Θεώρημα Cayley - Hamilton

Κάθε τετραγωνικός πίνακας ικανοποιεί την χαρακτηριστική του εξίσωση:

$$A \in \mathbb{R}^{n \times n}, \quad P(\lambda) = \det(\lambda \mathbb{1} - A) \Rightarrow P(A) = 0$$

- Χρήση για τον υπολογισμό της συνάρτησης:

$$e^{At} = \sum_{k=0}^{\infty} \left(\frac{t^k}{k!} \right) A^k, \quad t \in (-\infty, \infty)$$

$f(\lambda)$ πολυώνυμο $\beta_0, \beta_1 \dots$

$f(A) = \beta_0 \mathbb{1} + \beta_1 A + \cdots \beta_{n-1} A^{n-1}$

διότι: $f(\lambda) = p(\lambda) q(\lambda) + g(\lambda)$

με $\deg g(\lambda) \leq n-1$

επειδή $p(A) = 0 \Rightarrow$

$f(A) = g(A)$

γράφεται: $f(A) = e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i$

Υπολογισμός των $\alpha_i(t)$

έστω
$$\begin{cases} p(\lambda) = \det(\lambda \mathbb{1} - A) = \prod_{i=0}^{p} (\lambda_i - \lambda)^{m_i} \quad \text{(πολλ/τα ρίζας)} \\ f(\lambda),\ g(\lambda) = \text{αναλυτικές } \textbf{συναρτήσεις} \end{cases}$$

αν $f^{l}(\lambda_i) = g^{l}(\lambda_i), \quad l = 0, \dots m_i - 1, \quad i = 1, \dots p$

$$\left. \begin{array}{l} \text{με } f^{l}(\lambda_i) = \left. \dfrac{d^l}{d\lambda^l} f(\lambda) \right|_{\lambda = \lambda_i} \\ \sum_{i=1}^{p} m_i = n \end{array} \right\} \Rightarrow f(A) = g(A)$$

Παράδειγμα:

$$A = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad f(A) = e^{At}$$

$$f(\lambda) = e^{\lambda t} \qquad \lambda_1 = \lambda_2 = 0$$
$$g(\lambda) = \alpha_1 \lambda + \alpha_0 \qquad m_1 = 2$$

Άρα, από
$$f^{(l)}(\lambda_i) = g^{l}(\lambda_i) \Rightarrow \left. \begin{array}{l} f(\lambda_1) = g(\lambda_1) = 1 \\ f^{(1)}(\lambda_1) = g^{(1)}(\lambda_1) \end{array} \right\} \Rightarrow$$

$$\Rightarrow \alpha_0 = 1$$
$$\alpha_1 = t$$

Άρα, $e^{At} = f(A) = g(A) = \alpha_1 A + \alpha_0 \mathbb{1}$

$$= \begin{pmatrix} -\alpha_1 + \alpha_0 & \alpha_0 \\ -\alpha_1 & \alpha_1 + \alpha_0 \end{pmatrix} = \begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}$$

Πιό συγκεκριμένα: Παράδειγμα 2 : Έστω $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$

τότε · η Χ.Ε. του A : $P(\lambda) = \det(\lambda \mathbb{1} - A)$ και ισχύει $P(A) = 0$ (Θ. Cayley-Hamilton)

αυτό σημαίνει ότι : $P(\lambda) = \lambda^2 + 3\lambda + 2 \Rightarrow P(A) = A^2 + 3A + 2\mathbb{1} = [\emptyset]$

Εφαρμογή για τον υπολογισμό του e^{At}

ισχύει : $F(\lambda) = Q(\lambda)P(\lambda) + R(\lambda)$ (διαίρεση πολυωνύμων)

και επειδή $P(A) = 0 \Rightarrow \boxed{F(A) = R(A)}$

όμως $R(\lambda) = \alpha_0 + \alpha_1\lambda + \dots + \alpha_{n-1}\lambda^{(n-1)} \Rightarrow \forall$ ρίζα λ_i της Χ.Ε. ισχύει :

$F(\lambda_i) = R(\lambda_i)$

Τότε, στο παράδειγμά μας : $n = 2 \Rightarrow R(\lambda) = \alpha_0 + \alpha_1\lambda$

αλλά, $\lambda_1 = -1$, $\lambda_2 = -2$

$\Rightarrow F(\lambda_1) = R(\lambda_1) \Rightarrow e^{\lambda_1 t} = \alpha_0 + \alpha_1\lambda_1 \Rightarrow e^{-t} = \alpha_0 - \alpha_1$

