# Pulsatile flow in arterial bifurcations

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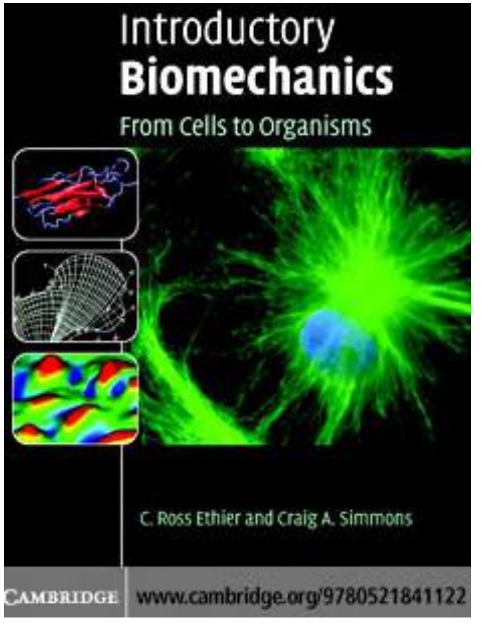
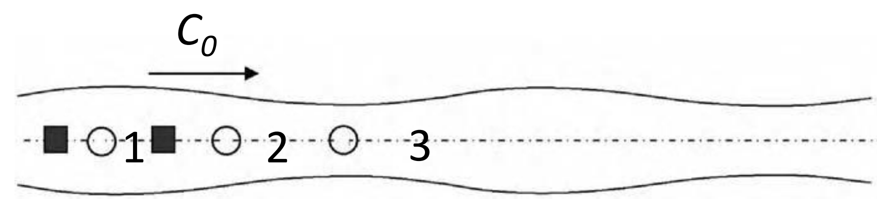
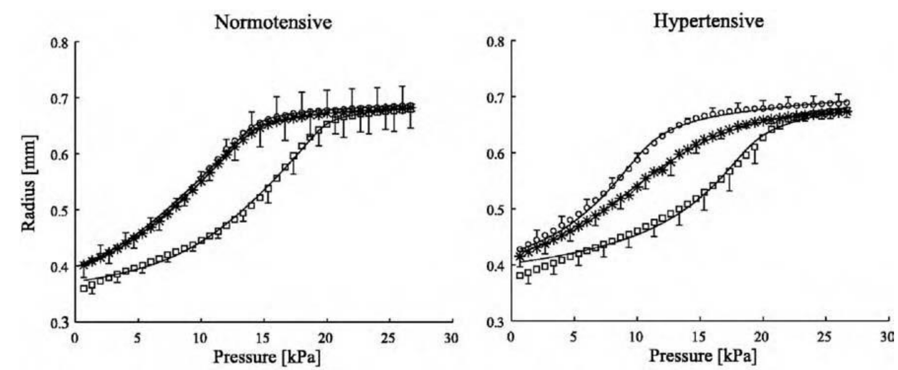
# Requirements: Basic fluid mechanics

- Chapters 3 (138-142) & 4 (179-204) of the book



# Unsteady inviscid flow across a tube




- Pressure pulses travel as transverse elastic waves along arterial tree

- Elastic energy is stored by arterial walls by elastic distensibility
- Non linear P/R

$$E_P = R_0 \frac{\Delta P}{\Delta R_0}$$

- Elastic tube filled with incompressible inviscid fluid, no flow (stationary state)

- Fluid pressure at entrance oscillates periodically (heart function)
- Elastic distension waves along the fluid in the tube
- Pressure elevation at 1 cause fluid to move towards 2
- Non-rigid tube walls, so fluid inertia forces wall at 2 to distend and pressure to rise, while in 1 pressure falls

- Distension travels from 1 to 2 with a velocity $C_0$
- It continuous from 2 to 3 next time
- Zero velocity means oscillation goes back and fourth
- Relative magnitudes of fluid inertia and wall elasticity
- Similar effects in oscillatory flow

- Korteweg-Moens wave speed (after many assumptions):

