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Unified Swing Up and Upright Position Stabilizing Controllers for Inverted-pendulum on a Cart

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***Abstract*—The challenging problem of stabilizing an inverted-pendulum on a cart system at the upright position via a nonlinear state feedback controller is analyzed. The proposed controller is designed in a manner that can guarantee local asymptotic stability for the up position equilibrium and instability for the down one. As shown in the paper by a stability analysis and verified numerically, a wide range for such a controller gains can be easily determined. Simultaneously, a systematic qualitative study of the system motion provides the required features the input has to fulfill in order to act in a unified form: initially with a series of quick cart movements and pendulum swings which may result in getting the pendulum stick away from the unstable down position, till it can be attracted to the upright stable equilibrium. Thus, a practically parameter-free and robust design is proposed. The overall scheme is examined by detailed simulations. The response of the system indicate an excellent performance with a fast swing up period following by a convergence to the stable up position with the desired displacement; during transient, small amplitude oscillations and limited overshoots are observed.**

I. Introduction

The inverted-pendulum system on a cart is one of the most famous nonlinear dynamical systems and constitutes a very attractive problem for many researchers in the area of control systems and robotics [1]. Many trends in robotic technology and control are directly based on the inverted-pendulum stabilization techniques. A characteristic example is the mobile wheeled inverted pendulum system that has already been a commercial product and has induced a lot of attention in research [2,3]. Other examples are on the humanoid robots field where many studies are based on the analogy between the bipedal gait and the inverted pendulum motion [4,5]. In any case, inverted-pendulum is a fundamental benchmark in robotics since, among others, it is an underdamped and underactuated mechanical system with many similarities to the underactuated robotic systems that are usually controlled by fewer independent actuators than the degrees of freedom.

It is well-known that the inverted-pendulum system has two equilibrium points. For the unforced system the first equilibrium point that corresponds to the down position of the pendulum can be easily proven by using Lyapunov's first method, to be asymptotically stable while the second equilibrium point that corresponds to the inverted up position is proven to be unstable. In contrary to the natural situation, the main aim of the design of an appropriate control law is to provide asymptotic stability at the upright equilibrium point regardless from the accurate knowledge of the system parameters. Thus, the main purpose of the design of the control input is the inversion and the stabilization of the pendulum around the upright position by using a force which is applied on the cart [6].

The solution to the problem of local stability of the upper equilibrium point was first presented in textbook [7]. However, the problem of both the inversion and the stabilization of the pendulum around the upper equilibrium point, appears to have some inherent difficulties caused, for example, by the fact that simple feedback controllers cannot be designed to ensure global asymptotic stability at the upright position, or, the nonlinear model of the inverted-pendulum system cannot be fully feedback linearizable [8]. In [9] and [10], after having used only partial feedback linearization techniques, a control scheme based on Lyapunov methods is proposed. Also, in [11], after applying partial feedback linearization based control, semi-global stabilization is proven. As presented in [12-14], partial feedback linearization can be avoided by using the passivity property. However, all the latter methods need to know the exact physical characteristics of the system in order to apply stable controls. In other attempts, adaptive fuzzy or some combined with robust approaches are proposed and examined [15,16]. Besides complexity, the methods can guarantee uniform ultimate boundedness of the tracking error and a good system performance.

Another significant endeavor, recently used in inverted-pendulum applications, is based on a rather heuristic method, known in the literature as the swing up technique [17]. The basic concept of this method is based on the fact that the total kinetic and potential energy of the system is smaller in the lower than the upper equilibrium point. Thus, the problem is divided into two separate parts. In the first part, a control scheme is applied on the cart in order to offer an appropriate amount of energy to the system until the pendulum reaches a region near the inverted position. Initially, a series of quick cart movements occur that result in pendulum swinging, in a fashion that efficiently adds energy to the pendulum. As the pendulum swings higher, gradually the cart movement amplitude is reduced so that the pendulum approaches the inverted up position in small

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increments and ultimately reaches vertical with small angular velocity. After approaching close to the inverted position, in the second part, a switching to a different control law occurs to stabilize the pendulum around the upper equilibrium point. Based on this technique, many researchers presented several solutions. One interesting solution based on energy control techniques was presented in [18] and similar approaches were also followed in [19-21]. Control laws combined with pumping-damping energy shaping are derived with control signal saturation taken into account. The method can ensure global stability, unfortunately only for the case of simple inverted pendulum, without a cart, where the control is applied directly on the pivot as accelerating input [21].