$F(\lambda_2) = R(\lambda_2) \Rightarrow e^{\lambda_2 t} = \alpha_0 + \alpha_1\lambda_2 \Rightarrow e^{-2t} = \alpha_0 - 2\alpha_1$

άρα, $\boxed{\begin{aligned} \alpha_0 &= 2e^{-t} - e^{-2t} \\ \alpha_1 &= e^{-t} - e^{-2t} \end{aligned}}$

Τέλος, $F(A) = e^{At} = \alpha_0\mathbb{1} + \alpha_1 A = \dots$

Καταστατικες εξισωσεις Διακριτου Χρονου

$x(k+1) = A(k)x(k) + B(k)u(k)$

$y(k) = C(k)x(k) + D(k)u(k)$

- Απόκριση Μηδενικής Εισόδου

$u(k) = 0 \Rightarrow x(k+1) = A(k)x(k)$

$y(k) = C(k)x(k)$

$x(k_0+1) = A(k_0)x(k_0)$

$x(k_0+2) = A(k_0+1)x(k_0+1) = A(k_0+1)[A(k_0)x(k_0)] = \dots$

$\vdots$

$\Rightarrow x(k) = [A(k-1)A(k-2)\cdots A(k_0+1)A(k_0)]x(k_0)$

$$x(k_0+2) = A(k_0+1)\, x(k_0+1) = A(k_0+1)\,[\ \dots$$
$$\vdots$$
$$\Rightarrow x(k) = \left[A(k-1)\, A(k-2) \cdots A(k_0+1)\, A(k_0)\right] x(k_0)$$

<u>Ορίζουμε</u>: $\Phi(k, k_0) = \begin{cases} A(k-1) \cdots A(k_0), & k > k_0 \\ \mathbb{1}_n & k = k_0 \end{cases}$

<u>Λύση</u>:

$$x(k) = \Phi(k, k_0)\, x_0 \quad , \quad y(k) = C(k)\, \Phi(k, k_0)\, x_0$$

<u>Ιδιότητες</u>:

$$\Phi(k+1, k_0) = A(k)\, \Phi(k, k_0)$$
$$\Phi(k_2, k_0) = \Phi(k_2, k_1)\, \Phi(k_1, k_0)$$

Αν <u>(Σ) Χ.Α.</u>: $\boxed{\Phi(k, k_0) = A^{k-k_0}}$

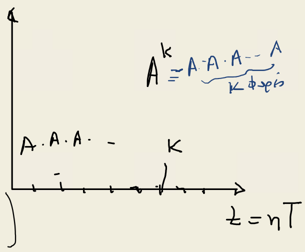
$$\Rightarrow y(k) = C A^{k-k_0} x_0$$

Ακόμα, ισχύουν:

1. $\Phi(k, k) = \mathbb{1}_n$
2. $\Phi(k+1, k_0) = A(k)\, \Phi(k, k_0)$

<u>Παράδειγμα</u>:

$$x(k+1) = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k)$$
$$y(k) = \begin{pmatrix} 1 & -1 \end{pmatrix} x(k) + 2u(k)$$



Επειδή $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \Rightarrow A^2 = 7\, \mathbb{1}_2 = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$

<u>γενικά</u>: $A^{2k-1} = 7^k A$

$$A^{2k} = 7^k\, \mathbb{1}_2 \quad , \quad k = 0, 1, 2, \dots$$

Άρα, $\Phi(k_2, k_1) = A^{k_2-k_1} = \begin{cases} 7^{k_2-k_1} A, & k_2-k_1 \text{ περιττός} \\ 7^{k_2-k_1}\, \mathbb{1} & \text{-\,\,-} \ \text{άρτιος} \end{cases}$