E=elastic modulus, t=wall thickness, ρ=blood density, D=aortic diameter

$$C_0 = \sqrt{\frac{Et}{\rho D}}$$

# Unsteady inviscid flow across a tube

- Different parts of arterial tree distended in different time.
- They differ in their wall elasticity
- Tapering, bifurcations, bends etc. change the morphology of pressure waves
- Windkessel model is so invalid to describe local pressure & flow values
- Elastic waves travel along arterial tree with velocity $C_0$

$$Q = K\frac{dp}{dt} + \frac{p}{R}$$

Q = Flow rate; p = pressure; R = Resistance

$$C_0 = \sqrt{\frac{Et}{\rho D}},$$

# Unsteady inviscid flow across a tube

- Elastic waves travel along arterial tree with velocity $\underline{C_0}$
- ➢ Pressure is a function of time t and position x, P = P(x,t)
- ➢ Suppose the simplest case: Oscillatory harmonic pressure (zero mean)
- ➢ Pressure wave propagates in positive x direction with velocity u

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$$P(x,t) = P_0 cos\left[\frac{2\pi}{\lambda}(ct - x)\right]$$

$$= P_0 cos\left[\omega t - \frac{2\pi x}{\lambda}\right] (4.37),$$

$$P_0, \lambda \; amplitude, wavelength \; of \; the \; oscilation$$

$$Angular \; velocity \; \omega = \frac{2\pi c}{\lambda}$$

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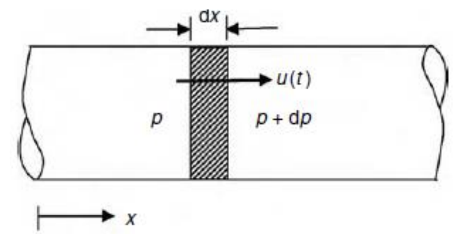
By differentiation:

$$\frac{\partial \mathrm{P}}{\partial x} = \frac{2\pi}{\lambda} P_0 sin\left[\omega t - \frac{2\pi x}{\lambda}\right] \; (4.39)$$

# Unsteady inviscid flow across a tube

$$\frac{\partial \mathrm{P}}{\partial x} = \frac{2\pi}{\lambda} P_0 sin\left[\omega t - \frac{2\pi x}{\lambda}\right] \ (4.39)$$

- Suppose a fluid element dx across a cross-sectional area A
- Inviscid flow means no velocity changes across the cross-section, velocity u
- According to Newton's law
- For inviscid flow u is uniform across A, so




$$\sum F_x = -dP\ A = \rho V \frac{\partial u}{\partial t} = \rho dx A \frac{\partial u}{\partial t} (4.40)$$

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} \ (4.41)$$

$Pressure\ gradient\ from\ previous\ (4.39)\ after\ substitution\ and\ integration$

$$u(x,t) = \frac{2\pi}{\rho\omega\lambda} P_0 cos\left[\omega t - \frac{2\pi x}{\lambda}\right] = \frac{2\pi}{\rho\omega\lambda} \mathrm{P(x,t)}, \omega = \frac{2pc}{\lambda}) \ (4.42)$$

# Unsteady inviscid flow across a tube

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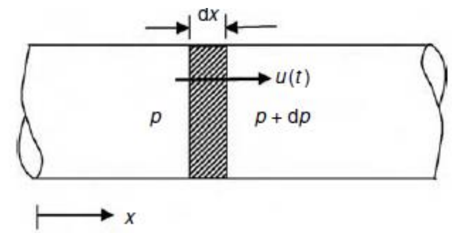


As ω = 2πc/λ (4.38), velocity u related to pressure P by

$$u(x.t) = \frac{P(x,t)}{\rho c} \ (4.43)$$

From 4.42: For a given P,
u = small if ρ or ω is large
(for large inertia or angular frequency)

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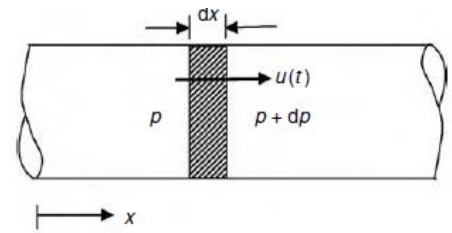