In this paper, an upper position stabilizing nonlinear state feedback smooth controller is proposed that acts as a force moving the cart while simultaneously operates as swing up driver when the pendulum stick is initially near the rest down position. Particularly, an extensive qualitative motion analysis of the pendulum rotational movement that takes into account the impact of the external force input characteristics, enables the successful implementation of such a unified closed-loop controller. The feedback control law is thus constructed with nonlinear terms contributing on one side to the selection of the desired closed-loop equilibriums exactly at the vertical up and down positions and on the other hand to an increased dissipation performance. Additionally, the proposed control scheme is capable to act as negative feedback for any angle between $-\pi/2$ and $\pi/2$ around the upright position wherein local asymptotic stability can also be proven by applying Lyapunov indirect method; outside this area, that also involves the down position equilibrium, the proposed controller represents a positive feedback. The analysis, as further confirmed by a numerical example, results in easily obtained ranges of appropriate gains that can ensure simultaneously asymptotic stability for the upright position equilibrium and instability for the down one.

Finally, the theoretical analysis and the proposed design as well as the expected system response are fully verified by extensive simulations.

II. Model Of The Inverted-Pendulum System

A. System Dynamics

The plane model of an inverted-pendulum system is shown in Fig.1, where x and $\theta$are the carrier's position and the pendulum's swing angle, respectively. Let M be the mass of the carrier, m the mass of the pendulum, while g is the gravitational acceleration, l the constant length of the pendulum and F is the external force that moves the carrier and the pendulum.

Now, in order to obtain the dynamic model of the system, the following assumptions are considered: i) the pendulum is taken as a point mass with or without an additional moment of inertia J considered for the pendulum stick, ii) frictional elements in pendulum angle rotation are not taken into account and iii) the effects of wind and other disturbances are not considered.

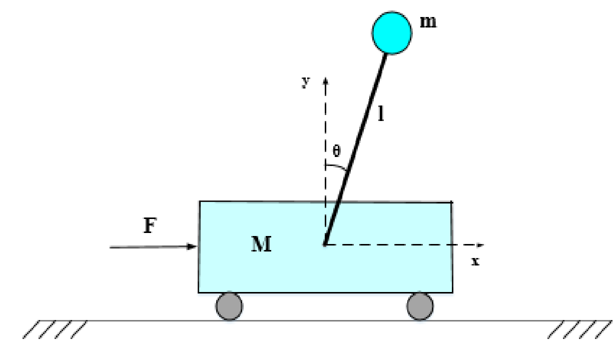


Fig.1. Schematic illustration of inverted-pendulum on a cart

Then, the differential equations of the system may be derived by using standard Euler-Lagrange methods or applying Newton's laws, as follows

$$(M+m)\ddot{x} + mlcos\theta\ddot{\theta} + \beta\dot{x} - mlsin\theta\dot{\theta}^2 = F \quad (1)$$

$$mlcos\theta\ddot{x} + (J+ml^2)\ddot{\theta} - mglsin\theta = 0 \quad (2)$$

where β is the friction coefficient on the cart wheels.

