SEEMOUS 2007
South Eastern European Mathematical Olympiad for University Students
Agros, Cyprus
7-12 March 2007

Mathematical Society of South Eastern Europe
Cyprus Mathematical Society

COMPETITION PROBLEMS

9 March 2007

Do all problems 1-4. Each problem is worth 10 points. All answers should be answered in the booklet provided, based on the rules written in the Olympiad programme. Time duration: 9.00 – 14.00

PROBLEM 1

Given $a \in (0,1) \cap \mathbb{Q}$ let $a = 0{,}a_1 a_2 a_3\ldots$ be its decimal representation. Define

$$f_a(x) = \sum_{n=1}^{\infty} a_n x^n,\ x \in (0\,,1).$$

Prove that f_a is a rational function of the form $f_a(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials with integer coefficients.

Conversely, if $a_k \in \{0, 1, 2, \ldots, 9\}$ for all $k \in \mathbb{N}$, and $f_a(x) = \sum_{n=1}^{\infty} a_n x^n$ for $x \in (0\,,1)$ is a rational function of the form $f_a(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials with integer coefficients, prove that the number $a = 0{,}a_1 a_2 a_3\ldots$ is rational.

PROBLEM 2

Let $f(x) = \underbrace{\max}_{i} |x_i|$ for $x = (x_1, x_2,\ldots, x_n)^T \in \mathbb{R}^n$ and let A be an nxn matrix such that $f(Ax) = f(x)$ for all $x \in \mathbb{R}^n$. Prove that there exists a positive integer m such that A^m is the identity matrix I_n.

PROBLEM 3

Let F be a field and let P: F x F $\rightarrow$ F be a function such that for every $x_0 \in$ F the function $P(x_0, y)$ is a polynomial in y and for every $y_0 \in$ F the function $P(x, y_0)$ is a polynomial in x.
Is it true that P is necessarily a polynomial in x and y, when

a) F = $\mathbb{Q}$, the field of rational numbers?

b) F is a finite field?

Prove your claims.

PROBLEM 4

For $x \in \mathbb{R}$, $y \geq 0$ and $n \in \mathbb{Z}$ denote by $w_n(x, y) \in [0, \pi)$ the angle in radians with which the segment joining the point (n , 0) to the point (n + y , 0) is seen from the point $(x, 1) \in \mathbb{R}^2$.

a) Show that for every $x \in \mathbb{R}$ and $y \geq 0$, the series $\sum_{n=-\infty}^{\infty} w_n(x, y)$ converges.

If we now set $w(x,y) = \sum_{n=-\infty}^{\infty} w_n(x, y)$, show that $w(x,y) \leq ([y] + 1)\pi$.
([y] is the integer part of y)

b) Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every y with $0 < y < \delta$ and every $x \in \mathbb{R}$ we have $w(x,y) < \varepsilon$.

c) Prove that the function $w : \mathbb{R} \times [0, +\infty) \rightarrow [0, +\infty)$ defined in (a) is continuous.

SEEMOUS 2008
South Eastern European Mathematical Olympiad
for University Students

Athens – March 7, 2008

Problem 1

Let $f : [1, \infty) \to (0, \infty)$ be a continuous function. Assume that for every $a > 0$, the equation $f(x) = ax$ has at least one solution in the interval $[1, \infty)$.

(a) Prove that for every $a > 0$, the equation $f(x) = ax$ has infinitely many solutions.

(b) Give an example of a strictly increasing continuous function f with these properties.

Problem 2

Let $P_0, P_1, P_2, \ldots$ be a sequence of convex polygons such that, for each $k \geq 0$, the vertices of P_{k+1} are the midpoints of all sides of P_k. Prove that there exists a unique point lying inside all these polygons.

Problem 3

Let $\mathcal{M}_n(\mathbb{R})$ denote the set of all real $n \times n$ matrices. Find all surjective functions $f : \mathcal{M}_n(\mathbb{R}) \to \{0, 1, \ldots, n\}$ which satisfy

$$f(XY) \leq \min\{f(X), f(Y)\}$$

for all $X, Y \in \mathcal{M}_n(\mathbb{R})$.

Problem 4

Let n be a positive integer and $f : [0, 1] \to \mathbb{R}$ be a continuous function such that

$$\int_0^1 x^k f(x)\, dx = 1$$

for every $k \in \{0, 1, \ldots, n-1\}$. Prove that

$$\int_0^1 (f(x))^2\, dx \geq n^2.$$

Answers

Problem 1

Solution. (a) Suppose that one can find constants $a > 0$ and $b > 0$ such that $f(x) \neq ax$ for all $x \in [b, \infty)$. Since f is continuous we obtain two possible cases:

1.) $f(x) > ax$ for $x \in [b, \infty)$. Define

$$c = \min_{x \in [1,b]} \frac{f(x)}{x} = \frac{f(x_0)}{x_0}.$$

Then, for every $x \in [1, \infty)$ one should have

$$f(x) > \frac{\min(a, c)}{2} x,$$

a contradiction.

2.) $f(x) < ax$ for $x \in [b, \infty)$. Define

$$C = \max_{x \in [1,b]} \frac{f(x)}{x} = \frac{f(x_0)}{x_0}.$$

Then,

$$f(x) < 2 \max(a, C) x$$

for every $x \in [1, \infty)$ and this is again a contradiction.

(b) Choose a sequence $1 = x_1 < x_2 < \cdots < x_k < \cdots$ such that the sequence $y_k = 2^{k \cos k\pi} x_k$ is also increasing. Next define $f(x_k) = y_k$ and extend f linearly on each interval $[x_{k-1}, x_k]$: $f(x) = a_k x + b_k$ for suitable a_k, b_k. In this way we obtain an increasing continuous function f, for which $\lim\limits_{n\to\infty} \frac{f(x_{2n})}{x_{2n}} = \infty$ and $\lim\limits_{n\to\infty} \frac{f(x_{2n-1})}{x_{2n-1}} = 0$. It now follows that the continuous function $\frac{f(x)}{x}$ takes every positive value on $[1, \infty)$.

Problem 2

Solution. For each $k \geq 0$ we denote by $A_i^k = (x_i^k, y_i^k)$, $i = 1, \ldots, n$ the vertices of P_k. We may assume that the center of gravity of P_0 is $O = (0, 0)$; in other words,

$$\frac{1}{n}(x_1^0 + \cdots + x_n^0) = 0 \ \text{ and } \ \frac{1}{n}(y_1^0 + \cdots + y_n^0) = 0.$$

Since $2x_i^{k+1} = x_i^k + x_{i+1}^k$ and $2y_i^{k+1} = y_i^k + y_{i+1}^k$ for all k and i (we agree that $x_{n+j}^k = x_j^k$ and $y_{n+j}^k = y_j^k$) we see that

$$\frac{1}{n}(x_1^k + \cdots + x_n^k) = 0 \ \text{ and } \ \frac{1}{n}(y_1^k + \cdots + y_n^k) = 0$$

for all $k \geq 0$. This shows that $O = (0, 0)$ is the center of gravity of all polygons P_k.

In order to prove that O is the unique common point of all P_k's it is enough to prove the following claim:

Claim. Let R_k be the radius of the smallest ball which is centered at O and contains P_k. Then, $\lim\limits_{k\to\infty} R_k = 0$.

Proof of the Claim. Write $\|\cdot\|_2$ for the Euclidean distance to the origin O. One can easily check that there exist $\beta_1, \ldots, \beta_n > 0$ and $\beta_1 + \cdots + \beta_n = 1$ such that

$$A_j^{k+n} = \sum_{i=1}^{n} \beta_i A_{j+i-1}^k$$

for all k and j. Let $\lambda = \min_{i=1,\ldots,n} \beta_i$. Since $O = \sum_{i=1}^n A_{j+i-1}^k$, we have the following:

$$\begin{aligned} \|A_j^{k+n}\|_2 &= \left\| \sum_{i=1}^{n} (\beta_i - \lambda) A_{j+i-1}^k \right\|_2 \\ &\le \sum_{i=1}^{n} (\beta_i - \lambda) \|A_{j+i-1}^k\|_2 \\ &\le R_k \sum_{i=1}^{n} (\beta_i - \lambda) = R_k (1 - n\lambda). \end{aligned}$$

This means that P_{k+n} lies in the ball of radius $R_k(1 - n\lambda)$ centered at O. Observe that $1 - n\lambda < 1$.

Continuing in the same way we see that P_{mn} lies in the ball of radius $R_0(1 - n\lambda)^m$ centered at O. Therefore, $R_{mn} \to 0$. Since $\{R_n\}$ is decreasing, the proof is complete.

Problem 3

Solution. We will show that the only such function is $f(X) = \operatorname{rank}(X)$. Setting $Y = I_n$ we find that $f(X) \le f(I_n)$ for all $X \in \mathcal{M}_n(\mathbb{R})$. Setting $Y = X^{-1}$ we find that $f(I_n) \le f(X)$ for all invertible $X \in \mathcal{M}_n(\mathbb{R})$. From these facts we conclude that $f(X) = f(I_n)$ for all $X \in GL_n(\mathbb{R})$.

For $X \in GL_n(\mathbb{R})$ and $Y \in \mathcal{M}_n(\mathbb{R})$ we have

$$\begin{aligned} &f(Y) = f(X^{-1}XY) \le f(XY) \le f(Y), \\ &f(Y) = f(YXX^{-1}) \le f(YX) \le f(Y). \end{aligned}$$

Hence we have $f(XY) = f(YX) = f(Y)$ for all $X \in GL_n(\mathbb{R})$ and $Y \in \mathcal{M}_n(\mathbb{R})$. For $k = 0, 1, \ldots, n$, let

$$J_k = \begin{pmatrix} I_k & O \\ O & O \end{pmatrix}.$$

It is well known that every matrix $Y \in \mathcal{M}_n(\mathbb{R})$ is equivalent to J_k for $k = \operatorname{rank}(Y)$. This means that there exist matrices $X, Z \in GL_n(\mathbb{R})$ such that $Y = XJ_kZ$. From the discussion above it follows that $f(Y) = f(J_k)$. Thus it suffices to determine the values of the function f on the matrices $J_0, J_1, \ldots, J_n$. Since $J_k = J_k \cdot J_{k+1}$ we have $f(J_k) \le f(J_{k+1})$ for $0 \le k \le n-1$. Surjectivity of f imples that $f(J_k) = k$ for $k = 0, 1, \ldots, n$ and hence $f(Y) = \operatorname{rank}(Y)$ for all $Y \in \mathcal{M}_n(\mathbb{R})$.

Problem 4

Solution. There exists a polynomial $p(x) = a_1 + a_2 x + \cdots + a_n x^{n-1}$ which satisfies

$$\text{(1)} \qquad \int_0^1 x^k p(x)\, dx = 1 \quad \text{for all } k = 0, 1, \ldots, n-1.$$

It follows that, for all $k = 0, 1, \dots, n-1$,

$$\int_0^1 x^k (f(x) - p(x))\, dx = 0,$$

and hence

$$\int_0^1 p(x)(f(x) - p(x))\, dx = 0.$$

Then, we can write

$$\begin{aligned} \int_0^1 (f(x) - p(x))^2 dx &= \int_0^1 f(x)(f(x) - p(x))\, dx \\ &= \int_0^1 f^2(x)\, dx - \sum_{k=0}^{n-1} a_{k+1} \int_0^1 x^k f(x)\, dx, \end{aligned}$$

and since the first integral is non-negative we get

$$\int_0^1 f^2(x)\, dx \geq a_1 + a_2 + \cdots + a_n.$$

To complete the proof we show the following:

Claim. For the coefficients $a_1, \dots, a_n$ of p we have

$$a_1 + a_2 + \cdots + a_n = n^2.$$

Proof of the Claim. The defining property of p can be written in the form

$$\frac{a_1}{k+1} + \frac{a_2}{k+2} + \cdots + \frac{a_n}{k+n} = 1, \qquad 0 \leq k \leq n-1.$$

Equivalently, the function

$$r(x) = \frac{a_1}{x+1} + \frac{a_2}{x+2} + \cdots + \frac{a_n}{x+n} - 1$$

has $0, 1, \dots, n-1$ as zeros. We write r in the form

$$r(x) = \frac{q(x) - (x+1)(x+2)\cdots(x+n)}{(x+1)(x+2)\cdots(x+n)},$$

where q is a polynomial of degree $n-1$. Observe that the coefficient of x^{n-1} in q is equal to $a_1 + a_2 + \cdots + a_n$. Also, the numerator has $0, 1, \dots, n-1$ as zeros, and since $\lim_{x \to \infty} r(x) = -1$ we must have

$$q(x) = (x+1)(x+2)\cdots(x+n) - x(x-1)\cdots(x-(n-1)).$$

This expression for q shows that the coefficient of x^{n-1} in q is $\frac{n(n+1)}{2} + \frac{(n-1)n}{2}$. It follows that

$$a_1 + a_2 + \cdots + a_n = n^2.$$

SEEMOUS 2009

South Eastern European Mathematical Olympiad for University Students

AGROS, March 6, 2009

COMPETITION PROBLEMS

Problem 1

a) Calculate the limit

$$\lim_{n\to\infty} \frac{(2n+1)!}{(n!)^2} \int_0^1 (x(1-x))^n x^k dx,$$

where $k \in \mathbb{N}$.

b) Calculate the limit

$$\lim_{n\to\infty} \frac{(2n+1)!}{(n!)^2} \int_0^1 (x(1-x))^n f(x) dx,$$

where $f : [0,1] \to \mathbb{R}$ is a continuous function.