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For inviscid flow, velocity is uniform across cross-section A, so:

$$Q(x,t) = A.u(x,t)$$

So

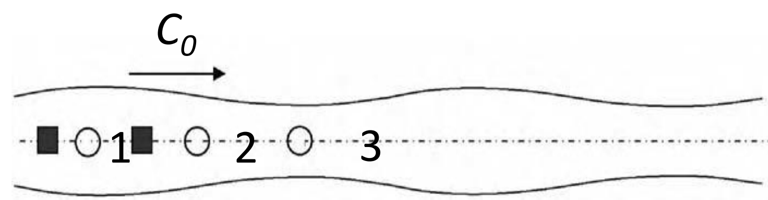
$$Q(x,t) = \frac{A}{\rho c} P(x,t)$$

In analogy with voltage and current in circuits

Characteristic impedance: $Z_0 = \frac{P}{Q} = \frac{\rho c}{A}$

# Pressure & flow
# Deviations from ideality

- **Ideality**: Infinitely long straight tube, inviscid fluid, Stationary oscillations (no mean Q)
- **In real case** Q = Qmean± Qosc, P = Pmean ± Posc




**Viscous losses**

- ✓ Viscosity of fluid dissipates energy, so decrease of pressure amplitude and wave propagation velocity $c_0$
- ✓ Real $Q(x,t)$ reduced, flow rate waveform lags pressure waveform

According to [Nichols & O'Rourke, 1990]

$Q(x,t) = \frac{AM'_{10}}{\rho c} P(x, t-\varphi)$ (4.47), instead of

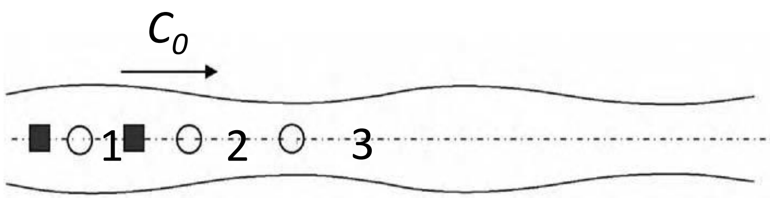
$Q(x,t) = \frac{A}{\rho c} P(x,t)$ (4.44)

$$\varphi = \frac{\varepsilon_{10}}{\omega}$$

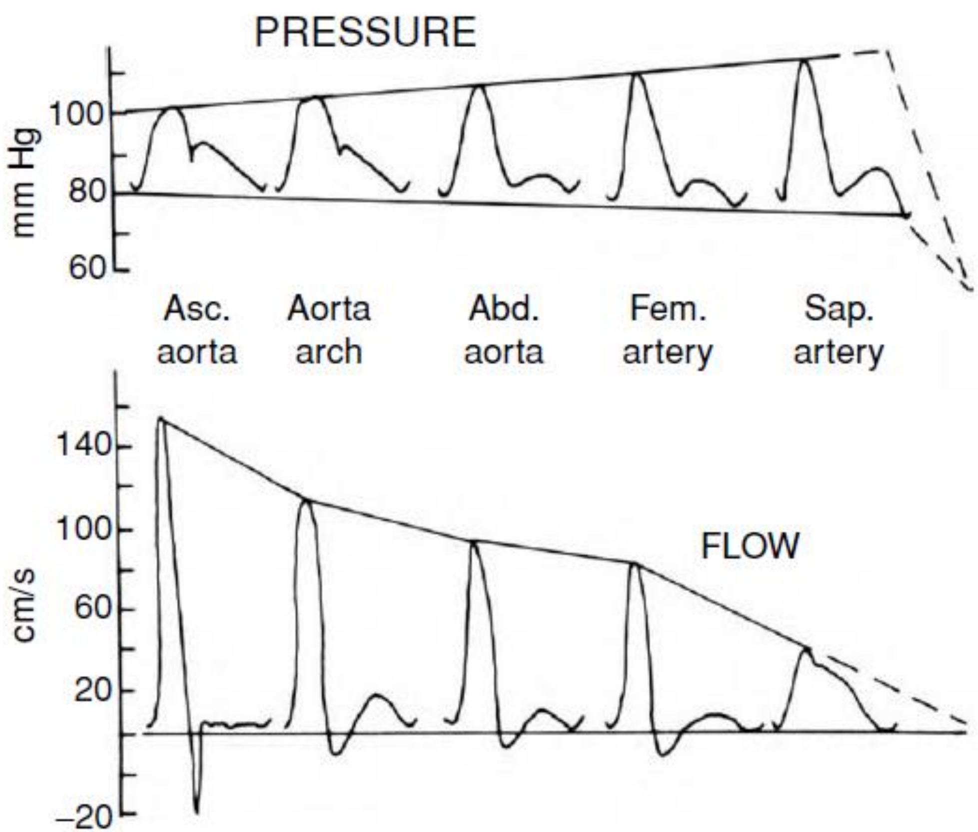
$\varepsilon_{10}$, $M'_{10}$ depending of flow parameters,

