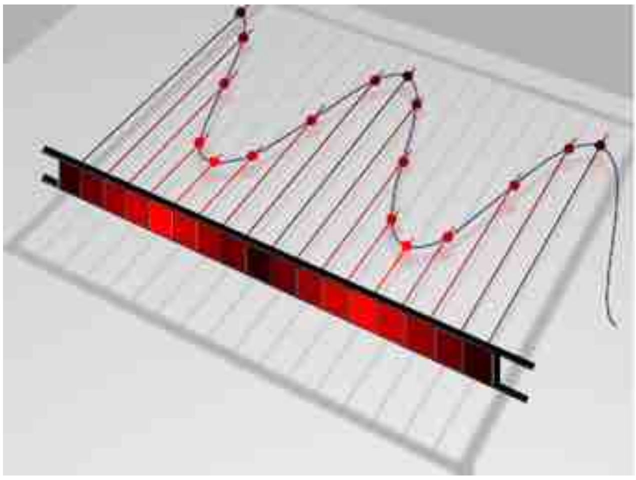
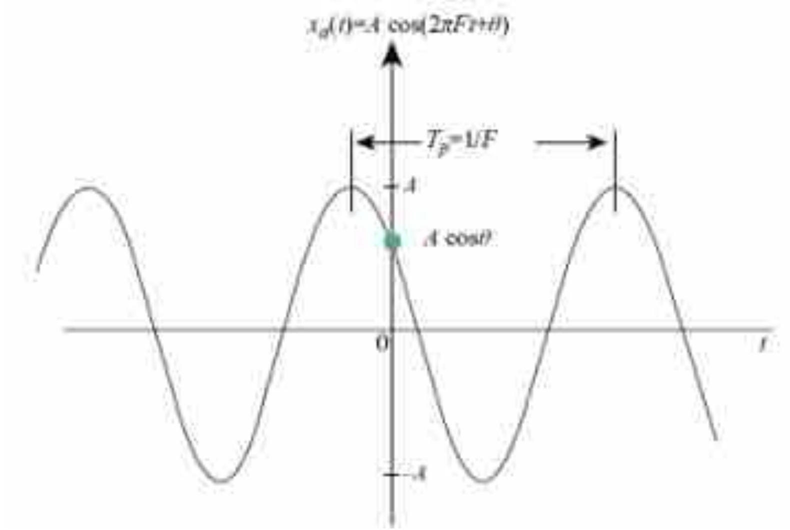
ΔΕΙΓΜΑΤΟΛΗΨΙΑ

- SAMPLING -

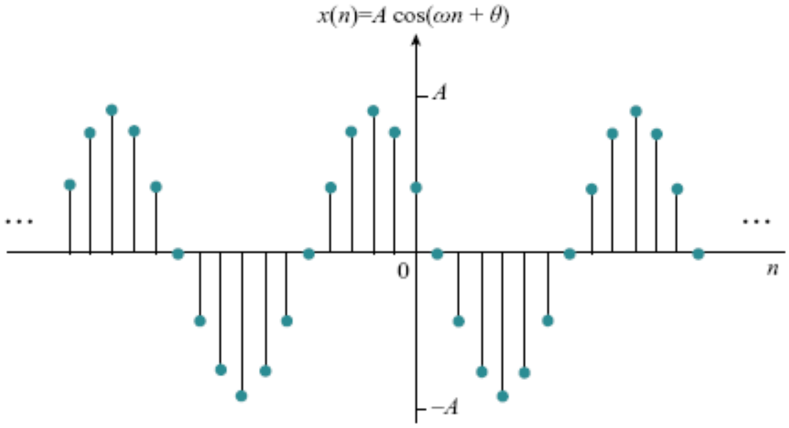


Continuous–Time Sinusoidal Signals



- It is periodic for every fixed value of *F*, I.e. $x_a(t+T_p)=x_a(t)$, where $T_p=1/F$
- For distinct (different) frequencies they are themselves distinct
- Increasing *F* results in an increase in the rate of oscillation

Discrete–Time Sinusoidal Signals



- It is **periodic only if f is a rational number**
- Discrete-Time sinusoids whose frequencies are separated by an integer multiple of 2π are identical
- The highest rate of oscillation is attained when ω=π (or ω=-π) or f=1/2 (or f=-1/2)

Continuous–Time Sinusoidal Signals

$$x_a(t) = A \cdot \cos(\Omega t + \theta), \qquad -\infty < t < \infty$$

where A is the amplitude

Ω is the frequency in rad/sec (Ω=2πF)

θ is the phase in rad

Discrete–Time Sinusoidal Signals

$$x(n) = A \cdot \cos(\omega n + \theta), \qquad -\infty < n < \infty$$

where n integer variable

A the amplitude

ω is the frequency in rad/sample (ω=2πf)

θ is the phase in rad

Discrete-time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.

Proof:

Let $x(n) = A \cdot \cos(\omega n + \theta)$

Then the signal $x_1(n)$ of frequency $\omega + 2k\pi$ is equal to

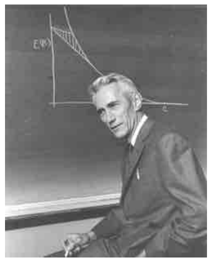
$$
\begin{aligned}
x_1(n) &= A \cdot \cos[(\omega + 2k\pi)n + \theta] = \\
&= A \cdot \cos(\omega n + \theta + 2k\pi n) = \\
&= A \cdot \cos(\omega n + \theta) = \\
&= x(n)
\end{aligned}
$$

$$
\begin{aligned}
f(t) &= \cos \Omega t \\
f(nT) &= \cos \Omega \overbrace{nT}^{t} \\
f(nT) &= \cos n \overbrace{\Omega T}^{\omega} \\
f[n] &= \cos n\omega \\
\omega &= \Omega T \quad \frac{radians}{second} \cdot \frac{seconds}{sample} = \frac{radians}{sample} \\
\Omega &\rightarrow \text{'analog frequency'} \quad \frac{radians}{second} \\
\omega &\rightarrow \text{'normalised frequency'} \quad \frac{radians}{sample}
\end{aligned}
$$

A Brief History of Sampling Research

- **1915** - **E.T. Whitaker** devised a proof showing that a band-limited function can be reconstructed from samples.
- **1920** - **K. Ogura** proved that if a function is sampled at a frequency at least twice the highest function frequency, it could be reconstructed from those samples.
- **1928** - Bell Labs engineer **Harry Nyquist** published an article entitled *Certain topics in Telegraph Transmission Theory*. In this article he provided proof that for complete signal construction, the frequency bandwidth is proportional to the signaling speed, and that the highest frequency is equal to half the number of code elements per second.
- **1949** - **Claude Shannon** unified many aspects of sampling, founded that larger science of information theory.

Harry Nyquist (1889–1976)



Claude Elwood Shannon (1916–2001)

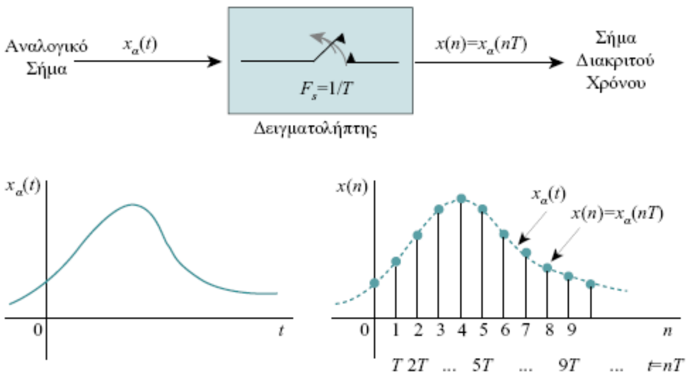
Sampling the continuous-time (analog) sinusoid signal at a frequency of $\mathbf{F_s=1/T}$, we get the discrete-time signal x(n):

$$x(n) = x_a(t)\big|_{t=nT} \equiv x_a(nT) = A\cdot\cos(2\pi F n T + \theta) =$$

$$= A\cdot\cos\left(2\pi n\frac{F}{F_S} + \theta\right) = A\cdot\cos(2\pi f n + \theta) = A\cdot\cos(\omega n + \theta)$$

i.e. $$f = \frac{F}{F_S}$$

or $$\omega = \Omega T$$



ALIASING

Continuous-Time | **Sampling** | **Discrete-Time**

$$x_a(t) = A \cdot \cos(2\pi F_0 t + \theta) \longrightarrow x(n) = A \cdot \cos(2\pi f_0 n + \theta)$$

$$x_a(t) = A \cdot \cos(2\pi F_k t + \theta) \nearrow \quad \text{where} \quad f_0 = F_0 / F_s$$

