Chapter 5. Flow in Large Arteries
Static & Steady flow models

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Requirements: Basic fluid mechanics

- Relevant Lectures in Biomechanics: basics & flow in arteries
- Chapters 1,3 & 5 of the book

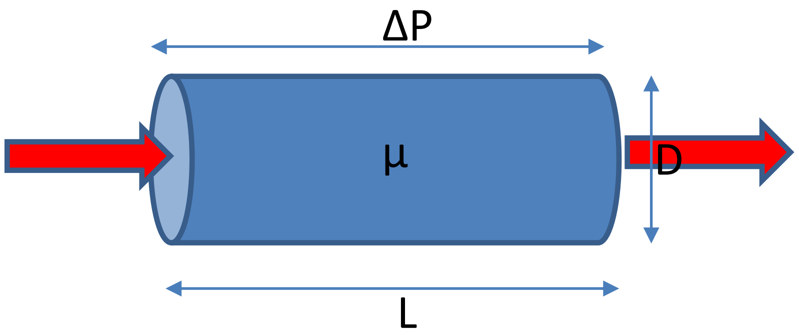
Biofluid Mechanics
The Human Circulation

Krishan B. Chandran
Ajit P. Yoganathan
Stanley E. Rittgers

From Basic fluid mechanics

- Similar to Ohm's law.
  Fluid Flow in a tube:

$$Q = \frac{\Delta \mathrm{P}}{R} = \frac{P_1 - P_2}{R}$$



Q=volumetric flow rate

P=Pressure at a point

R= tube resistance to fluid motion

Hydrostatics in the circulation

- Mean pressure in the adult 100 mmHg
- It fluctuates within 120-80 mmHg pulsatile aortic pressure
- Due to gravitation effects alone, a pressure difference can be recorded in different anatomic points: ΔP=rgh

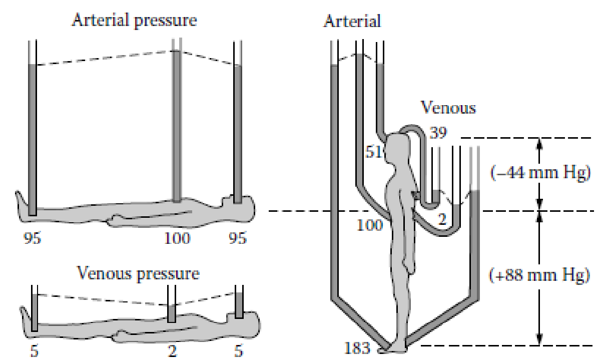


FIGURE 5.1
Hydrostatic pressure differences in the circulation. (Redrawn from Burton, A. C. (1971) *Physiology and Biophysics of the Circulation*. Year Book Medical Publishers, Chicago. With permission.)

Measuring static pressure *in vivo*

- Application of Bernoulli equation across a streamline of the fluid:

$$P + \rho V^2/2 + rgh = H,$$

H= Head or total energy per unit fluid volume

- Direct pressure measurements via catheter tip manometers.

$$P_e - P_l = \rho V^2/2,\ rgh=?$$

V=mean velocity across a cross-section

- In vitro measurements: pressure tube opening faces vessel wall, V=0 P=?, rgh=?

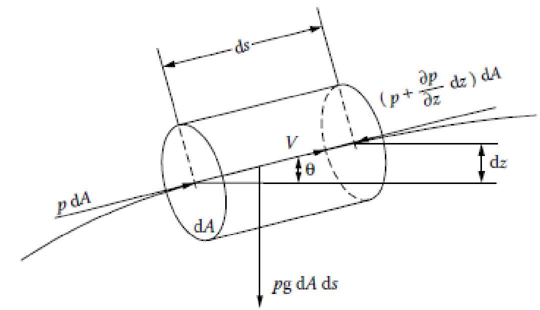


FIGURE 1.9
Force balance for a fluid particle along a stream line.