Solution Answer: $f\left(\frac{1}{2}\right)$. Proof: Set

$$L_n(f) = \frac{(2n+1)!}{(n!)^2} \int_0^1 (x(1-x))^n f(x)\, dx \,.$$

A straightforward calculation (integrating by parts) shows that

$$\int_0^1 (x(1-x))^n x^k dx = \frac{(n+k)!n!}{(2n+k+1)!} \,.$$

Thus, $\int_0^1 (x(1-x))^n dx = \frac{(n!)^2}{(2n+1)!}$ and desired limit is equal to $\lim_{n\to\infty} L_n(f)$. Next,

$$\lim_{n\to\infty} L_n(x^k) = \lim_{n\to\infty} \frac{(n+1)(n+2)\dots(n+k)}{(2n+2)(2n+3)\dots(2n+k+1)} = \frac{1}{2^k} \,.$$

According to linearity of the integral and of the limit, $\lim_{n\to\infty} L_n(P) = P\left(\frac{1}{2}\right)$ for every polynomial $P(x)$.

Finally, fix an arbitrary $\varepsilon > 0$. A polynomial P can be chosen such that $|f(x) - P(x)| < \varepsilon$ for every $x \in [0,1]$. Then

$$|L_n(f) - L_n(P)| \le L_n(|f-P|) < L_n(\varepsilon \cdot \mathbb{I}) = \varepsilon \,, \text{ where } \mathbb{I}(x) = 1, \text{ for every } x \in [0,1] \,.$$

There exists n_0 such that $\left|L_n(P) - P\left(\frac{1}{2}\right)\right| < \varepsilon$ for $n \geq n_0$. For these integers

$$\left|L_n(f) - f\left(\frac{1}{2}\right)\right| \leq |L_n(f) - L_n(P)| + \left|L_n(P) - P\left(\frac{1}{2}\right)\right| + \left|f\left(\frac{1}{2}\right) - P\left(\frac{1}{2}\right)\right| < 3\varepsilon\,,$$

which concludes the proof.

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Problem 2

Let P be a real polynomial of degree five. Assume that the graph of P has three inflection points lying on a straight line. Calculate the ratios of the areas of the bounded regions between this line and the graph of the polynomial P.

Solution Denote the inflection points by A, B, and C. Let $l : y = kx + n$ be the equation of the line that passes through them. If B has coordinates (x_0, y_0), the affine change

$$x' = x - x_0, \quad y' = kx - y + n$$

transforms l into the x-axis, and the point B—into the origin. Then without loss of generality it is sufficient to consider a fifth-degree polynomial $f(x)$ with points of inflection $(b, 0)$, $(0, 0)$ and $(a, 0)$, with $b < 0 < a$. Obviously $f''(x) = kx(x-a)(x-b)$, hence

$$f(x) = \frac{k}{20}x^5 - \frac{k(a+b)}{12}x^4 + \frac{kab}{6}x^3 + cx + d\,.$$

By substituting the coordinates of the inflection points, we find $d = 0$, $a + b = 0$ and $c = \frac{7ka^4}{60}$ and therefore

$$f(x) = \frac{k}{20}x^5 - \frac{ka^2}{6}x^3 + \frac{7ka^4}{60}x = \frac{k}{60}\,x(x^2 - a^2)(3x^2 - 7a^2)\,.$$

Since $f(x)$ turned out to be an odd function, the figures bounded by its graph and the x-axis are pairwise equiareal. Two of the figures with unequal areas are

$$\Omega_1 :\ 0 \leq x \leq a,\ 0 \leq y \leq f(x); \qquad \Omega_2 :\ a \leq x \leq a\sqrt{\frac{7}{3}},\ f(x) \leq y \leq 0.$$

We find

$$S_1 = S(\Omega_1) = \int_0^a f(x)\,dx = \frac{ka^6}{40}\,,$$

$$S_2 = S(\Omega_2) = -\int_a^{a\sqrt{\frac{7}{3}}} f(x)\,dx = \frac{4ka^6}{405}$$

and conclude that $S_1 : S_2 = 81 : 32$.

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Problem 3

Let $\mathbf{SL}_2(\mathbb{Z}) = \{A \mid A \text{ is a } 2\times 2 \text{ matrix with integer entries and } \det A = 1\}$.
a) Find an example of matrices $A, B, C \in \mathbf{SL}_2(\mathbb{Z})$ such that $A^2 + B^2 = C^2$.
b) Show that there do not exist matrices $A, B, C \in \mathbf{SL}_2(\mathbb{Z})$ such that $A^4 + B^4 = C^4$.

Solution a) Yes. Example:

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

b) No. Let us recall that every 2×2 matrix A satisfies $A^2 - (trA)\,A + (\det A)\,E = 0$ where $trA = a_{11} + a_{22}$.

Suppose that A, B, C $\in \mathbf{SL}_2(\mathbb{Z})$ and $A^4 + B^4 = C^4$. Let $a = trA,\ b = trB,\ c = trC$. Then $A^4 = (aA - E)^2 = a^2A^2 - 2aA + E = (a^3 - 2a)\,A + (1 - a^2)\,E$ and, after same expressions for B^4 and C^4 have been substituted,

$$\left(a^3 - 2a\right) A + \left(b^3 - 2b\right) B + \left(2 - a^2 - b^2\right) E = \left(c^3 - 2c\right) C + \left(1 - c^2\right) E\,.$$

Calculating traces of both sides we obtain $a^4 + b^4 - 4(a^2 + b^2) = c^4 - 4c^2 - 2$, so $a^4 + b^4 - c^4 \equiv -2 \pmod 4$. Since for every integer k: $k^4 \equiv 0 \pmod 4$ or $k^4 \equiv 1 \pmod 4$, then a and b are odd and c is even. But then $a^4 + b^4 - 4(a^2 + b^2) \equiv 2 \pmod 8$ and $c^4 - 4c^2 - 2 \equiv -2 \pmod 8$ which is a contradiction.

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Problem 4

Given the real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ we define the $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ by

$$a_{ij} = a_i - b_j \quad \text{and} \quad b_{ij} = \begin{cases} 1, & \text{if } a_{ij} \geq 0, \\ 0, & \text{if } a_{ij} < 0, \end{cases} \quad \text{for all } i, j \in \{1, 2, \ldots, n\}.$$

Consider $C = (c_{ij})$ a matrix of the same order with elements 0 and 1 such that

$$\sum_{j=1}^{n} b_{ij} = \sum_{j=1}^{n} c_{ij}, \quad i \in \{1, 2, \ldots, n\} \quad \text{and} \quad \sum_{i=1}^{n} b_{ij} = \sum_{i=1}^{n} c_{ij}, \quad j \in \{1, 2, \ldots, n\}.$$

Show that:

a) $$\sum_{i,j=1}^{n} a_{ij}(b_{ij} - c_{ij}) = 0 \quad \text{and } B = C.$$

b) B is invertible if and only if there exists two permutations σ and τ of the set $\{1, 2, \ldots, n\}$ such that

$$b_{\tau(1)} \leq a_{\sigma(1)} < b_{\tau(2)} \leq a_{\sigma(2)} < \cdots \leq a_{\sigma(n-1)} < b_{\tau(n)} \leq a_{\sigma(n)}.$$

Solution

(a) We have that

$$\sum_{i,j=1}^{n} a_{ij}(b_{ij} - c_{ij}) = \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{n} b_{ij} - \sum_{j=1}^{n} c_{ij} \right) - \sum_{j=1}^{n} b_j \left(\sum_{i=1}^{n} b_{ij} - \sum_{i=1}^{n} c_{ij} \right) = 0. \tag{1}$$

We study the sign of $a_{ij}(b_{ij} - c_{ij})$.

If $a_i \geq b_j$, then $a_{ij} \geq 0$, $b_{ij} = 1$ and $c_{ij} \in \{0, 1\}$, hence $a_{ij}(b_{ij} - c_{ij}) \geq 0$.

If $a_i < b_j$, then $a_{ij} < 0$, $b_{ij} = 0$ and $c_{ij} \in \{0, 1\}$, hence $a_{ij}(b_{ij} - c_{ij}) \geq 0$.

Using (1), the conclusion is that

$$a_{ij}(b_{ij} - c_{ij}) = 0, \quad \text{for all } i, j \in \{1, 2, \ldots, n\}. \tag{2}$$

If $a_{ij} \neq 0$, then $b_{ij} = c_{ij}$. If $a_{ij} = 0$, then $b_{ij} = 1 \geq c_{ij}$. Hence, $b_{ij} \geq c_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$ and since $\sum_{i,j=1}^{n} b_{ij} = \sum_{i,j=1}^{n} c_{ij}$ the final conclusion is that

$$b_{ij} = c_{ij}, \quad \text{for all } i, j \in \{1, 2, \ldots, n\}.$$

(b) We may assume that $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$ since any permutation of $a_1, a_2, \ldots, a_n$ permutes the lines of B and any permutation of $b_1, b_2, \ldots, b_n$ permutes the columns of B, which does not change whether B is invertible or not.

- If there exists i such that $a_i = a_{i+1}$, then the lines i and $i + 1$ in B are equal, so B is not invertible. In the same way, if there exists j such $b_j = b_{j+1}$, then the columns j and $j + 1$ are equal, so B is not invertible.
- If there exists i such that there is no b_j with $a_i < b_j \leq a_{i+1}$, then the lines i and $i + 1$ in B are equal, so B is not invertible. In the same way, if there exists j such that there is no a_i with $b_j \leq a_i < b_{j+1}$, then the columns j and $j + 1$ are equal, so B is not invertible.
- If $a_1 < b_1$, then $a_1 < b_j$ for any $j \in \{1, 2, \ldots, n\}$, which means that the first line of B has only zero elements, hence B is not invertible.

Therefore, if B is invertible, then $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ separate each other

$$b_1 \leq a_1 < b_2 \leq a_2 < \ldots \leq a_{n-1} < b_n \leq a_n. \tag{3}$$

It is easy to check that if (3), then

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

which is, obviously, invertible.

Concluding, B is invertible if and only if there exists a permutation $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ of $a_1, a_2, \ldots, a_n$ and a permutation $b_{j_1}, b_{j_2}, \ldots, b_{j_n}$ of $b_1, b_2, \ldots, b_n$ such that

$$b_{j_1} \leq a_{i_1} < b_{j_2} \leq a_{i_2} < \ldots \leq a_{i_{n-1}} < b_{j_n} \leq a_{i_n}.$$

South Eastern European Mathematical Olympiad for University Students
Plovdiv, Bulgaria
March 10, 2010

Problem 1. Let $f_0 : [0,1] \to \mathbb{R}$ be a continuous function. Define the sequence of functions $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \int_0^x f_{n-1}(t)\,dt$$

for all integers $n \geq 1$.

a) Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ is convergent for every $x \in [0,1]$.

b) Find an explicit formula for the sum of the series $\sum_{n=1}^{\infty} f_n(x)$, $x \in [0,1]$.

Solution 1. *a*) Clearly $f_n' = f_{n-1}$ for all $n \in \mathbb{N}$. The function f_0 is bounded, so there exists a real positive number M such that $|f_0(x)| \leq M$ for every $x \in [0,1]$. Then

$$|f_1(x)| \leq \int_0^x |f_0(t)|\,dt \leq Mx, \quad \forall x \in [0,1],$$

$$|f_2(x)| \leq \int_0^x |f_1(t)|\,dt \leq M\frac{x^2}{2}, \quad \forall x \in [0,1].$$

By induction, it is easy to see that

$$|f_n(x)| \leq M\frac{x^n}{n!}, \quad \forall x \in [0,1],\ \forall n \in \mathbb{N}.$$

Therefore

$$\max_{x\in[0,1]} |f_n(x)| \leq \frac{M}{n!}, \quad \forall n \in \mathbb{N}.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent, so the series $\sum_{n=1}^{\infty} f_n$ is uniformly convergent on $[0,1]$.

b) Denote by $F : [0,1] \to \mathbb{R}$ the sum of the series $\sum_{n=1}^{\infty} f_n$. The series of the derivatives $\sum_{n=1}^{\infty} f_n'$ is uniformly convergent on $[0,1]$, since

$$\sum_{n=1}^{\infty} f_n' = \sum_{n=0}^{\infty} f_n$$

and the last series is uniformly convergent. Then the series $\sum_{n=1}^{\infty} f_n$ can be differentiated term by term and $F' = F + f_0$. By solving this equation, we find $F(x) = e^x \left(\int_0^x f_0(t)e^{-t}\,dt\right)$, $x \in [0,1]$.

