Pulsatile flow in arterial circulation

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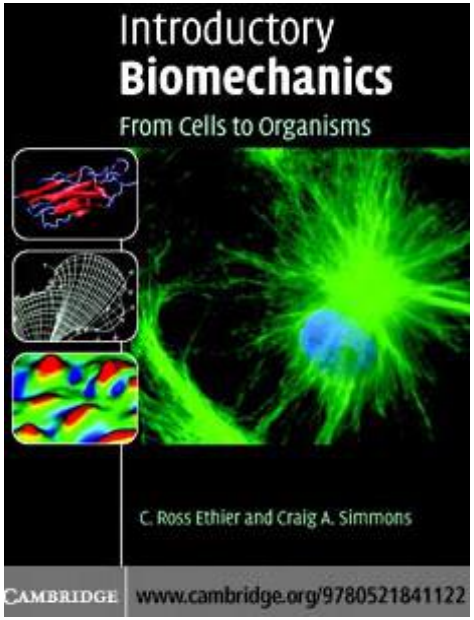
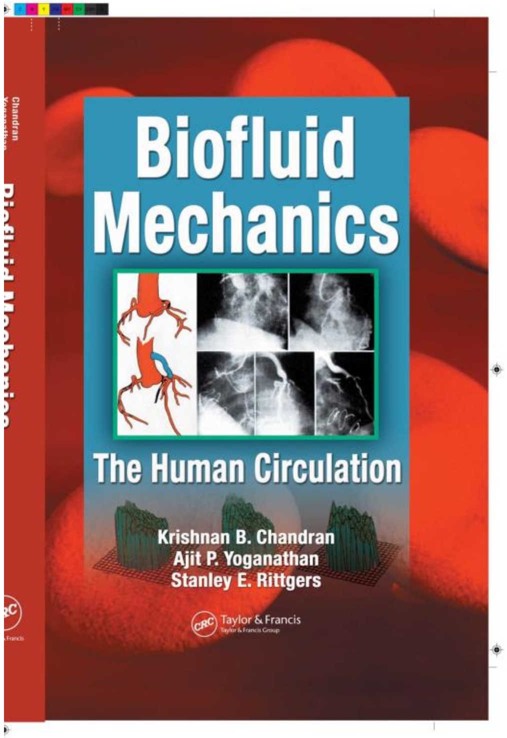
Requirements: Basic fluid mechanics

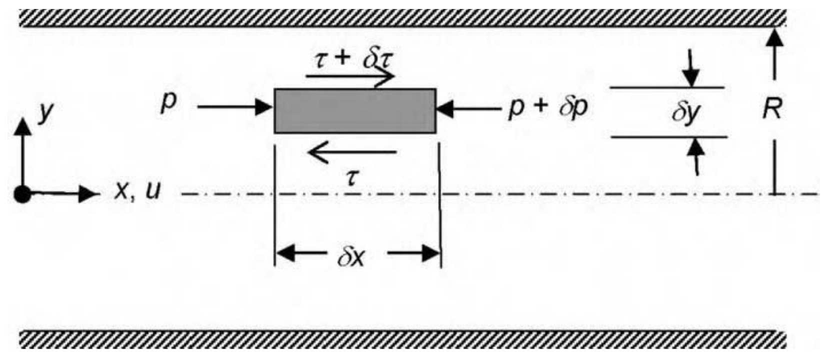




- Relevant Chapters:
- Chapters 3 (138-142) & 4 (179-204) (left), 6 (right)

From Basic fluid mechanics

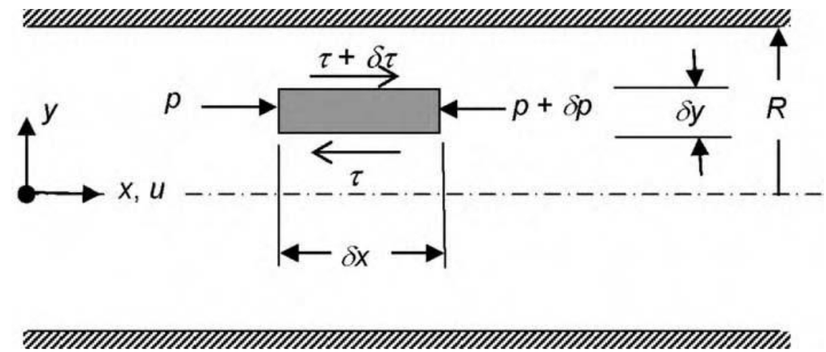
To prove a linear relation between U(x,t) and P (x,t)