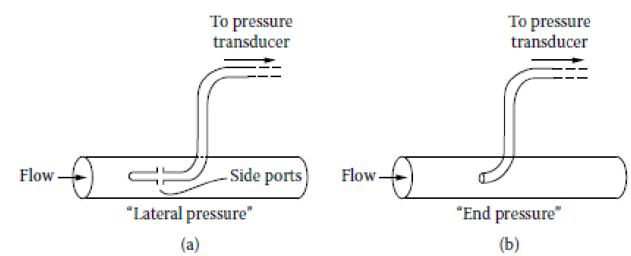


FIGURE 5.2
Schematic for the measurement of lateral and end pressure.

Arterial stenoses & aneurisms

- Conservation of mass:

$$A_1V_1 = A_2V_2$$

- Application of Bernoulli eq.:

$$P_1 + \rho V_1^2/2 = P_2 + \rho V_2^2/2 =, rgh=0$$

- Example page 158: Need corections
- The case of aneurism?

Disease of vascular tissue histology, effect of pressure

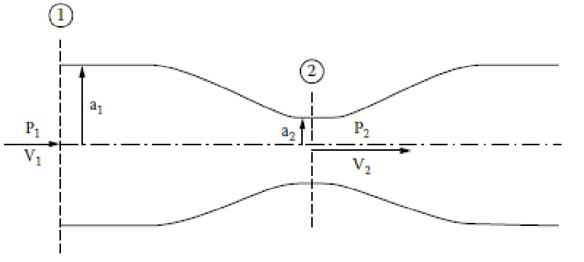


FIGURE 5.3
Application of Bernoulli equation to an arterial stenosis.

Pressure across a stenosed valve (Steady flow)

- Ideal valve: restriction due to the valve orifice alone. Leaflets not considered. Blood viscosity not considered
- Bernoulli eq. horizontal pos.:

$$P_1 - P_2 = \frac{\rho}{2}(V_2^2 - V_1^2) = \frac{\rho V_2^2}{2}\left(1 - \frac{V_1^2}{V_2^2}\right)$$

- Continuity eq.:

$$\left(\frac{V_1}{V_2}\right)^2 = \left(\frac{A_2}{A_1}\right)^2$$

- Finally:

$$P_1 - P_2 = \frac{\rho V_2^2}{2}\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)$$

$$V_{2ideal} = \sqrt{\frac{2(P_1 - P_2)}{\rho\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)}}$$

$$C_C = \frac{A_2}{A_0}$$

$$V_{2ideal} = \sqrt{\frac{2(P_1 - P_2)}{\rho\left(1 - C_C^2\left(\frac{A_0}{A_1}\right)^2\right)}}$$

- A_1 = Aortic Ring Area, A_0 = Valve orifice area

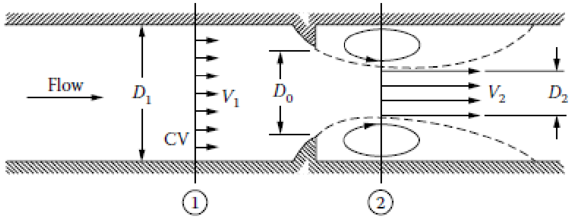


FIGURE 5.4
Schematic of the flow through a nozzle.

Flow across real valve

- Due to frictional loss:

$$C_V = V_{2actual}/V_{2ideal}$$

The volumetric flow rate:

$$Q = A_2 V_{2actual} = C_C A_0 C_V V_{2ideal}$$

$$Q = C_C A_0 C_V \sqrt{\frac{2(P_1 - P_2)}{\rho\left(1 - C_C^2\left(\frac{A_0}{A_1}\right)^2\right)}}$$

After squaring both parts

$$(P_1 - P_2) = \frac{\rho Q^2}{2}\left(\frac{1 - C_C^2\left(\frac{A_0}{A_1}\right)^2}{A_0^2 C_C^2 C_V^2}\right)$$

$$(P_1 - P_2) = \frac{\rho Q^2}{2}\frac{1}{A_0^2 C_d^2}$$

C_d = Discharge coefficient

Effective orifice area of the valve, using mean flow Q_m

$$EOA = A_0 = \frac{Q_m}{C_d}\sqrt{\frac{\rho}{2\Delta P}},$$ Gorlin equation

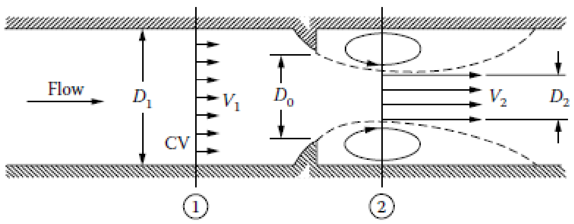


FIGURE 5.4
Schematic of the flow through a nozzle.