<u>Πλήρης Απόκριση</u>:

$$x(k_0+1) = A(k_0)\, x(k_0) + B(k_0)\, u(k_0) \quad , \quad y(k_0) = \dots$$
$$= \Phi(k_0+1, k_0)\, x_0 + \Phi(k_0+1, k_0+1)\, B(k_0)\, u(k_0)$$

$$x(k_0+1) = A(k_0)x(k_0) + B(k_0)u(k_0) \quad , \quad y(k_0) = \dots$$
$$= \Phi(k_0+1, k_0)x_0 + \Phi(k_0+1, k_0+1)B(k_0)u(k_0)$$

$$x(k_0+2) = A(k_0+1)\,x(k_0+1) + B(k_0+1)\,u(k_0+1) =$$
$$= \Phi(k_0+2, k_0)x_0 + \Phi(k_0+2, k_0+1)B(k_0)u(k_0) + \Phi(k_0+2, k_0+2)B(k_0+1)u(k_0+1)$$

$$\vdots$$

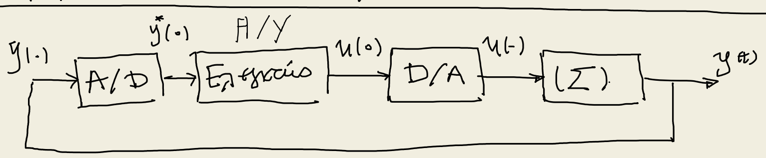
$$\boxed{x(k) = \Phi(k,k_0)x_0 + \sum_{j=k_0}^{k-1} \Phi(k,j+1)\,B(j)\,u(j)}$$ Χρον. Μεταβλ.

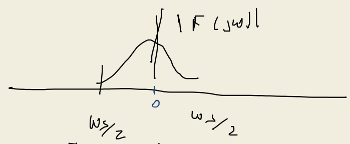
άρα, απόκριση:
$$y(k) = C(k)\Phi(k,k_0)x_0 + C(k)\sum_{j=k_0}^{k-1}\Phi(k,j+1)B(j)u(j) + D(k)u(k)$$

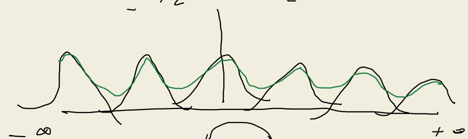
- Θεωρ. (Σ) - Χ.Α.: $\boxed{\Phi(k,k_0) = A^{k-k_0}}$
$$y(k) = CA^{k-k_0}x_0 + C\sum_{j=k_0}^{k-1} A^{k-j-1}Bu(j) + Du(k)$$
$$\forall\, k > k_0$$

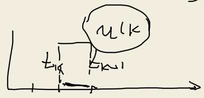
Υπολογισμός Μήτρας Κρουστικών Αποκρίσεων
$$H(k-k^*) = C\sum_{j=k_0}^{k-1} A^{k-j-1}Bu(j) + D\delta(k-k^*)$$
$$\Rightarrow \boxed{H(k-k^*) = CA^{k-k^*-1}B + D\delta(k-k^*)}$$

Μετατροπή Κ.Ε. συνεχούς χρόνου σε Κ.Ε. διακριτού χρόνου





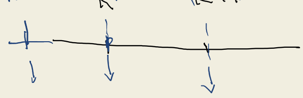




Έστω (Σ): $\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t) + D(t)u(t) \end{cases}$

& $u(t) = u(k) \quad , \forall\, t_k < t < t_{k+1}$

$$\Rightarrow x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$
$$= \Phi(t,t_0)\,x(t_0) + u(kT)\int_{t_0}^{t}\Phi(t,\tau)B(\tau)\,d\tau$$



αλλαγή μεταβλητών: $\left.\begin{aligned} t_0 &= kT \\ t &= (k+1)T \end{aligned}\right\} \Rightarrow$

$$\rightarrow x[(k+1)T] = \Phi[(k+1)T, kT]\,x(kT) + \int_{kT}^{(k+1)T}\Phi[(k+1)T,\tau]\,B(\tau)\,u(kT)\,d\tau$$

$$\rightarrow \boxed{x(k+1) = \Phi(k+1,k)\,x(k) + \Theta(k+1,k)\,u(k)} \quad , \quad \Theta(k+1,k) = \int_{kT}^{(k+1)T}\Phi(\quad)B(\tau)\,d\tau$$

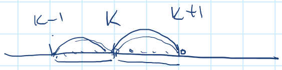
Κ.Ε. διακρ.: $x(k+1) = A(k)x(k) + B(k)u(k)$ $\boxed{kT \le t \le (k+1)T}$

$$\longrightarrow \boxed{x(k+1) = \Phi(k+1,k)\,x(k) + \Theta(k+1,k)\,u(k)} \quad , \quad \Theta(k+1,k) = \int_{kT} \Psi(\quad)\,B(\tau)\,d\tau$$

$$\text{Κ.Ε. δ}_{\text{χρ.}}: x(k+1) = A(k)\,x(k) + B(k)\,u(k) \qquad \boxed{kT \le t \le (k+1)T}$$

– Γ. Χ. Α.