Solution 2. We write

$$
\begin{aligned}
f_n(x) &= \int_0^x dt \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-2}} f_0(t_{n-1})\, dt_{n-1} \\
&= \underset{0\le t_{n-1}\le\ldots\le t_1\le t\le x}{\int\cdots\int} f_0(t_{n-1})\, dt\, dt_1 \, \dots \, dt_{n-1} \\
&= \underset{0\le t\le t_1\le\ldots\le t_{n-1}\le x}{\int\cdots\int} f_0(t)\, dt\, dt_1 \, \dots dt_{n-1} \\
&= \int_0^x f_0(t)\, dt \int_t^x dt_1 \int_{t_1}^x dt_2 \dots \int_{t_{n-3}}^x dt_{n-2} \int_{t_{n-2}}^x dt_{n-1} \\
&= \int_0^x f_0(t) \frac{(x-t)^{n-1}}{(n-1)!}\, dt.
\end{aligned}
$$

Thus

$$
\sum_{n=1}^N f_n(x) = \int_0^x f_0(t) \left(\sum_{n=1}^N \frac{(x-t)^{n-1}}{(n-1)!} \right) dt.
$$

We have

$$
e^{x-t} = \sum_{n=0}^{N-1} \frac{(x-t)^n}{n!} + e^{\theta}\, \frac{(x-t)^N}{N!}\,, \quad \theta \in (0, x-t)\,,
$$

$$
\sum_{n=0}^{N-1} \frac{(x-t)^n}{n!} \to e^{x-t}, \quad N \to \infty.
$$

Hence

$$
\begin{aligned}
\left| \int_0^x f_0(t) \left(\sum_{n=0}^{N-1} \frac{(x-t)^n}{n!} \right) dt - \int_0^x f_0(t) e^{x-t} dt \right| &\le \int_0^x |f_0(t)| e^{x-t} \frac{(x-t)^N}{N!} dt \\
&\le \frac{1}{N!} \int_0^x |f_0(t)| e^{x-t}\, dt \to 0, \quad N \to \infty.
\end{aligned}
$$

Problem 2. Inside a square consider circles such that the sum of their circumferences is twice the perimeter of the square.

a) Find the minimum number of circles having this property.

b) Prove that there exist infinitely many lines which intersect at least 3 of these circles.

Solution. *a*) Consider the circles $C_1, C_2, \dots, C_k$ with diameters $d_1, d_2, ..., d_k$, respectively. Denote by s the length of the square side. By using the hypothesis, we get

$$
\pi(d_1 + d_2 + \cdots + d_k) = 8s.
$$

Since $d_i \le s$ for $i = 1, \dots, k$, we have

$$
8s = \pi(d_1 + d_2 + \cdots + d_k) \le \pi k s,
$$

which implies $k \ge \dfrac{8}{\pi} \cong 2.54$. Hence, there are at least 3 circles inside the square.

b) Project the circles onto one side of the square so that their images are their diameters. Since the sum of the diameters is approximately $2.54s$ and there are at least three circles in the square, there exists an interval where at least three diameters are overlapping. The lines, passing through this interval and perpendicular to the side on which the diameters are projected, are the required lines.

Problem 3. Denote by $\mathcal{M}_2(\mathbb{R})$ the set of all 2×2 matrices with real entries. Prove that:

a) for every $A \in \mathcal{M}_2(\mathbb{R})$ there exist $B,\ C \in \mathcal{M}_2(\mathbb{R})$ such that $A = B^2 + C^2$;

b) there do not exist $B,\ C \in \mathcal{M}_2(\mathbb{R})$ such that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = B^2 + C^2$ and $BC = CB$.

Solution. *a*) Recall that every 2×2 matrix A satisfies $A^2 - (trA)\, A + (\det A)\, E = 0$. It is clear that

$$\lim_{t\to+\infty} tr\,(A+tE) = +\infty \quad \text{and} \quad \lim_{t\to+\infty} \frac{\det(A+tE)}{tr(A+tE)} - t = \lim_{t\to+\infty} \frac{\det A - t^2}{tr\,(A+tE)} = -\infty\,.$$

Thus, for t large enough one has

$$\begin{aligned} A &= (A+tE) - tE = \frac{1}{tr(A+tE)}\,(A+tE)^2 + \left(\frac{\det(A+tE)}{tr\,(A+tE)} - t\right) E \\ &= \left(\frac{1}{\sqrt{tr(A+tE)}}\,(A+tE)\right)^2 + \left(\sqrt{t - \frac{\det(A+tE)}{tr\,(A+tE)}}\right)^2 (-E) \\ &= \left(\frac{1}{\sqrt{tr(A+tE)}}\,(A+tE)\right)^2 + \left(\sqrt{t - \frac{\det(A+tE)}{tr\,(A+tE)}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)^2. \end{aligned}$$

b) No. For $B,\ C \in \mathcal{M}_2(\mathbb{R})$, consider $B+iC,\ B-iC \in \mathcal{M}_2(\mathbb{C})$. If $BC = CB$ then $(B+iC)\,(B-iC) = B^2 + C^2$. Thus

$$\det\left(B^2 + C^2\right) = \det(B+iC)\det(B-iC) = |\,B+iC\,|^{\,2} \geq 0,$$

which contradicts the fact that $\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$.

Problem 4. Suppose that A and B are $n\times n$ matrices with integer entries, and $\det B \neq 0$. Prove that there exists $m \in \mathbb{N}$ such that the product AB^{-1} can be represented as

$$AB^{-1} = \sum_{k=1}^{m} N_k^{-1},$$

where N_k are $n\times n$ matrices with integer entries for all $k = 1,\ldots,m$, and $N_i \neq N_j$ for $i \neq j$.

Solution. Suppose first that $n = 1$. Then we may consider the integer 1×1 matrices as integer numbers. We shall prove that for given integers p and q we can find integers $n_1,\ldots,n_m$ such that $\frac{p}{q} = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_m}$ and $n_i \neq n_j$ for $i \neq j$.

In fact this is well known as the "Egyptian problem". We write $\frac{p}{q} = \frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}$ (p times) and ensure different denominators in the last sum by using several times the equality $\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x(x+1)}$. For example, $\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$, where we keep the first fraction, we write $\frac{1}{5} = \frac{1}{6} + \frac{1}{30}$ for the second fraction, and $\frac{1}{5} = \frac{1}{7} + \frac{1}{42} + \frac{1}{31} + \frac{1}{930}$ for the third fraction. Finally,

$$\frac{3}{5} = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{30} + \frac{1}{31} + \frac{1}{42} + \frac{1}{930}.$$

Now consider $n > 1$.

CASE 1. Suppose that A is a nonsingular matrix. Denote by λ the least common multiple of the denominators of the elements of the matrix A^{-1}. Hence the matrix $C = \lambda B A^{-1}$ is integer and nonsingular, and one has

$$AB^{-1} = \lambda C^{-1}.$$

According to the case $n = 1$, we can write

$$\lambda = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_m},$$

where $n_i \neq n_j$ for $i \neq j$. Then

$$AB^{-1} = (n_1 C)^{-1} + (n_2 C)^{-1} + \cdots + (n_m C)^{-1}.$$

It is easy to see that $n_i C \neq n_j C$ for $i \neq j$.

CASE 2. Now suppose that A is singular. First we will show that

$$A = Y + Z,$$

where Y and Z are nonsingular. If $A = (a_{ij})$, for every $i = 1, 2, \dots, n$ we choose an integer x_i such that $x_i \neq 0$ and $x_i \neq a_{ii}$. Define

$$y_{ij} = \begin{cases} a_{ij}, & \text{if } i < j \\ x_i, & \text{if } i = j \\ 0, & \text{if } i > j \end{cases} \quad \text{and} \quad z_{ij} = \begin{cases} 0, & \text{if } i < j \\ a_{ii} - x_i, & \text{if } i = j \\ a_{ij}, & \text{if } i > j. \end{cases}$$

Clearly, the matrices $Y = (y_{ij})$ and $Z = (z_{ij})$ are nonsingular. Moreover, $A = Y + Z$.

From Case 1 we have

$$YB^{-1} = \sum_{r=1}^{k} (n_r C)^{-1}, \quad ZB^{-1} = \sum_{q=1}^{l} (m_q D)^{-1},$$

where

$$YB^{-1} = \lambda C^{-1}, \quad \lambda = \sum_{r=1}^{k} \frac{1}{n_r} \quad \text{and} \quad ZB^{-1} = \mu D^{-1}, \quad \mu = \sum_{q=1}^{l} \frac{1}{m_q},$$

C and D are integer and nonsingular. Hence,

$$AB^{-1} = \sum_{r=1}^{k} (n_r C)^{-1} + \sum_{q=1}^{l} (m_q D)^{-1}.$$

It remains to show that $n_r C \neq m_q D$ for $r = 1, 2, \dots, k$ and $q = 1, 2, \dots, l$. Indeed, assuming that $n_r C = m_q D$ and recalling that $m_q > 0$ we find $D = \frac{n_r}{m_q} C$. Hence $ZB^{-1} = \mu D^{-1} = \frac{\mu m_q}{n_r} C^{-1}$, and then $AB^{-1} = YB^{-1} + ZB^{-1} = \lambda C^{-1} + \frac{\mu m_q}{n_r} C^{-1} = \left(\lambda + \frac{\mu m_q}{n_r}\right) C^{-1}$. We have $\lambda + \frac{\mu m_q}{n_r} > 0$, and C^{-1} is nonsingular. Then AB^{-1} is nonsingular, and therefore A is nonsingular. This is a contradiction.

Bucharest, March 4th, 2011

SOUTH EASTERN EUROPEAN MATHEMATICAL OLYMPIAD FOR UNIVERSITY STUDENTS

PROBLEMS

Problem 1 *For a given integer $n \geq 1$, let $f : [0,1] \to \mathbb{R}$ be a non-decreasing function. Prove that*

$$\int_0^1 f(x)\,\mathrm{d}x \leq (n+1)\int_0^1 x^n f(x)\,\mathrm{d}x.$$

Find all non-decreasing continuous functions for which equality holds.

Problem 2 *Let $A = (a_{ij})$ be a real $n \times n$ matrix such that $A^n \neq 0$ and $a_{ij}a_{ji} \leq 0$ for all i, j. Prove that there exist two nonreal numbers among eigenvalues of A.*

Problem 3 *Given vectors $\overline{a}, \overline{b}, \overline{c} \in \mathbb{R}^n$, show that*

$$\left(||\overline{a}||\,\langle\overline{b},\overline{c}\rangle\right)^2 + \left(||\overline{b}||\,\langle\overline{a},\overline{c}\rangle\right)^2 \leq ||\overline{a}||||\overline{b}||\left(||\overline{a}||||\overline{b}|| + |\langle\overline{a},\overline{b}\rangle|\right)||\overline{c}||^2,$$

where $\langle\overline{x},\overline{y}\rangle$ denotes the scalar (inner) product of the vectors $\overline{x}$ and $\overline{y}$ and $||\overline{x}||^2 = \langle\overline{x},\overline{x}\rangle$.

Problem 4 *Let $f : [0,1] \to \mathbb{R}$ be a twice continuously differentiable increasing function. Define the sequences given by $L_n = \dfrac{1}{n}\sum\limits_{k=0}^{n-1} f\left(\dfrac{k}{n}\right)$ and $U_n = \dfrac{1}{n}\sum\limits_{k=1}^{n} f\left(\dfrac{k}{n}\right)$, $n \geq 1$. The interval $[L_n, U_n]$ is divided into three equal segments. Prove that, for large enough n, the number $I = \int\limits_0^1 f(x)\,\mathrm{d}x$ belongs to the middle one of these three segments.*

Each problem is 10 points worth.
Allowed time: 5 hours.

Sixth South Eastern European Mathematical Olympiad for University Students

Blagoevgrad, Bulgaria

March 8, 2012

Problem 1. Let $A = (a_{ij})$ be the $n \times n$ matrix, where a_{ij} is the remainder of the division of $i^j + j^i$ by 3 for $i, j = 1, 2, \ldots, n$. Find the greatest n for which $\det A \neq 0$.

Solution. We show that $a_{i+6,j} = a_{ij}$ for all $i, j = 1, 2, \ldots, n$. First note that if $j \equiv 0 \pmod 3$ then $j^i \equiv 0 \pmod 3$, and if $j \equiv 1$ or $2 \pmod 3$ then $j^6 \equiv 1 \pmod 3$. Hence, $j^i(j^6 - 1) \equiv 0 \pmod 3$ for $j = 1, 2, \ldots, n$, and

$$a_{i+6,j} \equiv (i+6)^j + j^{i+6} \equiv i^j + j^i \equiv a_{ij} \pmod 3,$$

or $a_{i+6,j} = a_{ij}$. Consequently, $\det A = 0$ for $n \geq 7$. By straightforward calculation, we see that $\det A = 0$ for $n = 6$ but $\det A \neq 0$ for $n = 5$, so the answer is $n = 5$.