Effective Orifice Area

- Assuming steady flow
- For aortic valves:

$$AVA = \frac{MSF}{44.5\sqrt{\Delta P_m}}$$

AVA (cm^2) Mean Systolic Flow (cm^3/sec),

ΔP (mmHg)

- For Mitral valves:

$$MVA = \frac{MDF}{31.0\sqrt{\Delta P_m}}$$

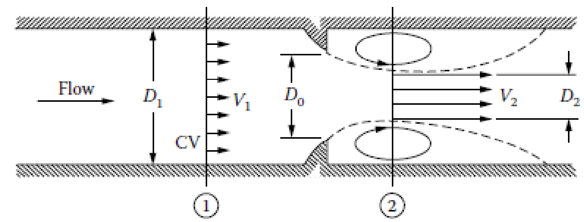


FIGURE 5.4
Schematic of the flow through a nozzle.

Rigid Tube modelling

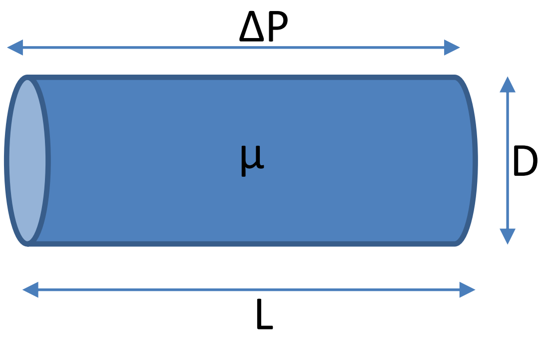
- Steady fully developed Newtonian fluid flow in a straight cylindrical tube:
- Poiseuille, Hagen independently:

$$Q = \frac{K\Delta \mathrm{P} D^4}{L}, Q = \frac{\pi \Delta \mathrm{PR}^4}{8\mu L},$$

$$\text{Haagen} - \text{Poiseuille:}$$

$Q = \frac{\pi \Delta \mathrm{P} D^4}{128\mu L}$, ΔP/L Pressure gradient across a linear rigid tube

- Laminar flow, no slip at wall (V=0)
- Need for different, more accurate/reliable approach (next chapter)



Problem
Application of Poiseuille formula

Example

Calculate the volumetric flow rate within an arteriole with a length of 100 μm and a radius of 35 μm. The pressure difference across the arteriole is 10 mmHg. Also calculate the change in diameter needed to reduce the volumetric flow rate by 5% and to increase the volumetric flow rate by 10%.

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Application of Poiseuille formula

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Solution

Using the Hagen-Poiseuille's formulation to calculate the volumetric flow rate, we get

$$Q = \frac{\pi \Delta P R^4}{8\,\mu L} = \frac{\pi (10\ \text{mmHg})(35\ \mu\text{m})^4}{8(3.5\ \text{cP})(100\ \mu\text{m})} = 0.135\ \text{mL/min}$$

Problem
Application of Poiseuille formula

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Calculate the volumetric flow rate within an arteriole with a length of 100 μm and a radius of 35 μm. The pressure difference across the arteriole is 10 mmHg. Also calculate the change in diameter needed to reduce the volumetric flow rate by 5% and to increase the volumetric flow rate by 10%.

