares theorem states that $r_4(n) > 0$ for every n. On the other hand there is a nice formula for $r_2(n)$ which is more complicated. Fermat proved that a prime p can be written as a sum of two squares iff it is $1(\mod 4)$.

It turns out that a closed formula for $r_2(n)$ is given by

$$r_2(n) = 4 \prod_{i=1}^{k} (a_i + 1)$$

where $n = 2^k p_1^{a_1}...p_k^{a_k} q_1^{2b_1}...q_m^{2b_m}$, with $p_i \equiv 1 \mod 4$ and $q_i \equiv 3 \mod 4$.

1.7. **Pell's equation.** Pell's equation is given by

$$x^2 - Dy^2 = 1.$$

Lagrange proved the fundamental theorem that if $D > 0$ is not a perfect square, then Pell's equation has infinitely many solutions in positive integers and the general solution (x_n, y_n) is computed from the relation

$$(x_n, y_n) = \left(x_1 + \sqrt{D} y_1\right)^n,$$

where (x_1, y_1) is the fundamental solution (the minimal solution different from the trivial solution $(1, 0)$).

1.8. **Descent method.** Fermat used the descent method to prove that there exists no solutions to $x^4 + y^4 = z^4$.

The main idea is the following: assume there is a solution to the given equation. Then construct one 'smaller' solution, which leads to a contradiction.

2. Warm-up problems

Use the above theory and techniques to solve the following warm-up problems.

1) Which one is larger, 444^{555} or 555^{444}?

2) If a and b are odd numbers, then $a^2 + b^2$ is not a square.

3) Prove that if n is not a prime, then $2^n - 1$ is not a prime. Conclude that for every natural number $n \geq 3$, at least one of $2^n - 1$ and $2^n + 1$ is composite.

4) If $n|(n-1)! + 1$ then n is a prime.

5) Prove that if $p \leq n$ then $p \nmid (n! + 1)$. Deduce there are infinitely many primes.

6) Can we find a polynomial $f(x)$ with integer coefficients such $f(231) = 554$ and $f(161) = 496$?

7) If $k \geq 1$ then $k(k+1)$ is not a power > 1.

8) For every $a, b \in \mathbb{N}$ we have

$$\gcd(a,b) \cdot \operatorname{lcm}(a,b) = a \cdot b.$$

9) Fix two natural numbers a, b. Then, there exists integers x, y such that

$$ax + by = \gcd(a,b).$$

10) Let r be a real number such that $r + r^{-1} \in \mathbb{N}$. Prove that for every $n \in \mathbb{Z}$, $r^n + r^{-n} \in \mathbb{N}$.

11) Let a, b be integers and p be a prime. If $p|ab$ then $p|a$ or $p|b$.

12) If n is a sum of two squares, then also $2n$ is.

13) Find all $n \in \mathbb{N}$ such that $[\sqrt{n}] \mid n$.

3. Challenging problems

More difficulet problems.

14) Prove that for any natural number $n \neq 2, 6$ we have

$$\phi(n) \geq \sqrt{n}.$$

15) Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers.

16) Prove that for no integer $n > 1$ does n divide $2^n - 1$.

17) Show that the equation

$$x^2 + 10y^2 = 3z^2$$

has no solution in the positive integers.

18) Let $m, n \in \mathbb{N}$ such that mn divides $m^2 + n^2 + m$. Then m is a square number.

19) Find all n such that $d(n) = n$ or $d(n)^2 = n$.

20) Prove that the product of four consecutive natural numbers cannot be the square of an integer.

21) Let $f(x)$ be a polynomial with all coefficients being natural numbers. Can we find $f(x)$ by determining only two values of $f(n), f(m)$ for two integers n, m?

22) If n is an integer, then

$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} \notin \mathbb{Z}.$$

23) Prove that there exists no n such that $\phi(n) = 14$. Are there infinitely many m such that $\phi(n) = m$ has no solution?

24) Show that if $n > 1$

$$\sum_{d|n} \mu(d) = 0.$$

25) We denote by $[x]$ the integral part of x. Prove that if $n \in \mathbb{N}$ and $a \geq 0$ real then

$$\sum_{k=0}^{n-1} \left[a + \frac{k}{n} \right] = [na].$$

26) Prove that if $n \in \mathbb{N}$ then

$$\sum_{k=0}^{\infty} \left[\frac{n + 2^k}{2^{k+1}} \right] = n.$$

27) Prove that for every $n \in \mathbb{N}$

$$\frac{1}{\zeta(2)} < \frac{\sigma(n)\phi(n)}{n^2} \leq 1.$$

28) For every $n > 1$ let $p(n)$ denote the number of partitions of n, i.e. the number of ways we can write n as a sum of natural numbers. For instance $p(3) = 3, p(4) = 5, ...$ Prove that $p(n) - p(n-1)$ is the number of ways to express n as a sum of integers each of which is strictly greater than 1.

29) Let $f : \mathbb{N} \to \mathbb{N}$ such that $f(2) = 2$, $f(mn) = f(m)f(n)$ and $f(m) > f(n)$ if $m > n$. Prove that $f(n) = n$.

30) Let $n > 6$ be a perfect number, and let $n = p^{a_1}...p^{a_k}$ its prime factorization, with $p_1 < ... < p_k$. Prove that a_1 is an even number.

31) Let A be the set of natural numbers representable in the form $a^2 + 2b^2$ for some integers a and b with $b \neq 0$. Show that if $p^2 \in A$ for a prime p, then $p \in A$.

32) A prime number p cannot be written as a sum of two squares in two different ways.

33) Is there a polynomial $f(x)$ with integer coefficients such that $f(n)$ is prime for every $n \in \mathbb{N} \cup \{0\}$?

34) Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2).$$

35) Let $f(x)$ be a polynomial of degree 2 with integer coefficients. Suppose that $f(k)$ is divisible by 5 for every integer k. Prove that all coefficients of f are divisible by 5.

36) If $29 \mid (x^4 + y^4 + z^4)$ then $29^4 \mid (x^4 + y^4 + z^4)$.

37) If $m = 1...12...25$ with n ones and $n + 1$ twos, then m is a perfect square.

38) If a, b, c are natural numbers satisfying $a^2 + b^2 + 1 = c^2$ then the quantity

$$\left[\frac{a}{2}\right] + \left[\frac{c}{2}\right]$$

is even.

39) If $n > 1$ then $n^4 + 4^n$ is not a prime.

40) Prove that n is perfect iff

$$\sum_{d|n} \frac{1}{d} = 2.$$

41) Show that if n has $p - 1$ digits all equal to 1, where $p > 7$ is a prime, then n is divisible by p.

43) Assume $n \geq 2$. If $f(n) = 2^n + n^2$ is prime, then $3|f(f(n))$.

44) Let m and n be positive integers such that $\operatorname{lcm}(m, n) + \gcd(m, n) = m + n$. Prove that one of the two numbers is divisible by the other.

45) Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$(f(m) + n)(m + f(n))$$

is a perfect square for all $m, n \in \mathbb{N}$.

46) Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n - 1$ divides $2^n - 1$, and $n - 2$ divides $2^n - 2$.

4. Solutions

We give here either full solutions or hints for the exercises.

1) It suffices to compare 444^5 with 555^4, i.e. $4^5 \times 111^5$ with $5^4 \times 111^4$, i.e. 111×4^5 with 5^4, and now the answer is obvious.

2) Since a and b are odds, we have $a^2 + b^2$ is even. Since $a \equiv 1 \pmod 2$ we get $a^2 \equiv 1 \pmod 4$. Hence we have $a^2 + b^2 \equiv 2 \pmod 4$. If $a^2 + b^2$ was a even square we should have $a^2 + b^2 \equiv 0 \pmod 4$, which is a contradiction.

3) Assume $a|n$ with $a > 1$. Then, using mod arithmetic: $2^a - 1|2^n - 1$ and $2^a - 1 > 1$.

4) Assume n is not prime. Then it has a prime divisor $p < n$. Then $p|(n-1)! + 1$ and $p|(n-1)!$ hence $p|1$, contradiction.

5) Obviously if $p < n$ then $p|n!$ and $\gcd(n!, n!+1) = 1$. If there were finite $p_1 < ... < p_k < N$, then none of them would divide $N! + 1$, contradiction.

6) We have $f(a) \equiv f(b) \mod (a-b)$. If there was such a polynomial we would deduce $554 \equiv 496 \mod 70$ i.e. $70|58$, contradiction.

7) Assume $k(k+1) = m^n$. Since $\gcd(k, k+1) = 1$, we have $k = a^n$, $k+1 = b^n$, hence $b^n - a^n = 1$, contradiction.

8) Standard theory, using prime factorization of a and b.

9)

10) Use induction in n.

11)

12) Notice that if $n = a^2 + b^2$ then $2n = (a+b)^2 + (a-b)^2$.

13) Assume $[\sqrt{n}] = k$. Then $k \leq \sqrt{n} < k+1$, i.e. $k^2 \leq n < k^2 + 2k + 1$. The condition $k|n$ means $n = k(k+1)$ or $n = k(k+2)$. These are exactly the integers satisfying the given condition.

14) Use the formula for $\phi(n)$ to deduce $\phi(n) = p_1^{a_1-1}(p_1 - 1)...p_k^{a_k-1}(p_k - 1)$ and the fact that

$$p_1^{a_i-1}(p_i - 1) \geq p_i^{a_i/2}.$$

To see this, check separately $a_i = 1$ and $a_i \geq 2$.

15) First solution: Assume solutions: $y^2 + y^2, x^2 + 0, x^2 + 1$. Solve Pell equation for $2y^2 + 1 = x^2$).

Second solution: Because $(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$, you can take any triplet (except $0, 1, 2$ which will give itself) and generate a new triplet. If $n, n+1, n+2$ is the chosen triplet then the following is also a triplet:

$$n(n+2) = n^2 + 2n, \quad (n+1)^2 + 0^2 = n^2 + 2n + 1, \quad (n+1)^2 + 1^2 = n^2 + 2n + 2.$$

Third solution: the numbers

$$4n^4 + 4n^2, \quad 4n^4 + 4n^2 + 1, \quad 4n^4 + 4n^2 + 2.$$

16)

17) Work mod 3. Then $x^2 + y^2 = 0 \mod 3$, giving $x = y = 0 \mod 3$. Then $x = 3u, y = 3v$ gives $3u^2 + 30v^2 = z^2$ hence $z = 0 \mod 3$, and the result follows using descent method.

18)

19)

20) Assume $n > 1$ and consider the product $(n-1)n(n+1)(n+2) = (n^2+n-1)^2–1$.

21)

22) Assume $n >$. Write the given sum as

$$1 + \frac{1}{2} + \frac{1}{3} + ... \frac{1}{n} = \frac{n!/1 + n!/2 + ... + n!/n}{n!}$$

Let m be the integer such that $2^m | n!$ but $2^{m+1} \nmid n!$ and s the integer such that $2^s \leq n$ but $2^{s+1} > n$. Since $n > 3$ we get $m > s > 0$. The only integer in the set $\{1, 2, ..., n\}$ divisible by 2^s is 2^s itself. We conclude that for $1 \leq k \leq n$

$$2^{m-s} \mid \frac{n!}{k},$$

and for every $k \neq 2^s$

$$2^{m-s+1} \mid \frac{n!}{k}.$$

Hence

$$\frac{n!/1 + n!/2 + ... + n!/n}{n!} = \frac{2^{m-s}(2t+1)}{2^m k} = \frac{2t+1}{2^s k}.$$

This is not an integer.

Second solution: Using Bertrand's Lemma. Pick the largest prime $p < n$ and assume

$$1 + \frac{1}{2} + \frac{1}{3} + ... \frac{1}{n} = k \in \mathbb{Z}.$$

By Bertrand's lemma $n < 2p$.

23)

24) For $n = 1$ it is obvious. Assume $n > 1$. Then, notice that the divisors $d|n$ contributing to the sum are only the squarefree divisors. Write $n = p^{a_1}...p^{a_k}$. Then

$$\sum_{d|n} \mu(n) = \sum_{m=1}^{k} \binom{k}{m} (-1)^k = (1-1)^k = 0.$$

25)

26)

27)

28) (IMC 2012) Let $\mathcal{P}_n = \{(a_1, ...a_k) : a_1 + ... + a_k = n\}$ denote the set of partitions of n and $\mathcal{Q}_n$ the set of partitions of n including 1 i.e. $\mathcal{Q}_n = \{(a_1, ...a_k, 1) : a_1 + ... + a_k + 1 = n\}$. We have to prove

$$\#\mathcal{P}_n - \#\mathcal{P}_{n-1} = \#(\mathcal{P}_n \backslash \mathcal{Q}_n).$$

It suffices to prove that $\#\mathcal{P}_{n-1} = \#\mathcal{Q}_n$ i.e. to find a bijection $\phi : \mathcal{P}_{n-1} \to \mathcal{Q}_n$. Define

$$\phi : (a_1, ...a_k) \to (a_1, ...a_k, 1).$$

29) We have $2 = f(2) = 2f(1)$ hence $f(1) = 1$. Further

$$f(2^k) = f(2 \cdot 2^{k-1}) = 2f(2^{k-1}) = ... = 2^k$$

by induction. For $n = 2^k + j \in [2^k, 2^{k+1}]$ we have

$$2^k = f(2^k) < f(2^k + 1) < f(2^k + 2) < ... < f(2^{k+1} - 1) < f(2^{k+1}) = 2^{k+1},$$

hence we have $2^k - 1$ numbers lying in an interval with $2^k - 1$ numbers. Since f is monotone, we conclude the statement.