- ➢ No realistic analytic solution for non-Newtonian pulsatile blood flow
- ➢ Newtonian approximation – Womerslay's solution
- ➢ Analogous solution – much simpler mathematics
- Consider a long **two-dimensional (x,y)** channel, half-height R, filled with Newtonian blood travelling to **x** direction with velocity **u**
- A fluid element length **δχ**, height **δy** is examined
- Pressures **p, p+δp** are applied to surfaces normal to x
- Shear stresses **τ, τ+δτ** applied to surfaces parallel to x

Non steady flow

Relationship of u/t,x →p/t,x



- A fluid element across a streamline:

$$u = u(y,t)$$

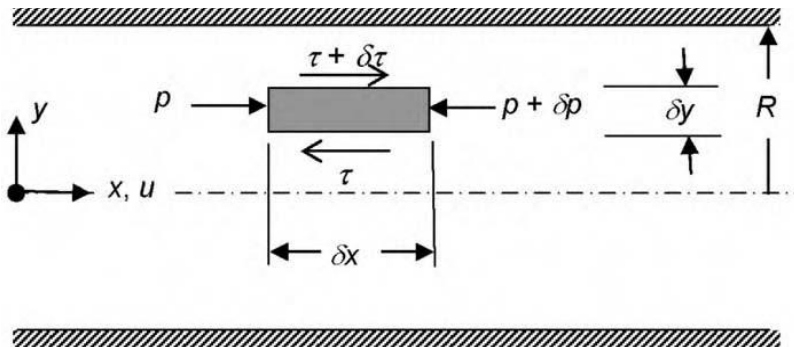
No **radial** u component (=u in axial direction alone), incompressibility

- P is **uniform across a cross-section** of the tube (parallel axial flow)
- **F** acting as a sum of pressure-viscous forces
- Newton's second law **Per unit depth** :

$$\sum F_x = ma_x \quad (3.20)$$

$$P\delta y + (\tau + \delta\tau)\delta x - (P + \delta P)\delta y - \tau\delta x = \rho\delta x\delta y\frac{\partial u}{\partial t} \quad (3.21)$$

From Basic fluid mechanics



$$\tau = \mu \frac{\partial u}{\partial y}$$

3.21 becomes

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} \ \textit{or after } \tau \textit{ replacement}$$

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \ 3.22$$

For periodic (oscillatory) blood flow & pressure gradient

$$\frac{-\partial p}{\partial x} = \Pi.\cos(\omega t) = (\text{Real})\{\Pi e^{i\omega t}\},3.23)$$

Hydrodynamic circulation

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \quad 3.22$$

From 3.22, u→$\frac{\partial P}{\partial x}$ is **linear**, so if we apply an harmonic (cos(ωt)) pressure P traveling to x then we expect to a similarly shaped velocity u at x direction, even with a phase shift (sin(ωt) = cos(ωt + ϕ) derivatives of u, p).

Hydrodynamic circulation

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$ 3.22

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$$u(y,t) = \frac{\Pi}{\rho\omega}(Real)\left\{\hat{u}(y)e^{i\omega t}\right\}$$ (3.24)

Suggestion: Read about simple mathematics for complex numbers

- See Chapter 7, pages 187-192 for application of complex numbers in pulsatile flow dynamics

Applied Biofluid Mechanics

Lee Waite, Ph.D., P.E.

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Singapore Sydney Toronto

Hydrodynamic circulation

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$u(y,t) = \frac{\Pi}{\rho\omega}(Real)\{\hat{u}(y)e^{i\omega t}\}$ (3.24)

Substituting 3.23 & 3.24 to 3.22 results finally to

$i\hat{u}(y) = \frac{\mu}{\rho\omega}\frac{d^2\hat{u}(y)}{dy^2} + 1$ (3.25) ,

Hydrodynamic circulation

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$i\hat{u}(y) = \frac{\mu}{\rho\omega}\frac{d^2\hat{u}(y)}{dy^2} + 1 \ (3.25)$,

$$\hat{y} = {}^{y}/_{R} \qquad \frac{1}{\alpha^2}\frac{d^2\hat{u}(y)}{d\hat{y}^2} - i\hat{u}(y) = -1$$

The Womersley parameter: $\alpha = \mathrm{R}\sqrt{\frac{\omega\rho}{\mu}}$

Hydrodynamic circulation

The Womersley parameter: $\alpha = \mathrm{R}\sqrt{\frac{\omega\rho}{\mu}}$

- It is a measure of oscillatory pressure or velocity

- It is also a measure of the inertial (caused from pressure) vs viscous force (caused from shear) applied in fluid during pulsed flow.