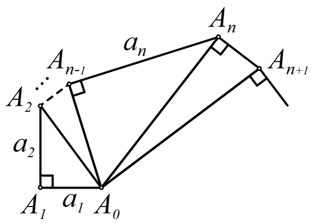
Grading of Problem 1.

5p: Concluding that $\Delta_n = 0$ for each $n \geq 7$

5p: Computing $\Delta_5 = 12,\ \Delta_6 = 0$

2p: Computing $\Delta_3 = -10,\ \Delta_4 = 4$ (in case none of the above is done)

Problem 2. Let $a_n > 0$, $n \geq 1$. Consider the right triangles $\triangle A_0A_1A_2$, $\triangle A_0A_2A_3, \ldots,$ $\triangle A_0A_{n-1}A_n, \ldots,$ as in the figure. (More precisely, for every $n \geq 2$ the hypotenuse A_0A_n of $\triangle A_0A_{n-1}A_n$ is a leg of $\triangle A_0A_nA_{n+1}$ with right angle $\angle A_0A_nA_{n+1}$, and the vertices A_{n-1} and A_{n+1} lie on the opposite sides of the straight line A_0A_n; also, $|A_{n-1}A_n| = a_n$ for every $n \geq 1$.)



Is it possible for the set of points $\{A_n \mid n \geq 0\}$ to be unbounded but the series $\sum\limits_{n=2}^{\infty} m(\angle A_{n-1}A_0A_n)$ to be convergent? Here $m(\angle ABC)$ denotes the measure of $\angle ABC$.

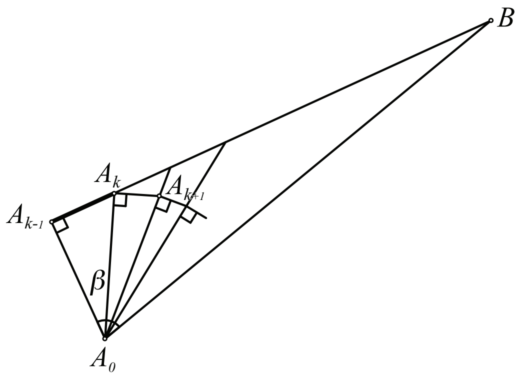
Note. A subset B of the plane is bounded if there is a disk D such that $B \subseteq D$.

Solution. We have $|A_0A_n| = \sqrt{\sum\limits_{i=1}^{n} a_i^2}$ and $\sum\limits_{n=2}^{k} m(\angle A_{n-1}A_0A_n) = \sum\limits_{n=2}^{k} \arctan \dfrac{a_n}{\sqrt{a_1^2 + \cdots + a_{n-1}^2}}$.
The set of points $\{A_n \mid n \geq 0\}$ will be unbounded if and only if the sequence of the lengths of the segments A_0A_n is unbounded. Put $a_i^2 = b_i$. Then the question can be reformulated as follows: Is it possible for a series with positive terms to be such that $\sum_{i=1}^{\infty} b_i = \infty$ and

$$\sum_{n=2}^{\infty} \arctan \sqrt{\frac{b_n}{b_1 + \cdots + b_{n-1}}} < \infty.$$

Denote $s_n = \sum_{i=1}^n b_i$. Since $\arctan x \sim x$ as $x \to 0$, the question we need to ask is whether one can have $s_n \to \infty$ as $n \to \infty$ and $\sum\limits_{n=2}^{\infty} \sqrt{\frac{s_n - s_{n-1}}{s_{n-1}}} < \infty$. Put $\sqrt{\frac{s_n - s_{n-1}}{s_{n-1}}} = u_n > 0$. Then $\frac{s_n}{s_{n-1}} = 1 + u_n^2$, $\ln s_n - \ln s_{n-1} = \ln(1+u_n^2)$, $\ln s_k = \ln s_1 + \sum_{n=2}^k \ln(1+u_n^2)$. Finally, the question is whether it is possible to have $\sum_{n=2}^\infty \ln(1+u_n^2) = \infty$ and $\sum_{n=2}^\infty u_n < \infty$. The answer is negative, since $\ln(1+x) \sim x$ as $x \to 0$ and $u_n^2 \le u_n \le 1$ for large enough n.

Different solution. Since $\sum_{n=2}^\infty m(\angle A_{n-1}A_0A_n) < \infty$, there exists some large enough k for which $\sum_{n=k}^\infty m(\angle A_{n-1}A_0A_n) \le \beta < \frac{\pi}{2}$. Then all the vertices A_n, $n \ge k-1$, lie inside the triangle $\triangle A_0A_{k-1}B$, where the side $A_{k-1}B$ of $\triangle A_0A_{k-1}B$ is a continuation of the side $A_{k-1}A_k$ of $\triangle A_0A_{k-1}A_k$ and $\angle A_{k-1}A_0B = \beta$. Consequently, the set $\{A_n \,|\, n \ge 0\}$ is bounded which is a contradiction.



Grading of Problem 2.

1p: Noting that $\{A_n \,|\, n \ge 0\}$ is unbounded $\Leftrightarrow$ $|A_0A_n|$ is unbounded **OR** expressing $|A_0A_n|$

1p: Observing that $\sum_{n=2}^\infty m(\angle A_{n-1}A_0A_n)$ is convergent $\Leftrightarrow$ A_0A_n tends to A_0B **OR** expressing the angles by arctan

8p: Proving the assertion

Problem 3.

a) Prove that if k is an even positive integer and A is a real symmetric $n \times n$ matrix such that $(\operatorname{Tr}(A^k))^{k+1} = (\operatorname{Tr}(A^{k+1}))^k$, then

$$A^n = \operatorname{Tr}(A)\, A^{n-1}.$$

b) Does the assertion from *a*) also hold for odd positive integers k?

Solution. *a*) Let $k = 2l$, $l \ge 1$. Since A is a symmetric matrix all its eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ are real numbers. We have,

$$\operatorname{Tr}(A^{2l}) = \lambda_1^{2l} + \lambda_2^{2l} + \cdots + \lambda_n^{2l} = a \tag{1}$$

and

$$\operatorname{Tr}(A^{2l+1}) = \lambda_1^{2l+1} + \lambda_2^{2l+1} + \cdots + \lambda_n^{2l+1} = b. \tag{2}$$

By (1) we get that $a \ge 0$, so there is some $a_1 \ge 0$ such that $a = a_1^{2l}$. On the other hand, the equality $a^{2l+1} = b^{2l}$ implies that $(a_1^{2l+1})^{2l} = b^{2l}$ and hence

$$b = \pm a_1^{2l+1} = (\pm a_1)^{2l+1} \quad \text{and} \quad a = a_1^{2l} = (\pm a_1)^{2l}.$$

Then equalities (1) and (2) become

$$\lambda_1^{2l} + \lambda_2^{2l} + \cdots + \lambda_n^{2l} = c^{2l} \tag{3}$$

and

$$\lambda_1^{2l+1} + \lambda_2^{2l+1} + \cdots + \lambda_n^{2l+1} = c^{2l+1}, \tag{4}$$

where $c = \pm a_1$. We consider the following cases.
Case 1. If $c = 0$ then $\lambda_1 = \cdots = \lambda_n = 0$, so $\mathrm{Tr}(A) = 0$ and we note that the characteristic polynomial of A is $f_A(x) = x^n$. We have, based on the Cayley-Hamilton Theorem, that

$$A^n = 0 = \mathrm{Tr}(A)\, A^{n-1}.$$

Case 2. If $c \neq 0$ then let $x_i = \lambda_i/c$, $i = 1, 2, \ldots, n$. In this case equalities (3) and (4) become

$$x_1^{2l} + x_2^{2l} + \cdots + x_n^{2l} = 1 \tag{5}$$

and

$$x_1^{2l+1} + x_2^{2l+1} + \cdots + x_n^{2l+1} = 1. \tag{6}$$

The equality (5) implies that $|x_i| \leq 1$ for all $i = 1, 2, \ldots, n$. We have $x^{2l} \geq x^{2l+1}$ for $|x| \leq 1$ with equality reached when $x = 0$ or $x = 1$. Then, by (5), (6), and the previous observation, we find without loss of generality that $x_1 = 1$, $x_2 = x_3 = \cdots = x_n = 0$. Hence $\lambda_1 = c$, $\lambda_2 = \cdots = \lambda_n = 0$, and this implies that $f_A(x) = x^{n-1}(x - c)$ and $\mathrm{Tr}(A) = c$. It follows, based on the Cayley-Hamilton Theorem, that

$$f_A(A) = A^{n-1}(A - cI_n) = 0 \quad \Leftrightarrow \quad A^n = \mathrm{Tr}(A)\, A^{n-1}.$$

b) The answer to the question is negative. We give the following counterexample:

$$k = 1, \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

Grading of Problem 3.

3p: Reformulating the problem through eigenvalues:

$$\left(\sum \lambda_i^{2l}\right)^{2l+1} = \left(\sum \lambda_i^{2l+1}\right)^{2l} \Rightarrow \forall i:\ \lambda_i^n = (\lambda_1 + \cdots + \lambda_n)\lambda_i^{n-1}$$

4p: Only $(\lambda_i) = (0, \ldots, 0, c, 0, \ldots, 0)$ or $(0, \ldots, 0)$ are possible

3p: Finding a counterexample

Problem 4.

a) Compute

$$\lim_{n\to\infty} n \int_0^1 \left(\frac{1-x}{1+x}\right)^n dx.$$

b) Let $k \geq 1$ be an integer. Compute

$$\lim_{n\to\infty} n^{k+1} \int_0^1 \left(\frac{1-x}{1+x}\right)^n x^k\, dx.$$

Solution. *a*) The limit equals $\dfrac{1}{2}$. The result follows immediately from *b*) for $k=0$.

b) The limit equals $\dfrac{k!}{2^{k+1}}$. We have, by the substitution $\dfrac{1-x}{1+x}=y$, that

$$\begin{aligned} n^{k+1}\int_0^1\left(\frac{1-x}{1+x}\right)^n x^k\,dx &= 2n^{k+1}\int_0^1 y^n(1-y)^k\frac{dy}{(1+y)^{k+2}} \\ &= 2n^{k+1}\int_0^1 y^n f(y)dy, \end{aligned}$$

where

$$f(y)=\frac{(1-y)^k}{(1+y)^{k+2}}\,.$$

We observe that

$$f(1)=f'(1)=\cdots=f^{(k-1)}(1)=0. \tag{7}$$

We integrate k times by parts $\displaystyle\int_0^1 y^n f(y)dy$, and by (7) we get

$$\int_0^1 y^n f(y)dy=\frac{(-1)^k}{(n+1)(n+2)\dots(n+k)}\int_0^1 y^{n+k}f^{(k)}(y)dy.$$

One more integration implies that

$$\begin{aligned} \int_0^1 y^n f(y)dy &= \frac{(-1)^k}{(n+1)(n+2)\dots(n+k)(n+k+1)} \\ &\quad\times\left(f^{(k)}(y)y^{n+k+1}\Big|_0^1-\int_0^1 y^{n+k+1}f^{(k+1)}(y)dy\right) \\ &= \frac{(-1)^k f^{(k)}(1)}{(n+1)(n+2)\dots(n+k+1)} \\ &\quad+\frac{(-1)^{k+1}}{(n+1)(n+2)\dots(n+k+1)}\int_0^1 y^{n+k+1}f^{(k+1)}(y)dy. \end{aligned}$$

It follows that

$$\lim_{n\to\infty}2n^{k+1}\int_0^1 y^n f(y)dy=2(-1)^k f^{(k)}(1),$$

since

$$\lim_{n\to\infty}\int_0^1 y^{n+k+1}f^{(k+1)}(y)dy=0,$$

$f^{(k+1)}$ being continuous and hence bounded. Using Leibniz's formula we get that

$$f^{(k)}(1)=(-1)^k\frac{k!}{2^{k+2}},$$

and the problem is solved.

Grading of Problem 4.

3p: For computing *a*)

7p: For computing *b*)

SEEMOUS 2013 PROBLEMS AND SOLUTIONS

Problem 1

Find all continuous functions $f : [1, 8] \to R$, such that

$$\int_1^2 f^2(t^3)dt + 2\int_1^2 f(t^3)dt = \frac{2}{3}\int_1^8 f(t)dt - \int_1^2 (t^2-1)^2 dt.$$

Solution. Using the substitution $t = u^3$ we get

$$\frac{2}{3}\int_1^8 f(t)dt = 2\int_1^2 u^2 f(u^3)du = 2\int_1^2 t^2 f(t^3)du.$$

Hence, by the assumptions,

$$\int_1^2 \left(f^2(t^3) + (t^2-1)^2 + 2f(t^3) - 2t^2 f(t^3)\right) dt = 0.$$

Since $f^2(t^3)+(t^2-1)^2+2f(t^3)-2t^2f(t^3) = (f(t^3))^2+(1-t^2)^2+2(1-t^2)f(t^3) = \left(f(t^3)+1-t^2\right)^2 \geq 0$, we get

$$\int_1^2 \left(f(t^3)+1-t^2\right)^2 dt = 0.$$

The continuity of f implies that $f(t^3) = t^2 - 1$, $1 \leq t \leq 2$, thus, $f(x) = x^{2/3} - 1$, $1 \leq x \leq 8$.