Dynamic equations (1), (2) of the inverted-pendulum system can be rewritten as:

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + R(q)\dot{q} + v(q) = f \quad (3)$$

where the generalized coordinate is $q(t) = [x(t)\ \theta(t)]^T$ and

$$D(q) = \begin{bmatrix} M+m & mlcos\theta \\ mlcos\theta & J+ml^2 \end{bmatrix}, C(q,\dot{q}) = \begin{bmatrix} 0 & -mlsin\theta\dot{\theta} \\ 0 & 0 \end{bmatrix}$$

$$R(q) = \begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}, v(q) = \begin{bmatrix} 0 \\ -mglsin\theta \end{bmatrix}, f = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

Notice that system (3) is an underactuated Euler-Lagrange system and $D(q)$ is symmetric and positive definite, since the parameters J, M, m, l are positive and

$$\det[D(q)] = (M+m)\,(J+ml^2) - (mlcos\theta)^2 =$$
$$= Mml^2 + (M+m)J + m^2l^2sin^2\theta > 0$$

Another well-known property of Euler-Lagrange systems is that the parameters of the model are such as the matrix: $\dot{D}(q) - 2C(q,\dot{q})$, is indeed skew-symmetric, where $\dot{D}(q)$ represents the time derivative of $D(q)$.

Finally, the potential energy associated to the pendulum may be defined as $P(\theta) = mgl(cos\theta - 1)$ such as $P(\theta) = 0$ when the pendulum is in the inverted position. Also, the term $v(q)$ is related to P as follows:

$$\frac{\partial P}{\partial \theta} = -mglsin\theta$$

i.e. through the second term of $v(q)$.

B. State Space Representation

The system dynamics as described by (3) can be easily represented in state space. Toward this end, without loss of generality, it is assumed that the pendulum stick moment of inertia J equals $\frac{1}{3}ml^2$, as has been adopted in [1]. Also, defining $a = 1/(m+M)$, equation (3) can be rewritten in state space, in the following 4th-order nonlinear form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \dfrac{a(\frac{4}{3}mlx_4{}^2 sinx_2 - \frac{1}{2}mgsin2x_2 - \beta x_3)}{\frac{4}{3} - amcos^2x_2} \\ \dfrac{gsinx_2 - \frac{l}{2}amx_4{}^2 sin2x_2 + a\beta x_3 cosx_2}{\frac{4}{3}l - lamcos^2x_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{\frac{4}{3}a}{\frac{4}{3} - amcos^2x_2} \\ \dfrac{-acosx_2}{\frac{4}{3}l - lamcos^2x_2} \end{bmatrix} u \tag{4}$$

where input u stands for the external force F and the state vector is$[x_1 x_2 x_3 x_4]^T = [x\ \theta\ \dot{x}\dot{\theta}]^T$.

III. Control Design Based On System Properties And Stability Concepts

A. Analysis of the Inverted-Pendulum System Motion

From the 4th state equation of (4), repeated here for the reader's convenience,

$$\ddot{\theta} = \frac{gsin\theta - \frac{l}{2}am\dot{\theta}^2 sin2\theta}{\frac{4}{3}l - lamcos^2\theta} + \frac{-acos\theta}{\frac{4}{3}l - lamcos^2\theta}u \equiv \varphi(\theta, \dot{\theta}) + b(\theta)u \tag{5}$$

wherein it is considered that the influence of the friction terms can be practically neglected, one can conclude the following: Input coefficient function $b(\theta)$ takes negative values and is bounded away from zero for all $-\pi/2 < \theta < \pi/2$ while it takes positive values outside this area; also for $\theta = \pi/2$ or $-\pi/2$ the system is uncontrollable. Therefore, a positive input may act on pendulum acceleration and rotation as negative feedback when the pendulum is around the upright position ($\theta = 0^o$), and as a positive one when the pendulum is around the down position ($\theta = 180^o$).

Similar comments can be made by considering the 3rd state equation of (4). In that equation it is observed that in all cases a positive input may act on cart acceleration and motion as positive feedback.