$$\text{where} \quad F_k = F_0 + kF_s, \quad k = \pm 1, \pm 2, \ldots$$

Proof:

$$x(n) \equiv x_a(nT) = A \cdot \cos\left(2\pi \frac{F_0 + kF_s}{F_s} n + \theta\right) =$$

$$= \mathrm{A} \cdot \cos\left(2\pi \frac{\mathrm{F}_0}{\mathrm{F}_s} n + \theta + 2\pi kn\right) =$$

$$= A \cdot \cos(2\pi f_0 n + \theta) =$$

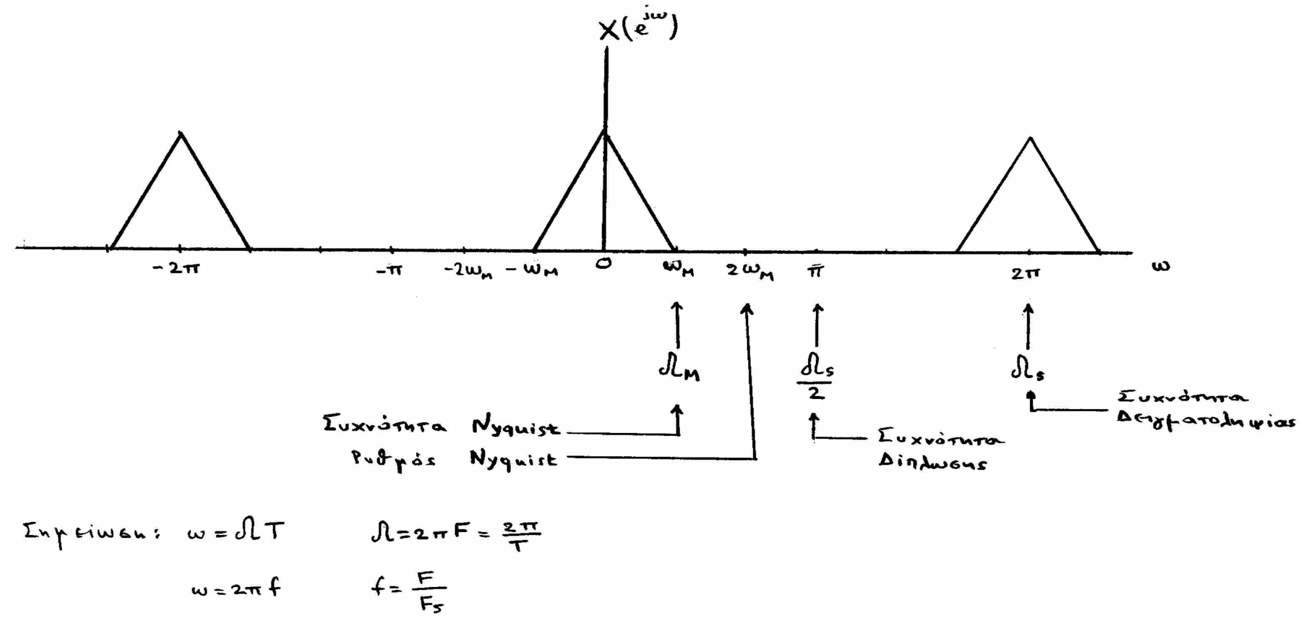
$$= x(n)$$

Frequencies $F_k=F_0+kF_s$ cannot be distinguished from F_0 after sampling. In other words, they are **aliases** of F_0.

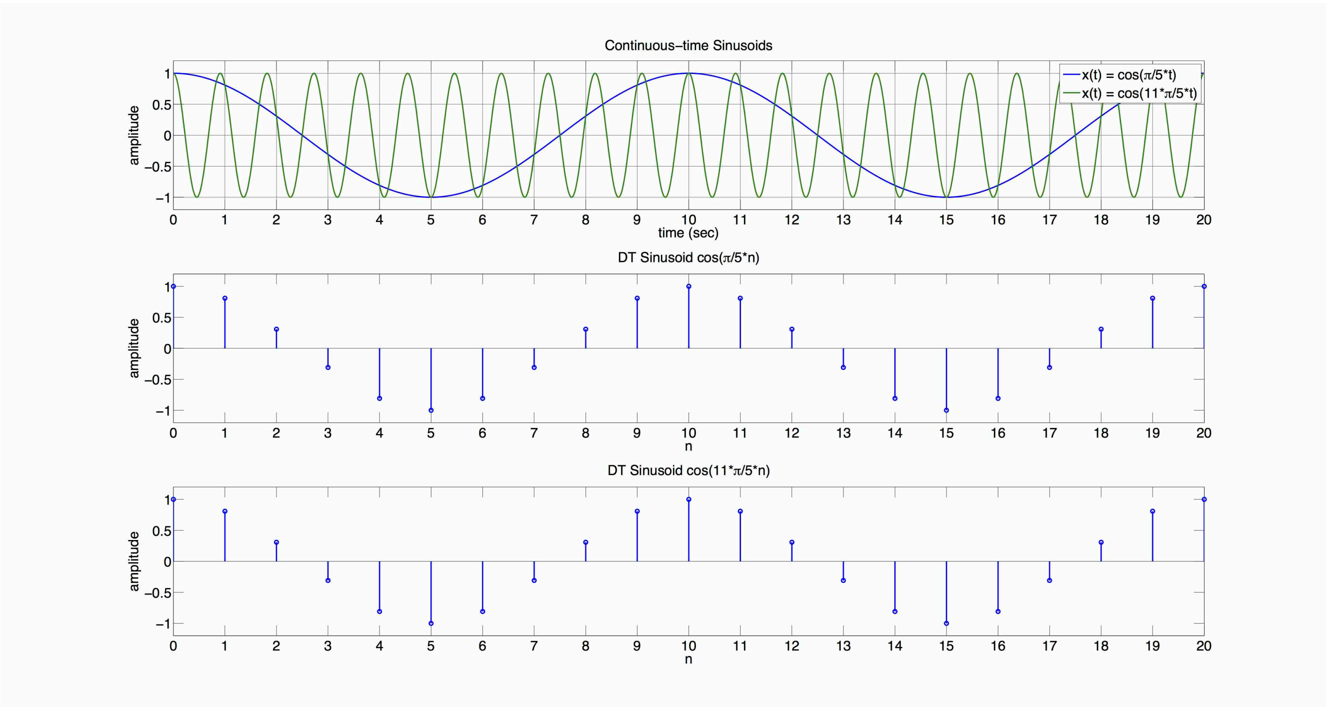
This phenomenon is called **aliasing** or **spectral overlap**.

Sampling Theorem or **Nyquist Criteria** or **Shannon Theorem**

If a signal contains no frequency components above a frequency $\mathbf{F_M}$ the signal can be uniquely represented by equally spaced samples if the sampling frequency $\mathbf{F_s}$ is greater than twice $\mathbf{F_M}$, i.e. $\mathbf{F_s > 2F_M}$



ALIASING



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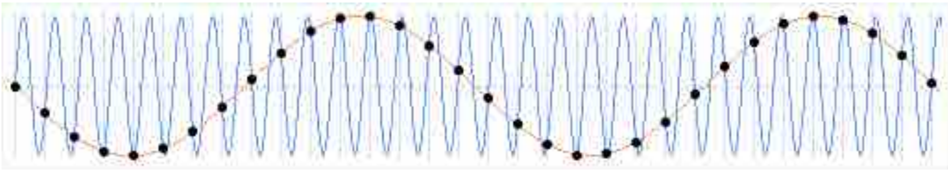