Remark. If the continuity assumption for f is replaced by Riemann integrability then infinitely many f's would satisfy the given equality. For example if C is any closed nowhere dense and of measure zero subset of $[1, 8]$ (for example a finite set or an appropriate Cantor type set) then any function f such that $f(x) = x^{2/3} - 1$ for every $x \in [1, 8]\setminus C$ satisfies the conditions.

Problem 2

Let $M, N \in M_2(\mathbb{C})$ be two nonzero matrices such that

$$M^2 = N^2 = 0_2 \text{ and } MN + NM = I_2$$

where 0_2 is the 2×2 zero matrix and I_2 the 2×2 unit matrix. Prove that there is an invertible matrix $A \in M_2(\mathbb{C})$ such that

$$M = A\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}A^{-1} \text{ and } N = A\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}A^{-1}.$$

First solution. Consider $f, g : \mathbb{C}^2 \to \mathbb{C}^2$ given by $f(x) = Mx$ and $g(x) = Nx$.
We have $f^2 = g^2 = 0$ and $fg + gf = \mathrm{id}_{\mathbb{C}^2}$; composing the last relation (to the left, for instance) with fg we find that $(fg)^2 = fg$, so fg is a projection of $\mathbb{C}^2$.
If fg were zero, then $gf = \mathrm{id}_{\mathbb{C}^2}$, so f and g would be invertible, thus contradicting $f^2 = 0$.
Therefore, fg is nonzero. Let $u \in \mathrm{Im}(fg) \setminus \{0\}$ and $w \in \mathbb{C}^2$ such that $u = fg(w)$. We obtain $fg(u) = (fg)^2(w) = fg(w) = u$. Let $v = g(u)$. The vector v is nonzero, because otherwise we obtain $u = f(v) = 0$.
Moreover, u and v are not collinear since $v = \lambda u$ with $\lambda \in \mathbb{C}$ implies $u = f(v) = f(\lambda u) = \lambda f(u) = \lambda f^2(g(w)) = 0$, a contradiction.
Let us now consider the ordered basis $\mathcal{B}$ of $\mathbb{C}^2$ consisting of u and v.
We have $f(u) = f^2(g(u)) = 0$, $f(v) = f(g(u)) = u$, $g(u) = v$ and $g(v) = g^2(u) = 0$.
Therefore, the matrices of f and g with respect to $\mathcal{B}$ are $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, respectively.
We take A to be the change of base matrix from the standard basis of $\mathbb{C}^2$ to $\mathcal{B}$ and we are done. $\square$

Second solution. Let us denote $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ by E_{12} and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ by E_{21}. Since $M^2 = N^2 = 0_2$ and $M, N \neq 0_2$, the minimal polynomials of both M and N are equal to x^2. Therefore, there are invertible matrices $B, C \in \mathcal{M}_2(\mathbb{C})$ such that $M = BE_{12}B^{-1}$ and $N = CE_{21}C^{-1}$.
Note that B and C are not uniquely determined. If $B_1E_{12}B_1^{-1} = B_2E_{12}B_2^{-1}$, then $(B_1^{-1}B_2)E_{12} = E_{12}(B_1^{-1}B_2)$; putting $B_1^{-1}B_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the last relation is equivalent to $\begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$. Consequently, $B_1E_{12}B_1^{-1} = B_2E_{12}B_2^{-1}$ if and only if there exist $a \in \mathbb{C} - \{0\}$ and $b \in \mathbb{C}$ such that

$$B_2 = B_1 \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}. \quad (*)$$

Similarly, $C_1E_{21}C_1^{-1} = C_2E_{21}C_2^{-1}$ if and only if there exist $\alpha \in \mathbb{C} - \{0\}$ and $\beta \in \mathbb{C}$ such that

$$C_2 = C_1 \begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix}. \quad (**)$$

Now, $MN + NM = I_2$, $M = BE_{12}B^{-1}$ and $N = CE_{21}C^{-1}$ give

$$BE_{12}B^{-1}CE_{21}C^{-1} + CE_{21}C^{-1}BE_{12}B^{-1} = I_2,$$

or

$$E_{12}B^{-1}CE_{21}C^{-1}B + B^{-1}CE_{21}C^{-1}BE_{12} = I_2.$$

If $B^{-1}C = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, the previous relation means

$$\begin{pmatrix} z & t \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ t & -y \end{pmatrix} + \begin{pmatrix} y & 0 \\ t & 0 \end{pmatrix}\begin{pmatrix} 0 & t \\ 0 & -z \end{pmatrix} = (xt - yz)I_2 \neq 0_2.$$

After computations we find this to be equivalent to $xt - yz = t^2 \neq 0$. Consequently, there are $y, z \in \mathbb{C}$ and $t \in \mathbb{C} - \{0\}$ such that

$$C = B \begin{pmatrix} t + \frac{yz}{t} & y \\ z & t \end{pmatrix}. \quad (***)$$

According to $(*)$ and $(**)$, our problem is equivalent to finding $a, \alpha \in \mathbb{C} - \{0\}$ and $b, \beta \in \mathbb{C}$ such that $C \begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix} = B \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. Taking relation $(***)$ into account, we need to find $a, \alpha \in \mathbb{C} - \{0\}$ and $b, \beta \in \mathbb{C}$ such that

$$B \begin{pmatrix} t + \frac{yz}{t} & y \\ z & t \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix} = B \begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

or, B being invertible,

$$\begin{pmatrix} t + \frac{yz}{t} & y \\ z & t \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

This means $\begin{cases} \alpha t + \alpha \dfrac{yz}{t} + \beta y = a \\ \alpha y = b \\ \alpha z + \beta t = 0 \\ \alpha t = a \end{cases}$,

and these conditions are equivalent to $\begin{cases} \alpha y = b \\ \alpha z = -\beta t \\ \alpha t = a \end{cases}$.

It is now enough to choose $\alpha = 1$, $a = t$, $b = y$ and $\beta = -\dfrac{z}{t}$. $\square$

Third Solution. Let f, g be as in the first solution. Since $f^2 = 0$ there exists a nonzero $v_1 \in \mathrm{Ker} f$ so $f(v_1) = 0$ and setting $v_2 = g(v_1)$ we get

$$f(v_2) = (fg + gf)(v_1) = v_1 \neq 0$$

by the assumptions (and so $v_2 \neq 0$). Also

$$g(v_2) = g^2(v_1) = 0$$

and so to complete the proof it suffices to show that v_1 and v_2 are linearly independent, because then the matrices of f, g with respect to the ordered basis (v_1, v_2) would be E_{12} and E_{21} respectively, according to the above relations. But if $v_2 = \lambda v_1$ then $0 = g(v_2) = \lambda g(v_1) = \lambda v_2$ so since $v_2 \neq 0$, λ must be 0 which gives $v_2 = 0v_1 = 0$ contradiction. This completes the proof. □

Remark. A nonelementary solution of this problem can be given by observing that the conditions on M, N imply that the correspondence $I_2 \to I_2, M \to E_{12}$ and $N \to E_{21}$ extends to an isomorphism between the subalgebras of $\mathcal{M}_2(\mathbb{C})$ generated by I_2, M, N and I_2, E_{12}, E_{21} respectively, and then one can apply Noether-Skolem Theorem to show that this isomorphism is actually conjugation by an $A \in Gl_2(\mathbb{C})$ etc.

Problem 3

Find the maximum value of

$$\int_0^1 |f'(x)|^2 \, |f(x)| \, \frac{1}{\sqrt{x}} \, dx$$

over all continuously differentiable functions $f : [0,1] \to \mathbb{R}$ *with* $f(0) = 0$ *and*

$$\int_0^1 |f'(x)|^2 \, dx \leq 1. \qquad (*)$$

Solution. For $x \in [0,1]$ let

$$g(x) = \int_0^x |f'(t)|^2 \, dt.$$

Then for $x \in [0,1]$ the Cauchy-Schwarz inequality gives

$$|f(x)| \leq \int_0^x |f'(t)| \, dt \leq \left(\int_0^x |f'(t)|^2 \, dt \right)^{1/2} \sqrt{x} = \sqrt{x} g(x)^{1/2}.$$

Thus

$$\int_0^1 |f'(x)|^2 |f(x)| \frac{1}{\sqrt{x}} \, dx \leq \int_0^1 g(x)^{1/2} g'(x) dx = \frac{2}{3}[g(1)^{3/2} - g(0)^{3/2}]$$
$$= \frac{2}{3} \left(\int_0^1 |f'(t)|^2 \, dt \right)^{3/2} \leq \frac{2}{3}.$$

by $(*)$. The maximum is achieved by the function $f(x) = x$. □

Remark. If the condition $(*)$ is replaced by $\int_0^1 |f'(x)|^p dx \leq 1$ with $0 < p < 2$ fixed, then the given expression would have supremum equal to $+\infty$, as it can be seen by considering continuously differentiable functions that approximate the functions $f_M(x) = Mx$ for $0 \leq x \leq \frac{1}{M^p}$ and $\frac{1}{M^{p-1}}$ for $\frac{1}{M^p} < x \leq 1$, where M can be an arbitrary large positive real number.

Problem 4

Let $A \in M_2(Q)$ *such that there is* $n \in N$, $n \neq 0$, *with* $A^n = -I_2$. *Prove that either* $A^2 = -I_2$ *or* $A^3 = -I_2$.

First Solution. Let $f_A(x) = det(A - xI_2) = x^2 - sx + p \in \mathbb{Q}[x]$ be the characteristic polynomial of A and let λ_1, λ_2 be its roots, also known as the eigenvalues of matrix A. We have that $\lambda_1 + \lambda_2 = s \in \mathbb{Q}$ and $\lambda_1\lambda_2 = p \in \mathbb{Q}$. We know, based on Cayley-Hamilton theorem, that the matrix A satisfies the relation $A^2 - sA + pI_2 = 0_2$. For any eigenvalue $\lambda \in \mathbb{C}$ there is an eigenvector $X \neq 0$, such that $AX = \lambda X$. By induction we have that $A^n X = \lambda^n X$ and it follows that $\lambda^n = -1$. Thus, the eigenvalues of A satisfy the equalities

$$\lambda_1^n = \lambda_2^n = -1 \qquad (*).$$

Is $\lambda_1 \in \mathbb{R}$ then we also have that $\lambda_2 \in \mathbb{R}$ and from $(*)$ we get that $\lambda_1 = \lambda_2 = -1$ (and note that n must be odd) so A satisfies the equation $(A + I_2)^2 = A^2 + 2A + I_2 = 0_2$ and it follows that $-I_2 = A^n = (A + I_2 - I_2)^n = n(A + I_2) - I_2$ which gives $A = -I_2$ and hence $A^3 = -I_2$.

If $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$ then $\lambda_2 = \overline{\lambda_1} \in \mathbb{C} \setminus \mathbb{R}$ and since $\lambda_1^n = -1$ we get that $|\lambda_{1,2}| = 1$ and this implies that $\lambda_{1,2} = \cos t \pm i \sin t$. Now we have the equalities $\lambda_1 + \lambda_2 = 2\cos t = s \in \mathbb{Q}$ and $\lambda_1^n = -1$ implies that $\cos nt + i \sin nt = -1$ which in turn implies that $\cos nt = -1$. Using the equality $\cos(n+1)t + \cos(n-1)t = 2\cos t \cos nt$ we get that there is a polynomial $P_n = x^n + \cdots$ of degree n with integer coefficients such that $2\cos nt = P_n(2\cos t)$. Set $x = 2\cos t$ and observe that we have $P_n(x) = -2$ so $x = 2\cos t$ is a rational root of an equation of the form $x^n + \cdots = 0$. However, the rational roots of this equation are integers, so $x \in \mathbb{Z}$ and since $|x| \leq 2$ we get that $2\cos t = -2, -1, 0, 1, 2$.

When $2\cos t = \pm 2$ then $\lambda_{1,2}$ are real numbers (note that in this case $\lambda_1 = \lambda_2 = 1$ or $\lambda_1 = \lambda_2 = -1$) and this case was discussed above.

When $2\cos t = 0$ we get that $A^2 + I_2 = 0_2$ so $A^2 = -I_2$.

When $2\cos t = 1$ we get that $A^2 - A + I_2 = 0_2$ which implies that $(A + I_2)(A^2 - A + I_2) = 0_2$ so $A^3 = -I_2$.