5% of this flow rate is 0.00657 mL/min; therefore,

5% Reduction	10% Increase
$Q = 0.128\ \text{mL/min}$	$Q = 0.148\ \text{mL/min}$
$R = \sqrt[4]{\dfrac{8Q\mu L}{\pi \Delta P}}$	$R = \sqrt[4]{\dfrac{8Q\mu L}{\pi \Delta P}}$
$= \sqrt[4]{\dfrac{8(0.128\ \text{mL/min})(3.5\ \text{cP})(100\ \mu\text{m})}{\pi(10\ \text{mmHg})}}$	$= \sqrt[4]{\dfrac{8(0.148\ \text{mL/min})(3.5\ \text{cP})(100\ \mu\text{m})}{\pi(10\ \text{mmHg})}}$
$= 34.5\ \mu\text{m}$	$= 35.8\ \mu\text{m}$
$\Delta R = 0.5\ \mu\text{m} (1.5\%\ \text{change})$	$\Delta R = 0.8\ \mu\text{m}\ (2.3\%\ \text{change})$

Vascular resistance

$$Q = \frac{\Delta \text{P}}{R_s}, Rs = \frac{\Delta \text{P}}{Q}$$

From Poiseuille

$$R_S = \frac{8\mu L}{\pi R^4}$$

- Vessels of different configurations in series or in parallel: Total resistance like electrical resistance
- Alterations in vascular muscle tension may decrease or increase vessel diameter. Resistance is changed in the **fourth power** of the new dimensions
- Diagnostic limitations between pressure points: which pathway changed, the cause of change, net (not local) changes, dilatation or angiogenesis etc.

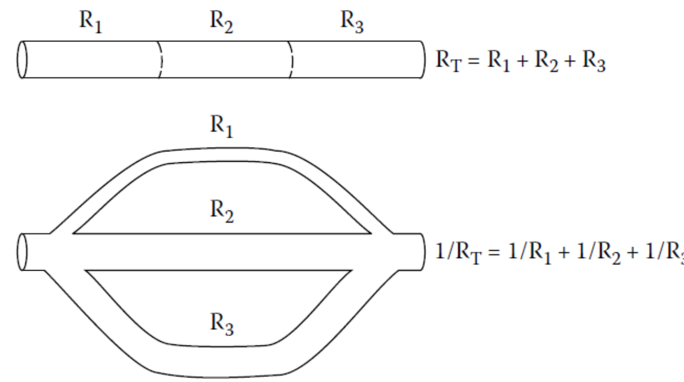


FIGURE 5.5
Total resistance with tubes in series or parallel configuration.

Problem vascular resistance

Example

Calculate the resistance to blood flow within the descending aorta and the inferior vena cava. Assume that the pressure difference between the distal portion of the aortic arch and the iliac arteries is 20 mmHg. The pressure difference within the inferior vena cava is 3 mmHg. Assume that the flow rate through both vessels is 4.5 L/min.

Problem vascular resistance

Example

Calculate the resistance to blood flow within the descending aorta and the inferior vena cava. Assume that the pressure difference between the distal portion of the aortic arch and the iliac arteries is 20 mmHg. The pressure difference within the inferior vena cava is 3 mmHg. Assume that the flow rate through both vessels is 4.5 L/min.

$$R_S = \frac{8\mu L}{\pi R^4} = \frac{\Delta P}{Q}$$

Problem vascular resistance

Example

Calculate the resistance to blood flow within the descending aorta and the inferior vena cava. Assume that the pressure difference between the distal portion of the aortic arch and the iliac arteries is 20 mmHg. The pressure difference within the inferior vena cava is 3 mmHg. Assume that the flow rate through both vessels is 4.5 L/min.

$$R_S = \frac{8\mu L}{\pi R^4} = \frac{\Delta P}{Q}$$

Solution

Descending Aorta	Inferior Vena Cava
$R = \frac{\Delta P}{Q} = \frac{20\ \text{mmHg}}{4.5\ \text{L/min}}$	$R = \frac{\Delta P}{Q} = \frac{3\ \text{mmHg}}{4.5\ \text{L/min}}$
$R = 355.5 \frac{\text{dyne} * \text{s}}{\text{cm}^5}$	$R = 53.3 \frac{\text{dyne} * \text{s}}{\text{cm}^5}$