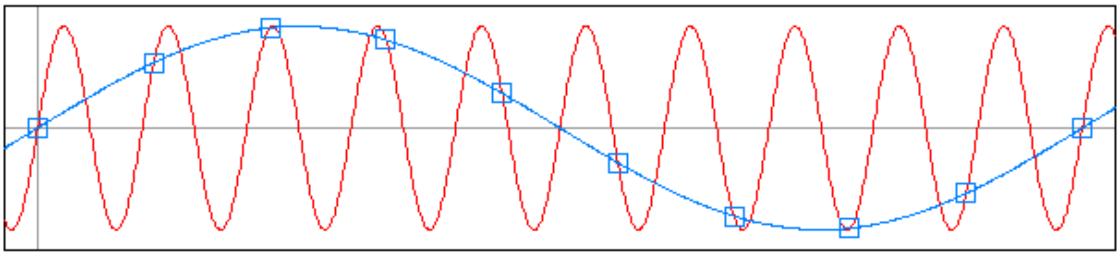

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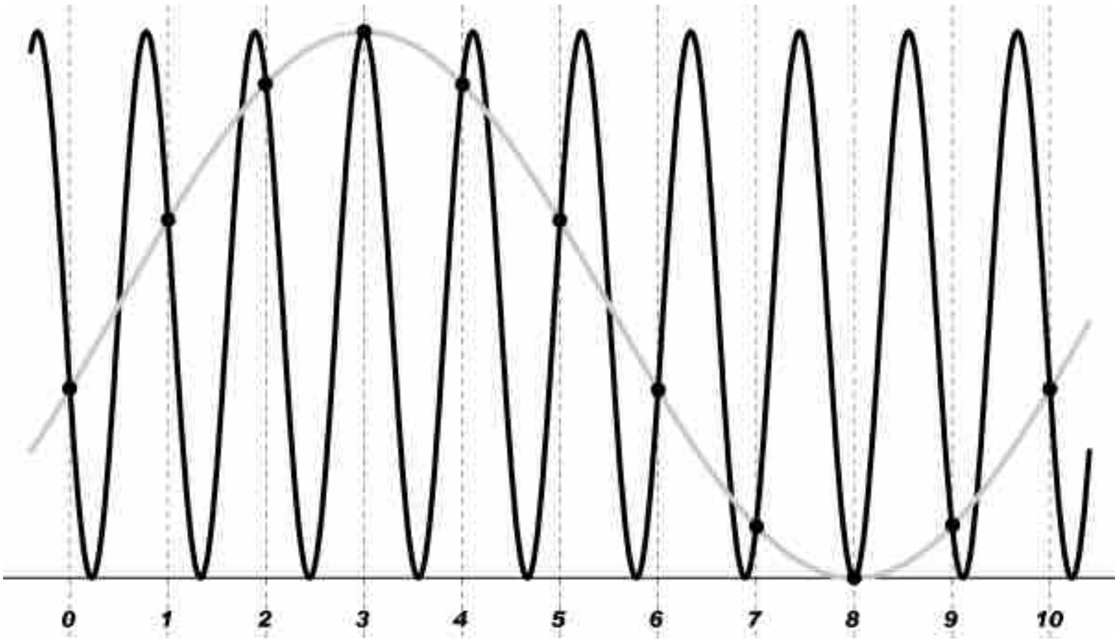

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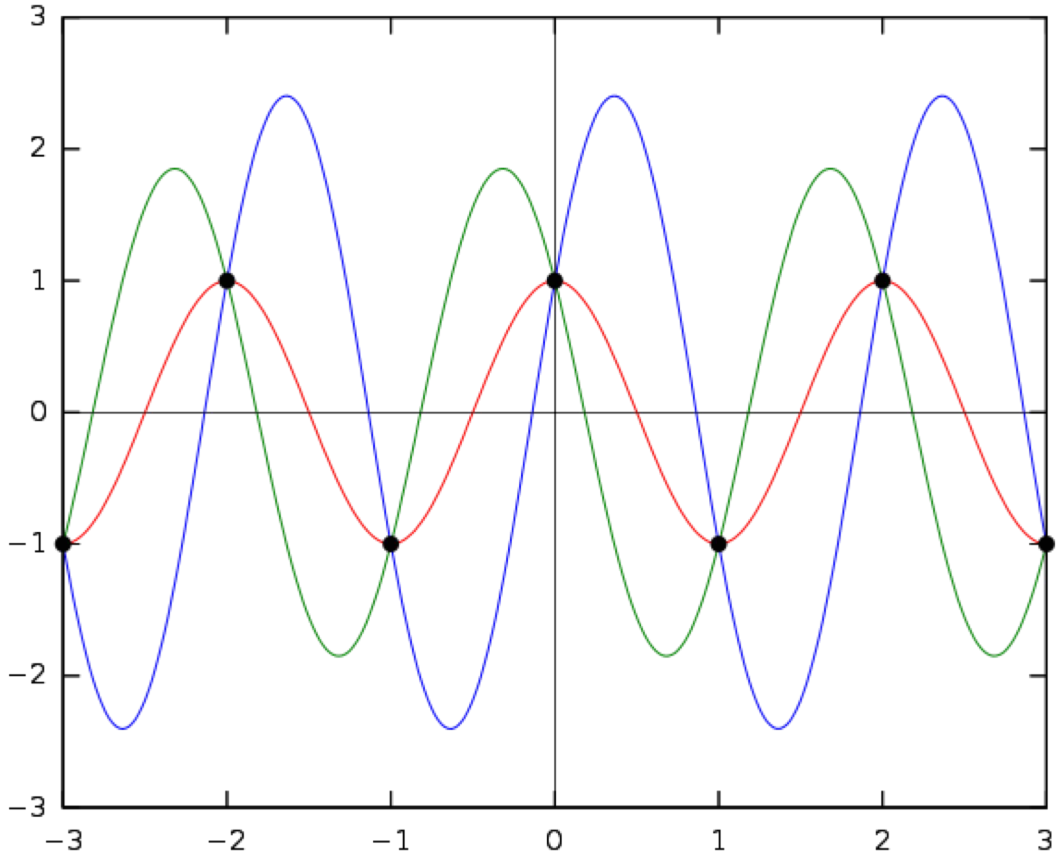

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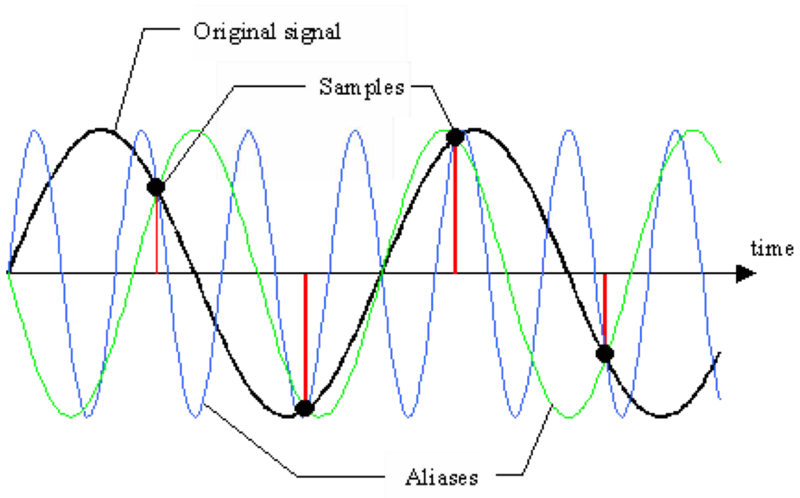
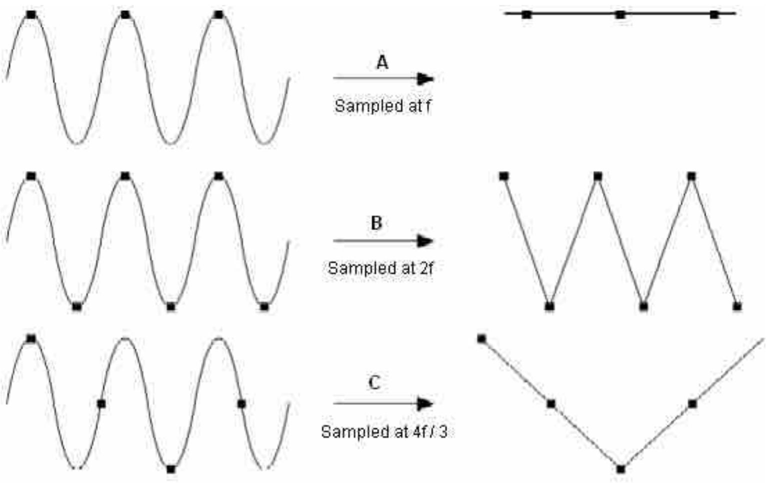
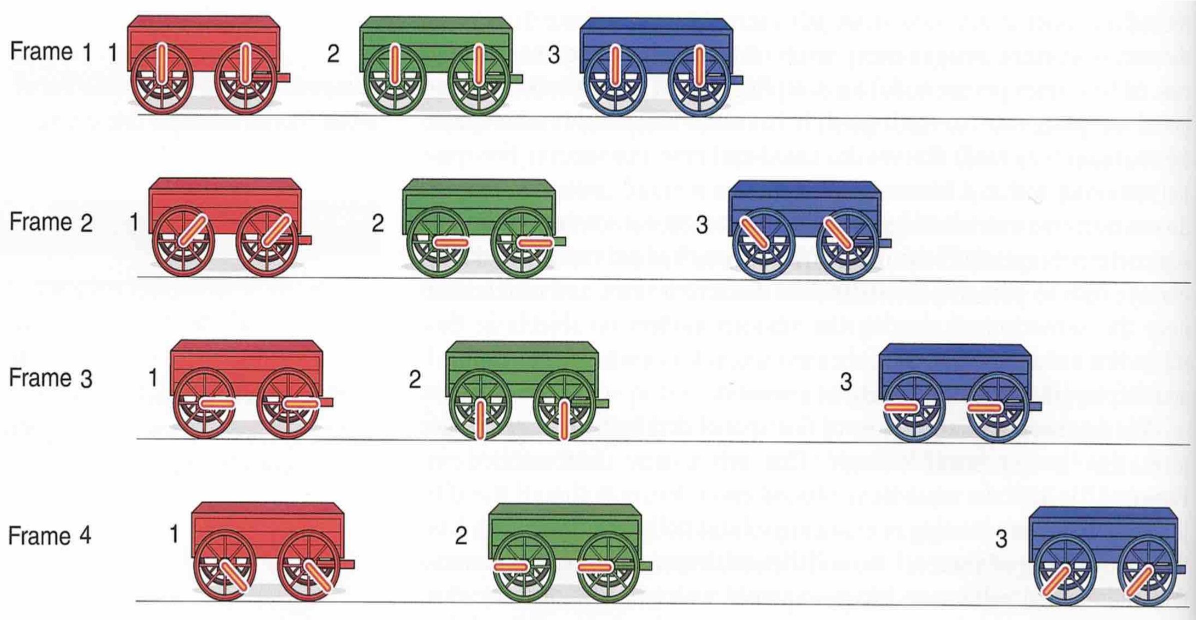