30) (IMC 2014) Suppose a_1 is odd. Then

$$\sigma(n) = \prod_{i=1}^{i=k}(1 + p_i + ... + p_i^{a_i}) = 2n = 2p_1^{a_1}p_2^{a_2}...p_k^{a_k}.$$

Since a_1 is odd,

$$p_1 + 1|(1 + p_1 + ... + p_1^{a_1}).$$

Hence $p_1 + 1|2n$. But $p_1 + 1 > 2$ which means that some p_j divides $p_1 + 1$. We have $p_3 > p_1 + 1$ and we conclude $p_2|p_1 + 1$. But $p_1 + 1 < 2p_2$ which implies $p_2 = p_1 + 1$, i.e. $p_1 = 2, p_2 = 3$. That means $6|n$. Since $n > 6$ we conclude that $n, n/2, n/3, n/6, 1$ are dinstict divisors of n and they sum up to

$$n + \frac{n}{2} + \frac{n}{3} + \frac{n}{6} + 1 = 2n + 1 > 2n,$$

which means that $\sigma(n) > 2n$, contradiction.

31) The case $p = 2$ is trivial. Assume p is odd. If $p^2 = a^2 + 2b^2$ then $2b^2 = (p + a)(p - a)$. Thus a is odd, $p > a$ and a is not divisible by p. We get $\gcd(p - a, p + a) = \gcd(p - a, 2p) = 2$.If $4 \mid p - a$ then $4 \nmid p + a$, hence we can assume $4 \nmid p - a$ (symmetry under $a \to -a$). Hence $p + a = 2m^2$, $p - a = 2n^2$. We obtain $2p = 2m^2 + n^2$, i.e. n is even, $n = 2k$. We conclude $p = m^2 + 2k^2$.

32) engel 131

33)

34)

35)

36)

37) Our number can be written as $5 + 2(10 + 10^2 + ... + 10^{n+1}) + 1(10^{n+2} + ... + 10^{2n+1}) = 3 + (1 + 10 + 10^2 + ... + 10^{n+1}) + (1 + 10 + ... + 10^{2n+1})$. Now this is equal to

$$3 + \frac{10^{n+2} - 1}{9} + \frac{10^{2n+2} - 1}{9} = \frac{10^{2n+2} + 10^{n+2} + 25}{9} = \left(\frac{10^{n+1} + 5}{3}\right)^2.$$

38) Working mod 4, we see that the left side is $1, 2$ or 3 and the right side is 0 or 1. Hence both are 1 mod 4, i.e. a, b are even and c is odd. Hence $a = 2k$, $b = 2m$, $c = 2n + 1$. We get $k^2 + m^2 = n^2 + n$. We want to show that $k + n$ is even.

The side $n^2 + n = n(n + 1)$ is even, hence k, m are either both odd or both even. If they are both odd then $k^2 + m^2 = 2 \mod 8$. But then n is odd and the desired result follows.

If both k, m are even, then $k^2 + m^2 = 0 \mod 4$, and then $n = 0$ or $3 \mod 4$. To exclude the last case, work again $\mod 8$. The left side is 0 or $4 \mod 8$, the right side is 0 or $1 \mod 8$. The last case is excluded, and we have the desired result.

39) Clearly if n is even then $n^4 + 4^n$ is even. Assume n is odd. Write $n = 2k + 1$ with $k > 0$. We have the Germain's identity:

$$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2).$$

Thus we conclude

$$n^4 + 4^n = (2k + 1)^4 + 4 \cdot (2^k)^4$$

is composite if $(2k + 1)^2 + 2 \cdot 2^{2k} > 2(2k + 1)2^k$, which holds for $k \geq 1$.

40) Since n is perfect we have

$$2n = \sigma(n) = \sum_{d|n} d = \sum_{d|n} \frac{n}{d} = n \sum_{d|n} \frac{1}{d}.$$

41) We write $n = 111...1$ with $p - 1$ digits in the form

$$n = 10^{p-2} + 10^{p-3} + ... + 100 + 10 + 1 = \frac{10^{p-1} - 1}{10 - 1} = \frac{10^{p-1} - 1}{9}.$$

In order to conclude $p|n$ we need $p|10^{p-1} - 1$ i.e. $10^{p-1} \equiv 1 \pmod{p}$. This holds iff $\gcd(p, 10) = 1$ i.e. $p \neq 2, 5$.

42)

43) We work (mod 6). Since $f(n) = 2^n + n^2$ is prime we have $2^n + n^2 \equiv 1 \pmod{6}$ or $2^n + n^2 \equiv 5 \pmod{6}$. Assume first $2^n + n^2 = 6k + 1$. Then

$$f(f(n)) = 2^{6k+1} + (6k + 1)^2 = 2 \cdot 4^{3k} + 36k^2 + 12k + 1 \equiv 2 + 0 + 0 + 1 \equiv 0 \pmod{3}.$$

If $2^n + n^2 = 6k + 5$. Then

$$f(f(n)) = 2^{6k+5} + (6k + 5)^2 = 2 \cdot 4^{3k+2} + 36k^2 + 30k + 25 \equiv 2 + 0 + 0 + 1 \equiv 0 \pmod{3}.$$

44) Let $d = \gcd(m, n)$. Then $m = ad, n = bd$ with $\gcd(a, b) = 1$. We have

$$\gcd(m, n) = \frac{mn}{\cdot \operatorname{lcm}(m, n)} = abd$$

hence

$$abd + d = ad + bd$$

i.e. $ab - a - b + 1 = 0$ or $(a-1)(b-1) = 0$. Thus $a = 1$ or $b = 1$.

45) Add IMO 2010, Problem 3.

46) (Putnam 2018, B3) Since $n|2^n$, n must be a power of 2, say $n = 2^r$.

Then prove the following Lemma: If $(2^a - 1)|(2^b - 1)$, then $a|b$.

To see this, write $b = aq + s$ with $s < a$. Then $2^s \equiv 1 \mod (2^a - 1)$. Since $s < a$ we get $s = 0$. So $a|b$.

Now $(n-1) = (2^r - 1)|(2^n - 1)$ means $r|n$. But $n = 2^r$ hence $r = 2^k$ and $n = 2^{2^k}$.

Since $n - 2 = 2(2^{r-1} - 1)$ divides $2^n - 2 = 2(2^{n-1} - 1)$ we use the Lemma and get $(r-1)|(n-1)$, i.e. $(2^k - 1)|(2^r - 1)$, i.e. $k|r = 2^k$, hence $k = 2^m$ and $n = 2^{2^{2^m}}$. We care for m such that $2^{2^{2^m}} < 10^{100}$ and that means $0 \leq m \leq 3$.

Παρουσίαση προβλημάτων για την προετοιμασία στους μαθηματικούς διαγωνισμούς

8 Μαρτίου 2022

Πρόβλημα(1): Δημιουργούμε τυχαία έναν αριθμό χρησιμοποιώντας όλους τους διψήφιους από τον $19, 20, 21, \ldots, 92$, ο οποίος σχηματίζεται με παράθεση αυτών των αριθμών. Να εξεταστεί αν ο αριθμός που προκύπτει είναι τέλειο τετράγωνο φυσικού αριθμού.

Λύση

1. Ο σχηματιζόμενος αριθμός κάθε φορά θα έχει σταθερό άθροισμα ψηφιών. (Εντοπίζουμε τι παραμένει αναλλοίωτο κάθε φορά σε αυτόν τον αριθμό)

2. Οπότε το άθροισμα των ψηφίων του είναι
$17 \cdot (2 + 3 + 4 + 5 + 6 + 7 + 8) + (1 + 9) + (9 + 0) + (9 + 1) + (9 + 2) + 7 \cdot 1 = 642.$

3. Δηλαδή το άθροισμα των ψηφίων αυτού του αριθμού διαιρείται με το 3 και δεν διαιρείται όμως με το 9.

4. Επομένως, αριθμός αυτός δεν μπορεί να είναι σε καμία περίπτωση τέλειο τετράγωνο φυσικού αριθμού.

Πρόβλημα(2): Έστω $x, y, z \in \mathbb{Z}$ ώστε $29\big|x^4 + y^4 + z^4$. Να δειχθεί ότι $29^4\big|x^4 + y^4 + z^4$.

Λύση

1. Θέλουμε να δείξουμε ότι $29^4\big|x^4 + y^4 + z^4$. Οπότε θα επιθυμούσαμε να αληθεύει $29\big|x$, $29\big|y$ και $29\big|z$.

2. **Ισχυρισμός**: $29\big|x$, $29\big|y$, $29\big|z$.

3. **Απόδειξη Ισχυρισμού**: Γνωρίζουμε ότι

$$x,y,z \equiv \begin{cases} 0 \\ \pm 1 \\ \pm 2 \\ \pm 3 \\ \pm 4 \\ \pm 5 \\ \pm 6 \\ \pm 7 \\ \pm 8 \\ \pm 9 \\ \pm 10 \\ \pm 11 \\ \pm 12 \\ \pm 13 \\ \pm 14 \end{cases} \big(mod\ 29\big) \Rightarrow x^4, y^4, z^4 \equiv \begin{cases} 0 \\ 1 \\ 16 \\ 23 \\ 24 \\ 16 \\ 20 \\ 23 \\ 7 \\ 7 \\ 24 \\ 25 \\ 1 \\ 25 \\ 20 \end{cases} \big(mod\ 29\big) \Rightarrow x^4, y^4, z^4 \equiv \begin{cases} 0 \\ 1 \\ 7 \\ 16 \\ 20 \\ 23 \\ 24 \\ 25 \end{cases} \big(mod\ 29\big).$$

   Επειδή ισχύει $29\big|x^4+y^4+z^4$, διαπιστώνουμε ότι αυτό επιτυγχάνεται μόνο όταν $x,y,z \equiv 0\big(mod\ 29\big)$.

4. Συνεπώς, $x^4, y^4, z^4 \equiv 0\big(mod\ 29^4\big) \Rightarrow x^4+y^4+z^4 \equiv 0\big(mod\ 29^4\big)$.

Πρόβλημα(3): Υπάρχει φυσικός αριθμός n ώστε να ισχύει η ισότητα $\big(\sqrt{2021}-\sqrt{2020}\big)^{2022} = \sqrt{n}-\sqrt{n-1}$;

Λύση

1. $$\big(\sqrt{2021}-\sqrt{2020}\big)^{2022} = \sum_{k=0}^{2022}\binom{2022}{k}\sqrt{2021}^{k}(-\sqrt{2020})^{2022-k} =$$
$$= \sum_{k=0}^{1011}\binom{2022}{2k}\sqrt{2021}^{2k}(-\sqrt{2020})^{2022-2k} + \sum_{k=0}^{1010}\binom{2022}{2k+1}\sqrt{2021}^{2k+1}(-\sqrt{2020})^{2022-2k-1} =$$
$$= \underbrace{\sum_{k=0}^{1011}\binom{2022}{2k}2021^{k}\cdot 2020^{1011-k}}_{=A\in\mathbb{N}} - \underbrace{\left(\sum_{k=0}^{1010}\binom{2022}{2k+1}2021^{k}\cdot 2020^{1011-k-1}\right)}_{=B\in\mathbb{N}}\cdot\sqrt{2021\cdot 2020} =$$

$= A - B\sqrt{2021 \cdot 2020}.$

2. $\left(\sqrt{2021} + \sqrt{2020}\right)^{2022} = A + B\sqrt{2021 \cdot 2020}.$

3. $A^2 - 2021 \cdot 2020 \cdot B^2 = \left(A - B\sqrt{2021 \cdot 2020}\right)\left(A + B\sqrt{2021 \cdot 2020}\right) =$
$= \left(\sqrt{2021} - \sqrt{2020}\right)^{2022}\left(\sqrt{2021} + \sqrt{2020}\right)^{2022} = \left(\sqrt{2021}^2 - \sqrt{2020}^2\right)^{2022} = 1.$

4. Αν θέσουμε $n = A^2 \in \mathbb{N}$ έχουμε $\sqrt{n} - \sqrt{n-1} = \sqrt{A^2} - \sqrt{A^2 - 1} =$
$= A - \sqrt{2021 \cdot 2020 \cdot B^2} = A - B\sqrt{2021 \cdot 2020} = \left(\sqrt{2021} - \sqrt{2020}\right)^{2022}.$

Πρόβλημα(4): Έστω P σημείο τριγώνου ABC και έστω επίσης D, E, Z οι προβολές από το P στις πλευρές BC, CA, AB, αντίστοιχα. Βρείτε όλα τα P ώστε το $\dfrac{BC}{PD} + \dfrac{CA}{PE} + \dfrac{AB}{PZ}$ να είναι ελάχιστο.