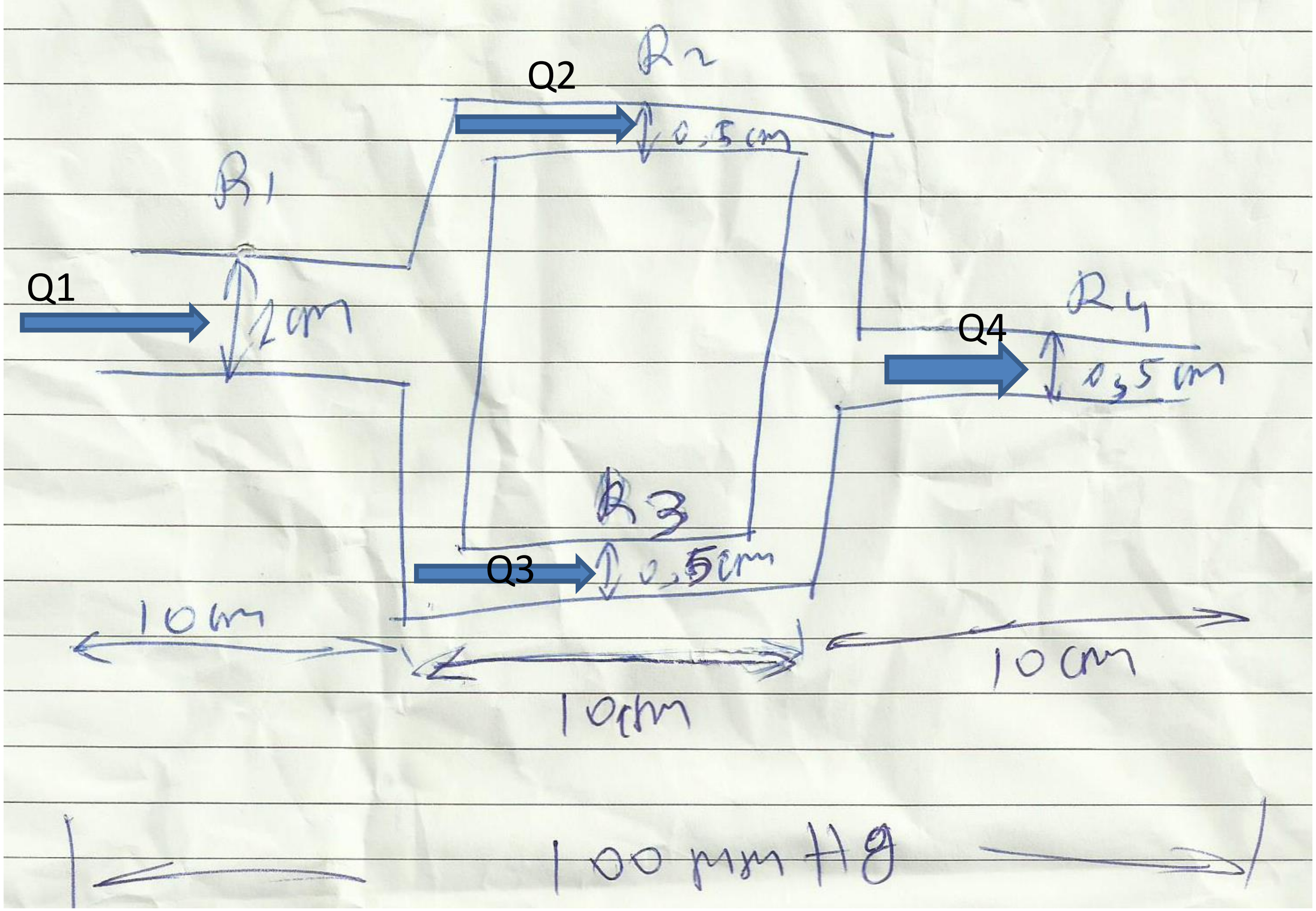
Homework

Please try problems 5.1 – 5.9

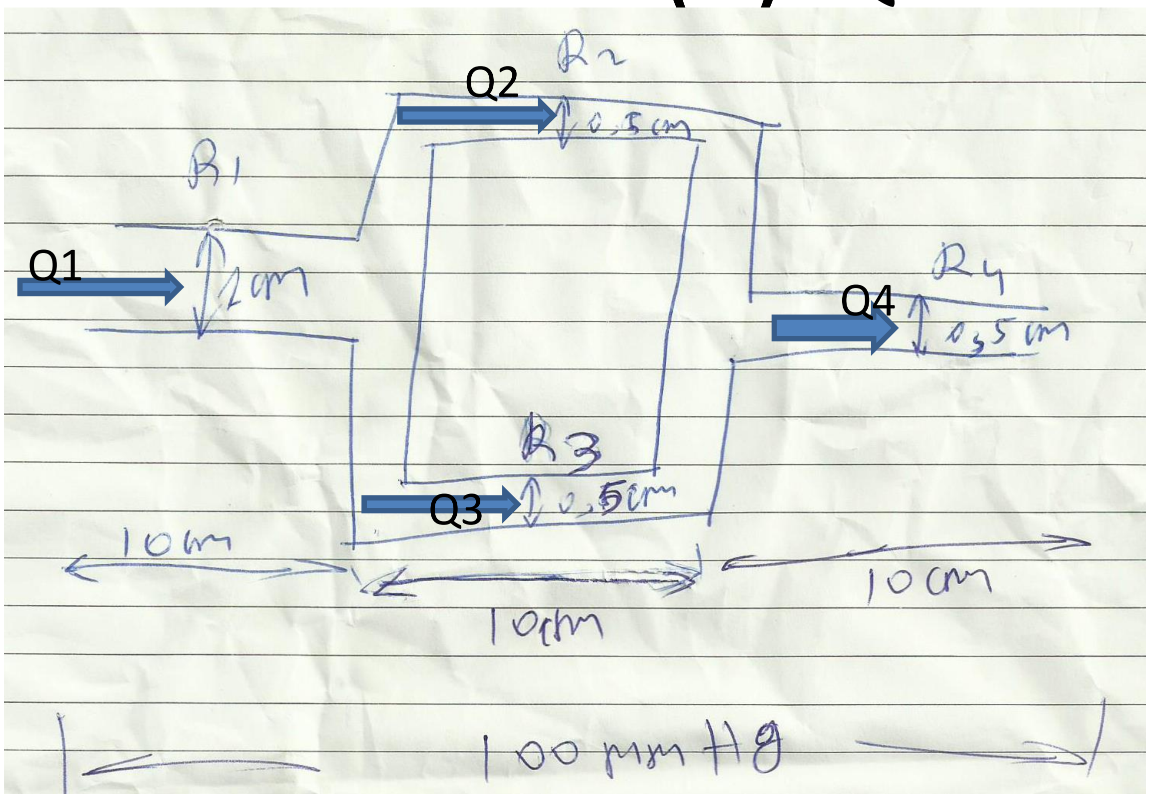
Pages 175-177

Example: Solution of problem 5.8 (a).

Problem 5.8 (a)
ΔP total 100 mmHg, Q1,2,3,4?



Solution 5.8 (a) Qi=?



$$R_s = \frac{\Delta P}{Q} = \frac{8\mu L}{\pi R^4}$$

Solution 5.8 (a)
Finding R1