ALIASING

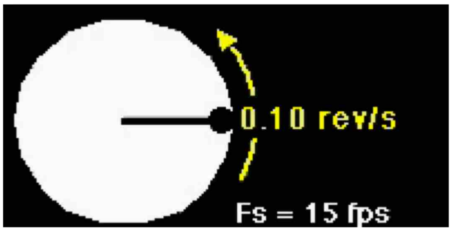


Aliasing example

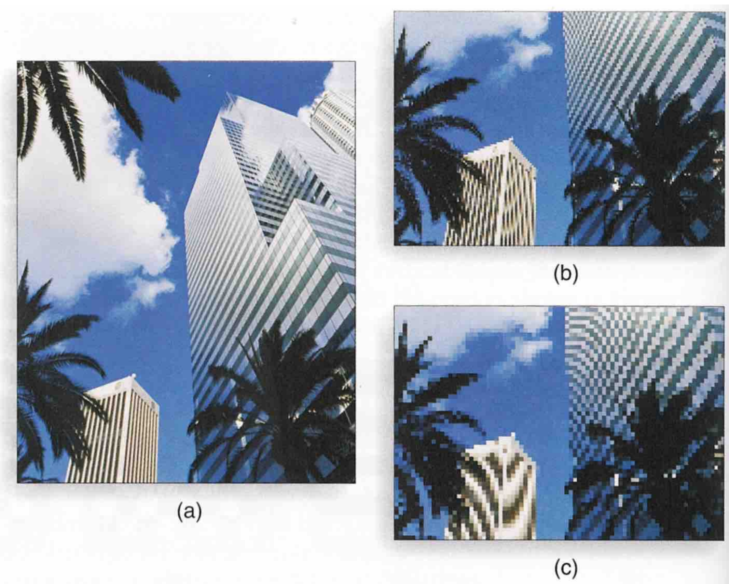


Four frames of a movie showing three wagons moving in the same direction at different speeds. Wagon 2 is traveling twice as fast as wagon 1, and wagon 3 is traveling seven times as fast as wagon 1. Temporal aliasing causes the wheels of wagon 3 to appear to be rotating at the same rate as the wheels of wagon 1, but in the opposite direction.

Aliasing demos



Aliasing example

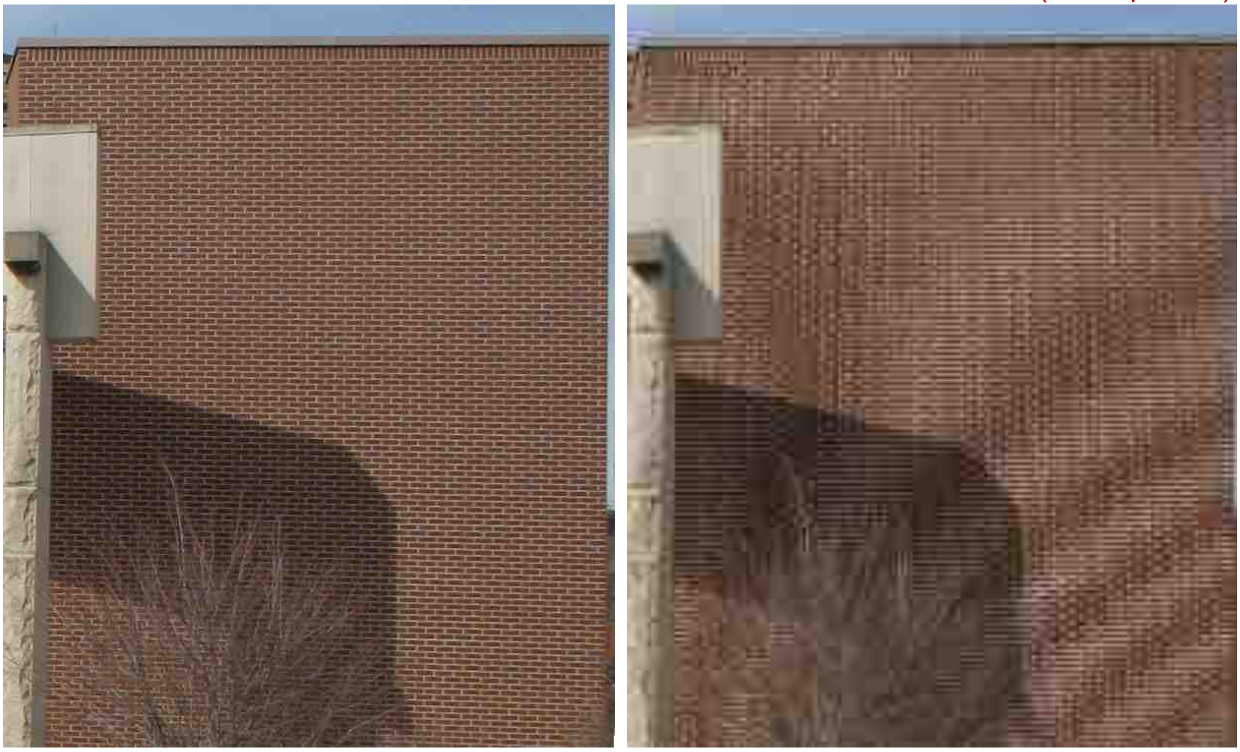


The original image of two buildings in (a) has 812 rows and 650 columns. When the linear sampling rate is reduced by a factor of four in (b), the sampling rate is too low to accurately represent the vertical structures in the more distant building in the lower left, and diagonal bands begin to appear. In (c), the sampling rate is reduced by another factor of two, and these aliasing effects become more obvious.

Aliasing demo

Similar to one-dimensional discrete-time signals, images can also suffer from aliasing if the sampling resolution or pixel density, is inadequate.

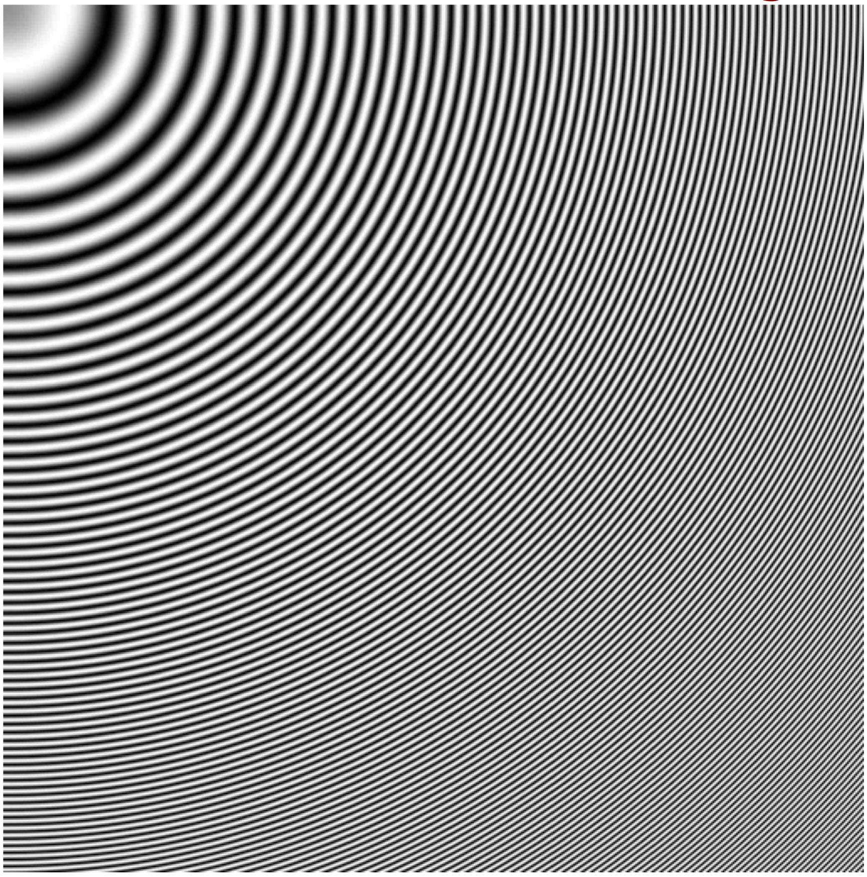
(Moiré pattern)