When $2\cos t = -1$ we get that $A^2 + A + I_2 = 0_2$ and this implies that $(A - I_2)(A^2 + A + I_2) = 0_2$ so $A^3 = I_2$. It follows that $A^n \in \{I_2, A, A^2\}$. However, $A^n = -I_2$ and this implies that either $A = -I_2$ or $A^2 = -I_2$ both of which contradict the equality $A^3 = I_2$. This completes the proof. □

Remark. The polynomials P_n used in the above proof are related to the Chebyshev polynomials, $T_n(x) = \cos(n \arccos x)$. One could also get the conclusion that $2\cos t$ is an integer by considering the sequence $x_m = 2\cos(2^m t)$ and noticing that since $x_{m+1} = x_m^2 - 2$, if x_0 were a noninteger rational $\frac{a}{b}$ $(b > 1)$ in lowest terms then the denominator of x_m *in lowest terms* would be b^{2^m} and this contradicts the fact that x_m must be periodic since t is a rational multiple of π.

Second Solution. Let $m_A(x)$ be the minimal polynomial of A. Since $A^{2n} - I_2 = (A^n + I_2)(A^n - I_2) = 0_2$, $m_A(x)$ must be a divisor of $x^{2n} - 1$ which has no multiple roots. It is well known that the monic irreducible over $\mathbb{Q}$ factors of $x^{2n} - 1$ are exactly the cyclotomic polynomials $\Phi_d(x)$ where d divides $2n$. Hence the irreducible over $\mathbb{Q}$ factors of $m_A(x)$ must be cyclotomic polynomials and since the degree of $m_A(x)$ is at most 2 we conclude that $m_A(x)$ itself must be a cyclotomic polynomial, say $\Phi_d(x)$ for some positive integer d with $\phi(d) = 1$ or 2 (where ϕ is the Euler totient function), $\phi(d)$ being the degree of $\Phi_d(x)$. But this implies that $d \in \{1, 2, 3, 4, 6\}$ and since A, A^3 cannot be equal to I_2 we get that $m_A(x) \in \{x + 1, x^2 + 1, x^2 - x + 1\}$ and this implies that either $A^2 = -I_2$ or $A^3 = -I_2$. □

South Eastern European Mathematical
Olympiad for University Students
Iaşi, România - March 7, 2014

Problem 1. Let n be a nonzero natural number and $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be a function such that $f(2014) = 1 - f(2013)$. Let $x_1, x_2, x_3, \dots, x_n$ be real numbers not equal to each other. If

$$\begin{vmatrix} 1+f(x_1) & f(x_2) & f(x_3) & \dots & f(x_n) \\ f(x_1) & 1+f(x_2) & f(x_3) & \dots & f(x_n) \\ f(x_1) & f(x_2) & 1+f(x_3) & \dots & f(x_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f(x_1) & f(x_2) & f(x_3) & \dots & 1+f(x_n) \end{vmatrix} = 0,$$

prove that f is not continuous.

Problem 2. Consider the sequence (x_n) given by

$$x_1 = 2, \qquad x_{n+1} = \frac{x_n + 1 + \sqrt{x_n^2 + 2x_n + 5}}{2}, \quad n \geq 2.$$

Prove that the sequence $y_n = \sum_{k=1}^{n} \frac{1}{x_k^2 - 1}, \quad n \geq 1$ is convergent and find its limit.

Problem 3. Let $A \in \mathcal{M}_n(\mathbb{C})$ and $a \in \mathbb{C},\ a \neq 0$ such that $A - A^* = 2aI_n$, where $A^* = (\bar{A})^t$ and $\bar{A}$ is the conjugate of the matrix A.

(a) Show that $|\det A| \geq |a|^n$

(b) Show that if $|\det A| = |a|^n$ then $A = aI_n$.

Problem 4. a) Prove that $\lim\limits_{n\to\infty} n \int\limits_0^n \frac{\operatorname{arctg} \frac{x}{n}}{x\,(x^2+1)}\, dx = \frac{\pi}{2}$.

b) Find the limit $\lim\limits_{n\to\infty} n \left(n \int\limits_0^n \frac{\operatorname{arctg} \frac{x}{n}}{x\,(x^2+1)}\, dx - \frac{\pi}{2} \right)$.

SEEMOUS 2015 Contest Problems and Solutions

Problem 1. Prove that for every $x \in (0,1)$ the following inequality holds:

$$\int_0^1 \sqrt{1+(\cos y)^2}\,dy > \sqrt{x^2+(\sin x)^2} \;.$$

Solution 1. Clearly

$$\int_0^1 \sqrt{1+(\cos y)^2}\,dy \geq \int_0^x \sqrt{1+(\cos y)^2}\,dy \,.$$

Define a function $F:[0,1] \to \mathbb{R}$ by setting:

$$F(x) = \int_0^x \sqrt{1+(\cos y)^2}\,dy - \sqrt{x^2+(\sin x)^2} \,.$$

Since $F(0)=0$, it suffices to prove $F'(x) \geq 0$. By the fundamental theorem of Calculus, we have

$$F'(x) = \sqrt{1+(\cos x)^2} - \tfrac{x+\sin x \cos x}{\sqrt{x^2+(\sin x)^2}} \,.$$

Thus, it is enough to prove that

$$(1+(\cos x)^2)(x^2+(\sin x)^2) \geq (x+\sin x \cos x)^2 .$$

But this is a straightforward application of the Cauchy-Schwarz inequality.

Solution 2. Clearly $\int_0^1 \sqrt{1+(\cos y)^2}\,dy \geq \int_0^x \sqrt{1+(\cos y)^2}\,dy$ for each fixed $x \in (0,1)$. Observe that $\int_0^x \sqrt{1+(\cos y)^2}\,dy$ is the arc length of the function $f(y) = \sin y$ on the interval $[0,x]$ which is clearly strictly greater than the length of the straight line between the points $(0,0)$ and $(x, \sin x)$ which in turn is equal to $\sqrt{x^2+(\sin x)^2}$.

Problem 2. For any positive integer n, let the functions $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_{n+1}(x) = f_1(f_n(x))$, where $f_1(x) = 3x-4x^3$. Solve the equation $f_n(x)=0$.

Solution. First, we prove that $|x|>1 \Rightarrow |f_n(x)|>1$ holds for every positive integer *n*. It suffices to demonstrate the validity of this implication for $n=1$. But, by assuming $|x|>1$, it readily follows that $|f_1(x)| = |x||3-4x^2| \geq |3-4x^2| > 1$, which completes the demonstration. We conclude that every solution of the equation $f_n(x)=0$ lies in the closed interval $[-1,1]$. For an arbitrary such x, set $x=\sin t$ where $t=\arcsin x\in[-\frac{\pi}{2},\frac{\pi}{2}]$. We clearly have $f_1(\sin t)=\sin 3t$, which gives

$$f_n(x)=\sin 3^n t=\sin(3^n \arcsin x).$$

Thus, $f_n(x)=0$ if and only if $\sin(3^n \arcsin x)=0$, i.e. only when $3^n \arcsin x=k\pi$ for some $k\in\mathbf{Z}$. Therefore, the solutions of the equation $f_n(x)=0$ are given by

$$x=\sin\frac{k\pi}{3^n},$$

where k acquires every integer value from $\frac{1-3^n}{2}$ up to $\frac{3^n-1}{2}$.

Problem 3. For an integer $n > 2$, let $A, B, C, D \in M_n(\mathbb{R})$ be matrices satisfying:

$$AC - BD = I_n,$$
$$AD + BC = O_n,$$

where I_n is the identity matrix and O_n is the zero matrix in $M_n(\mathbb{R})$.

Prove that:

a) $CA - DB = I_n$ and $DA + CB = O_n$,

b) $\det(AC) \geq 0$ and $(-1)^n \det(BD) \geq 0$.

Solution. a) We have

$$AC - BD + i(AD + BC) = I_n \Leftrightarrow (A + iB)(C + iD) = I_n,$$

which implies that the matrices $A + iB$ and $C + iD$ are inverses to one another. Thus,

$$(C + iD)(A + iB) = I_n \Leftrightarrow CA - DB + i(DA + CB) = I_n$$
$$\Leftrightarrow CA - DB = I_n, DA + CB = O_n.$$

b) We have

$$\det((A + iB)C) = \det(AC + iBC)$$
$$\overset{AD+BC=O_n}{=} \det(AC - iAD)$$
$$= \det(A(C - iD)).$$

On the other hand,

$$\det C \overset{(C+iD)(A+iB)=I_n}{=} \det((C + iD)(A + iB)C) = \det((C + iD)A(C - iD))$$
$$= \det(A) \,|\det(C + iD)|^2.$$

Thus,

$$\det(AC) = (\det A)^2 |\det(C + iD)|^2 \geq 0.$$

Similarly

$$\det((A + iB)D) = \det(AD + iBD)$$
$$\overset{AD+BC=O_n}{=} \det(-BC + iBD)$$
$$= (-1)^n \det(B(C - iD)).$$

This implies that

$$\det D \overset{(C+iD)(A+iB)=I_n}{=} \det((C + iD)(A + iB)D) = (-1)^n \det((C + iD)B(C - iD))$$
$$= (-1)^n \det(B) \,|\det(C + iD)|^2.$$

Thus, $(-1)^n \det(BD) = (\det B)^2 \,|\det(C + iD)|^2 \geq 0$.

Problem 4. Let $I \subset \mathbb{R}$ be an open interval which contains 0, and $f : I \to \mathbb{R}$ be a function of class $C^{2016}(I)$ such that $f(0)=0,\ f'(0)=1,\ f''(0)=f'''(0)=\ldots=f^{(2015)}(0)=0,\ f^{(2016)}(0)<0.$

i) Prove that there is $\delta > 0$ such that

$$0 < f(x) < x, \quad \forall x \in (0,\delta). \tag{1.1}$$

ii) With δ determined as in *i*), define the sequence (a_n) by

$$a_1 = \frac{\delta}{2},\ a_{n+1} = f(a_n),\ \forall n \ge 1. \tag{1.2}$$

Study the convergence of the series $\sum_{n=1}^{\infty} a_n^r$, for $r \in \mathbb{R}$.

Solution. ***i)*** We claim that there exists $\alpha > 0$ such that $f(x) > 0$ for any $x \in (0,\alpha)$. For this, observe that, since f is of class C^1 and $f'(0)=1>0$, there exists $\alpha > 0$ such that $f'(x) > 0$ on $(0,\alpha)$. Since $f(0)=0$ and f is strictly increasing on $(0,\alpha)$, the claim follows.

Next, we prove that there exists $\beta > 0$ such that $f(x) < x$ for any $x \in (0,\beta)$. Since $f^{(2016)}(0) < 0$ and f is of class C^{2016}, there is $\beta > 0$ such that $f^{(2016)}(t) < 0$, for any $t \in (0,\beta)$. By the Taylor's formula, for any $x \in (0,\beta)$, there is $\theta \in [0,1]$ such that

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \ldots + \frac{f^{(2015)}(0)}{2015!}x^{2015} + \frac{f^{(2016)}(\theta x)}{2016!}x^{2016}, \tag{1.3}$$

hence

$$g(x) = \frac{f^{(2016)}(\theta x)}{2016!}x^{2016} < 0, \quad \forall x \in (0,\beta).$$

Taking $\delta = \min\{\alpha,\beta\} > 0$, the item *i)* is completely proven.

ii) We will prove first that the sequence (a_n) given by (1.2) converges to 0. Indeed, by relation (1.1) it follows that

$$0 < a_{n+1} < a_n,\ \forall n \ge 1,$$

hence the sequence (a_n) is strictly decreasing and lower bounded by 0. It follows that (a_n) converges to some $\ell \in [0,\frac{\delta}{2})$. Passing to the limit in (1.2) , one gets $\ell = f(\ell)$. Taking into account (1.1), we deduce that $\ell = 0$.

In what follows, we calculate

$$\lim_{n\to\infty} n a_n^{2015}.$$

From $a_n \downarrow 0$, using the Stolz-Cesàro Theorem, we conclude that

$$\lim_{n\to\infty} n a_n^{2015} = \lim_{n\to\infty} \frac{n}{\frac{1}{a_n^{2015}}} = \lim_{n\to\infty} \frac{(n+1)-n}{\frac{1}{a_{n+1}^{2015}} - \frac{1}{a_n^{2015}}} = \lim_{n\to\infty} \frac{1}{\frac{1}{f(a_n)^{2015}} - \frac{1}{a_n^{2015}}}$$

$$= \lim_{x\to 0} \frac{1}{\frac{1}{f(x)^{2015}} - \frac{1}{x^{2015}}} = \lim_{x\to 0} \frac{(xf(x))^{2015}}{x^{2015} - f(x)^{2015}}.$$

Observe that, by (1.3) $\frac{\left(xf\left(x\right)\right)^{2015}}{x^{2015}-f(x)^{2015}} = \frac{\left(x^2+\frac{f^{(2016)}(\theta x)}{2016!}x^{2017}\right)^{2015}}{-\frac{f^{(2016)}(\theta x)}{2016!}x^{2016}(x^{2014}+x^{2013}f(x)+\ldots+f(x)^{2014})}$.