Λύση

1. Παρατηρούμε ότι $PD \cdot BC + PE \cdot AC + PZ \cdot AB = 2 \cdot (ABC)$.

2. Εφαρμόζουμε την ανισότητα $Cauchy - Schwarz$ ως εξής:

$$2 \cdot (ABC) \cdot \left(\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PZ}\right) =$$
$$\Big(PD \cdot BC + PE \cdot AC + PZ \cdot AB\Big) \cdot \left(\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PZ}\right) =$$
$$= \left(\sqrt{PD \cdot BC}^2 + \sqrt{PE \cdot AC}^2 + \sqrt{PZ \cdot AB}^2\right) \cdot \left(\sqrt{\frac{BC}{PD}}^2 + \sqrt{\frac{CA}{PE}}^2 + \sqrt{\frac{AB}{PZ}}^2\right) \geqslant$$
$$\geqslant \sqrt{PD \cdot BC} \cdot \sqrt{\frac{BC}{PD}} + \sqrt{PE \cdot AC} \cdot \sqrt{\frac{CA}{PE}} + \sqrt{PZ \cdot AB} \cdot \sqrt{\frac{AB}{PZ}} =$$
$$= \left(BC + CA + AB\right)^2 \Rightarrow \frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PZ} \geqslant \frac{\left(BC + CA + AB\right)^2}{2 \cdot (ABC)}.$$

3. Οπότε η ελάχιστη τιμή λαμβάνεται όταν έχουμε

$$\frac{\sqrt{PD \cdot BC}}{\sqrt{\frac{BC}{PD}}} = \frac{\sqrt{PE \cdot AC}}{\sqrt{\frac{CA}{PE}}} = \frac{\sqrt{PZ \cdot AB}}{\sqrt{\frac{AB}{PZ}}} \Leftrightarrow PD = PE = PZ.$$

Πρόβλημα(5): Όλοι οι φυσικοί αριθμοί τοποθετούνται σε τριγωνική σειρά όπως παρακάτω:

$$\begin{array}{cccccccc}
1 & 3 & 6 & 10 & 15 & 21 & 28 & \cdots \\
2 & 5 & 9 & 14 & 20 & 27 & \cdots & \\
4 & 8 & 13 & 19 & 26 & \cdots & & \\
7 & 12 & 18 & 25 & \cdots & & & \\
11 & 17 & 24 & \cdots & & & & \\
16 & 23 & \cdots & & & & & \\
22 & \cdots & & & & & &
\end{array}$$

Να βρεθεί η γραμμή και η στήλη που βρίσκεται ο αριθμός 2022.

Λύση

1. Ας είναι

$$\begin{array}{lcccccccc}
 & 1^{\eta}\text{στήλη} & 2^{\eta}\text{στήλη} & 3^{\eta}\text{στήλη} & 4^{\eta}\text{στήλη} & 5^{\eta}\text{στήλη} & 6^{\eta}\text{στήλη} & 7^{\eta}\text{στήλη} & \\
 & \overbrace{\qquad}^{=b_1} & \overbrace{\qquad}^{=b_2} & \overbrace{\qquad}^{=b_3} & \overbrace{\qquad}^{=b_4} & \overbrace{\qquad}^{=b_5} & \overbrace{\qquad}^{=b_6} & \overbrace{\qquad}^{=b_7} & \\
1^{\eta}\text{γραμμή} & 1 & 3 & 6 & 10 & 15 & 21 & 28 & \cdots \\
2^{\eta}\text{γραμμή} & 2 & 5 & 9 & 14 & 20 & 27 & \cdots & \\
3^{\eta}\text{γραμμή} & 4 & 8 & 13 & 19 & 26 & \cdots & & \\
4^{\eta}\text{γραμμή} & 7 & 12 & 18 & 25 & \cdots & & & \\
5^{\eta}\text{γραμμή} & 11 & 17 & 24 & \cdots & & & & \\
6^{\eta}\text{γραμμή} & 16 & 23 & \cdots & & & & & \\
7^{\eta}\text{γραμμή} & 22 & \cdots & & & & & &
\end{array}$$

2. Μετά από αυτό παρατηρούμε πως η πρώτη γραμμή περιέχει τους τριγωνικούς αριθμούς, δηλαδή $b_n = 1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}, \forall n \in \mathbb{N}$.

3. Πλησιάζουμε όσο το δυνατόν περισσότερο τον αριθμό 2022, παρατηρώντας ότι $b_{63} = \dfrac{63 \cdot 64}{2} = 2016$.

4. Οπότε στην 64^{η} γραμμή και 1^{η} στήλη έχουμε τον αριθμό 2017.

5. Επομένως, στην 59^{η} γραμμή και 6^{η} στήλη έχουμε τον αριθμό 2022.

Πρόβλημα(6): Θεωρούμε τους τυχαίους αριθμούς $a, b, c \in \mathbb{R} \setminus \{0\}$ με $a + b + c \neq 0$ ώστε να ισχύει $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{a+b+c}$. Έστω ένας οποιοσδήποτε περιττός φυσικός αριθμός n, τότε να δειχθεί ότι $\dfrac{1}{a^n} + \dfrac{1}{b^n} + \dfrac{1}{c^n} = \dfrac{1}{a^n + b^n + c^n}$.

Λύση

1. Γράφουμε ότι $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c} \Leftrightarrow (a+b+c)(ab+bc+ca)=abc$.
2. Θεωρούμε το πολυώνυμο $P(x)$ το οποίο έχει ρίζες τους αριθμούς a, b και c, δηλαδή $P(x)=(x-a)(x-b)(x-c)$.
3. Από την άλλη μεριά γράφουμε $P(x)=x^3-px^2+qx-r$, όπου $p=a+b+c$, $q=ab+bc+ca$ και $r=abc$.
4. Αληθεύει πως: $pq=r$.
5. Παρατηρούμε ότι:
$$P(x)=x^3-px^2+qx-r=x^3-px^2+qx-pq=x^2(x-p)+q(x-p)=(x-p)(x^2+q).$$
6. Εντοπίζουμε ότι το $P(x)$ έχει για πραγματική ρίζα τον αριθμό p.
7. Συνεπώς, $\begin{cases} p=a \\ \text{ή} \\ p=b \\ \text{ή} \\ p=c \end{cases} \Rightarrow \begin{cases} a+b+c=a \\ \text{ή} \\ a+b+c=b \\ \text{ή} \\ a+b+c=c \end{cases} \Rightarrow \begin{cases} b+c=0 \\ \text{ή} \\ a+c=0 \\ \text{ή} \\ a+b=0 \end{cases}$.
8. Αν υποθέσουμε ότι $b+c=0$, τότε $\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{(-b)^n}=$ $=\frac{1}{a^n}+\frac{1}{b^n}-\frac{1}{b^n}=\frac{1}{a^n}=\frac{1}{a^n+b^n-b^n}=\frac{1}{a^n+b^n+(-b)^n}=\frac{1}{a^n+b^n+c^n}$, για οποιονδήποτε περιττό φυσικό αριθμό n.

Σημείωση: Η σχέση $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$ είναι τελικά ισοδύναμει με την σχέση $(a+b)(b+c)(c+a)=0$, παρεπιπτώντος αυτή η παραγοντοποίηση είναι "δύσκολη" όχι όμως ανέφικτη. Επιπλέον, το Πρόβλημα αυτό το έχει δημιουργήσει ο *Terence Tao* και η ιδέα του είναι ακριβώς αυτή που περιγράφεται.

Ελάχιστο Πολυώνυμο: Έστω ένας τετραγωνικός πίνακας $A \in M_{n\times n}(\mathbb{F})$ με χαρακτηριστικό πολυώνυμο $x_A(t)$, τότε υπάρχει μοναδικό πολυώνυμο $m(t)$, το οποίο έχει όλες τις ρίζες του πολυωνύμου $x_A(t)$, τέτοιο ώστε:

1. $m(t) \neq 0$, $m(A) = \mathbb{O}_n$ και το $m(t)$ έχει μεγιστοβάθμιο συντελεστή ένα δηλαδή είναι μονικό πολυώνυμο,
2. αν $\varphi(t)$ πολυώνυμο ώστε $\varphi(A) = \mathbb{O}_n$ τότε $m(t) \big| \varphi(t)$.

Σχόλιο : Αν $x_A(t) = (x - \lambda_1)^{a_1} \cdot (x - \lambda_2)^{a_2} \cdots (x - \lambda_n)^{a_n}$, τότε

1. $m(t) = (x - \lambda_1)^{b_1} \cdot (x - \lambda_2)^{b_2} \cdots (x - \lambda_n)^{b_n}$,
2. όπου $\forall i \in \{1, 2, \cdots, n\} : 1 \leqslant b_i \leqslant a_i$ και
3. επιπλέον είναι $m(t) \big| x_A(t)$ με $deg\big(m(t)\big) \leqslant deg\big(x_A(t)\big)$.

Πρόβλημα(7) : Έστω n ένας θετικός ακέραιος, $k \in \mathbb{C}$ και $A \in M_n\big(\mathbb{C}\big)$ τέτοιο ώστε $tr(A) \neq 0$ και $rank(A) + rank\big(tr(A) \cdot \mathbb{I}_n - k \cdot A\big) = n$. Να βρεθεί το $rank(A)$.

Λύση

1. Αρχικά γνωρίζουμε ότι: $rank\big(tr(A) \cdot \mathbb{I}_n - k \cdot A\big) = rank\Big(\frac{tr(A)}{k} \cdot \mathbb{I}_n - A\Big)$, αφού $k \neq 0$.
2. Τότε έχουμε ότι:

$$n = rank(A) + rank\big(tr(A) \cdot \mathbb{I}_n - k \cdot A\big) = rank(A) + rank\Big(\frac{tr(A)}{k} \cdot \mathbb{I}_n - A\Big) =$$

$$= dim\big[Im(A)\big] + dim\Big[\frac{tr(A)}{k} \cdot \mathbb{I}_n - A\Big] =$$

$$= dim\Bigg[Im\bigg(A + \Big(\frac{tr(A)}{k} \cdot \mathbb{I}_n - A\Big)\bigg)\Bigg] + dim\Bigg[Im(A) \cap Im\Big(\frac{tr(A)}{k} \cdot \mathbb{I}_n - A\Big)\Bigg] \geqslant$$

$$\geqslant dim\Bigg[Im\bigg(A + \Big(\frac{tr(A)}{k} \cdot \mathbb{I}_n - A\Big)\bigg)\Bigg] = dim\Bigg[Im\Big(\frac{tr(A)}{k} \cdot \mathbb{I}_n\Big)\Bigg] = rank\Big(\frac{tr(A)}{k} \cdot \mathbb{I}_n\Big) = n$$

3. Συνεπώς η παραπάνω ανισότητα ισχύει σαν ισότητα, και άρα προκύπτει ότι: $dim\Bigg[Im(A) \cap Im\bigg(\frac{tr(A)}{k} \cdot \mathbb{I}_n - A\bigg)\Bigg] = 0 \Rightarrow Im(A) \cap Im\bigg(\frac{tr(A)}{k} \cdot \mathbb{I}_n - A\bigg) = \big\{\vec{0}\big\}$.

4. Παρατηρούμε ότι: $A\cdot\left(\dfrac{tr(A)}{k}\cdot\mathbb{I}_n-A\right)=\dfrac{tr(A)}{k}\cdot A-A^2=\left(\dfrac{tr(A)}{k}\cdot\mathbb{I}_n-A\right)\cdot A.$

5. Αν επιλέξουμε $\vec{x}\in\mathbb{C}^n$, είναι:
$Im(A)\ni A\cdot\left[\left(\dfrac{tr(A)}{k}\cdot\mathbb{I}_n-A\right)\cdot\vec{x}\right]=\left(\dfrac{tr(A)}{k}\cdot\mathbb{I}_n-A\right)\cdot\big(A\cdot\vec{x}\big)\in Im\left(\dfrac{tr(A)}{k}\cdot\mathbb{I}_n-A\right),$
δηλαδή έχουμε πως $A\cdot\left(\dfrac{tr(A)}{k}\cdot\mathbb{I}_n-A\right)\cdot\vec{x}\in Im(A)\cap Im\left(\dfrac{tr(A)}{k}\cdot\mathbb{I}_n-A\right)=\big\{\vec{0}\big\}.$
Τελικά έχουμε ότι $A\cdot\left(\dfrac{tr(A)}{k}\cdot\mathbb{I}_n-A\right)\cdot\vec{x}=\vec{0},\forall\vec{x}\in\mathbb{C}^n.$

6. Τότε προκύπτει η εξής σχέση: $A\cdot\left(\dfrac{tr(A)}{k}\cdot\mathbb{I}_n-A\right)=\mathbb{O}_n.$

7. Το ελάχιστο πολυώνυμο του πίνακα A,έστω να είναι το $m_A(t)$, διαιρεί το πολυώνυμο $t\cdot\Big(\dfrac{tr(A)}{k}-t\Big)$, δηλαδή $m_A(t)\Big|t\cdot\Big(\dfrac{tr(A)}{k}-t\Big)$.