On the other hand, from (2) with J as previously considered and the friction term negligible again, we obtain

$$\ddot{\theta} = \frac{3}{4}\frac{g}{l}sin\theta - \frac{3}{4}\frac{\ddot{x}}{l}cos\theta \tag{6}$$

From (6), a qualitative analysis of the inverted-pendulum motion can be conducted. Without cart acceleration, only gravity causes pendulum rotational acceleration at a magnitude increasing with swing angle. Particularly, starting from an initial angle $\theta = 180^o$ (pendulum down), obviously no rotational acceleration is produced. Clearly, as it can be seen from (6), cart acceleration causes pendulum rotation in addition to the effects of gravity. Specifically, it can be observed that the direction of pendulum rotational acceleration is opposite to the direction of the cart acceleration for all $-\pi/2 < \theta < \pi/2$ while is acting on the same direction outside this area. The effect of cart acceleration on pendulum rotation is negative greatest when the pendulum is in up position ($\theta = 0^o$), and is positive greatest when the pendulum is in down position ($\theta = 180^o$). Therefore, if one applies positive cart acceleration (by applying a positive input) when pendulum is in an initial down position, then θ is positively accelerated and as θ becomes greater than 180^o it moves toward the -90^o angle with the effect of cart acceleration to be continuously decreased. Simultaneously, the gravity effect causes an additional reduction on the rotational acceleration since it acts on the opposite direction of that of the cart acceleration.

B. Controller Design

Taking into account the above remarks, a cart motion strategy is considered and implemented by suitably selecting the feedback control. The main task is to incorporate in a unified and as possible simple formula, both, the swing up process and the stabilization of the pendulum at the inverted up position, $\theta = 0^o$. Particularly, our aim is to apply a feedback law in order to force cart in such a way that starting from the down position ($\theta = 180^o$) with all other state initial conditions zero, to achieve after a limited swing period, convergence to the upright equilibrium where the pendulum will be stabilized thereafter. In accordance to the previously presented analysis, the following simple linear *positive* feedback control law may be examined:

$$u = k_1 x + k_2 \theta + k_3 \dot{x} + k_4 \dot{\theta} \tag{7}$$

where k_1, k_2, k_3 and k_4 are positive constant gains.

For the full state feedback controller (7), our aim is the positive values of the gains to be selected in a manner that interchanges the nature of the upright and down equilibria of the pendulum from locally unstable and stable to become locally stable and unstable, respectively. This is expected to be satisfied for gains lying on specific ranges coming from the system local analysis around both the equilibria. In any case, the attempt is the permitted ranges to be independent from an accurate knowledge of the system parameters, as for example feedback linearization techniques require.

By applying this type of feedback control, one can easily see that the closed-loop equilibria are given as

$$\left[x^*\theta^*\dot{x}^*\dot{\theta}^*\right] = \left[-\frac{k_2}{k_1}p\pi \;\; p\pi \;\; 0 \;\; 0\right] \qquad (8)$$

with p any integer number.

Nevertheless, it is obvious from (8), that, a necessary condition for the proposed control law to be effective is all the equilibrium points with p zero or any integer even number to be stable, while the rest equilibrium points with p any integer odd number to be unstable. Moreover, as one can see from (8), x^* element cannot be zero for any nonzero p. This is an undesirable fact that can be eliminated by modifying the second term of the control law (7) from $k_2\theta$ into $k_2 sin\theta$.

It is worth noting that usually, track length is limited and therefore, the cart position displacement during transients should be suppressed. To this end, the coupled dissipation technique is adopted as described in [22, 23] by considering the gain k_3 to include a time-varying quadratic function of the angle derivative, i.e., k_3 is substituted by $k_3 - c\dot{\theta}^2$, with some $c > 0$. Hence, the finally proposed feedback control law takes on the form:

$$u = k_1 x + k_2 sin\theta + \left(k_3 - c\dot{\theta}^2\right)\dot{x} + k_4\dot{\theta} \qquad (9)$$

where all the equilibria are now calculated to be

$$\left[x^*\theta^*\dot{x}^*\dot{\theta}^*\right] = [0 \;\; p\pi \;\; 0 \;\; 0] \qquad (10)$$

C. Controller Gains Selection based on Stability Concepts

It is again worth noting that in order to apply the control law (9), it is necessary all the equilibrium points (10) with p zero or integer even number to be stable, while the rest equilibrium points with p integer odd number to be unstable.