Since f is of class C^{2016}, $\lim_{x\to 0} f^{(2016)}(\theta x) = f^{(2016)}(0)$ and

$$\lim_{x\to 0}\frac{(xf(x))^{2015}}{x^{2015}-f(x)^{2015}} = -\frac{2016!}{2015 f^{(2016)}(0)} > 0.$$

It means, by the comparison criterion, that the series $\sum_{n=1}^{\infty} a_n^r$ and $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{r}{2015}}}$ converge and/or diverge simultaneously, hence the series $\sum_{n=1}^{\infty} a_n^r$ converges for $r > 2015,$ and diverges for $r \le 2015.$

	SEEMOUS 2016 **South Eastern European** **Mathematical Olympiad for University Students** **Protaras, Cyprus** **1-6 March 2016**
Mathematical Society of South Eastern Europe **Cyprus Mathematical Society**	

LANGUAGE: ENGLISH

COMPETITION PROBLEMS

Do all problems 1-4. Each problem is worth 10 points. All answers are written in the booklet provided, following the rules written in the Olympiad programme.

Problem1.

Let f be a continuous and decreasing real valued function, defined on $\left[0, \frac{\pi}{2}\right]$.

Prove the inequalities

$$\int_{\frac{\pi}{2}-1}^{\frac{\pi}{2}} f(x)dx \leq \int_{0}^{\frac{\pi}{2}} f(x)\cos x\, dx \leq \int_{0}^{1} f(x)dx$$

when do equalities hold?

Problem 2.

a) Prove that for every matrix $X \in M_2(\mathbb{C})$ there exists a matrix $Y \in M_2(\mathbb{C})$ such that $Y^3 = X^2$.
b) Prove that there exists a matrix $A \in M_3(\mathbb{C})$ such that $Z^3 \neq A^2$ for all $Z \in M_3(\mathbb{C})$.

Problem3.

Let $A_1, A_2, \dots, A_k$ be idempotent matrices ($A_i^2 = A_i$) in $M_n(\mathbb{R})$. Prove that

$$\sum_{i=1}^{k} N(A_i) \geq \operatorname{rank}\left(I - \prod_{i=1}^{k} A_i\right)$$

where $N(A_i) = n - \operatorname{rank}(A_i)$ and $M_n(\mathbb{R})$ is the set of square $n \times n$ matrices with real entries.

Problem4.

Let $n \geq 1$ be an integer and let

$$I_n = \int_0^{\infty} \frac{\arctan x}{(1+x^2)^n}\, dx$$

Prove that

a) $\displaystyle\sum_{n=1}^{\infty} \frac{I_n}{n} = \frac{\pi^2}{6}$

b) $\displaystyle\int_0^{\infty} \arctan x \cdot \ln\left(1+\frac{1}{x^2}\right) dx = \frac{\pi^2}{6}$

Seemous 2017, February 28 – March 5, 2017, Ohrid, Republic of Macedonia

Problems (time for work: 5 hours).

1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$ such that $a^2 + b^2 + c^2 + d^2 < \frac{1}{5}$.
Show that $I + A$ is invertible.

2. Let $A, B \in \mathcal{M}_n(\mathbb{R})$.
a) Show that there exists $a > 0$ such that for every $\varepsilon \in (-a, a)$, $\varepsilon \neq 0$, the matrix equation

$$AX + \varepsilon X = B, \quad X \in \mathcal{M}_n(\mathbb{R})$$

has a unique solution $X(\varepsilon) \in \mathcal{M}_n(\mathbb{R})$.
b) Prove that if $B^2 = I_n$, and A is diagonalizable then

$$\lim_{\varepsilon \to 0} \mathrm{Tr}(BX(\varepsilon)) = n - \mathrm{rank}(A).$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^4 f(x(x-3)^2)\,\mathrm{d}x = 2\int_1^3 f(x(x-3)^2)\,\mathrm{d}x$$

4. **a)** Let $n \geq 0$ be an integer. Calculate $\int_0^1 (1-t)^n e^t\,\mathrm{d}t$.
b) Let $k \geq 0$ be a fixed integer and let $(x_n)_{n \geq k}$ be a sequence defined by

$$x_n = \sum_{i=k}^{n} \binom{i}{k} \left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{i!} \right).$$

Prove that the sequence converges and find its limit.

Solutions

1. Obvious. $\|A\| < 1 \implies (I \pm A)$ are invertible. Or, prove directly: $\det(A + I) > 0$.
Note: $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$.

2. **a)** The equation is $(A + \varepsilon I)X = B$. Take $a = \min\{|\lambda| : \lambda \in \sigma_p(A) \backslash \{0\}\}$. Then $\exists (A + \varepsilon I)^{-1}$ for $0 < |\varepsilon| < a$.
b) $X(\varepsilon) = (A + \varepsilon I)^{-1}B \implies BX(\varepsilon) = B(A + \varepsilon I)^{-1}B = B^{-1}(A + \varepsilon I)^{-1}B \implies$
$\mathrm{Tr}(BX(\varepsilon)) = \mathrm{Tr}\left((A + \varepsilon I)^{-1}\right) = \sum_{k=1}^{n} \frac{1}{\lambda_k + \varepsilon}$, where λ_k are the eigenvalues (with multiplicities).
$\lim_{\varepsilon \to 0} \varepsilon\,\mathrm{Tr}(BX(\varepsilon)) = \lim_{\varepsilon \to 0} \sum_{k=1}^{n} \frac{\varepsilon}{\lambda_k + \varepsilon} = \mathrm{card}\{k \in \{1, \ldots, n\} : \lambda_k = 0\}$.
If A is diagonalizable this is exactly $n - \mathrm{rank}(A)$.
[It is enough for A to satisfy the condition that the algebraic and geometric multiplicity for 0 be the same]

3. Denote $p : [0, 4] \to [0, 4]$, $p(x) = x(x-3)^2$ and the intervals $J_1 = [0, 1]$, $J_2 = [1, 3]$ and $J_3 = [3, 4]$.
Let p_k be the restriction of p to J_k. p_k is bijective, and denote by $q_k : [0, 4] \to J_k$ its inverse.
Notice that $p_1' > 0, p_2' < 0, p_3' > 0$ in $\mathring{J}_k$, so $q_1' > 0, q_2' < 0, q_3' > 0$ in the open interval $(0, 4)$.
Changing the variables $p_k(x) = y$ we obtain:

$$\int_{J_k} f(p(x))\,\mathrm{d}x = \int_0^4 f(y)\,\left|q_k'(y)\right|\,\mathrm{d}y, \ k = 1 \ldots 3. \tag{*}$$

Now, $q_k(y)$ are the roots of the equation $p(x) = y$, i.e. $x^3 - 6x^2 + 9x - y = 0$.
It follows that $q_1 + q_2 + q_3 = 6$, hence $q_1' + q_2' + q_3' = 0$ in $(0, 4)$
Since only q_2' is negative: $|q_k'| - |q_k'| + |q_k'| = 0$ in $(0, 4)$
Using (*) it results:
$\int_{J_1} f(p(x))\,\mathrm{d}x - \int_{J_2} f(p(x))\,\mathrm{d}x + \int_{J_3} f(p(x))\,\mathrm{d}x = 0$
which is exactly the desired conclusion.

Note. The integrals in the right hand side of (*) are improper. If we want to avoid this, then we may apply the change of variables only for smaller intervals e.g. $J_k(\varepsilon) = p_k^{-1}([\varepsilon, 4 - \varepsilon])$ and take $\varepsilon \to 0_+$ at the end.

$\boxed{\textbf{4.}}$ **a)** Denote by I_k the integral.
$I_0 = e - 1$. An integration by parts gives $I_n = -1 + nI_{n-1}$, $n \geq 1$.
$I_n/n! = -1/n! + I_{n-1}/(n-1)!$ After a telescoping summation we obtain
$I_n/n! - I_0 = -\sum_{k=1}^{n} 1/k!$

$$I_n = n!\left(e - \sum_{k=0}^{n} \frac{1}{k!}\right)$$

b) $x_n = \sum_{i=k}^{n} \binom{i}{k} I_i/i! = \sum_{i=k}^{n} \frac{I_i}{k!(i-k)!}$. The sequence is increasing, so its limit exists and
$\lim_{n\to\infty} x_n = \sum_{i=k}^{\infty} \frac{I_i}{k!(i-k)!} =$
$\frac{1}{k!} \sum_{i=0}^{\infty} \frac{I_{k+i}}{i!} =$
$\frac{1}{k!} \sum_{i=0}^{\infty} \frac{I_{k+i}}{i!} =$
$\frac{1}{k!} \sum_{i=0}^{\infty} \frac{1}{i!} \int_0^1 (1-t)^{k+i} e^t\,\mathrm{d}t =$ [by the theorem of monotone convergence]
$\frac{1}{k!} \int_0^1 \sum_{i=0}^{\infty} \frac{(1-t)^{k+i}}{i!} e^t\,\mathrm{d}t =$
$\frac{1}{k!} \int_0^1 (1-t)^k \sum_{i=0}^{\infty} \frac{(1-t)^i}{i!} e^t\,\mathrm{d}t =$
$\frac{1}{k!} \int_0^1 (1-t)^k e^{1-t} e^t\,\mathrm{d}t = \frac{e}{(k+1)!}$.

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SEEMOUS 2018

FEBRUARY 27-MARCH 4, IAŞI, ROMÂNIA

South Eastern European Mathematical Olympiad for University Students
Iaşi, Romania
March 1, 2018

COMPETITION PROBLEMS

Problem 1. Let $f:[0,1]\to(0,1)$ be a Riemann integrable function. Show that

$$\frac{2\displaystyle\int_0^1 xf^2(x)\,\mathrm{d}x}{\displaystyle\int_0^1 (f^2(x)+1)\,\mathrm{d}x} < \frac{\displaystyle\int_0^1 f^2(x)\,\mathrm{d}x}{\displaystyle\int_0^1 f(x)\,\mathrm{d}x}.$$

Problem 2. Let $m,n,p,q\geq 1$ and let the matrices $A\in\mathcal{M}_{m,n}(\mathbb{R})$, $B\in\mathcal{M}_{n,p}(\mathbb{R})$, $C\in\mathcal{M}_{p,q}(\mathbb{R})$, $D\in\mathcal{M}_{q,m}(\mathbb{R})$ be such that

$$A^t = BCD,\quad B^t = CDA,\quad C^t = DAB,\quad D^t = ABC.$$

Prove that $(ABCD)^2 = ABCD$.

Problem 3. Let $A,\ B\in\mathcal{M}_{2018}(\mathbb{R})$ such that $AB=BA$ and $A^{2018}=B^{2018}=I$, where I is the identity matrix. Prove that if $\operatorname{Tr}(AB)=2018$, then $\operatorname{Tr}A=\operatorname{Tr}B$.

Problem 4. (a) Let $f:\mathbb{R}\to\mathbb{R}$ be a polynomial function. Prove that

$$\int_0^\infty \mathrm{e}^{-x}f(x)\,\mathrm{d}x = f(0)+f'(0)+f''(0)+\cdots.$$

(b) Let f be a function which has a Taylor series expansion at 0 with radius of convergence $R=\infty$. Prove that if $\displaystyle\sum_{n=0}^{\infty} f^{(n)}(0)$ converges absolutely then $\displaystyle\int_0^\infty \mathrm{e}^{-x}f(x)\,\mathrm{d}x$ converges and

$$\sum_{n=0}^{\infty} f^{(n)}(0) = \int_0^\infty \mathrm{e}^{-x}f(x)\,\mathrm{d}x.$$

SEEMOUS
BULGARIA • DEVIN • MARCH 12-17 • 2019

Problem (1). *We shall call the numerical sequence $\{x_n\}$ a "Devin" sequence if $0 \le x_n \le 1$ and for each function $f \in \mathrm{C}[0,1]$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \int_0^1 f(x)\mathrm{d}x.$$

Prove that the numerical sequence $\{x_n\}$ is a "Devin" sequence if and only if $\forall k \ge 0$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i^k = \frac{1}{k+1}.$$

Problem (2). *Let m and n be positive integers. Prove that for any matrices $A_1, A_2, \ldots, A_m \in \mathcal{M}_n(\mathbb{R})$ there exist $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \in \{-1, 1\}$ such that*

$$\mathrm{Tr}\left((\varepsilon_1 A_1 + \varepsilon_2 A_2 + \ldots + \varepsilon_m A_m)^2\right) \ge \mathrm{Tr}\left(A_1^2\right) + \mathrm{Tr}\left(A_2^2\right) + \ldots + \mathrm{Tr}\left(A_m^2\right).$$

Problem (3). *Let $n \ge 2$ and $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $B^2 = B$. Prove that*

$$\mathrm{rank}(AB - BA) \le \mathrm{rank}(AB + BA).$$

Problem (4). *(a) Let $n \ge 1$ be an integer. Calculate $\int_0^1 x^{n-1} \ln x \, dx$.*

(b) Calculate

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} - \cdots\right).$$

SEEMOUS 2020 SOLUTIONS

Thessaloniki, Greece
March 3–8, 2020

Problem 1. Consider $A \in \mathcal{M}_{2020}(\mathbb{C})$ such that

$$(1) \qquad \begin{aligned} A + A^{\times} &= I_{2020}, \\ A \cdot A^{\times} &= I_{2020}, \end{aligned}$$

where $A^{\times}$ is the adjugate matrix of A, i.e., the matrix whose elements are $a_{ij} = (-1)^{i+j} d_{ji}$, where d_{ji} is the determinant obtained from A, eliminating the line j and the column i.