8. Οπότε ο πίνακας A είναι διαγωνοποιήσιμος, επειδή έχει διακεκριμμένες ρίζες. Άρα υπάρχει αντιστρέψιμος πίνακας $P\in M_n\big(\mathbb{C}\big)$ τέτοιος ώστε:

$$A=P\cdot\begin{bmatrix}\dfrac{tr(A)}{k}\cdot\mathbb{I}_r & \mathbb{O}_{n-r}\\ \mathbb{O}_{n-r} & \mathbb{O}_r\end{bmatrix}\cdot P^{-1},\ r=rank(A).$$

9. Στη συνέχεια λαμβάνουμε ότι:

$$rank(A)=rank\left(\begin{bmatrix}\dfrac{tr(A)}{k}\cdot\mathbb{I}_r & \mathbb{O}_{n-r}\\ \mathbb{O}_{n-r} & \mathbb{O}_r\end{bmatrix}\right)$$

10. Επιπλέον έχουμε ότι:

$$tr(A)=tr\left(\begin{bmatrix}\dfrac{tr(A)}{k}\cdot\mathbb{I}_r & \mathbb{O}_{n-r}\\ \mathbb{O}_{n-r} & \mathbb{O}_r\end{bmatrix}\right)=tr\left(\dfrac{tr(A)}{k}\cdot r\right)=\dfrac{tr(A)}{k}\cdot r=\dfrac{tr(A)}{k}\cdot rank(A)\Leftrightarrow$$

$$\Leftrightarrow tr(A)=\dfrac{tr(A)}{k}\cdot rank(A)\xLeftrightarrow{tr(A)\neq 0}rank(A)=k$$

Σχόλια

1. **Ιδέα** : Η προφανή ανισότητα $rank\big(C\big)+rank\big(D\big)\geqslant rank\big(C+D\big),\forall C,D\in M_n\big(\mathbb{C}\big)$ παρατηρούμε ότι ισχύει ως ισότητα, όπου η ισότητα ισχύει όταν $dim\Big[Im(C)\cap Im(D)\Big]=0$ και άρα είναι $Im(C)\cap Im(D)=\big\{\vec{0}\big\}$ και αν ισχύει ότι $C\cdot D=D\cdot C$ τότε $C\cdot D=\mathbb{O}_n$.

2. Το πρόβλημα κατασκευάστηκε με την εξής λογική : Πρώτον, δίνουμε σαν δεδομένο δύο πίνακες οι οποίοι μετατίθονται μεταξύ τους, χωρίς να παρουσιάζουμε την μεταθετικότητα αυτών των δύο πινάκων, και Δεύτερον είναι η ισότητα σε μια ανισότητα. Παρατηρήστε ότι αυτά τα δύο είναι ότι πρέπει για την επίλυση του προβλήματος.

3. $$A\cdot\left(\frac{tr(A)}{k}\cdot\mathbb{I}_n-A\right)=\mathbb{O}_n\Leftrightarrow\frac{tr(A)}{k}\cdot A-A^2=\mathbb{O}_n\xLeftrightarrow{B=\dfrac{k}{tr(A)}\cdot A}B-B^2=\mathbb{O}_n\Leftrightarrow$$
$$\Leftrightarrow B^2=B\Leftrightarrow B^m=B,\forall m\in\mathbb{N}\Leftrightarrow k^{m-1}\cdot A^m=\big(tr(A)\big)^{m-1}\cdot A,\forall m\in\mathbb{N}.$$

4. Ο πίνακας $A\in M_n\big(\mathbb{C}\big)$ που δίνεται είναι διαγωνοποιήσιμος με ιδιοτιμή $\dfrac{tr(A)}{k}$ και πιθανόν την 0.

5.
   - $Im(A)=Ker\Big(tr(A)\cdot\mathbb{I}_n-k\cdot A\Big)$,
   - $Im\Big(tr(A)\cdot\mathbb{I}_n-k\cdot A\Big)=Ker(A)$ και
   - $\mathbb{C}^n=Ker(A)\oplus Ker\Big(tr(A)\cdot\mathbb{I}_n-k\cdot A\Big)$.

Πρόβλημα(8): Θεωρούμε τους πίνακες $A,B\in M_{2022\times2022}\big(\mathbb{R}\big)$ έτσι ώστε $A\cdot B=B\cdot A$ και $A^{2022}=B^{2022}=\mathbb{I}_{2022}$. Να δείξετε ότι εάν $tr\big(A\cdot B\big)=2022$, τότε $tr(A)=tr(B)$.

Λύση

1. Παρατηρούμε ότι τα ελάχιστα πολυώνυμα των πινάκων A και B, έστω $m_A(t)$ και $m_B(t)$, αντίστοιχα διαιρούν το πολυώνυμο $t^{2022}-1$, δηλαδή ισχύει ότι: $m_A(t)\big|t^{2022}-1$, $m_B(t)\big|t^{2022}-1$.

2. Διαπιστώνουμε ότι τα ελάχιστα πολυώνυμα των πινάκων A και B έχουν διακεκριμμένες ρίζες, και μαλιστα αυτες θα ειναι σε συζυγη ζευγη των 2022 ριζων της μοναδας, οπότε οι πίνακες A και B είναι διαγωνοποιήσιμοι.

3. Επιπλέον αφού οι πίνακες A και B είναι πραγματικοί έπεται ότι οι ιδιοτιμές τους θα είναι σε ζεύγη συζυγών μιγαδικών αριθμών.

4. Υποθέτουμε ότι οι ιδιοτιμές του πίνακα A είναι οι αριθμοί $\lambda_1, \lambda_2, \ldots, \lambda_{2022}$ οι οποίοι δεν είναι απαραίτητα διαφορετικοί μεταξύ τους, και ας είναι οι ιδιοτιμές του πίνακα B οι αριθμοί $\mu_1, \mu_2, \ldots, \mu_{2022}$ οι οποίοι δεν είναι απαραίτητα διαφορετικοί μεταξύ τους.

5. Μάλιστα επειδή οι πίνακες αυτοί μετατίθονται έχουμε ως αποτέλεσμα ότι υπάρχει αντιστρέψιμος πίνακας $P \in M_{2022\times 2022}\big(\mathbb{R}\big)$ τέτοιος ώστε να ισχύει ότι

$$A = P\cdot \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{2022} \end{bmatrix}}_{=D} \cdot P^{-1} \ , \ B = P\cdot \underbrace{\begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu_{2022} \end{bmatrix}}_{=Q} \cdot P^{-1}$$

$$AB = \big(PDP^{-1}\big)\big(PQP^{-1}\big) = PDQP^{-1} = P \begin{bmatrix} \lambda_1\mu_1 & 0 & \cdots & 0 \\ 0 & \lambda_2\mu_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{2022}\mu_{2022} \end{bmatrix} P^{-1}$$

6. Επομένως επειδή $2022 = \big|tr\big(A\cdot B\big)\big| = \left|\sum\limits_{k=1}^{2022} \lambda_k \cdot \mu_k\right| \leqslant \sum\limits_{k=1}^{2022} \big|\lambda_k \cdot \mu_k\big| = \sum\limits_{k=1}^{2022} \big|\lambda_k\big| \cdot \big|\mu_k\big| = 2022.$

7. Συνεπώς διαπιστώνουμε ότι ισχύει η ισότητα στην τριγωνική ανισότητα. Οπότε έχουμε πως: $\lambda_k \cdot \mu_k = 1 \ , \ \forall k \in \big\{1, 2, \cdots, 2022\big\} \Rightarrow \lambda_k = \dfrac{1}{\mu_k} = \overline{\mu_k} \Rightarrow$

$$\Rightarrow tr(A) = \sum_{k=1}^{2022} \lambda_k = \sum_{k=1}^{2022} \overline{\mu_k} = \overline{\sum_{k=1}^{2022} \mu_k} = \overline{tr(B)} = tr(B) \xRightarrow{tr(B)\in\mathbb{R}} tr(A) = tr(B).$$

Πρόβλημα(9): Έστω $\big(x_n\big)_{n=1}^{+\infty}$ μια ακολουθία πραγματικών αριθμών η οποία ικανοποιεί τη σχέση $x_{n+1} = \dfrac{\sqrt{3}x_n - 1}{x_n + \sqrt{3}}, \forall n \in \mathbb{N}$. Να δειχθεί ότι η ακολουθία x_n είναι περιοδική.

Λύση

1. Παρατηρούμε ότι $tan(a-b) = \dfrac{tan(a) - tan(b)}{1 - tan(a) \cdot tan(b)}$
2. $x_{n+1} = \dfrac{x_n - \frac{1}{\sqrt{3}}}{1 + x_n \frac{1}{\sqrt{3}}} = \dfrac{x_n - tan\big(\frac{\pi}{6}\big)}{1 + x_n \cdot tan\big(\frac{\pi}{6}\big)}, \forall n \in \mathbb{N}$
3. $x_1 \in \mathbb{R} \Rightarrow \exists t \in \Big(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\Big) : x_1 = tan(t)$
4. $x_2 = \dfrac{x_1 - tan\big(\frac{\pi}{6}\big)}{1 + x_1 \cdot tan\big(\frac{\pi}{6}\big)} = \dfrac{tan(t) - tan\big(\frac{\pi}{6}\big)}{1 + tan(t) \cdot tan\big(\frac{\pi}{6}\big)} = tan\big(t - \dfrac{\pi}{6}\big)$
5. $x_3 = \dfrac{x_2 - tan\big(\frac{\pi}{6}\big)}{1 + x_2 \cdot tan\big(\frac{\pi}{6}\big)} = \dfrac{tan\big(t - \frac{\pi}{6}\big) - tan\big(\frac{\pi}{6}\big)}{1 + tan\big(t - \frac{\pi}{6}\big) \cdot tan\big(\frac{\pi}{6}\big)} = tan\big(t - 2\dfrac{\pi}{6}\big)$
6. $x_n = tan\Big(t - \big(n-1\big)\dfrac{\pi}{6}\Big), \forall n \in \mathbb{N}$
7. $x_n = tan\Big(t - \big(n-1\big)\dfrac{\pi}{6}\Big) = tan\Big(t - \big(n-1\big)\dfrac{\pi}{6} - \pi\Big) = tan\Big(t - n\dfrac{\pi}{6} - 5\dfrac{\pi}{6}\Big) =$
   $= tan\Big(t - \big(n+5\big)\dfrac{\pi}{6}\Big) = tan\Big(t - \big((n+6)-1\big)\dfrac{\pi}{6}\Big) = x_{n+6}, \forall n \in \mathbb{N}$

Πρόβλημα(10): Δίνεται η συνεχής συνάρτηση $f : \mathbb{R} \to \mathbb{R}$. Να αποδειχθεί ότι $\displaystyle\int_0^4 f\big(x(x-3)^2\big)\, dx = 2\int_1^3 f\big(x(x-3)^2\big)\, dx$.

Λύση

1. Θεωρούμε $g : \mathbb{R} \to \mathbb{R}$ με τύπο $g(x) = x(x-3)^2 = x^3 - 6x^2 + 9x$, όπου
   $$g'(x) = 3(x-1)(x-3) \Rightarrow \begin{cases} x \in (-\infty, 1) \cup (3, +\infty) : g'(x) > 0 \Rightarrow g \text{ γνησίως αύξουσα} \\ \text{ή} \\ x \in (1,3) : g'(x) < 0 \Rightarrow g \text{ γνησίως φθίνουσα} \end{cases}$$
2. $$\begin{cases} g \equiv g_1 : [0,1] \xrightarrow{\text{αύξουσα}} [0,4] \quad g^{-1} \equiv g_1^{-1} \equiv h_1 : [0,4] \to [0,1] \\ \text{ή} \\ g \equiv g_2 : [1,3] \xrightarrow{\text{φθίνουσα}} [0,4] \quad g^{-1} \equiv g_2^{-1} \equiv h_2 : [0,4] \to [1,3] \\ \text{ή} \\ g \equiv g_3 : [3,4] \xrightarrow{\text{αύξουσα}} [0,4] \quad g^{-1} \equiv g_3^{-1} \equiv h_3 : [0,4] \to [3,4] \end{cases} \text{, με } g_i\big(h_i(x)\big) = x,$$
   $\forall x \in [0,4], \forall i \in \{1,2,3\}$, επιπλέον οι τρεις αυτοί κλάδοι της αντίστροφης

συνάρτησης g ικανοποιούν τη σχέση $g(x)=t \Leftrightarrow x^3-6x^2+9x=t$ την οποία την επιλύουμε ως προς x και βρίσκουμε τους τρεις κλάδους $h_1(t), h_2(t), h_3(t)$, οι οποίοι ικανοποιούν τους τύπους $Vieta$, δηλαδή $h_1(t)+h_2(t)+h_3(t)=6, \forall t \in [0,4]$