- ❖ $M'_{10} < 1$
- ❖ P & Q phase difference, **$Z_0$ complex**

# Pressure & flow across arteries

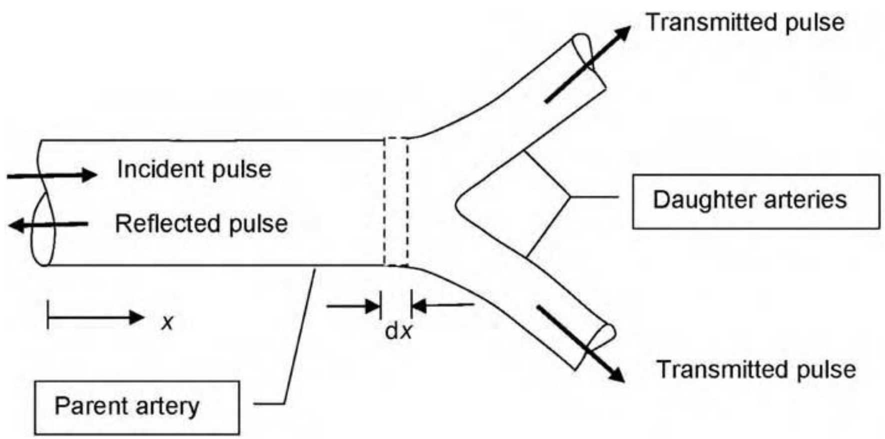


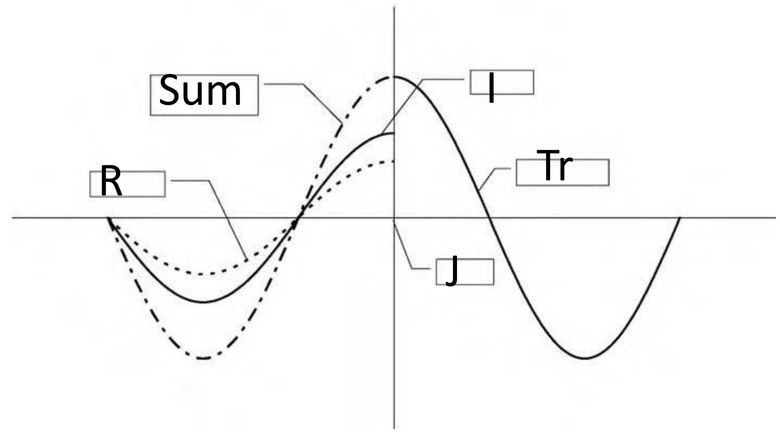

$$C_0 = \sqrt{\frac{Et}{\rho D}}$$




- Distally variations in arterial **stiffness** and **caliber**
- In point 2 larger pressure (if Q constant) or smaller Q (if P constant)
- In practice variations in Q and P compromise above cases
- Small pressure increment, small flow decrease