Therefore, to proceed with our analysis, it is required to determine if such a condition can be met by selecting suitably the range where the gains k_1, k_2, k_3 and k_4 can take their values. Particularly, we incorporate the proposed control law (9) into the original system model (4) wherein it is assumed $\beta = 0$ and $c = 0$ in order to relax the calculation effort. Our aim is to apply the indirect Lyapunov's method in order to examine the local stability or instability of both the equilibria of the closed-loop system. To this end, a linear approximation of the closed-loop system is obtained by applying Taylor expansion around the up and the down equilibrium, i.e.

$$\dot{x}_{cl} = A_i x_{cl} \qquad i = 1,2 \qquad (11)$$

where $A_i =$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{4k_1}{4M+m} & \frac{(-1)^{i+1}4k_2 - 3mg}{4M+m} & \frac{4k_3}{4M+m} & \frac{4k_4}{4M+m} \\ \frac{(-1)^i 3k_1}{l(4M+m)} & \frac{(-1)^{i+1}3g/a - 3k_2}{l(4M+m)} & \frac{(-1)^i 3k_3}{l(4M+m)} & \frac{(-1)^i 3k_4}{l(4M+m)} \end{bmatrix}$$

and x_{cl} is the closed-loop state vector and matrix A_1 corresponds to the upright equilibrium point and A_2 to the down one.

In accordance to the indirect Lyapunov's method [24], asymptotic stability of the up equilibrium is guaranteed if there exist gains k_1, k_2, k_3 and k_4 such that all the eigenvalues of matrix A_1 have negative real parts; simultaneously, if for these gains, at least one of the eigenvalues of matrix A_2 has positive real part, the down equilibriun is unstable. In practice, the above eigenvalue limits is expected to impose the ranges (if any) wherein the gains k_1, k_2, k_3 and k_4 should be lying instead of determining exact values for them. Therefore, in the procedure of selecting suitable gains, a direct dependence from the system parameters is avoided, thus contributing to the robustness of the inverted pendulum performance, against its natural characteristics.

Furthermore, recalling (6) with $\ddot{x}$ taken as input, and following the analysis presented in [19], it can be ensured that after swinging, the pendulum will insert in the upper region $-\pi/2 < \theta < \pi/2$. It is expected that as larger k_2, k_4, are, as faster the pendulum starts swinging from the rest down position and continuous to move on the down semicircle towards on the inverted up range. Inside the upper semicircle the controller creates a negative feedback and it is expected that suitable conditions can be found such that the pendulum to converge at the stable region of attraction. Simultaneously, the small values of k_1, k_3 may keep the cart displacement close enough to the final equilibrium.

Certainly, the overall qualitative analysis presented is a necessary supplementary tool for practical design solutions in sight of the fact that only local stability can be proved.

IV. The System Under Consideration

An inverted pendulum system with the proposed nonlinear state feedback controller, and parameters $M = m = 1$, $l = 0.3$, $\beta = 0.0001$ $(g = 9.81)$ is considered and simulated. To examine if there exist suitable ranges wherein the gains k_1, k_2, k_3 and k_4 are lying, the corresponding characteristic polynomials of A_1 and A_2 are calculated as functions of the controller gains. Clearly, to determine the suitable gains and their values' ranges is a cumbersome task. Towards this end, the Descartes' rule of signs [25] is used as guidance and an algorithmic procedure based on the trial and error concept is developed. This is due to the fact that Descartes' rule provides only necessary conditions for the signs of the roots of a polynomial. Therefore, one cannot be *a priori* ensured that all the roots of matrix A_1 have negative real parts and simultaneously at least one root of A_2 lies for sure on the right half-plane, ensuring the stability of the up and the instability of the down equilibrium point, respectively.

Nevertheless, the practical implementation of the calculation procedure seems to easily result in appropriate values for the controller gains, perhaps due to a correct qualitative analysis used for designing the controller and studying stability. In this frame, a possible set of gains for the proposed controller are chosen to be $k_1 = 2$, $k_2 = 140$, $k_3 = 6$ and $k_4 = 60$. Also, a value of $c = 3$ is chosen in an endeavor to further improve the system performance.