3. $$\begin{cases} \int\limits_0^1 f\big(g(x)\big)\ dx = \int\limits_0^1 f\big(g_1(x)\big)\ dx = \int\limits_0^4 f(t)h_1'(t)\ dt, \text{όπου θέσαμε } x=h_1(t) \\ \text{και} \\ \int\limits_1^3 f\big(g(x)\big)\ dx = \int\limits_0^1 f\big(g_2(x)\big)\ dx = -\int\limits_0^4 f(t)h_2'(t)\ dt, \text{όπου θέσαμε } x=h_2(t) \\ \text{και} \\ \int\limits_3^4 f\big(g(x)\big)\ dx = \int\limits_3^4 f\big(g_3(x)\big)\ dx = \int\limits_0^4 f(t)h_3'(t)\ dt, \text{όπου θέσαμε } x=h_3(t) \end{cases}$$

4. $$\int\limits_0^4 f\big(x(x-3)^2\big)\ dx - 2\int\limits_1^3 f\big(x(x-3)^2\big)\ dx = \int\limits_0^4 f\big(g(x)\big)\ dx - 2\int\limits_1^3 f\big(g(x)\big)\ dx =$$
$$= \int\limits_0^1 f\big(g(x)\big)\ dx + \int\limits_1^3 f\big(g(x)\big)\ dx + \int\limits_3^4 f\big(g(x)\big)\ dx - 2\int\limits_1^3 f\big(g(x)\big)\ dx =$$
$$= \int\limits_0^1 f\big(g(x)\big)\ dx - \int\limits_1^3 f\big(g(x)\big)\ dx + \int\limits_3^4 f\big(g(x)\big)\ dx =$$
$$= \int\limits_0^1 f\big(g_1(x)\big)\ dx - \int\limits_1^3 f\big(g_2(x)\big)\ dx + \int\limits_3^4 f\big(g_3(x)\big)\ dx =$$
$$= \int\limits_0^4 f(t)h_1'(t)\ dt + \int\limits_0^4 f(t)h_2'(t)\ dt + \int\limits_0^4 f(t)h_3'(t)\ dt =$$
$$= \int\limits_0^4 f(t)\big(h_1'(t)+h_2'(t)+h_3'(t)\big)\ dt = 0$$

<u>Πρόβλημα</u>(11): Δίνεται ένας τυχαίος φυσικός αριθμός $n \in \mathbb{N}$ και ολοκληρώσιμη συνάρτηση κατά $Riemann$ $f:[0,1]\to\mathbb{R}$ στο $[0,1]$. Να δειχθεί ότι υπάρχει $a_n \in \left[0, 1-\dfrac{1}{n}\right]$ ώστε $\displaystyle\int\limits_{a_n}^{a_n+\frac{1}{n}} f(x)\ dx = 0$ ή $\displaystyle\int\limits_0^{a_n} f(x)\ dx = \int\limits_{a_n+\frac{1}{n}}^{1} f(x)\ dx.$

<u>Λύση</u>

1. Θεωρούμε την παραγωγίσιμη συνάρτηση $F : [0,1] \to \mathbb{R}$ με τύπο $F(x) = \displaystyle\int\limits_0^x f(t)\ dt$

2. Θελουμε να δείξουμε ότι υπάρχει $a_n \in \Big[0, 1 - \dfrac{1}{n}\Big]$ ώστε $\begin{cases} \int\limits_{a_n}^{a_n+\frac{1}{n}} f(x)\ dx = 0 \\ \text{ή} \\ \int\limits_0^{a_n} f(x)\ dx = \int\limits_{a_n+\frac{1}{n}}^{1} f(x)\ dx \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} 0 = \int\limits_{a_n}^{a_n+\frac{1}{n}} f(x)\ dx = \int\limits_0^{a_n+\frac{1}{n}} f(x)\ dx - \int\limits_0^{a_n} f(x)\ dx \\ \text{ή} \\ \int\limits_0^{a_n} f(x)\ dx = \int\limits_{a_n+\frac{1}{n}}^{1} f(x)\ dx = \int\limits_0^1 f(x)\ dx - \int\limits_0^{a_n+\frac{1}{n}} f(x)\ dx \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} F\Big(a_n + \frac{1}{n}\Big) = F(a_n) \\ \text{ή} \\ F(a_n) = F(1) - F\Big(a_n + \frac{1}{n}\Big) \end{cases}$$

3. Επιπλέον, έστω η παραγωγίσιμη συνάρτηση $G : [0,1] \to \mathbb{R}$ με τύπο
$$G(x) = F(x) \cdot (F(1) - F(x)) = \int\limits_0^x f(t)\ dt \cdot \int\limits_x^1 f(t)\ dt$$

4. Θελουμε να δείξουμε ότι υπάρχει $a_n \in \Big[0, 1 - \dfrac{1}{n}\Big]$ ώστε $G(a_n) = G\Big(a_n + \dfrac{1}{n}\Big) \Leftrightarrow$
$$\Leftrightarrow F(a_n) \cdot \big(F(1) - F(a_n)\big) = F\Big(a_n + \frac{1}{n}\Big) \cdot \Bigg(F(1) - F\Big(a_n + \frac{1}{n}\Big)\Bigg)$$

5. Έχουμε ότι $G(0) = G(1) = 0$

6. $0 = G(1) - G(0) =$
$$= \Big[G(1) - G\Big(1 - \frac{1}{n}\Big)\Big] + \Big[G\Big(1 - \frac{1}{n}\Big) - G\Big(1 - \frac{2}{n}\Big)\Big] + \cdots + \Big[G\Big(1 - \frac{n-1}{n}\Big) - G(0)\Big] =$$

$$= \sum_{k=0}^{n-1} \Big[G\Big(\frac{k}{n} + \frac{1}{n}\Big) - G\Big(\frac{k}{n}\Big) \Big]$$

7. Υποθέτουμε ότι δεν υπάρχει $x \in \Big[0, \dfrac{1}{n}\Big]$ ώστε $G(x) = G\Big(x + \dfrac{1}{n}\Big)$.

8. Επειδή η G είναι συνεχής στο $[0,1]$ θα διατηρεί σταθερό πρόσημο στο $[0,1]$.

9. Άρα $\begin{cases} G\Big(x+\frac{1}{n}\Big) - G(x) < 0, \forall x \in [0,1] \\ \text{ή} \\ G\Big(x+\frac{1}{n}\Big) - G(x) > 0, \forall x \in [0,1] \end{cases} \Rightarrow$

   $\Rightarrow \begin{cases} 0 = G(1) - G(0) = \sum\limits_{k=0}^{n-1} \Big[G\Big(\frac{k}{n} + \frac{1}{n}\Big) - G\Big(\frac{k}{n}\Big)\Big] < 0 \\ \text{ή} \\ 0 = G(1) - G(0) = \sum\limits_{k=0}^{n-1} \Big[G\Big(\frac{k}{n} + \frac{1}{n}\Big) - G\Big(\frac{k}{n}\Big)\Big] > 0 \end{cases}$, όπου σε κάθε πείπτωση καταλήγουμε σε άτοπο.

Εφαρμογή: Έχουμε ως αποτέλεσμα για παράδειγμα ότι υπάρχει πραγματικός αριθμός $\xi \in \Big[0, \dfrac{2021}{2022}\Big]$ ώστε $\displaystyle\int_{\xi}^{\xi+\frac{1}{2022}} \frac{e^x}{x^2+1}\, dx = 0$ ή $\displaystyle\int_0^{\xi} \frac{e^x}{x^2+1}\, dx = \int_{\xi+\frac{1}{2022}}^{1} \frac{e^x}{x^2+1}\, dx$.

Θεώρημα Μονότονης Σύγκλισης: Έστω $f_n : A \to \mathbb{R}$ μια ακολουθία συναρτήσεων ώστε:

1. $f_1(x) \leqslant f_2(x) \leqslant \cdots \leqslant f_n(x) \leqslant f_{n+1}(x) \leqslant \cdots, \forall n \in \mathbb{N}, \forall x \in A \subseteq \mathbb{R}$
2. $f_n \xrightarrow{\text{κ.σ.}} f$ στο A
3. $\displaystyle\int_A f_n(x)\, dx < +\infty$
4. $\displaystyle\int_A f(x)\, dx < +\infty$

Τότε $\lim\limits_{n\to+\infty}\int\limits_A f_n(x)\ dx = \int\limits_A f(x)\ dx.$

Θεώρημα Κυριαρχούμενης Σύγκλισης: Έστω $f_n : A \to \mathbb{R}$ μια ακολουθία συναρτήσεων ώστε:

1. $\exists g(x) : \big|f_n(x)\big| \leqslant g(x), \forall n \in \mathbb{N}, \forall x \in A \subseteq \mathbb{R}$
2. $\exists f(x) : f_n \xrightarrow{\text{χ.σ.}} f$ στο A
3. $\int\limits_A f_n(x)\ dx < +\infty$
4. $\int\limits_A f(x)\ dx < +\infty$
5. $\int\limits_A g(x)\ dx < +\infty$

Τότε $\lim\limits_{n\to+\infty}\int\limits_A f_n(x)\ dx = \int\limits_A f(x)\ dx.$

Θεώρημα *Bepo − Levi*: Έστω $f_n : A \to \mathbb{R}$ είναι μια ακολουθία συναρτήσεων, με

1. $\exists f(x) : \sum\limits_{n=1}^{+\infty} f_n \xrightarrow{\text{χ.σ.}} f$ στο A
2. $\sum\limits_{n=1}^{+\infty}\int\limits_A \big|f_n(x)\big|\ dx < +\infty$
3. $\int\limits_A \sum\limits_{n=1}^{+\infty} f_n(x)\ dx < +\infty$

Τότε $\sum\limits_{n=1}^{+\infty}\int\limits_A f_n(x)\ dx = \int\limits_A f(x)\ dx.$

Πρόβλημα(12): Έστω $f : [0,1] \to \mathbb{R}$ συνεχής συνάρτηση. Να βρεθεί το όριο

$$\lim_{n\to+\infty}\int\limits_0^1 (n+1)x^n f(x)\ dx.$$

<u>Λύση</u>

1. $\int\limits_0^1 (n+1)x^n f(x)\ dx = \int\limits_0^1 f\big(\sqrt[n+1]{t}\big)\ dt$, θέτοντας $t = x^{n+1}$ με $dt = (n+1)x^n dx$
2. Έστω $f_n : [0,1] \to \mathbb{R}$ μια ακολουθία συναρτήσεων με τύπο
3. $f_n(t) = f\big(\sqrt[n+1]{t}\big), t \in [0,1], \forall n \in \mathbb{N}$
4. $f_n(t)$ είναι ολοκληρώσιμες κατά *Riemann* στο $[0,1]$
5. $\exists M > 0 : \big|f_n(t)\big| \leqslant M, \forall x \in [0,1], \forall n \in \mathbb{N}$, επειδή η f είναι συνεχής στο $[0,1]$
6. $\mu([0,1]) = l([0,1]) = 1$
7. $f_n \xrightarrow{\text{κ.σ.}} f(1)$ σχεδόν παντού στο $[0,1]$
8. Τότε ισχύει: $\lim\limits_{n\to+\infty}\int\limits_0^1 (n+1)x^n f(x)\ dx = \lim\limits_{n\to+\infty}\int\limits_0^1 f\big(\sqrt[n+1]{t}\big)\ dt = \lim\limits_{n\to+\infty}\int\limits_0^1 f_n(t)\ dt =$
   $= \lim\limits_{n\to+\infty}\int\limits_{[0,1]} f_n\ d\mu = \int\limits_{[0,1]} f(1)\ d\mu = f(1).$

<u>Παρατηρήσεις</u>:

1. **<u>Θεώρημα *Weierstrass*</u>**: Για κάθε συνεχή συνάρτηση $f : [a,b] \to \mathbb{R}$ και για κάθε $\epsilon > 0$ υπαρχει πολυώνυμο $P(x)$ τέτοιο ώστε $\big|f(x) - P(x)\big| < \epsilon$, $x \in [a,b]$. Επιπλέον σχολιάζουμε ότι αν η συνάρτηση $f(x)$ είναι συνεχής στο $[a,b]$ τότε υπάρχει ακολουθία πολυωνύμων $P_n(x)$ η οποία συγκλίνει ομοιόμορφα, δηλαδή $\lim\limits_{n\to+\infty} P_n(x) = f(x)$, $x \in [a,b]$.
2. (a) Ας είναι $L_n\big(f(x)\big) = \int\limits_0^1 (n+1)x^n f(x)\ dx = (n+1)\int\limits_0^1 x^n f(x)\ dx, \forall n \in \mathbb{N}$

(b) Έχουμε ότι η συνάρτηση $f : [0,1] \to \mathbb{R}$ είναι συνεχής στο κλειστό διάστημα $[0,1]$

(c) Τότε από το Προσεγγιστικό Θεώρημα του $Karl - Weierstrass$ έχουμε ότι υπάρχει πολυώνυμο $p(x) = p_m x^m + p_{m-1}x^{m-1} + \cdots + p_1 x + p_0, \forall x \in [0,1]$, για κάποιο $m \in \mathbb{N}$ και για κάθε $\epsilon > 0$ να ισχύει $\big|f(x) - p(x)\big| < \frac{\epsilon}{3}, \forall x \in [0,1]$