➢ Great α means inertial force dominate
➢ Small α means the viscous forces dominates
➢ α = 20 in aorta, so inertia dominates vs viscous force

- At wall internal, u=0 so viscous force dominates
- At centerline u=max so inertial force dominates
- At distance δ, close to wall (boundary layer limit) inertial force equal viscous force.

$$\rho\omega u = \frac{\mu u}{\delta^2} \rightarrow \delta = \sqrt{\frac{\mu}{\rho\omega}} \rightarrow \alpha = \frac{R}{\delta}$$

Physiological values

TABLE 6.1

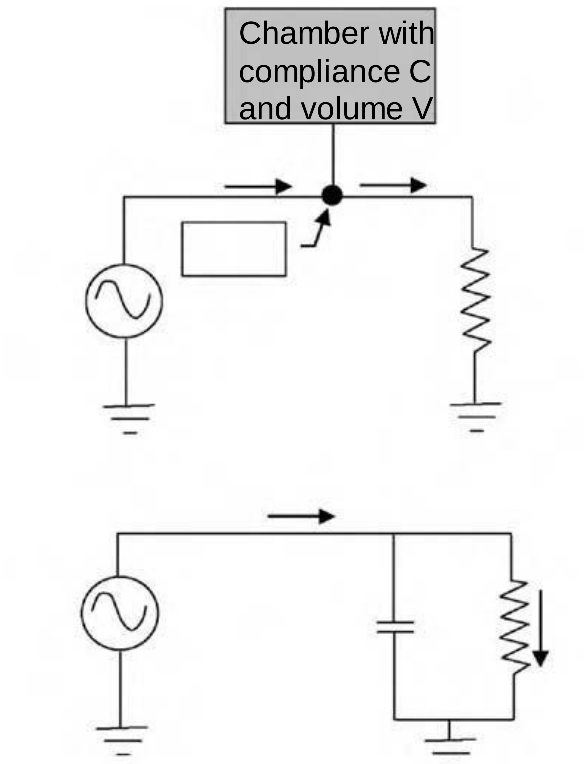
Radius of the Vessel, Heart Rate, and the Womersley Parameter for the Various Species

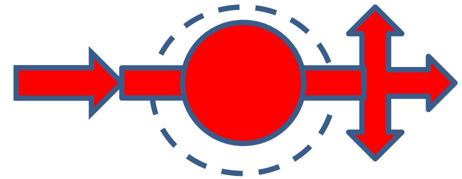
Species	Weight (kg)	Vessel	Radius (cm)	Heart Rate	α^a (min^{-1})
Mouse	0.017	Aorta	0.035	500	1.4
Rat	0.6	Aorta	0.13	350	4.3
Cat	3.0	Aorta	0.21	140	4.4
Rabbit	4.0	Aorta	0.23	280	6.8
Dog	20.0	Aorta	0.78	90	13.1
Man	75.0	Aorta	1.5	70	22.2
Ox	500.0	Aorta	2.0	52	25.6
Elephant	2000.0	Aorta	4.5	38	49.2
Rat	0.6	Femoral	0.04	350	1.5
Rabbit	4.0	Femoral	0.08	280	2.4
Dog	20.0	Femoral	0.23	90	3.9
Man	75.0	Femoral	0.27	70	4.0