Two different cases are simulated. In the first case the control law is applied when the inverted-pendulum system is initially near its natural down position equilibrium point, with task to swing up the stick and stabilize the pendulum at the upright position with the final displacement on x to be zero, i.e. the system returns exactly to the initial position with the stick up inverted. In the second case, the only difference is that the final displacement on x is arbitrarily selected at a desired value different to zero (then obviously the first term of the control law ($k_1 x$) is substituted by $k_1(x - x_{ref})$, where now it is selected x_{ref}=3m). It is noting that for the first case, for practical reasons, the actual initial conditions of the inverted-pendulum system are taken to be $x(0) = 0.01, \theta(0) = 180^o, \dot{x}(0) = \dot{\theta}(0) = 0$; a very small nonzero $x(0)$ is chosen since otherwise, the control input is zero and the system remains unforced at the initial rest point for all future time.

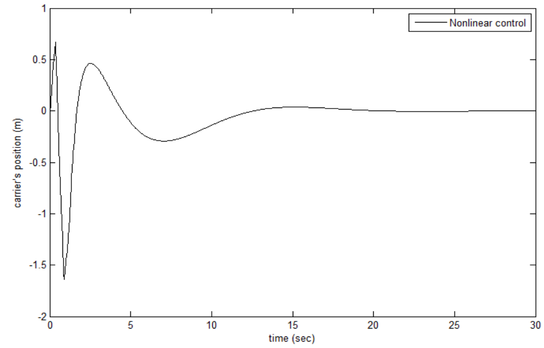
Figure 2(a) represents the carrier's position and Figure 2(b) the pendulum's angle response. Figures 2(c) and 2(d) represent the carrier's velocity and the pendulum's angular velocity, respectively for the first case. The corresponding responses for the second case are shown in Figs. 3(a) to 3(d).

In both cases, one can observe the swing up period that is quickly finished in less than 2s, as shown in Figures 2(a) and 3(a). The system response appears at most two oscillations (see Figures 2(b) and 3(b), respectively) before its slower convergence to the inverted upright position. This indicates a clear improvement with respect to other methods where usually more oscillations are observed [17-21]. As seen in Figures 2(a) and 3(a), small overshoots are observed after swing up while the final displacements are on the command values, at the origin or at x_{ref}=3m, respectively.

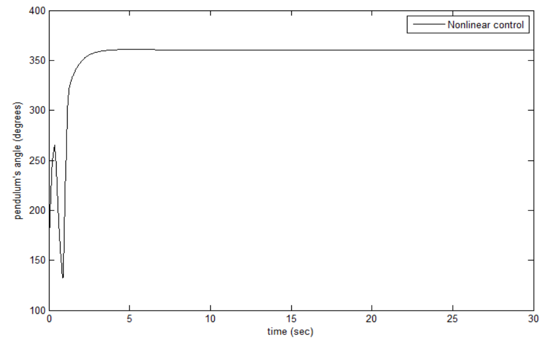
Finally, Figures 2(c)-(d) and 3(c)-(d), respectively, represent the cart and pendulum angular velocities for each case. It is again verified the excellent system performance and the effective unified manner in which the controller acts on the system, with the velocities to take feasible values.

V. Conclusion

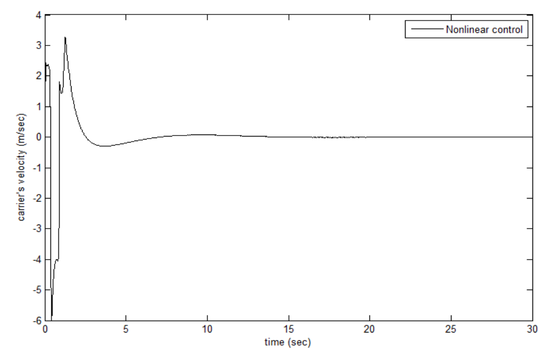
A novel controller design for inverted-pendulum systems has been developed in the basis of a careful motion and stability analysis. As it is theoretically analyzed and confirmed by simulations, the proposed nonlinear state feedback controller can effectively act in a twofold sequential mode: Firstly, a quick swing up processing is actuated which is followed by an asymptotic convergence at the inverted position in the stable region of attraction. Though the theoretical analysis is based on local stability studies and qualitative concepts, both the design and the system response appear to be very satisfactory.