(d) Παρατηρούμε ότι:

$$L_n\big(p(x)\big) = (n+1)\int_0^1 x^n p(x)\ dx = (n+1)\int_0^1 x^n \sum_{k=0}^{m} p_k x^k\ dx =$$

$$= (n+1)\int_0^1 \sum_{k=0}^{m} p_k x^{n+k}\ dx = (n+1)\sum_{k=0}^{m} p_k \int_0^1 x^{n+k}\ dx =$$

$$= (n+1)\sum_{k=0}^{m} \frac{p_k}{n+k+1} = \sum_{k=0}^{m} p_k \frac{n+1}{n+k+1} = \sum_{k=0}^{m} \frac{p_k}{1+\frac{k}{n+1}} =$$

$$= p_0 + \frac{p_1}{1+\frac{1}{n+1}} + \frac{p_2}{1+\frac{2}{n+1}} + \cdots + \frac{p_{m-1}}{1+\frac{m-1}{n+1}} + \frac{p_m}{1+\frac{m}{n+1}}, \forall n \in \mathbb{N}$$

(e) $$\lim_{n\to+\infty} L_n\big(p(x)\big) = p_0 + p_1 + \cdots + p_{m-1} + p_m = p(1) \Leftrightarrow$$

$$\Leftrightarrow \forall \epsilon > 0, \exists n_0 \in \mathbb{N} : \Big|L_n\big(p(x)\big) - p(1)\Big| < \frac{\epsilon}{3}, \forall n \geqslant n_0$$

(f) $$\forall \epsilon > 0, \exists n_0 \in \mathbb{N} : \Big|L_n\big(f(x)\big) - f(1)\Big| \leqslant$$

$$\leqslant \Big|L_n\big(f(x)\big) - L_n\big(p(x)\big)\Big| + \Big|L_n\big(p(x)\big) - p(1)\Big| + \big|p(1) - f(1)\big| <$$

$$< \Big|(n+1)\int_0^1 x^n f(x)\ dx - (n+1)\int_0^1 x^n p(x)\ dx\Big| + \frac{\epsilon}{3} + \frac{\epsilon}{3} \leqslant$$

$$\leqslant (n+1)\int_0^1 x^n \big|f(x) - p(x)\big|\ dx + \frac{2\cdot\epsilon}{3} < (n+1)\int_0^1 x^n \cdot \frac{\epsilon}{3}\ dx + \frac{2\cdot\epsilon}{3} =$$

$$= \frac{\epsilon}{3} + \frac{2\cdot\epsilon}{3} = \epsilon, \forall n \geqslant n_0 \Leftrightarrow \lim_{n\to+\infty} L_n\big(f(x)\big) = f(1)$$

3. (a) $$L_n\big(f(x)\big) = \int_0^1 (n+1)x^n f(x)\ dx = (n+1)\int_0^1 x^n f(x)\ dx =$$

$$= \frac{\int\limits_0^1 x^n f(x)\ dx}{\frac{1}{n+1}} = \frac{\overbrace{\int\limits_0^1 x^n f(x)\ dx}^{=a_n}}{\underbrace{\int\limits_0^1 x^n\ dx}_{=b_n}} = \frac{a_n}{b_n}$$

(b) Η ακολουθία $\big(b_n\big)_{n=1}^{+\infty}$ είναι γνήσια θετική, φθίνουσα και φραγμένη

(c) $\dfrac{a_{n+1} - a_n}{b_{n+1} - b_n}$

(d) Σαν να φαίνεται ότι μπορεί να επιλυθεί με το Θεώρημα $Stolz - Cesaro$

4. Όμοιο αποτέλεσμα έχουμε αν

   (a) Δίδεται η συνεχής συνάρτηση $f : [0, 1] \to \mathbb{R}$. Τότε $\lim\limits_{n\to+\infty} \int\limits_0^1 nx^n f(x)\ dx = f(1)$.

   (b) Δίδεται η συνεχής συνάρτηση $f : [0, 1] \to \mathbb{R}$. Τότε $\lim\limits_{n\to+\infty} \int\limits_0^1 x^n f(x)\ dx = 0$.

   (c) Δίδεται η συνεχής συνάρτηση $f : [0, 1] \to \mathbb{R}$. Τότε $\lim\limits_{n\to+\infty} \int\limits_0^1 f(x^n)\ dx = f(0)$.

Πρόβλημα(13): Έστω μια παραγωγίσιμη συνάρτηση $f : [0, 1] \to \mathbb{R}$ με $f(1) = 1$ και $f(t) + tf'(t) \neq 0, \forall t \in (0, 1)$. Τότε να αποδείξετε ότι: $\lim\limits_{n\to+\infty} \int\limits_0^1 \big(f(\sqrt[n]{x})\big)^n\ dx = \dfrac{1}{1 + f'(1)}$.

Λύση

Έστω ότι $\boxed{I_n = \int\limits_0^1 \big(f(\sqrt[n]{x})\big)^n\ dx\ ,\ n \geqslant 2}$ με το n να είναι ένας φυσικός αριθμός. Τότε

$$I_n = \int_0^1 \big(f(\sqrt[n]{x})\big)^n \ dx \xRightarrow[x=0\,:\,t=0,x=1,t=1]{t=\sqrt[n]{x}\Leftrightarrow x=t^n,dx=nt^{n-1}dt} I_n = \int_0^1 \big(f(t)\big)^n nt^{n-1}\ dt =$$

$$= n\int_0^1 f(t)\big(tf(t)\big)^{n-1}\ dt \xRightarrow[g'(t)=f(t)+tf'(t)\neq 0,\forall t\in(0,1)]{\boxed{g(t)=tf(t),t\in[0,1]}} I_n = n\int_0^1 f(t)\big(g(t)\big)^{n-1}\ dt \Rightarrow$$

$$\xRightarrow[t=0\,:\,y=g(0)=0,t=1\,:\,y=g(1)=f(1)=1]{t=g^{-1}(y)=h(y)\Leftrightarrow y=g(t),dt=h'(y)dy} I_n = n\int_0^1 f\big(h(y)\big)y^{n-1}h'(y)\ dy =$$

$$= n\int_0^1 y^{n-1}f\big(h(y)\big)h'(y)\ dy \xRightarrow{\boxed{F(y)=f\big(h(y)\big)h'(y),y\in[0,1]}}$$

$$\Rightarrow I_n = n\int_0^1 y^{n-1}F(y)\ dy \xRightarrow[y=0\,:\,x=0,y=1\,:\,x=1]{y=\sqrt[n]{x}\Leftrightarrow x=y^n,dx=ny^{n-1}dy} \boxed{I_n = \int_0^1 F\big(\sqrt[n]{x}\big)\ dx\ ,\ n\geqslant 2},$$

και στη συνέχεια γνωρίζουμε ότι $\lim\limits_{n\to+\infty} I_n = \lim\limits_{n\to+\infty} F\big(\sqrt[n]{x}\big)\ dx = F(1) = f\big(h(1)\big)h'(1) = \dfrac{1}{1+f'(1)}$.

<u>Παρατηρησεις</u>:

1. Εφαρμόζουμε $f(t) = \dfrac{k}{t+k-1}, t\in[0,1]$, όπου $f(1) = \dfrac{k}{1+k-1} = \dfrac{k}{k} = 1$
   και η συνάρτηση $f(t)$ είναι παραγωγίσιμη με $f'(t) = -\dfrac{k}{\big(t+k-1\big)^2} = -\dfrac{\big(f(t)\big)^2}{k}, t\in[0,1]$
   με
   $f(t)+tf'(t) = \dfrac{k}{t+k-1} - \dfrac{kt}{\big(t+k-1\big)^2} = \dfrac{kt+k(k-1)-kt}{\big(t+k-1\big)^2} = \dfrac{k(k-1)}{\big(t+k-1\big)^2} \neq 0, t\in[0,1]$,
   και $\dfrac{1}{1+f'(1)} = \dfrac{1}{1-\frac{1}{k}} = \dfrac{k}{k-1}$, οπότε το ζητούμενο όριο ισούται με
   $$L = \lim_{n\to+\infty}\int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n dx = \frac{k}{k-1}, k>1.$$

2. Έστω τώρα $f(t) = t^2-2t+2, \forall t\in[0,1]$ με $f(1)=1$, $f'(t) = 2t-2, \forall t\in[0,1]$
   και
   $f(t)+tf'(t) = t^2-2t+2+t\big(2t-2\big) = t^2-2t+2+2t^2-2t = 3t^2-4t+2 =$
   $= t^2+2\big(t-1\big)^2 \neq 0, \forall t\in[0,1]$, επιπλέον είναι $\dfrac{1}{1+f'(1)} = \dfrac{1}{1+0} = 1$,

συνεπώς έχουμε ότι $L = \lim\limits_{n\to+\infty} \int\limits_0^1 \left(\sqrt[n]{x^2} - 2\sqrt[n]{x} + 2\right)^n \, dx = 1.$

3. <u>**Ερώτημα**</u> : Να υπολογιστεί το όριο $\lim\limits_{n\to+\infty} \left[n\Big(L - \int\limits_0^1 \left(f(\sqrt[n]{x})\right)^n \, dx \Big) \right].$

<u>**Πρόβλημα**(13)</u>: Να υπολογιστούν τα παρακάτω όρια:

1. $\lim\limits_{n\to+\infty} \int\limits_0^1 \frac{1+nx^2}{(1+x^2)^n} \, dx$

2. $\lim\limits_{n\to+\infty} \int\limits_0^{+\infty} \frac{e^{-x}\cos x}{nx^2 + \frac{1}{n}} \, dx$

3. $\lim\limits_{n\to+\infty} n \int\limits_0^n \frac{arctan(\frac{x}{n})}{x(x^2+1)} dx$

4. $\lim\limits_{n\to+\infty} n\Bigg(n \int\limits_0^n \frac{arctan(\frac{x}{n})}{x(x^2+1)} dx - \frac{\pi}{2} \Bigg)$

5. $\lim\limits_{n\to+\infty} n^{k+1} \int\limits_0^1 x^k \Big(\frac{1-x}{1+x}\Big)^n \, dx, \forall k \in \mathbb{N} \cup \{0\}$

<u>**Λύση**</u>

1. (a) Έστω $f_n : [0,1] \to \mathbb{R}$ μια ακολουθία συναρτήσεων με τύπο

   (b) $f_n(x) = \frac{1+nx^2}{(1+x^2)^n}, x \in [0,1], \forall n \in \mathbb{N}$

   (c) $f_n(x)$ είναι ολοκληρώσιμες κατά *Riemann* στο $[0,1]$

   (d) $\left| \frac{1+nx^2}{(1+x^2)^n} \right| \leqslant 1, \forall n \in \mathbb{N}$ σχεδόν παντού στο $[0,1]$

   (e) $\mu([0,1]) = l([0,1)) = 1$

(f) $f_n \xrightarrow{\text{κ.σ.}} 0$

(g) Τότε ισχύει: $\displaystyle\lim_{n\to+\infty}\int\limits_0^1 \frac{1+nx^2}{(1+x^2)^n}\,dx = \lim_{n\to+\infty}\int\limits_0^1 f_n(x)\,dx =$

$$= \lim_{n\to+\infty}\int\limits_{[0,1]} f_n\,d\mu = \int\limits_{[0,1]} 0\,d\mu = 0.$$

2. (a) $\displaystyle\int\limits_0^{+\infty} \frac{e^{-x}\cos x}{nx^2+\frac{1}{n}}\,dx = n\int\limits_0^{+\infty} \frac{e^{-x}\cos x}{(nx)^2+1}\,dx = \int\limits_0^{+\infty} \big(e^{-x}\cos x\big)\big(arctan(nx)\big)'\,dx =$

$$= \Big[\cancelto{0}{e^{-x}\cdot\cos x\cdot arctan(nx)}\Big]_{x=0}^{x\to+\infty} - \int\limits_0^{+\infty} \big(-e^{-x}\cos x - e^{-x}\sin x\big)arctan(nx)\,dx =$$

$$= \int\limits_0^{+\infty} e^{-x}\cdot\big(\cos x+\sin x\big)\cdot arctan(nx)\,dx$$

(b) Έστω $f_n : [0,+\infty) \to \mathbb{R}$ μια ακολουθία συναρτήσεων με τύπο

(c) $f_n(x) = e^{-x}\cdot\big(\cos x+\sin x\big)\cdot arctan(nx), x\in[0,+\infty), \forall n\in\mathbb{N}$

(d) Η συνάρτηση $g(x) = \frac{\pi}{2}e^{-x}\cdot\big(\cos x+\sin x\big), \forall x\in[0,+\infty)$ είναι ολοκληρώσιμη κατά *Riemann* σε κάθε κλειστό διάστημα της μρφής $[0,M], M>0$ και $\displaystyle\int\limits_0^{+\infty} g(x)\,dx = \Big[-\frac{\pi}{2}e^{-x}\cdot\cos x\Big]_{x=0}^{x\to+\infty} = \frac{\pi}{2} < +\infty$

(e) $\big|f_n(x)\big| \leqslant \frac{\pi}{2}e^{-x}\cdot\big(\cos x+\sin x\big), \forall x\in[0,+\infty), \forall n\in\mathbb{N}$

(f) $f_n(x)$ είναι ολοκληρώσιμες κατά *Riemann* στο $[0,+\infty)$

(g) $f_n \xrightarrow{\text{κ.σ.}} g$

(h) $\displaystyle\lim_{n\to+\infty}\int\limits_0^{+\infty} \frac{e^{-x}\cos x}{nx^2+\frac{1}{n}}\,dx = \lim_{n\to+\infty}\int\limits_0^{+\infty} e^{-x}\cdot\big(\cos x+\sin x\big)\cdot arctan(nx)\,dx =$

$$= \lim_{n\to+\infty}\int\limits_0^{+\infty} f_n(x)\,dx = \lim_{n\to+\infty}\int\limits_{[0,+\infty)} f_n\,d\mu = \int\limits_{[0,+\infty)} g\,d\mu = \int\limits_0^{+\infty} g(x)\,dx = \frac{\pi}{2}$$

3. (a) Θεωρούμε την ακολουθία συναρτήσεων $f_n : \mathbb{R} \to \mathbb{R}$ με τύπο $f_n(x) = n\dfrac{arctan(\frac{x}{n})}{x(x^2+1)}\chi_{(0,n]}(x)$.