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du \quad \Rightarrow \; ?$$



$$x(t) = e^{At}x(0) + e^{At}\int_0^t e^{-A\tau}Bu(\tau)\,d\tau \quad , \quad u(\tau) = u(kT) \; , \; kT \le t \le (k+1)T$$

$$\longrightarrow \; x(kT) = e^{AkT}x(0) + e^{AkT}\int_0^{kT} e^{-A\tau}Bu(\tau)\,d\tau$$

$$\vdots$$

$$\left[\; x(k+1) = \overset{\Phi(T,0)}{\Phi(T)}\,x(k) + \Theta(T)\,u(k) \;\right] \longrightarrow \text{Π.Κ.Ε. διακριτού χρόνου}$$

$$\rightarrow \quad \underline{\Phi(T) = e^{AT}} \quad , \quad \underline{\Theta(T) = \left(\int_0^T e^{A\lambda}\,d\lambda\right)B}$$

Π.χ.

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} x(t) + \begin{pmatrix} 2 \\ 0 \end{pmatrix} u(t) \quad , \quad y(t) = \begin{pmatrix} -2 & 1 \end{pmatrix} x(t) + 3\,u(t)$$

$$x(k+1) = \Phi(T)\,x(k) + \Theta(T)\,u(k)$$

$$\exp\left[\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} T\right] = \begin{bmatrix} 3e^{-2T} - 2e^{-3T} & e^{-2T} - e^{-3T} \\ 6 - e^{-3T} - 6e^{-2T} & 3e^{-3T} - e^{-2T} \end{bmatrix} = \Phi(T) = A(k)$$

$$\Theta(T) = \int_0^T \exp\left[\begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}\tau\right] d\tau\, B = 3\begin{bmatrix} e^{-3T} - 3e^{-2T} + 2 \\ 6e^{-2T} - 4e^{-3T} - 2 \end{bmatrix} = B(k)$$

$$C(k) = [2 \quad 1]$$

$$D(k) = 3$$

<u>Συμπλήρωμα</u>: Υπολογισμός της $\Phi(k+1,k)$ για <u>Διακριτού Χρ. Συστήματα</u>

– Γενική περίπτωση (Χρον. Μεταβλητά) → όπως στα Συνεχούς Χρόνου

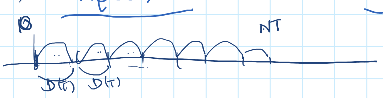
– Γ.Χ.Α. , $\Phi(k+1,k) = \Phi(kT) = A^k$

Γενικά $\quad \Phi(k+1,k) = \Phi(T) = e^{AT} = \mathbb{1} + AT + \dfrac{A^2T^2}{2!} + \cdots$

$$\rightarrow \Phi(T) = Z^{-1}\left\{ z(z\mathbb{1} - A)^{-1} \right\}$$

(*)

$\longrightarrow$ Προσοχή : $\Phi(NT) = \underbrace{\Phi(T) \cdots \Phi(T)}_{N \text{ φορές}} = \Phi^N(T)$



$$\boxed{\Phi(NT) \ne \Phi(t)\big|_{t=NT}}$$

<u>Χρήση Θ. Cayley-Hamilton</u> – Παράδειγμα

$$x(k+1) = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} x(k)$$

$$|\lambda\mathbb{1} - A| = 0 \Rightarrow \begin{cases} \lambda_1 = -2 \\ \lambda_2 = -3 \end{cases} \qquad R(\lambda) = \text{1ης τάξης} , = \alpha_0 + \alpha_1\lambda$$

$$(\lambda^2 + 5\lambda - 6 = 0)$$

$$\Rightarrow \left.\begin{aligned} F(\lambda_1) &= \lambda_1^k = (-2)^k = \alpha_0 + \alpha_1\lambda_1 = \alpha_0 - 2\alpha_1 \\ F(\lambda_2) &= \lambda_2^k = (-3)^k = \alpha_0 + \alpha_1\lambda_2 = \alpha_0 - 3\alpha_1 \end{aligned}\right\} \Rightarrow$$

$$\alpha_0 = 3(-2)^k - 2(-3)^k$$

$\alpha_1 = (-2)^k - (-3)^k$

$A_{\varepsilon\alpha}$, $F(A) = A^k = \alpha_0 \mathbb{I} + \alpha_1 A = \ldots$