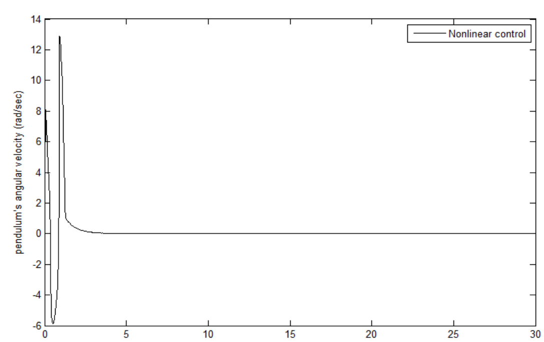
(a) Carrier's position



(b) Pendulum's angle

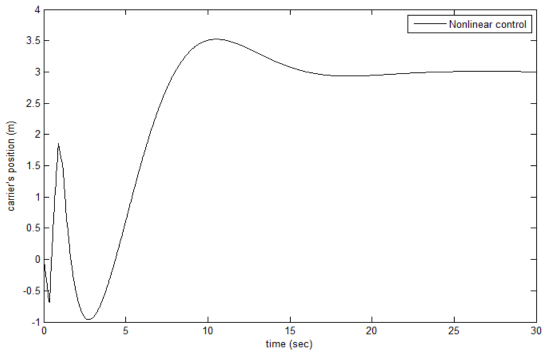


(c) Carrier's velocity

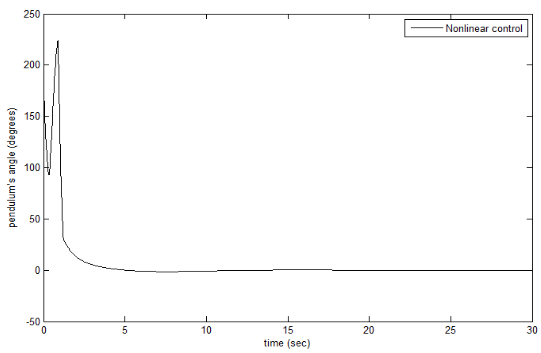


(d) Pendulum's angular velocity

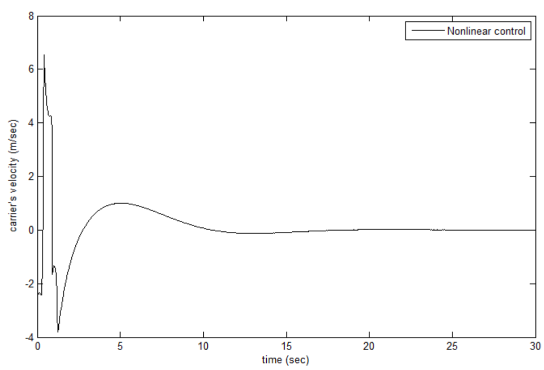
Fig. 2. System response for case 1: (a) carrier's position, (b) pendulum's angle, (c) carrier's velocity, and (d) pendulum's angular velocity.



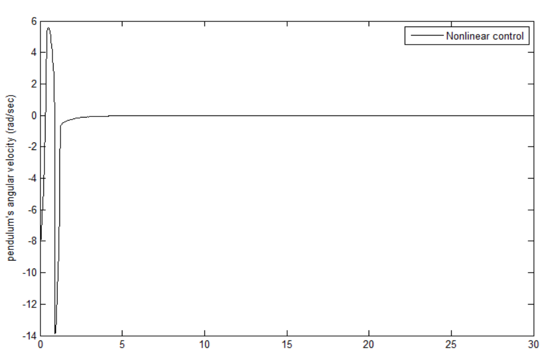
(a) Carrier's position



(b) Pendulum's angle



(c) Carrier's velocity



(d) Pendulum's angular velocity

Fig. 3. System response for case 2: (a) carrier's position, (b) pendulum's angle, (c) carrier's velocity, and (d) pendulum's angular velocity.

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