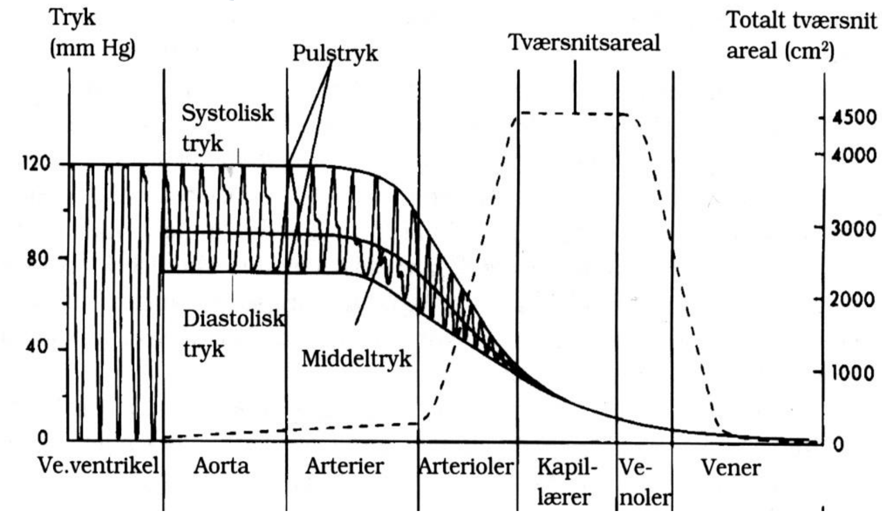
[a] At fundamental frequency.

Source: Milnor, W.R. (1989) *Hemodynamics*, 2nd ed., Williams and Wilkins, Baltimore. With permission.

Arterial Compliance







- Mechanical & Electrical analog of Windkessel model
- Gradual dissipation of oscillatory pressure and velocity

Windkessel model of circulation

- Oscillatory blood flow
- Arterial elasticity (dumbing of pulsation)
- Peripheral Resistance (bifurcations, arterioles, capillaries, venules, venous valves)
- Athreosclerosis increases vessel wall elasticity, reduces arterial compliance, increases Resistance, reduces dumbing of arterial tree

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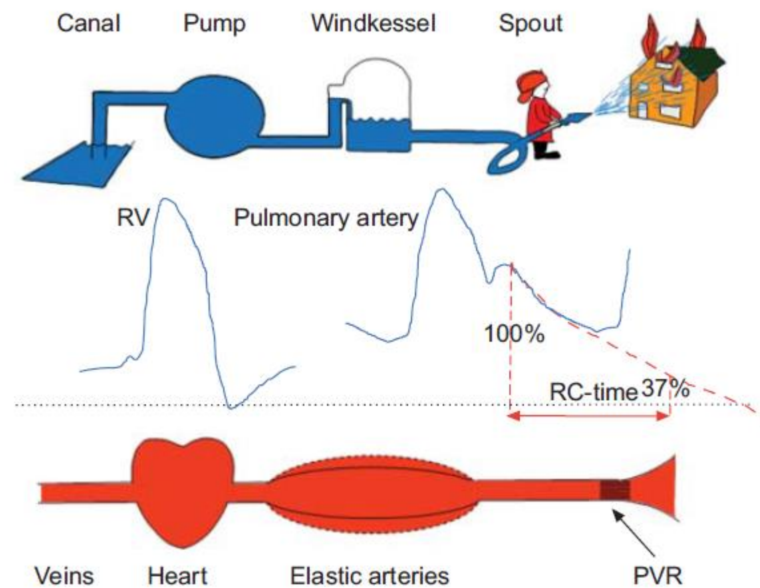
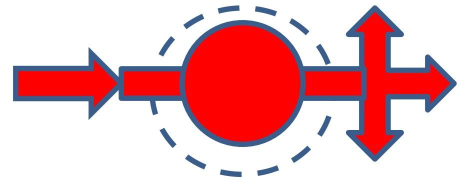


Otto Frank, 1899

$$Q = K\frac{dp}{dt} + \frac{p}{R}$$

Q = Flow rate; p = pressure; R = Resistance

Arterial compliance



- For volume V, assuming dV proportional to dP, then for Compliance C:

$$C = \frac{dV}{dP_{art}}$$

if assume $P_{art}=\Delta P_{periph}$

C, R constant, $P_{art}(t) = Q(t)R$

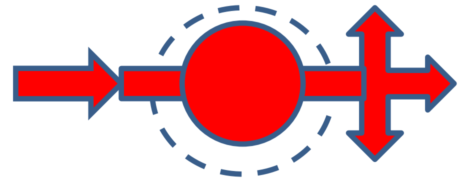
Q_H=Cardiac flow, Q=Peripheral flow

Conservation of mass:

$$\frac{dV}{dt} = Q_H - Q$$

$$RC\frac{dQ}{dt} + Q = Q_H$$

Arterial compliance



See the box 4.1, page 183 to get

- For volume V, assuming dV proportional to dP, then for Compliance C:

$$C = \frac{dV}{dP_{art}} \ (4.8)$$

C, R constant, $P_{art}(t) = Q(t)R$

Q_H=Cardiac flow, Q=Peripheral flow

Conservation of mass:

$$\frac{dV}{dt} = Q_H - Q \ (4.9)$$

$$RC\frac{dQ}{dt} + Q = Q_H \ (4.10)$$

If $Q_H(t) = Q_0 + Q_1 \sin(\omega t)$

$$Q(t) = \frac{Q_1}{\sqrt{1 + (\mathrm{RC}\omega)^2}} \sin(\omega t - \varphi) + Q_0 \ (4.11)$$

It is a phase shift between cardiac output and arterial circulation, due to arterial compliance.

- A part of the blood is stored in arteries and returns to circulation with a phase shift
- Complicated true volumetric flow deviates from simple harmonic oscillation, been the sum of a Fourier series periodic function of simple harmonic terms.
- NOT all parts of the arterial tree are distended in synchronization: finite blood pressure wave velocity, reflectance of wave in geometric disturbances, like bifurcations.
- Differences in wall elasticity across arterial tree

Pressure and velocities across arterial tree

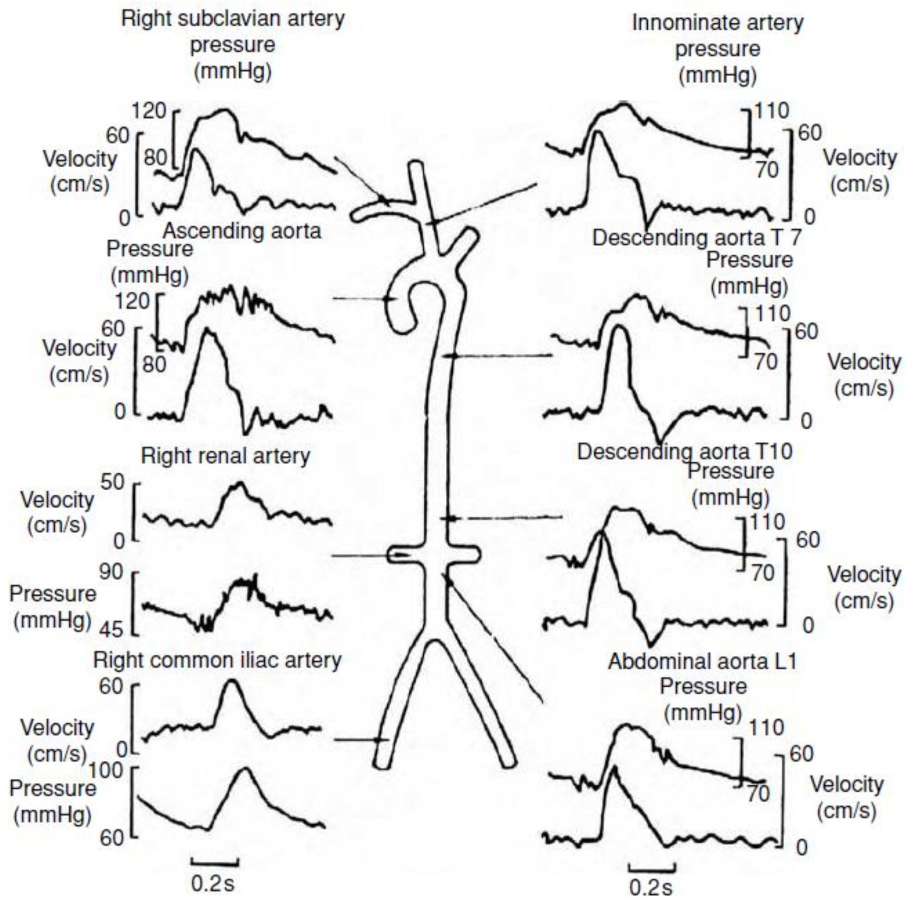


Figure 4.14

Elasticity and pulsation across arterial tree

Table 4.2. Values of the "pressure–strain modulus", E_p, and other arterial parameters in humans. From more complete listings in Milnor [4] and Nichols and O' Rourke [8], except the values for aortic root.