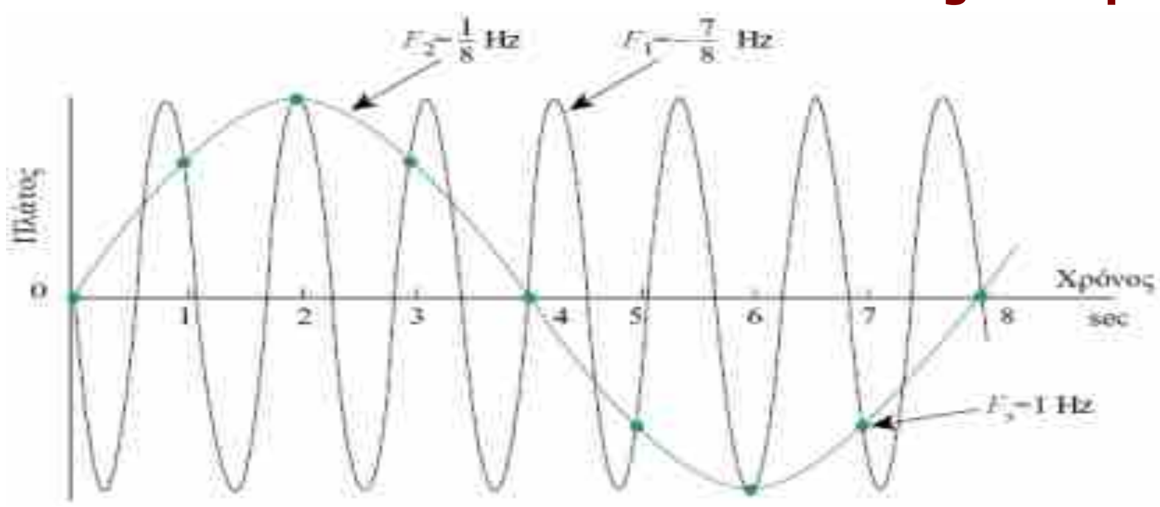
Aliasing example

Aliasing demo

Aliasing demo



Aliasing example



$$x_1(t) = \sin 2\pi \left(\frac{1}{8}\right) t$$

$$x_2(t) = \sin 2\pi \left(-\frac{7}{8}\right) t$$

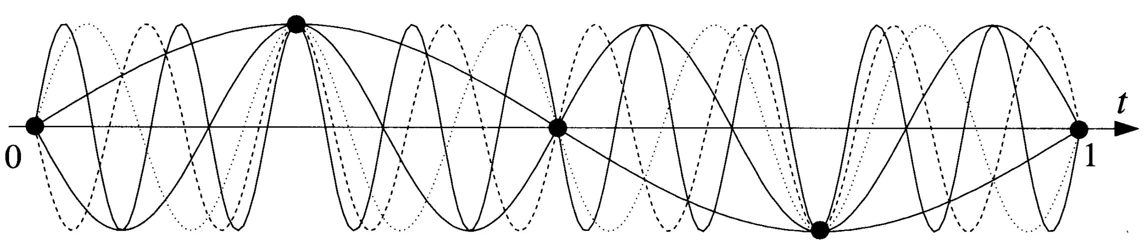
Proof

$$x_1(n) = \sin 2\pi \left(\frac{1}{8}\right) nT = \sin 2\pi \left(\frac{1}{8}\right) n = \sin \frac{\pi}{4} n$$

$$x_2(n) = \sin 2\pi \left(-\frac{7}{8}\right) nT = \sin 2\pi \left(-\frac{7}{8}\right) n = \sin \frac{-7\pi}{4} n = \sin\left(-\frac{8\pi}{4} + \frac{\pi}{4}\right) n =$$

$$= \sin\left(-2\pi n + \frac{\pi}{4} n\right) = \sin \frac{\pi}{4} n = x_1(n)$$

In conclusion ALIASING is …



$$F_k = F_0 + kF_S, \qquad k = \pm 1, \pm 2, \ldots$$

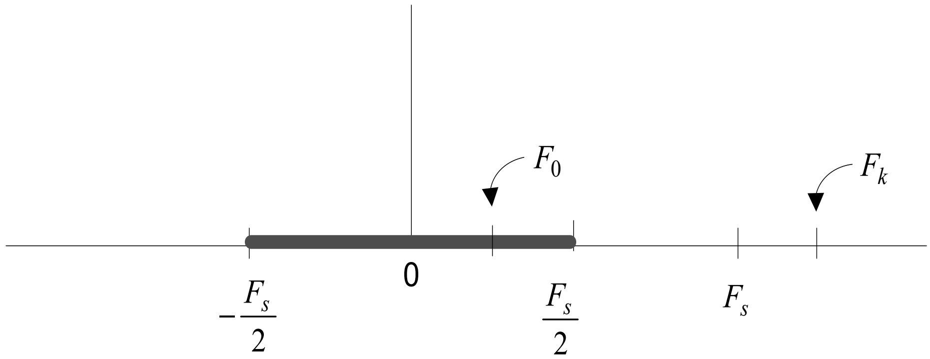
$$x(n) \equiv x_a(nT) = A\cos\left(2\pi \frac{F_0 + kF_s}{F_s} n + \theta\right) =$$

$$= A\cos\left(2\pi n \frac{F_0}{F_s} + \theta + 2\pi k n\right) =$$

$$= A\cos(2\pi f_0 n + \theta)$$

… higher frequency impersonating lower frequencies due to the sampling rate not satisfying the Nyquist sampling criteria.