$$R_s = \frac{8 \mu L}{\pi R^4}$$

$$R_1 = \frac{8 \times (3,5\ cP) \times 10\ cm}{\pi\ (0,5\ cm)^4} \Rightarrow$$

$$\Rightarrow R_1 = \frac{8 \times \left(3,5 \times 10^{-2}\ \frac{gr}{cm \cdot sec}\right) \times 10\ cm}{\pi\ (0,0625)\ cm^4} \Rightarrow$$

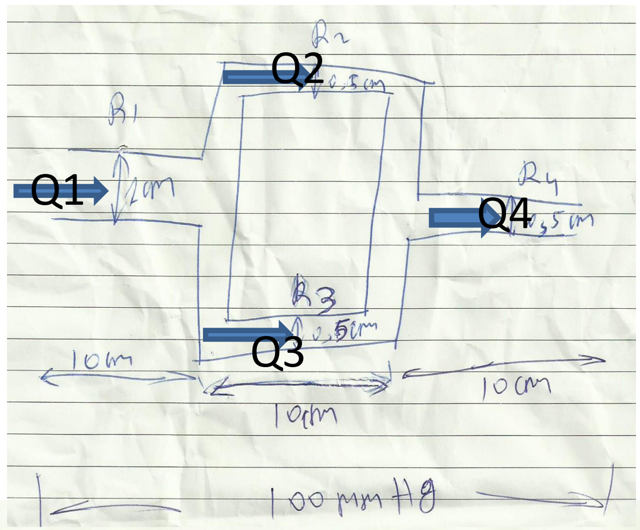
$$\Rightarrow R_1 = 1426,03\ gr / cm^4 \cdot sec$$