   (b) $\big(f_n\big)_{n=1}^{+\infty}$ είναι *Riemann* ολοκληρώσιμη στο $(0, +\infty)$.

   (c) $f(x) = \dfrac{1}{1+x^2}\chi_{(0,+\infty)}(x)$ είναι *Riemann* ολοκληρώσιμη στο $(0, +\infty)$ με $f_n \xrightarrow{\varkappa.\sigma.} f$.

   (d) $\exists f \in L^1 : \big|f_n(x)\big| \leqslant f(x), \forall n \in \mathbb{N}, \forall x \in (0, +\infty)$.

   Τότε $$\lim_{n\to+\infty} n\int\limits_0^n \frac{arctan(\frac{x}{n})}{x(x^2+1)}dx = \lim_{n\to+\infty}\int\limits_{\mathbb{R}} f_n(x)\ dx = \lim_{n\to+\infty}\int\limits_{\mathbb{R}} f_n\ d\mu = \int\limits_{\mathbb{R}} f\ d\mu =$$
   $$= \int\limits_{\mathbb{R}} f(x)\ dx = \int\limits_0^{+\infty}\frac{1}{1+x^2}\ dx = \frac{\pi}{2}.$$

4. (a) $x_n = n^2\displaystyle\int\limits_0^n \frac{arctan(\frac{x}{n})}{x(1+x^2)}\ dx - n\frac{\pi}{2}, \forall n \in \mathbb{N}$

   (b) $z_n = n\displaystyle\int\limits_0^n \frac{1}{1+x^2}\ dx = n\Big(\frac{\pi}{2} - arctan(n)\Big), \forall n \in \mathbb{N}$ με $\displaystyle\lim_{n\to+\infty} z_n = 1$

   (c) $$y_n = n^2\int\limits_0^n \frac{arctan(\frac{x}{n})}{x(1+x^2)}\ dx - n\int\limits_0^n \frac{1}{1+x^2}\ dx = \int\limits_0^n \frac{n^2\cdot arctan(\frac{x}{n}) - nx}{x(1+x^2)}\ dx =$$
   $$= n\int\limits_0^1 \frac{n^2\cdot arctan(t) - n^2t}{nt(1+n^2t^2)}\ dt = \int\limits_0^1 \frac{arctan(t)-t}{t^3}\cdot\frac{n^2t^2}{1+n^2t^2}\ dt, \forall n \in \mathbb{N}$$

      i. Θεωρούμε την ακολουθία συναρτήσεων $f_n : (0, 1] \to \mathbb{R}$ με τύπο $f_n(t) = \dfrac{arctan(t)-t}{t^3}\cdot\dfrac{n^2t^2}{1+n^2t^2}\chi_{(0,1]}(t)$.

      ii. $\big(f_n\big)_{n=1}^{+\infty}$ είναι *Riemann* ολοκληρώσιμη στο $(0, 1]$.

      iii. $f(t) = \dfrac{arctan(t)-t}{t^3}\chi_{(0,1]}(t)$ είναι *Riemann* ολοκληρώσιμη στο $(0, 1]$ με $f_n \xrightarrow{\varkappa.\sigma.} f$ και $\displaystyle\lim_{t\to0^+} f(t) = -\frac{1}{3}$.

      iv. $\exists f \in L^1 : \big|f_n(t)\big| \leqslant -f(t), \forall n \in \mathbb{N}, \forall t \in (0, 1]$

Τότε $\lim\limits_{n\to+\infty} y_n = \lim\limits_{n\to+\infty} \int\limits_{\mathbb{R}} f_n(x)\ dx = \lim\limits_{n\to+\infty} \int\limits_{\mathbb{R}} f_n\ d\mu = \int\limits_{\mathbb{R}} f\ d\mu = \int\limits_0^1 f(t)\ dt =$

$$= \int\limits_0^1 \frac{arctan(t) - t}{t^3}\ dt = \left[\frac{t - arctan(t)}{2t^2} - \frac{arctan(t)}{2}\right]_{t\to 0}^{t=1} = \frac{1}{2} - \frac{\pi}{4}$$

(d) $x_n = y_n - z_n, \forall n \in \mathbb{N}$

(e) $\lim\limits_{n\to+\infty} x_n = \lim\limits_{n\to+\infty} y_n - \lim\limits_{n\to+\infty} z_n = -\frac{1}{2} - \frac{\pi}{4}$

5. (a) $n^{k+1} \int\limits_0^1 x^k \Big(\frac{1-x}{1+x}\Big)^n\ dx = 2n^{k+1} \int\limits_0^1 \Big(\frac{1-t}{1+t}\Big)^k \frac{t^n}{(1+t)^2}\ dt = 2n^{k+1} \int\limits_0^1 t^n \frac{(1-t)^k}{(1+t)^{k+2}}\ dt$

(b) Θεωρούμε τη συνάρτηση $f : [0,1] \to \mathbb{R}$ με τύπο $f(t) = \frac{(1-t)^k}{(1+t)^{k+2}}$

με $f^{(i)}(1) = 0, \forall i \in \{0, 1, 2, \dots, k-1\}$ και $f^{(k)}(1) = \frac{(-1)^k k!}{2^{k+2}}$

(c) Θέλουμε να υπολογίσουμε το όριο $2 \cdot \lim\limits_{n\to+\infty} n^{k+1} \int\limits_0^1 t^n f(t)\ dt$

(d) $n^{k+1} \int\limits_0^1 t^n f(t)\ dt = n^{k+1} \int\limits_0^1 \Big[\frac{t^{n+1}}{n+1}\Big]' f(t)\ dt = -\frac{n^{k+1}}{n+1} \int\limits_0^1 t^{n+1} f'(t)\ dt =$

$$= -\frac{n^{k+1}}{n+1} \int\limits_0^1 \Big[\frac{t^{n+2}}{n+2}\Big]' f'(t)\ dt = (-1)^2 \frac{n^{k+1}}{(n+1)(n+2)} \int\limits_0^1 t^{n+2} f''(t)\ dt = \cdots =$$

$$= \cdots = (-1)^k \frac{n^{k+1}}{(n+1)(n+2)\dots(n+k)} \int\limits_0^1 t^{n+k} f^{(k)}(t)\ dt =$$

$$= (-1)^k \frac{n^{k+1}}{(n+1)(n+2)\dots(n+k)} \int\limits_0^1 \Big[\frac{t^{n+k+1}}{n+k+1}\Big]' f^{(k)}(t)\ dt =$$

$$= \frac{(-1)^k n^{k+1} f^{(k)}(1)}{(n+1)(n+2)\dots(n+k)(n+k+1)} +$$

$$+ \frac{(-1)^{k+1} n^{k+1}}{(n+1)(n+2)\dots(n+k)(n+k+1)} \int\limits_0^1 t^{n+k+1} f^{(k+1)}(t)\ dt =$$

$$= \frac{(-1)^k f^{(k)}(1)}{(1+\frac{1}{n})(1+\frac{2}{n})\dots(1+\frac{k}{n})(1+\frac{k+1}{n})} +$$

$$+ \frac{(-1)^{k+1}}{(1+\frac{1}{n})(1+\frac{2}{n})\dots(1+\frac{k}{n})(1+\frac{k+1}{n})} \int_0^1 t^{n+k+1} f^{(k+1)}(t)\ dt$$

(e) Η συνάρτηση $f^{(k+1)}(t)$ είναι συνεχής στο κλειστό διάστημα $t \in [0,1]$. Άρα λαμβάνει μέγιστη και ελάχιστη τιμή σε αυτό το διάστημα. Οπότε $\exists M > 0 : \left|f^{(k+1)}(t)\right| \leqslant M, \forall t \in [0,1]$

(f) $0 \leqslant \left| \int_0^1 t^{n+k+1} f^{(k+1)}(t)\ dt \right| \leqslant \int_0^1 t^{n+k+1} \left|f^{(k+1)}(t)\right|\ dt \leqslant M \int_0^1 t^{n+k+1}\ dt = \frac{M}{n+k+2}$ για κάθε $n \in \mathbb{N}$

(g) $\lim\limits_{n\to+\infty} \int_0^1 t^{n+k+1} f^{(k+1)}(t)\ dt = 0$

(h) $\lim\limits_{n\to+\infty} n^{k+1} \int_0^1 x^k \Big(\frac{1-x}{1+x}\Big)^n\ dx = 2 \cdot \lim\limits_{n\to+\infty} n^{k+1} \int_0^1 t^n f(t)\ dt =$

$= 2(-1)^k f^{(k)}(1) = 2(-1)^k \frac{(-1)^k k!}{2^{k+2}} = \frac{k!}{2^{k+1}}, \forall k \in \mathbb{N} \cup \{0\}$

Παρατήρηση: Δίδεται συνεχή συνάρτηση $f : [0,1] \to \mathbb{R}$ με $f^{(i)}(1) = 0, \forall i \in \{0, 1, \dots, k-1\}$ και $\exists f^{(k)}(1)$, για κάποιο $k \in \mathbb{N}$. Τότε $\lim\limits_{n\to+\infty} n^{k+1} \int_0^1 x^n f(x)\ dx = (-1)^k f^{(k)}(1)$.

Πρόβλημα(14): Δίδεται η ακολουθία συνεχών συναρτήσεων $K_n : [0,1] \to [0,+\infty)$ και ο γραμμικός τελεστής $L_n : C[0,1] \to \mathbb{R}$ με τύπο $L_n\big(f\big) = \int_0^1 K_n(x) f(x)\ dx$.

Αν υπάρχει $x_0 \in [0,1]$ ώστε $L_n\big(x^i\big) = x_0^i, \forall i = 0, 1, 2, \dots$(για πεπερασμενο i) για κάθε $n \in \mathbb{N}$. Τότε $\lim\limits_{n\to+\infty} L_n\big(f\big) = f(x_0), \forall f \in C[0,1]$.

Λύση

1. Έστω μια τυχαία συνεχή συνάρτηση $f : [0,1] \to \mathbb{R}$

2. Γνωρίζουμε ότι υπάρχει πολυώνυμο $p(x) = p_m x^m + p_{m-1} x^{m-1} + \cdots + p_1 x + p_0, \forall x \in [0,1]$, για κάποιο $m \in \mathbb{N}$ και για κάθε $\epsilon > 0$ ισχύει $\big|f(x) - p(x)\big| < \frac{\epsilon}{3}, \forall x \in [0,1]$

3. $\forall n \in \mathbb{N} : L_n\big(p\big) = \int\limits_0^1 K_n(x) p(x)\ dx = \int\limits_0^1 K_n(x) \sum\limits_{i=0}^m p_i x^i\ dx = \sum\limits_{i=0}^m p_i \int\limits_0^1 K_n(x) x^i\ dx =$
   $= p_0 \int\limits_0^1 K_n(x)\ dx + p_1 \int\limits_0^1 K_n(x) x\ dx + p_2 \int\limits_0^1 K_n(x) x^2\ dx + \cdots + p_m \int\limits_0^1 K_n(x) x^m\ dx =$
   $= p_0 + p_1 x_0 + p_2 x_0^2 + \cdots + x_0^m\ dx = p(x_0)$

4. $\lim\limits_{n \to +\infty} L_n\big(p\big) = p(x_0) \Leftrightarrow \forall \epsilon > 0, \exists n_0 \in \mathbb{N} : \Big|L_n\big(p\big) - p(x_0)\Big| < \frac{\epsilon}{3}, \forall n \geqslant n_0$

5. $\forall \epsilon > 0, \exists n_0 \in \mathbb{N} : \Big|L_n\big(f(x)\big) - f(x_0)\Big| \leqslant$
   $\leqslant \Big|L_n\big(f(x)\big) - L_n\big(p(x)\big)\Big| + \Big|L_n\big(p(x)\big) - p(x_0)\Big| + \big|p(x_0) - f(x_0)\big| <$
   $< \left| \int\limits_0^1 K_n(x) f(x)\ dx - \int\limits_0^1 K_n(x) p(x)\ dx \right| + \frac{\epsilon}{3} + \frac{\epsilon}{3} \leqslant$
   $\leqslant \int\limits_0^1 K_n(x) \big|f(x) - p(x)\big|\ dx + \frac{2 \cdot \epsilon}{3} < \int\limits_0^1 K_n(x) \cdot \frac{\epsilon}{3}\ dx + \frac{2 \cdot \epsilon}{3} =$
   $= \frac{\epsilon}{3} + \frac{2 \cdot \epsilon}{3} = \epsilon, \forall n \geqslant n_0 \Leftrightarrow \lim\limits_{n \to +\infty} L_n\big(f(x)\big) = f(x_0).$

Εφαρμογη: Να υπολογιστεί το όριο $\lim\limits_{n \to +\infty} \frac{(2n+1)!}{(n!)^2} \int\limits_0^1 \big(x(1-x)\big)^n f(x)\ dx$,

όπου $f : [0,1] \to \mathbb{R}$ είναι μια συνεχής συνάρτηση.