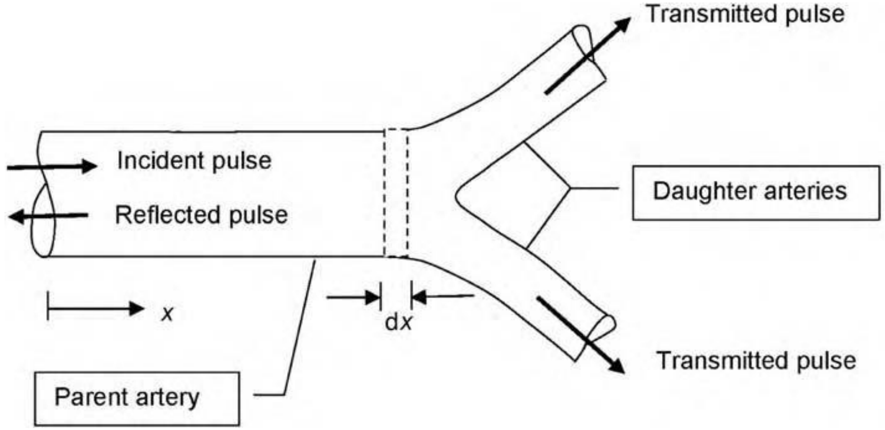
# Branching & taper

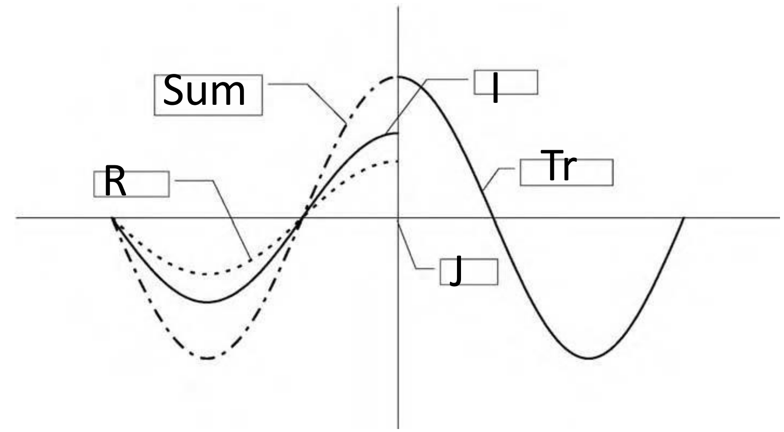



- As an incident, I, pressure wave reaches the junction a part, R, is reflected back into the parent artery, and another, T, divided and transmitted across the two daughter branches
- Parent and daughter arteries have different characteristic impendences (?)
- Pressure to flow relationship changes due to mechanical mismatch
- Due to energy loses between I and T (however no energy is lost) R waves reflected to parent artery
- R is added to income wave in phase at junction
- Sum of R + I = T wave

# Branching & taper




- For pressure waves:

$$P_i(x_j,t) = P_{0,i}\cos(\omega t+\theta)$$
$$P_r(x_j,t) = P_{0,r}\cos(\omega t+\theta)$$
$$P_t(x_j,t) = P_{0,t}\cos(\omega t+\theta)$$
$$(P_{0,i}+P_{0,r})\cos(\omega t+\theta)=P_{0,t}\cos(\omega t+\theta)$$

So

$$(P_{0,i}+P_{0,r}) = P_{0,t}$$

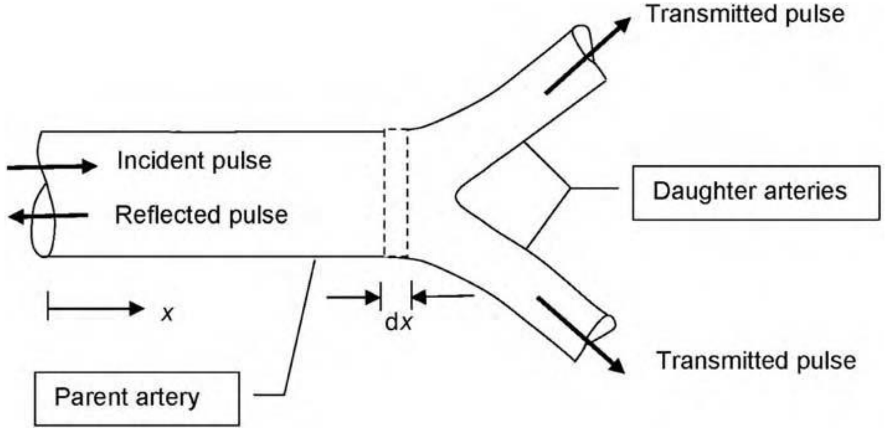
- For flow waves similarly:

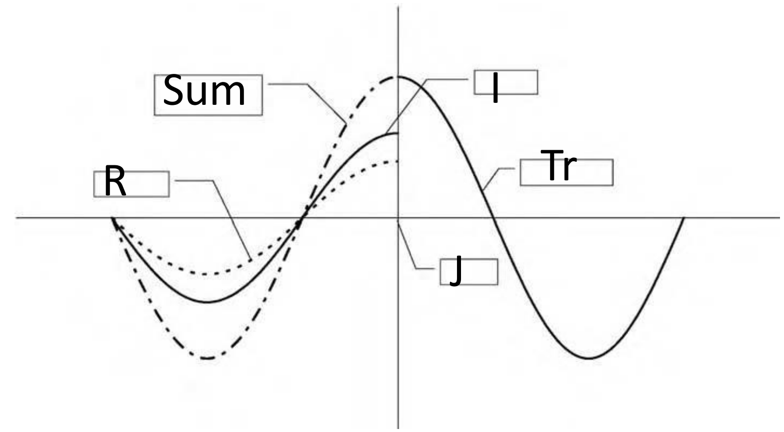
$$Q_i(x_j,t) = Q_{0,i}\cos(\omega t+\theta)$$
$$Q_r(x_j,t) = Q_{0,r}\cos(\omega t+\theta)$$
$$Q_t(x_j,t) = Q_{0,t}\cos(\omega t+\theta)$$
$$(Q_{0,i}-Q_{0,r})\cos(\omega t+\theta)=Q_{0,t}\cos(\omega t+\theta)$$

So

$$(Q_{0,i}-Q_{0,r}) = 2Q_{0,t}$$

# Branching & taper




- For characteristic impedance of parent and daughter tubes:

$$Z_{0,p} = \frac{P_{0,i}}{Q_{0,i}} = \frac{P_{0,r}}{Q_{0,r}}$$

$$Z_{0,d} = \frac{P_{0,t}}{Q_{0,t}}$$

$As\ (P_{0,i}+P_{0,r}) = P_{0,t}\ then$

$$\frac{Z_{0,p}}{Z_{0,d}}\left[Q_{0,i} + Q_{0,r}\right] = Q_{0,t}$$

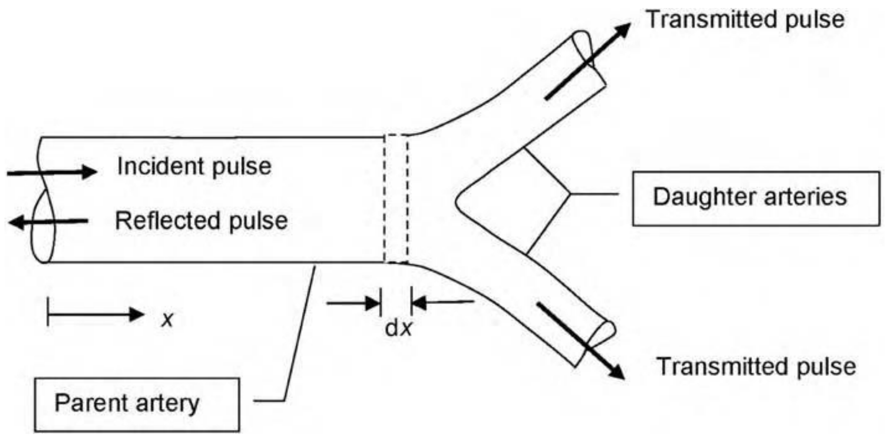
- Transmision ratio T:

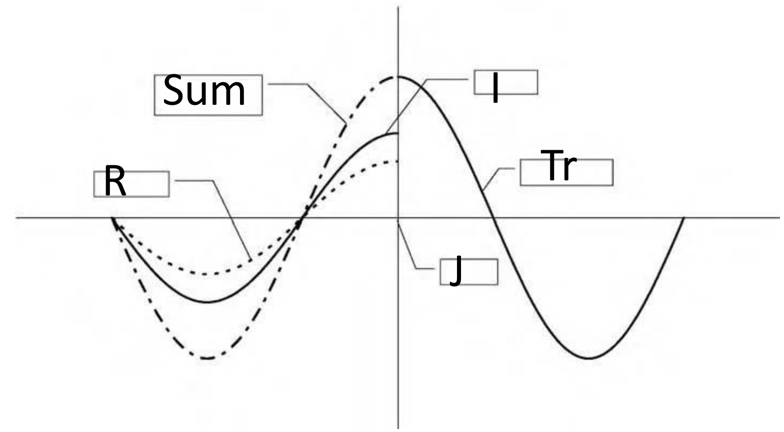
$$T = \frac{P_{0,t}}{P_{0,i}} = \frac{Z_{0,d}}{Z_{0,p}}\frac{Q_{0,t}}{Q_{0,i}} = \frac{2Z_{0,d}}{2Z_{0,p} + Z_{0,d}} \quad (4.61)$$