Find the maximum number of matrices verifying (1) such that any two of them are not similar.

Solution. It is known that

$$A \cdot A^{\times} = \det A \cdot I_{2020},$$

hence, from the second relation we get $\det A = 1$, so A is invertible. Next, multiplying in the first relation by A, we get

$$A^2 - A + I_{2020} = \mathcal{O}_{2020}.$$

It follows that the minimal polynomial of A divides

$$X^2 - X + 1 = (X - \omega)(X - \bar{\omega}),$$

where

$$\omega = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}.$$

Because the factors of the minimal polynomial of A are of degree 1, it follows that A is diagonalizable, so A is similar to a matrix of the form

$$A_k = \begin{pmatrix} \omega I_k & \mathcal{O}_{k,2020-k} \\ \mathcal{O}_{2020-k,k} & \bar{\omega} I_{n-k} \end{pmatrix}, \quad k \in \{0, 1, ..., 2020\}.$$

But $\det A = 1$, so we must have

$$\begin{aligned} \omega^k \bar{\omega}^{2020-k} &= 1 \Leftrightarrow \omega^{2k-2020} = 1 \Leftrightarrow \cos\frac{(2k-2020)\pi}{3} + i\sin\frac{(2k-2020)\pi}{3} = 1 \\ &\Leftrightarrow \cos\frac{(2k+2)\pi}{3} + i\sin\frac{(2k+2)\pi}{3} = 1 \\ &\Leftrightarrow k = 3n + 2 \in \{0, ..., 2020\} \Leftrightarrow k \in \{2, 5, 8, ..., 2018\} \end{aligned}$$

Two matrices that verify the given relations are not similar if and only if the numbers k_1, k_2 corresponding to those matrices are different, so the required maximum number of matrices is 673. $\square$

Problem 2. Let $k > 1$ be a real number. Calculate:

(a) $L = \lim_{n\to\infty} \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x$.

(b) $\lim_{n\to\infty} n \left[L - \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x\right]$.

Proof. **(a)** The limit equals $\boxed{\frac{k}{k-1}}$.

Using the substitution $x = y^n$ we have that

$$I_n = \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x = nk^n \int_0^1 \left(\frac{y}{y+k-1}\right)^{n-1} \frac{\mathrm{d}y}{y+k-1}.$$

Using the substitution $\frac{y}{y+k-1} = t \Rightarrow y = \frac{(k-1)t}{1-t}$ we get, after some calculations, that

$$I_n = nk^n \int_0^{\frac{1}{k}} \frac{t^{n-1}}{1-t}\,\mathrm{d}t.$$

We integrate by parts and we have that

$$I_n = \frac{k}{k-1} - k^n \int_0^{\frac{1}{k}} \frac{t^n}{(1-t)^2}\,\mathrm{d}t.$$

It follows that $\lim_{n\to\infty} I_n = \frac{k}{k-1}$ since

$$0 < k^n \int_0^{\frac{1}{k}} \frac{t^n}{(1-t)^2}\,\mathrm{d}t < \frac{k^{n+2}}{(k-1)^2} \int_0^{\frac{1}{k}} t^n\,\mathrm{d}t = \frac{k}{(n+1)(k-1)^2}.$$

(b) The limit equals $\boxed{\frac{k}{(k-1)^2}}$.

We have that

$$\frac{k}{k-1} - I_n = k^n \int_0^{\frac{1}{k}} \frac{t^n}{(1-t)^2}\,\mathrm{d}t.$$

We integrate by parts and we have that

$$\frac{k}{k-1} - I_n = \frac{1}{n+1} \cdot \frac{k}{(k-1)^2} - \frac{2k^n}{n+1} \int_0^{\frac{1}{k}} \frac{t^{n+1}}{(1-t)^3}\,\mathrm{d}t.$$

This implies that

$$\lim_{n\to\infty} n \left[\frac{k}{k-1} - \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x\right] = \lim_{n\to\infty} \left[\frac{n}{n+1} \cdot \frac{k}{(k-1)^2} - \frac{2k^n n}{n+1} \int_0^{\frac{1}{k}} \frac{t^{n+1}}{(1-t)^3}\,\mathrm{d}t\right].$$

Thus

$$\lim_{n\to\infty} n \left[\frac{k}{k-1} - \int_0^1 \left(\frac{k}{\sqrt[n]{x}+k-1}\right)^n \mathrm{d}x\right] = \frac{k}{(k-1)^2},$$

since

$$0 < k^n \int_0^{\frac{1}{k}} \frac{t^{n+1}}{(1-t)^3}\,\mathrm{d}t < \frac{k^{n+3}}{(k-1)^3} \int_0^{\frac{1}{k}} t^{n+1}\,\mathrm{d}t = \frac{k}{(k-1)^3(n+2)}.$$

□

Problem 3. Let n be a positive integer, $k \in \mathbb{C}$ and $A \in \mathcal{M}_n(\mathbb{C})$ such that $\operatorname{Tr} A \neq 0$ and

$$\operatorname{rank} A + \operatorname{rank}\left((\operatorname{Tr} A) \cdot I_n - kA\right) = n.$$

Find $\operatorname{rank} A$.

Proof. For simplicity, denote $\alpha = \operatorname{Tr} A$. Consider the block matrix:

$$M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & \alpha I_n - kA \end{array}\right].$$

We perform on M a sequence of elementary transformations on rows and columns (that do not change the rank) as follows:

$$M \xrightarrow{R_1} \left[\begin{array}{c|c} A & 0 \\ \hline A & \alpha I_n - kA \end{array}\right] \xrightarrow{C_1} \left[\begin{array}{c|c} A & kA \\ \hline A & \alpha I_n \end{array}\right] \xrightarrow{R_2}$$
$$\xrightarrow{R_2} \left[\begin{array}{c|c} A - \frac{k}{\alpha}A^2 & 0 \\ \hline A & \alpha I_n \end{array}\right] \xrightarrow{C_2} \left[\begin{array}{c|c} A - \frac{k}{\alpha}A^2 & 0 \\ \hline 0 & \alpha I_n \end{array}\right] = N$$

where

$R_1:$ is the left multiplication by $\left[\begin{array}{c|c} I_n & 0 \\ \hline I_n & I_n \end{array}\right]$;

$C_1:$ is the right multiplication by $\left[\begin{array}{c|c} I_n & kI_n \\ \hline 0 & I_n \end{array}\right]$;

$R_2:$ is the left multiplication by $\left[\begin{array}{c|c} I_n & -\frac{k}{\alpha}A \\ \hline 0 & I_n \end{array}\right]$;

$C_2:$ is the right multiplication by $\left[\begin{array}{c|c} I_n & 0 \\ \hline -\frac{1}{\alpha}A & I_n \end{array}\right]$.

It follows that

$$\operatorname{rank} A + \operatorname{rank}(\alpha I_n - kA) = \operatorname{rank} M = \operatorname{rank} N = \operatorname{rank}\left(A - \frac{k}{\alpha}A^2\right) + n.$$

Note that

$$\operatorname{rank}\left(A - \frac{k}{\alpha}A^2\right) = 0 \Leftrightarrow A - \frac{k}{\alpha}A^2 = 0 \Leftrightarrow \underbrace{\frac{k}{\alpha}A}_{B} = \left(\frac{k}{\alpha}A\right)^2 \Leftrightarrow B = B^2$$
$$\Rightarrow \operatorname{rank} B = \operatorname{Tr} B = \operatorname{Tr}\left(\frac{k}{\alpha}A\right) = \frac{k}{\alpha}\operatorname{Tr} A = k$$

so finally $\operatorname{rank} A = \operatorname{rank} B = k$. $\square$

Problem 4. Consider $0 < a < T$, $D = \mathbb{R} \setminus \{kT + a \mid k \in \mathbb{Z}\}$, and let $f : D \to \mathbb{R}$ a $T-$periodic and differentiable function which satisfies $f' > 1$ on $(0, a)$ and

$$f(0) = 0, \ \lim_{\substack{x \to a \\ x < a}} f(x) = +\infty \text{ and } \lim_{\substack{x \to a \\ x < a}} \frac{f'(x)}{f^2(x)} = 1.$$

(a) Prove that for every $n \in \mathbb{N}^*$, the equation $f(x) = x$ has a unique solution in the interval $(nT, nT + a)$, denoted x_n.

(b) Let $y_n = nT + a - x_n$ and $z_n = \displaystyle\int_0^{y_n} f(x)dx$. Prove that $\lim_{n \to \infty} y_n = 0$ and study the convergence of the series $\sum_{n=1}^{\infty} y_n$ and $\sum_{n=1}^{\infty} z_n$.

Proof. **(1)** Observe first that, for every $n \in \mathbb{N}^*$, $f(nT) = 0$ and $\lim_{\substack{x \to nT+a \\ x < nT+a}} f(x) = +\infty$, hence the equation $f(x) = x$ has at least one solution in the interval $(nT, nT + a)$.

Now, consider the function $g(x) = f(x) - x$ on $(nT, nT + a)$ and observe that if there would exist two solutions of the equation $f(x) = x$, say $x_n^1 < x_n^2$, by Rolle's Theorem, there exists $r_n \in (x_n^1, x_n^2) \subset (nT, nT + a)$ such that $g'(r_n) = f'(r_n) - 1 = 0$, a contradiction, since $f' > 1$ on $(nT, nT + a)$ by periodicity.
(2) Observe that for any n, f is strictly increasing on $(nT, nT + a)$. We prove that (y_n) is decreasing. By contradiction, suppose that $y_n < y_{n+1}$ for some n. Then $T + x_n > x_{n+1}$, and by the monotonicity of f that

$$x_n = f(x_n) = f(x_n + T) > f(x_{n+1}) = x_{n+1},$$

an obvious contradiction.

Since $y_n \in (0, a)$ for every n, it follows that (y_n) it converges. Then there exists $\overline{y} \geq 0$ such that $y_n \to \overline{y}$. Suppose, by contradiction, that $\overline{y} > 0$. Observe that $\overline{y} < a$. Since $x_n - nT \to a - \overline{y}$ for $n \to \infty$, it follows by the continuity of f on $(-T, a)$ that $f(x_n - nT) \to f(a - \overline{y}) \in \mathbb{R}$ for $n \to \infty$. But $f(x_n - nT) = f(x_n) = x_n \to \infty$, hence we obtain a contradiction. Therefore, $y_n \to 0$.

Next, we will prove that

$$\lim_{n \to \infty} n \cdot y_n = \frac{1}{T},$$

hence $\sum_{n=1}^{\infty} y_n$ diverges by a comparison test.

For that, observe that

$$\lim_{n \to \infty} n \cdot y_n = \lim_{n \to \infty} \frac{nT}{Tx_n} \cdot x_n y_n = \frac{1}{T} \lim_{n \to \infty} x_n y_n.$$

Moreover,

$$\begin{aligned} \lim_{n \to \infty} x_n y_n &= \lim_{n \to \infty} f(x_n) \cdot y_n = \lim_{n \to \infty} f(nT + a - y_n) \cdot y_n \\ &= \lim_{n \to \infty} \frac{y_n}{\frac{1}{f(a - y_n)}} = - \lim_{n \to \infty} \frac{(a - y_n) - a}{\frac{1}{f(a - y_n)}}. \end{aligned}$$

But $a - y_n$ converges increasingly to a so the previous limit is

$$- \lim_{\substack{x \to a \\ x < a}} \frac{x - a}{\frac{1}{f(x)}} = - \lim_{\substack{x \to a \\ x < a}} \frac{1}{-\frac{f'(x)}{f^2(x)}} = 1.$$

For the second series, observe that for every n, there is $c_n \in (0, y_n)$ such that $z_n = y_n \cdot f(c_n)$. Since f is increasing on $(0, a)$,

$$z_n \leq y_n \cdot f(y_n) = y_n^2 \cdot \frac{f(y_n)}{y_n}.$$

But f is differentiable at 0, and $\dfrac{f(y_n)}{y_n} \to f'(0) \geq 0$ for $n \to \infty$, hence there exists $M > 0$ such that, for any large n,

$$\frac{f(y_n)}{y_n} \leq M.$$

Then there exist $n_0 \in \mathbb{N}$ and $K > 0$ such that

$$0 \leq z_n \leq \frac{K}{n^2}, \quad \forall n \geq n_0.$$

By a comparison test, $\sum_{n=1}^{\infty} z_n$ converges. $\square$