Aliased frequencies

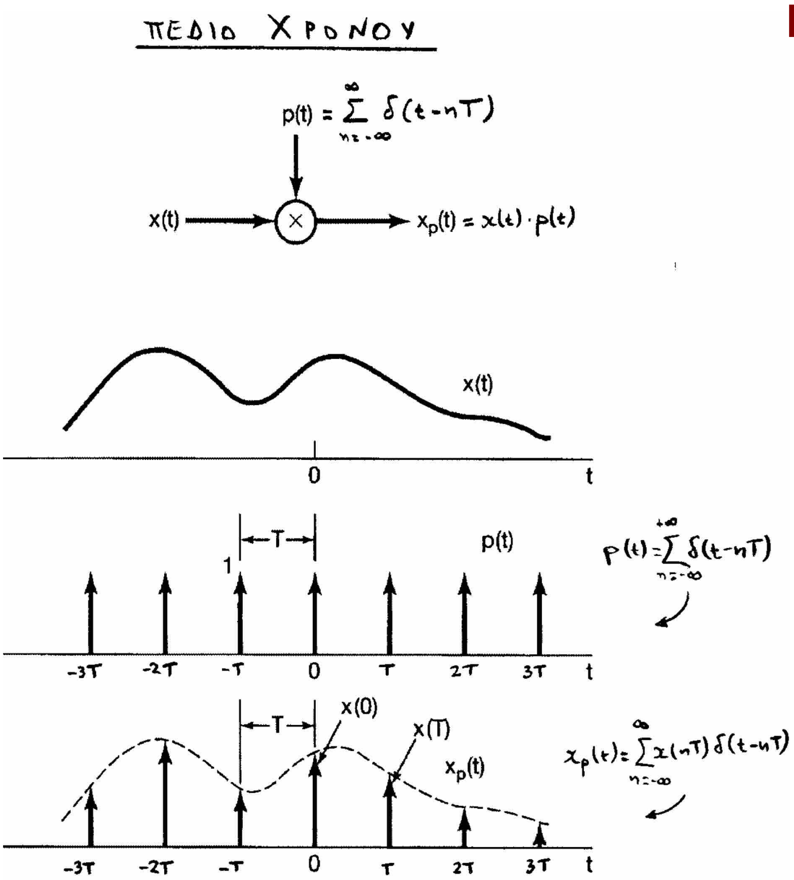


If $F_k > F_s/2$ then the actual frequency obtained is given by

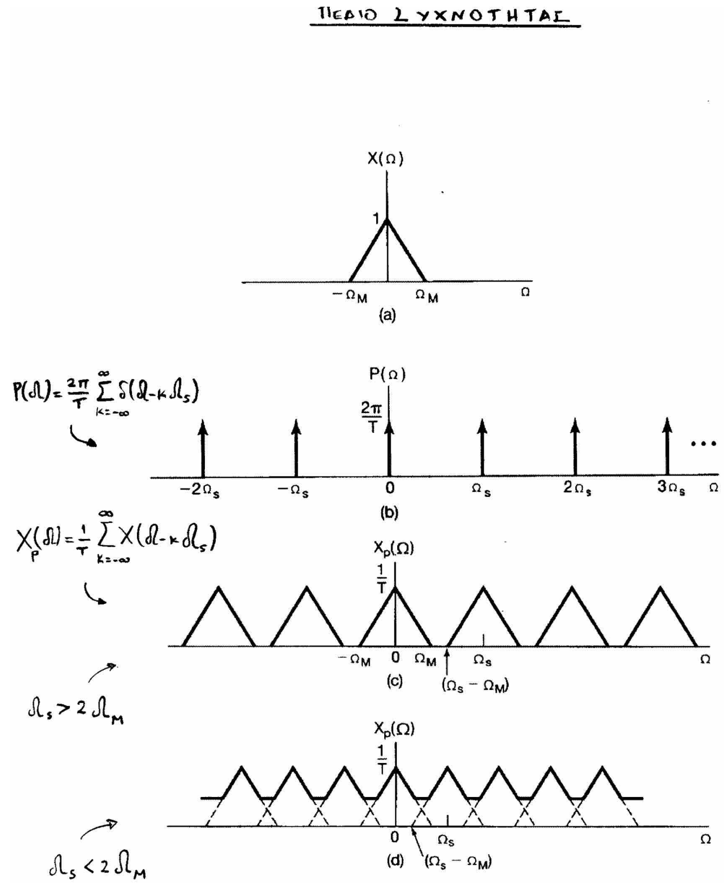
$$F_0 = F_k - kF_s$$

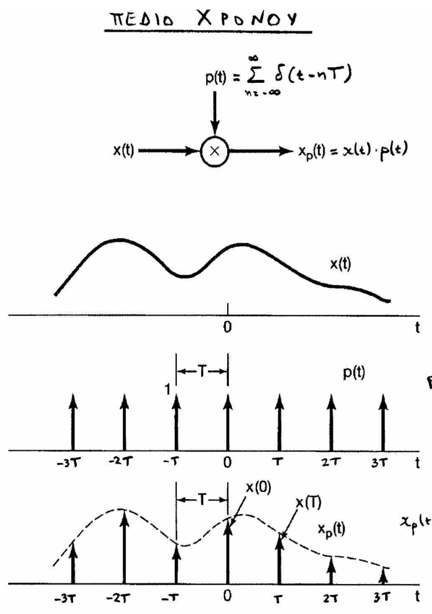
where k is any integer $(-\infty < k < \infty)$ such that $-\frac{F_s}{2} \leq F_0 \leq \frac{F_s}{2}$

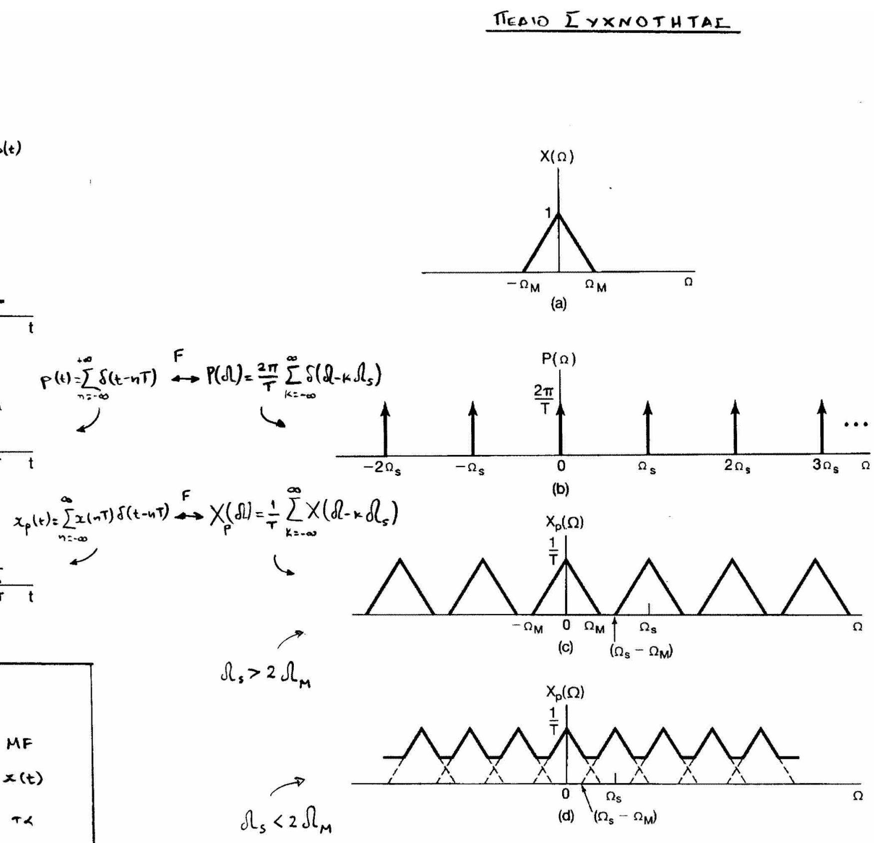
Δειγματοληψία στο Πεδίο του Χρόνου



Δειγματοληψία στο Πεδίο της Συχνότητας







ΘΕΩΡΗΜΑ ΔΕΙΓΜΑΤΟΛΗΨΙΑΣ

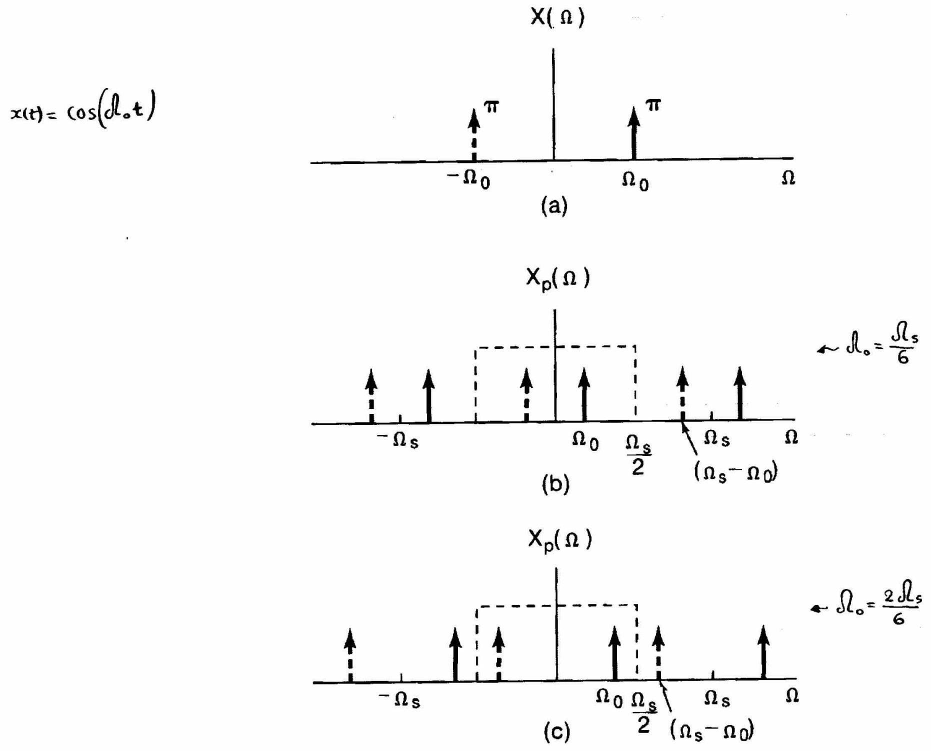
Έστω το περιορισμένου εύρους σήμα $x(t)$ με ΜΣ $X(\Omega)=0$ για $|\Omega|>\Omega_M$. Τότε το σήμα $x(t)$ μπορεί μονοσήμαντα να προσδιοριστεί από τα δείγματά του $x(nT)$, $n=0,\pm1,\pm2,\dots$ εάν