Artery	No.[a]	R_0 (cm)[b]	Pressure (mmHg) (mmHg)	Radial pulsation (%)[c]	E_p (dyn/cm^2)[d]	Source
Aortic root[e]	1	1.6	–	±4.7	–	Jin *et al.* [23]
Ascending aorta	10	1.42	79–111	±2.9	0.76×10^6	Patel and Fry [24]
Thoracic aorta	12	1.17	98–174	±2.6	1.26×10^6	Luchsinger *et al.* [25]
Femoral	6	0.31	85–113	±0.6	4.33×10^6	Patel *et al.* [24]
Carotid	11	0.44	126–138	±0.5	6.08×10^6	Patel *et al.* [24]
Carotid	16	0.40	96	±7.4	0.49×10^6	Arndt [26]
Carotid	109	–	–	–	0.63×10^6	Riley *et al.* [27]
Pulmonary (main)	8	1.35	16	±5.6	0.16×10^6	Greenfield and Griggs [28]
Pulmonary (left)	5	1.07	25	±6.2	0.17×10^6	Luchsinger *et al.* [25]
Pulmonary (right)	13	1.13	27	±5.8	0.16×10^6	Luchsinger *et al.* [25]
Pulmonary (main)	8	1.43	18–22	±5.4	0.16×10^6	Patel *et al.* [24]

[a] Number of arteries studied.
[b] Mean outer radius.
[c] Pulsation about the mean radius from normal pulse pressures (i.e., 100 × one-half the total radial excursion in each cardiac cycle (systolic–diastolic) divided by the average radius).
[d] Calculated from Equation (4.15) using total excursion of pressure and radius during natural pulsations; therefore represents a dynamic modulus.
[e] Measured using MRI.

Wave propagation along elastic tubes

- Pressure pulses travel as transverse elastic waves along arterial tree

- Elastic energy is stored by arterial walls by elastic distensibility
- Nonlinear P/R

$$E_P = R_0 \frac{\Delta P}{\Delta R_0}$$

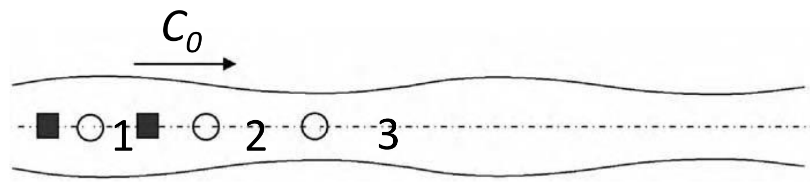
- Elastic tube filled with incompressible inviscid fluid, no flow (stationary state)

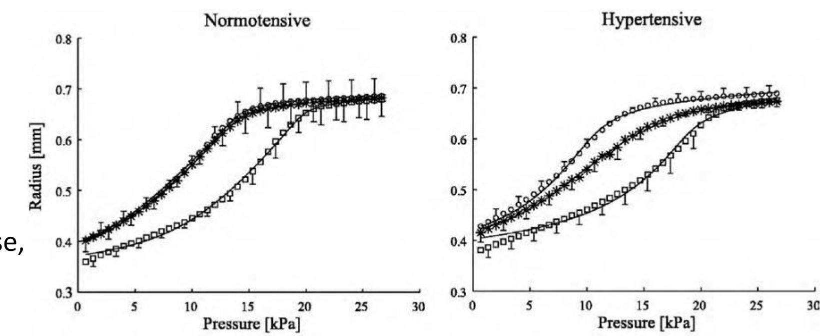
- Fluid pressure at entrance oscillates periodically (heart function)
- Elastic distension waves along the fluid in the tube
- Pressure elevation at 1 cause fluid to move towards 2
- Non-rigid tube walls, so fluid inertia forces wall at 2 to distend and pressure to rise, while in 1 pressure falls

- Distension travels from 1 to 2 with a velocity C_0
- It continuous from 2 to 3 next time
- Zero velocity means oscillation goes back and forth
- Relative magnitudes of fluid inertia and wall elasticity
- Similar effects in oscillatory flow

- Korteweg-Moens wave speed (after many assumptions):

E=elastic modulus, t=wall thickness, ρ=blood density, D=aortic diameter





$$C_0 = \sqrt{\frac{Et}{\rho D}}$$