Solution 5.8 (a)
Finding R2,3, R4

$$\frac{8\mu L}{\pi R^4}$$

- The only difference compared with R1 is the reduction in diameter from 1cmm to 0.5cm.
- So its radius is half that of vessel 1, it is in the denominator of the above fraction
- It means that R2,3,4 are equal each other, greater from R1 by 2^4=16
- So, R2=R3=R4=22828 gr/cm^4.sec

Solution 5.8 (a)
Finding Rtotal



$$R_{0s} = R_1 + \frac{R_2 \times R_3}{R_2 + R_3} + R_4$$

$$R_{0s} = 0{,}1426 \times 10^4 + 2{,}2828 \times 10^4 + \frac{2{,}2828}{2} \times 10^4$$

$$R_{0s} = 3{,}512\ldots \times 10^4 \ \text{gr/cm}^4\text{.sec}$$

Solution 5.8 (a)
Finding Qtotal

$$Q_{os} = \frac{\Delta P_{od}}{R_{os}} \quad , \quad 7,5 \times 10^{-4}\ mmHg = 1\ gr/cm \cdot sec^2$$

$$Q_{os} = \frac{(100\ /\ 7,5) \times 10^{4}\ gr/cm \cdot sec^2}{3,512 \times 10^{4}\ gr/cm^4 \cdot sec}$$

$$Q_{os} = 3,80\ cm^3/sec = 228\ \frac{mL}{min}$$

Solution 5.8 (a)
Finding Q1,2,3,4

$$Q_1 = Q_0 = Q_4$$

$$Q_2 = Q_3 = Q_0 / 2$$

- For the solution of part b) you must consider:

1. The reduction to diameter of segment 2 (new R2, Rtotal, Qtotal)
2. $Q2/Q3 = (R2/R3)^4$ according to Poiseuille formula

Entrance effects

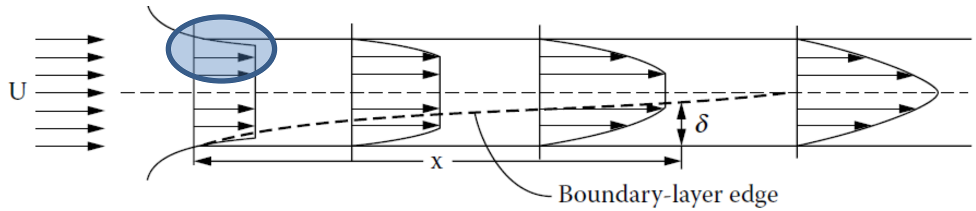


FIGURE 5.6
Concept of entry length before the flow becomes fully developed.

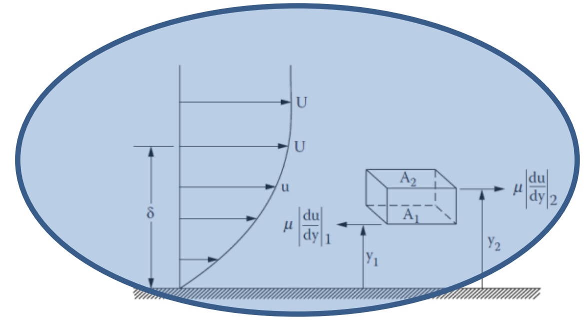


FIGURE 5.7
A force balance on a fluid element in the boundary layer.