Integrals*

(i)

$$\int_1^\infty \frac{dx}{x+x^m} = \frac{\ln(2)}{m-1}, \quad m > 1.$$

(ii)

$$\int_0^\infty \frac{\ln(x)}{x^2+1} dx = 0.$$

(iii)

$$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)}+\sqrt{\cos(x)}} dx = \frac{\pi}{4}.$$

(iv)

$$\int_0^1 x^a (\ln(x))^2 dx = \frac{2}{(a+1)^3}, \quad a > -1.$$

(v)

$$\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) = \sin^{-1}(x), \quad x > 0.$$

(vi)

$$\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}.$$

(vii)

$$\int_0^\infty \frac{dx}{x^4+2x^2\cos(2a)+1} = \frac{\pi}{4|\cos(a)|}, \quad a \in \mathbb{R}\backslash\{k\pi+\pi/2 \,|\, k \in \mathbb{Z}\}.$$

(viii)

$$\int_0^1 x^x dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots.$$

(ix)

$$\int_1^\infty \frac{\{x\}-\frac{1}{2}}{x} dx = -1 + \ln(\sqrt{2\pi}),$$

where $\{x\} \in [0,1)$ stands for the fractional part of $x \geq 0$.

(x)

$$\int_0^1 \frac{x^a-1}{\ln(x)} dx = \ln(a+1), \quad a \geq 0.$$

$*$ The list is taken from:

- • G. Boros, V. Moll. *Irresistible integrals.* Cambridge University Press (2004).
- • P. J. Nahin. *Inside Interesting Integrals.* Undergraduate Lecture Notes in Physics, Springer Verlag (2015)

Solutions

(i) We have

$$\int_1^\infty \frac{dx}{x+x^m} = \int_1^\infty \frac{x^{-m}}{x^{1-m}+1}dx = [(1-m)^{-1}\ln(x^{1-m}+1)]_1^\infty = \frac{\ln(2)}{m-1}.$$

(ii) We have

$$\int_0^\infty \frac{\ln(x)}{x^2+1}dx = \int_0^1 \frac{\ln(x)}{x^2+1}dx + \int_1^\infty \frac{\ln(x)}{x^2+1}dx.$$

Setting $x = 1/t$ to the first integral on the right hand side, we get

$$\int_0^1 \frac{\ln(x)}{x^2+1}dx = \int_1^\infty \frac{\ln(t^{-1})}{t^{-2}+1}t^{-2}dt = -\int_1^\infty \frac{\ln(x)}{x^2+1}dx.$$

(iii) We have

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin(x)}}{\sqrt{\sin(x)}+\sqrt{\cos(x)}}dx \overset{y=\frac{\pi}{2}-x}{=} \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos(y)}}{\sqrt{\cos(y)}+\sqrt{\sin(y)}}dy.$$

Thus

$$2I = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}.$$

(iv) We have

$$\int_0^1 x^a \ln(x)dx = \frac{d}{da}\int_0^1 x^a dx = \frac{d}{da}(a+1)^{-1} = -(a+1)^{-2}.$$

Thus,

$$\int_0^1 x^a(\ln(x))^2dx = \frac{d}{da}\int_0^1 x^a\ln(x)dx = -\frac{d}{da}(a+1)^{-2} = 2(a+1)^{-3}.$$

(v) It suffices to show that

$$\int_0^{x/\sqrt{1-x^2}} \frac{dt}{1+t^2} = \int_0^x \frac{dy}{\sqrt{1-y^2}},$$

which holds true by changing $t = y/\sqrt{1-y^2}$.

(vi) Let $r, R > 0$ and consider the counterclockwise path $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ in $\mathbb{C}$, where $\Gamma_1 = \{(x,0)\,|\,x \in [r,R]\}$, $\Gamma_2 = \{Re^{i\theta}\,|\,\theta \in [0.\pi/2]\}$, $\Gamma_3 = \{(0,iy)\,|\,y \in [r,R]\}$ and $\Gamma_4\{re^{i\theta}\,|\,\theta \in [0.\pi/2]\}$. If $f(z) = e^{iz}/z$, $z \in \mathbb{C}\backslash\{0\}$, then

$$0 = \oint_\Gamma f(z)dz = \int_r^R \frac{e^{ix}}{x}dx + \int_0^{\frac{\pi}{2}} \frac{e^{ie^{iR\theta}}}{Re^{i\theta}}iRe^{i\theta}d\theta - \int_r^R \frac{e^{i(iy)}}{iy}idy - \int_0^{\frac{\pi}{2}} \frac{e^{ie^{ir\theta}}}{re^{i\theta}}ire^{i\theta}d\theta,$$

which gives

$$\int_r^R \frac{e^{ix}-e^{-x}}{x}dx + i\Big(\int_0^{\frac{\pi}{2}}\Big(e^{iRe^{i\theta}} - e^{ire^{i\theta}}\Big)d\theta\Big) = 0. \tag{1}$$

We have

$$e^{iRe^{i\theta}} - e^{ire^{i\theta}} = e^{-R\sin(\theta)}e^{iR\cos(\theta)} - e^{-r\sin(\theta)}e^{ir\cos(\theta)},$$

so that, by taking $r \to 0$ and $R \to \infty$ in (1), we get

$$\int_0^\infty \frac{\cos(x) - i\sin(x) - e^{-x}}{x}dx + i\int_0^{\frac{\pi}{2}}(-1)d\theta = 0.$$

(vii) We have

$$I=\int_0^\infty \frac{dx}{x^4+2x^2\cos(2a)+1} \overset{x=1/y}{=} \int_0^\infty \frac{y^2dx}{y^4+2y^2\cos(2a)+1}.$$

Therefore,

$$I=\frac{1}{2}\int_0^\infty \frac{(x^2+1)dx}{x^4+2x^2\cos(2a)+1},$$

and since the integrand is even, we get

$$I=\frac{1}{4}\int_{-\infty}^{+\infty} \frac{(x^2+1)dx}{x^4+2x^2\cos(2a)+1}.$$

By using $\cos(2a)=1-2\sin^2(a)$, we observe that

$$x^4+2x^2\cos(2a)+1=(x^2-2x\sin(a)+1)(x^2+2x\sin(a)+1).$$

Hence,

$$\begin{aligned} I&=\frac{1}{4}\int_{-\infty}^{\infty} \frac{(x^2+1)dx}{(x^2-2x\sin(a)+1)(x^2+2x\sin(a)+1)}\\ &=\frac{1}{4}\int_{-\infty}^{\infty} \frac{(x^2-2x\sin(a)+1)dx}{(x^2-2x\sin(a)+1)(x^2+2x\sin(a)+1)}\\ &=\frac{1}{4}\int_{-\infty}^{\infty} \frac{dx}{x^2+2x\sin(a)+1}=\frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^2+2x\sin(a)+\sin^2(a)+\cos^2(a)}\\ &=\frac{1}{4}\int_{-\infty}^{\infty} \frac{dx}{(x+\sin(a))^2+\cos^2(a)}, \end{aligned}$$

where at the second equality we used the fact that $2x\sin(a)$ is odd. Letting $x+\sin(a)=u$, we deduce

$$I=\frac{1}{4}\int_{-\infty}^{\infty} \frac{du}{u^2+\cos^2(a)}=\frac{1}{4|\cos(a)|}\big[\tan^{-1}\big(\frac{u}{\cos(a)}\big)\big]_{-\infty}^{+\infty}=\frac{\pi}{4|\cos(a)|}.$$

(viii) We start with

$$e^y=1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\cdots,\quad y\in\mathbb{R},$$

where, by setting $y=cx^\nu\ln(x)$, $x>0$, for certain $\nu\in\mathbb{N}$ and $c\in\mathbb{R}$, we obtain

$$x^{cx^\nu}=1+cx^\nu\ln(x)+\frac{c^2}{2!}x^{2\nu}\ln^2(x)+\frac{c^3}{3!}x^{3\nu}\ln^3(x)+\cdots.$$

Hence

$$\int_0^1 x^{cx^\nu}dx=\int_0^1\Big(\sum_{k=0}^\infty \frac{c^k}{k!}x^{k\nu}\ln^k(x)\Big)dx. \tag{2}$$

For any $m,n\in\mathbb{N}\cup\{0\}$,we have

$$\int_0^1 x^m\ln^n(x)dx=\int_0^1\Big(\frac{x^{m+1}}{m+1}\Big)'\ln^n(x)dx=-\frac{n}{m+1}\int_0^1 x^m\ln^{n-1}(x)dx=\cdots=(-1)^n\frac{n!}{(m+1)^{n+1}}.$$

Thus, we can interchange the integral and the sum in (2) to obtain

$$\int_0^1 x^{cx^\nu}dx=\sum_{k=0}^\infty(-1)^k\frac{c^k}{(k\nu+1)^{k+1}}.$$

(ix) We start with

$$\ln(n!)=\sum_{k=2}^n\ln(k)=\sum_{k=2}^n\int_1^k\frac{dx}{x}=\sum_{k=2}^n\sum_{j=1}^{k-1}\int_j^{j+1}\frac{dx}{x},$$

where $n \in \mathbb{N}$, $n \geq 2$. For the double sum above, we write

$$\begin{array}{ccccccccccc}
k=2 & : & \int_1^2 & & & & & & & & \\
k=3 & : & \int_1^2 & + & \int_2^3 & & & & & & \\
k=4 & : & \int_1^2 & + & \int_2^3 & + & \int_3^4 & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & & \\
k=n & : & \int_1^2 & + & \int_2^3 & + & \int_3^4 & + & \cdots & + & \int_{n-1}^n
\end{array}$$

and then summing each column to obtain

$$\begin{aligned}
\ln(n!) &= (n-1)\int_1^2 \frac{dx}{x} + (n-2)\int_2^3 \frac{dx}{x} + \cdots + \int_{n-1}^n \frac{dx}{x} \\
&= \int_1^2 \frac{n-\lfloor x\rfloor}{x}dx + \int_2^3 \frac{n-\lfloor x\rfloor}{x}dx + \cdots + \int_{n-1}^n \frac{n-\lfloor x\rfloor}{x}dx \ = \ \int_1^n \frac{n-\lfloor x\rfloor}{x}dx,
\end{aligned}$$

where by $\lfloor\cdot\rfloor$ we mean integer part, i.e. $x = \lfloor x\rfloor + \{x\}$, for each $x \in \mathbb{R}$. Therefore

$$\begin{aligned}
\ln(n!) &= \int_1^n \frac{n}{x}dx - \int_1^n dx + \int_1^n \frac{\{x\}}{x}dx \\
&= n\ln(n) - n + 1 + \int_1^n \frac{\{x\}}{x}dx \ = \ n\ln(n) - n + 1 + \frac{\ln(n)}{2} + \int_1^n \frac{\{x\}-1/2}{x}dx \\
&= (n+1/2)\ln(n) - n + 1 + \int_1^n \frac{\{x\}-1/2}{x}dx \\
&= \ln(n^{n+\frac{1}{2}}) + \ln(e^{-n}) + 1 + \int_1^n \frac{\{x\}-1/2}{x}dx,
\end{aligned}$$

which implies that

$$n! = n^{n+\frac{1}{2}}e^{-n}e^{1+\int_1^n \frac{\{x\}-1/2}{x}dx}.$$

Hence,

$$e^{1+\int_1^n \frac{\{x\}-1/2}{x}dx} = \frac{n!}{n^{n+\frac{1}{2}}e^{-n}},$$

where, by taking the limit as $n \to \infty$, and using Stirling's asymptotic formula for $n!$, namely

$$\lim_{n\to\infty} \frac{n!}{n^{n+\frac{1}{2}}e^{-n}} = \sqrt{2\pi},$$

we find the result.

(x) We have

$$\int_0^a x^y dy = \int_0^a e^{y\ln(x)}dy = \frac{x^a-1}{\ln(x)}.$$

Consequently,

$$\begin{aligned}
\int_0^1 \frac{x^a-1}{\ln(x)}dx &= \int_0^1\int_0^a x^y dydx \\
&= \int_0^a\int_0^1 x^y dxdy = \int_0^a [(y+1)^{-1}x^{y+1}]_0^1 dy = \int_0^a (y+1)^{-1}dy = [\ln(u)]_1^{a+1} = \ln(a+1).
\end{aligned}$$