$$\Omega_s > 2\,\Omega_M$$

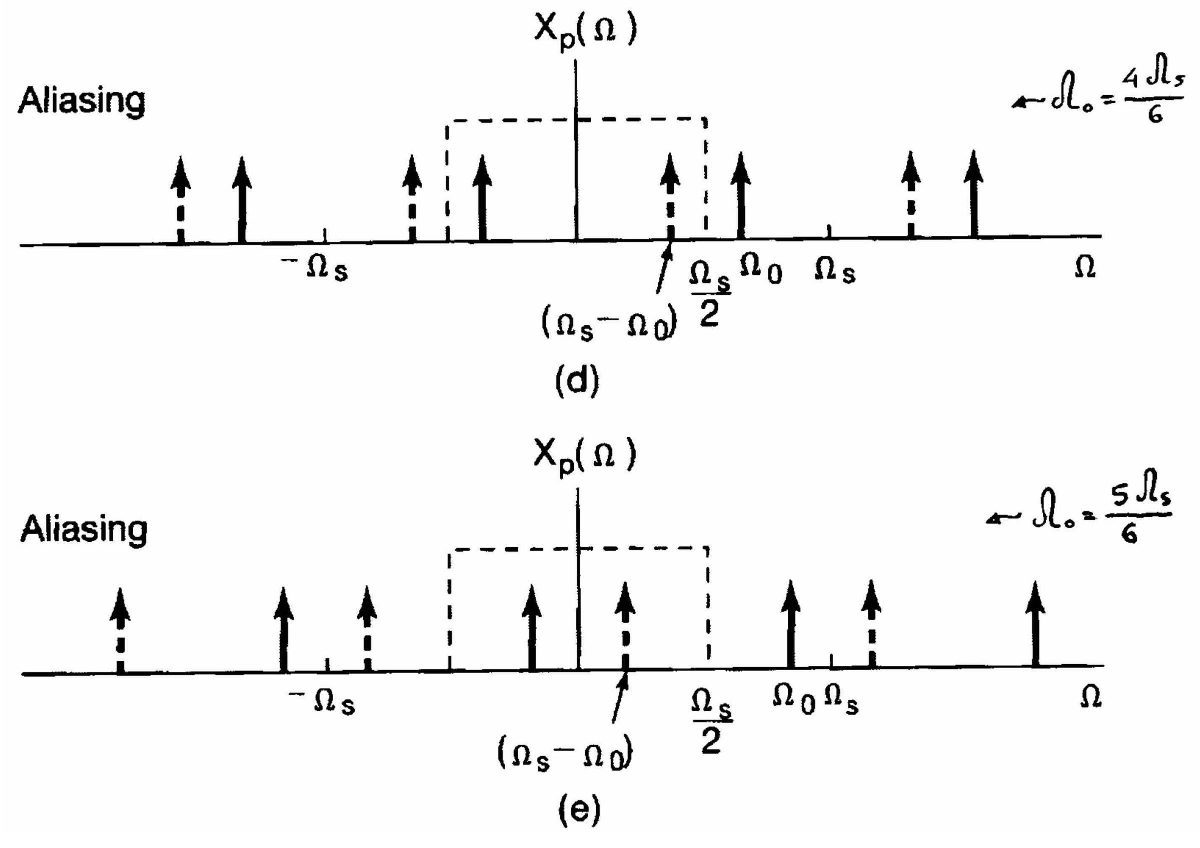
όπου $\Omega_s=\frac{2\pi}{T}$

Παράδειγμα

Παράδειγμα δειγματοληψίας με συχνότητα Ω_s ενός σήματος $x(t)=\cos(\Omega_0 t)$ για διαφορετικές συχνότητες Ω_0, δηλ. για $\Omega_0=\frac{\Omega_s}{6}$, $\Omega_0=\frac{2\Omega_s}{6}$, $\Omega_0=\frac{4\Omega_s}{6}$, $\Omega_0=\frac{5\Omega_s}{6}$



Παράδειγμα
(συνέχεια)



Aliasing in the Frequency Domain

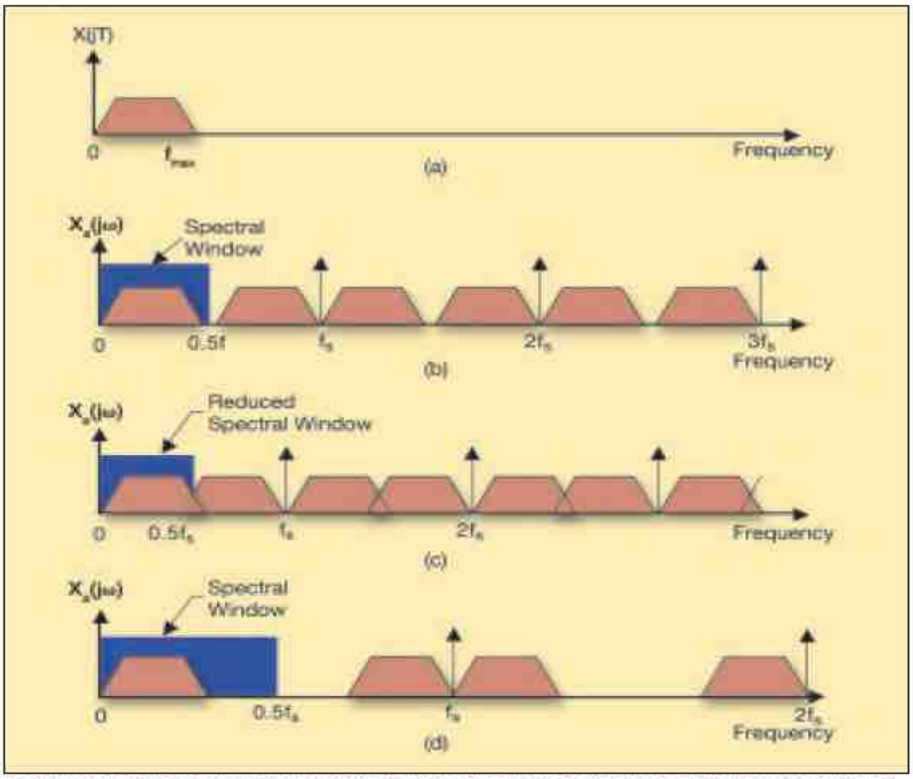
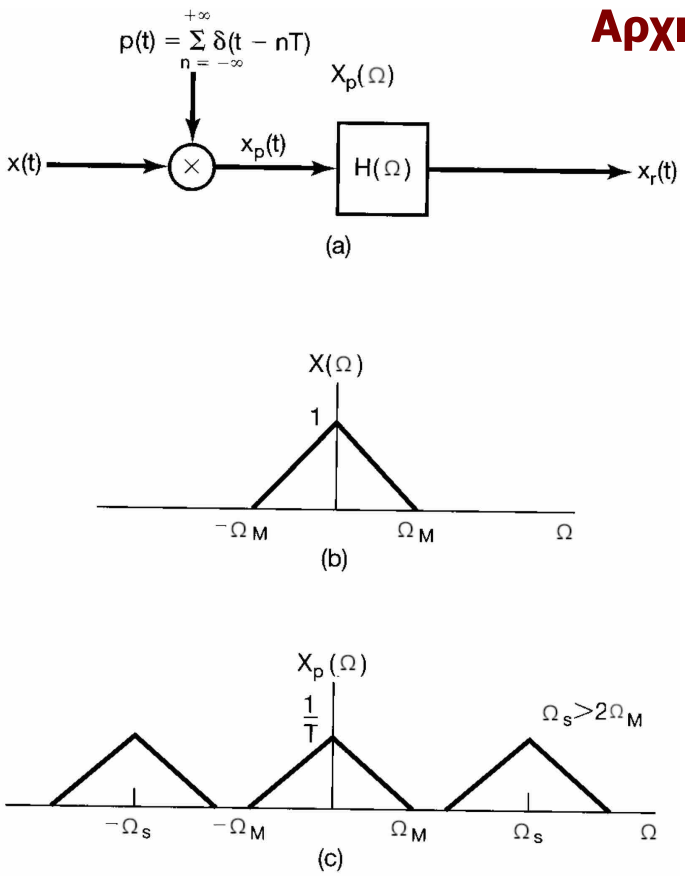
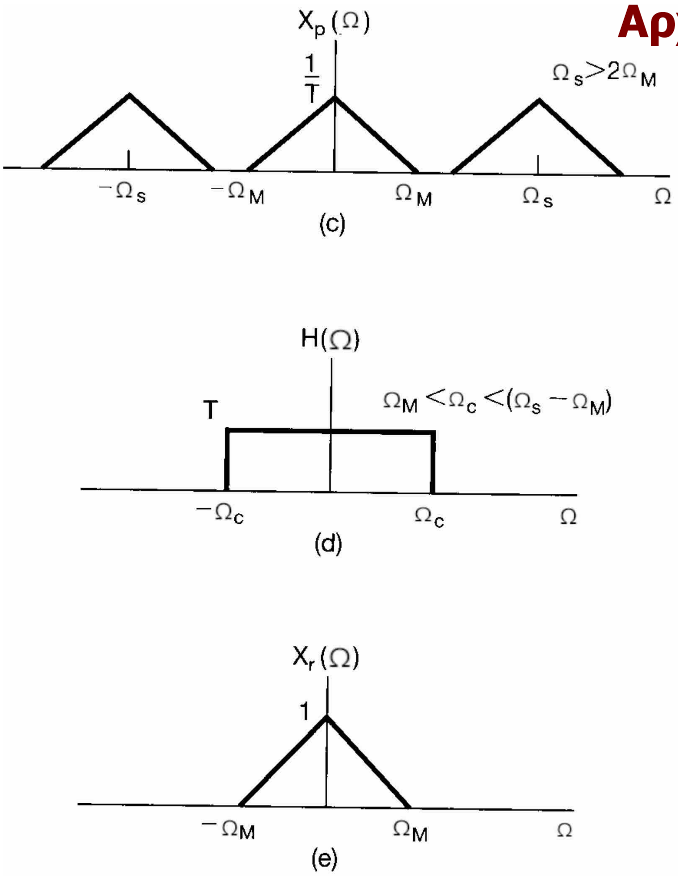