- Coming from a larger reservoir, the blood is forced to a small diameter, hence acceleration and radial gradient of the velocity
- Initially a blunt velocity profile.
- Parabolic velocity profile (see V/R relationship) is obtained gradually, after a certain vessel length from entrance
- Close to walls velocity gradually minimized towards zero in contact, due to wall friction - deceleration
- At center region acceleration due to continuity and incompressibility
- Big shear forces at entrance, gradually minimized together with velocity gradient

Entrance effects

- A fluid element at distances Y_1 and Y_2 imposed at shear stresses/forces:

$$\tau = \mu \frac{du}{dy}, \mathrm{F} = \mathrm{A}\tau$$

- The net viscus force on the element:

$$\Delta\tau = \mu \frac{d}{dy}\left(\frac{du}{dy}\right) A(y_2 - y_1)$$

- For boundary layer thickness δ at distance X from entrance and U = free stream velocity we **estimate** viscus total force analog to:

$$\frac{\mu U}{\delta^2} A(y_2 - y_1) \qquad (1)$$

- Inertial total force (ργV) estimated at distance X

$$\rho \frac{U^2}{X} A(y_2 - y_1) \qquad (2)$$

After equation (1)=(2) (steady flow)

$$\rho \frac{U^2}{X} = k \frac{\mu U}{\delta^2}, \delta \propto \sqrt{\frac{\mu X}{\rho U}}$$

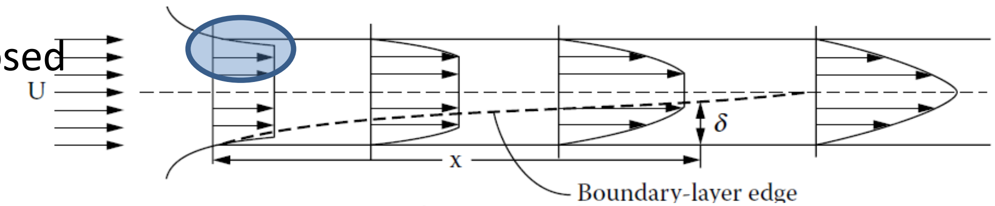


FIGURE 5.6
Concept of entry le

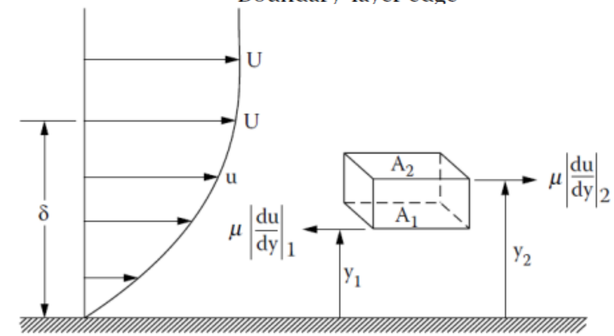


FIGURE 5.7
A force balance on a fluid element in the boundary layer.

At t we have δ,U at distance X

t=X/U, γ=U/t=/(X/U)=U^2/X

Entrance effects

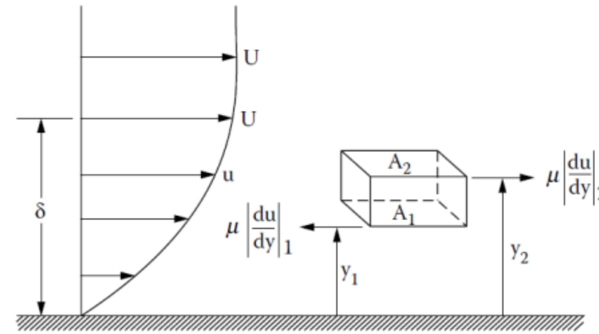
$$\rho \frac{U^2}{X} = k \frac{\mu U}{\delta^2}, \delta \propto \sqrt{\frac{\mu X}{\rho U}}$$

- In fully developed flow U is the central velocity (top of parabola)
- δ = R = D/2

$$X = kD^2 \frac{U}{\nu}, \nu = \frac{\mu}{\rho}, kinematic\ viscocity$$

$$X = kD\left(\frac{DU}{\nu}\right) = kD(Re)$$

- K experimentally = 0.06, Eq. valid for Re>50
- X=long at arteries (large D & Re) so increased shear forces proximal for long length. (What to lumen endothelial layer?)
- Shear forces also increased in stenoses, bifurcations, curvatures (big U near vessel walls) (next lectures)



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