- Reflected pulse ratio R:

$$R = \frac{P_{0,r}}{P_{0,i}} = \frac{Z_{0,d} - 2Z_{0,p}}{Z_{0,d} + 2Z_{0,p}} \quad (4.62)$$

# Final comments

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**From (4.61 & 4.62)**

- T is always positive, R can be positive, negative or zero
- If R zero no reflected pressure is happened, junction is said to be "Matched junction)
- R=1 when $Z_{0,d}$ >> $Z_{o.p}$ if $A_d$<<$A_{p,}$, then
- $Q_{0,t}$=0 & $Q_{0,I} \approx Q_{0,r}$ ("blocked artery")

$$P_{net}(x_j, t) = 2P_{0,i}\cos(\omega t + \theta)$$

This is the case in the capillary bed, P in phase, Q in 180$^o$ out of phase.

- R=-1, Zo,d<<Zo,p, P & Q waves inverse as above. Not realistic *in vivo.*

# Final comments

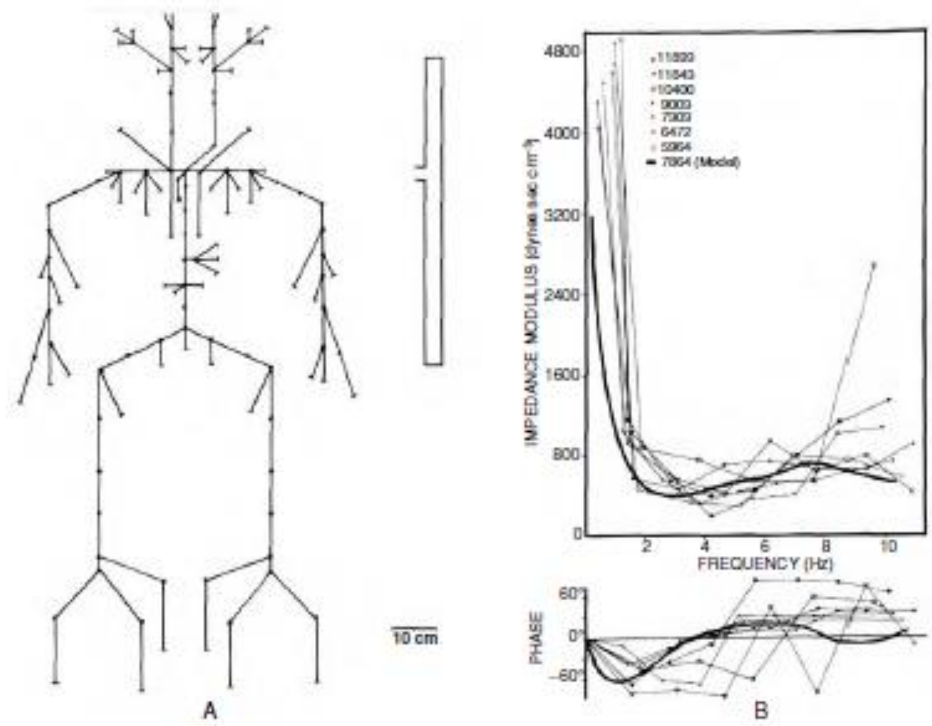


Figure 4.24

- Wave reflection is met at distal capillary beds
- In most of the others, vessels are physiologically “matched”
- At each point of consideration (e.g. aortic root) need to model all the downstream elements by a lumped quantity, called “input impedance”.
- Real pulse waveforms are very different from simple sinusoids. It must be decomposed into simple harmonic components (Fourier series, FFT)