Figure 5. Sampling seen in frequency domain (a) spectrum of the analog signal (b) spectrum of the signal sampled just above the Nyquist rate (c) spectrum of the signal sampled below the Nyquist rate (d) spectrum of the signal sampled much above the Nyquist rate.

Ανακατασκευή Αρχικού Σήματος



Ανακατασκευή Αρχικού Σήματος



Ανακατασκευή Αρχικού Σήματος

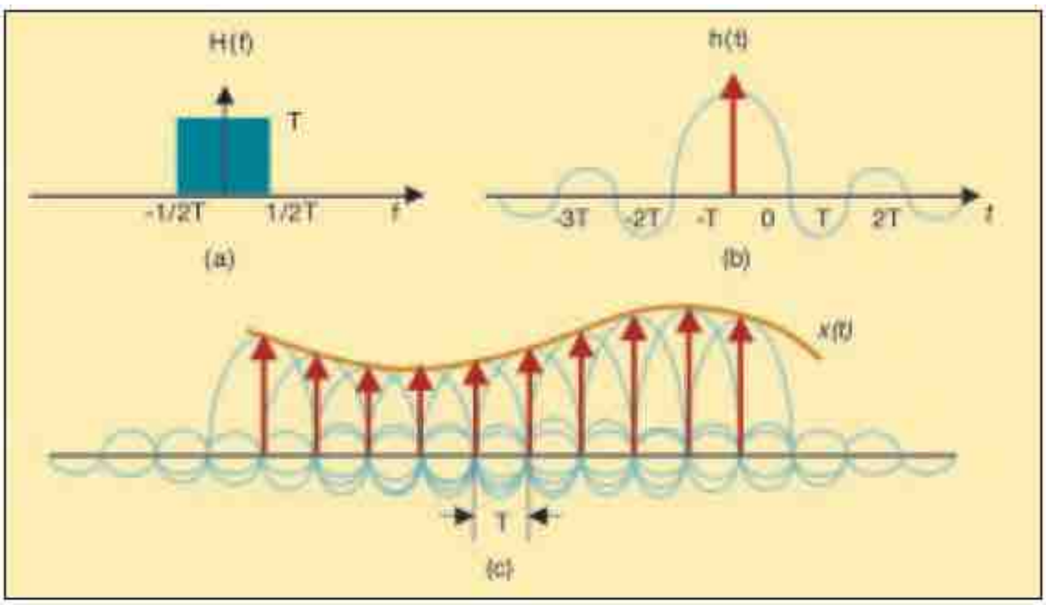


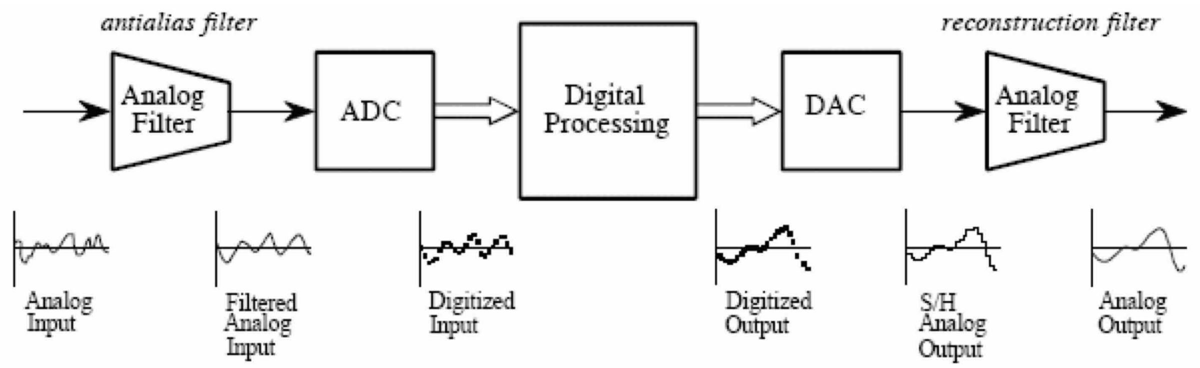
Figure 15. (a) Frequency response of ideal low-pass filter for signal reconstruction, (b) impulse response of the filter, (c) signal recovery through convolution of samples with impulse response.

◆ Analog Anti-Aliasing Filter (Lowpass Filter)

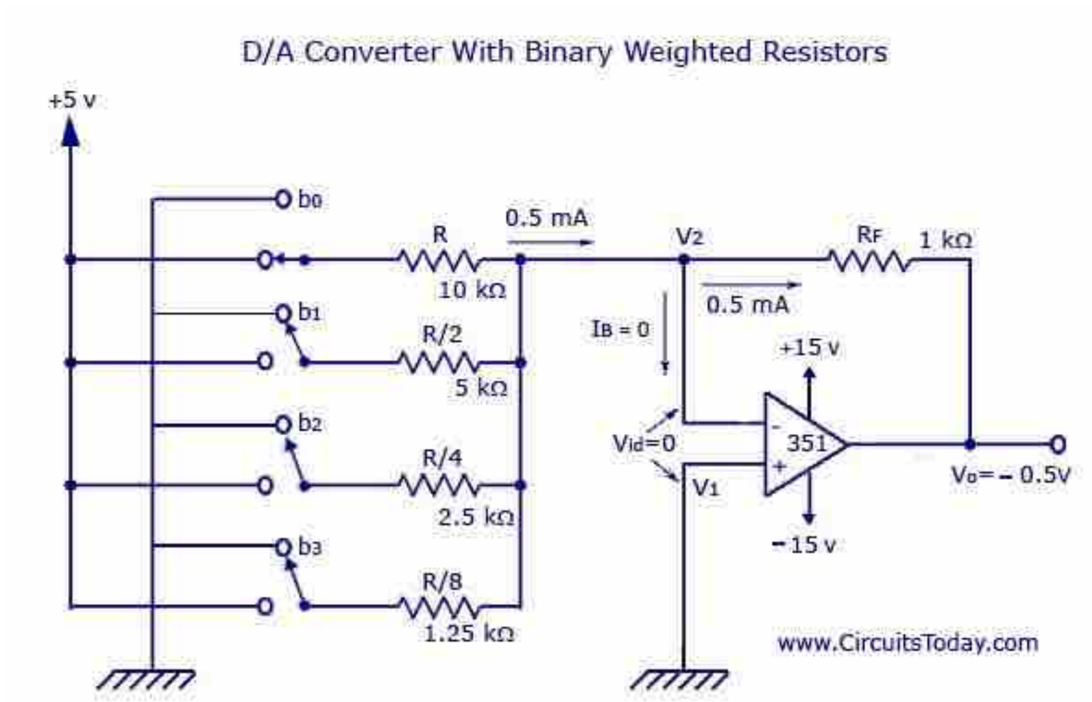
Analog signals must be band-limited to proper frequency before sampling, because:

a. Input signal is time-limited and therefore cannot be band-limited

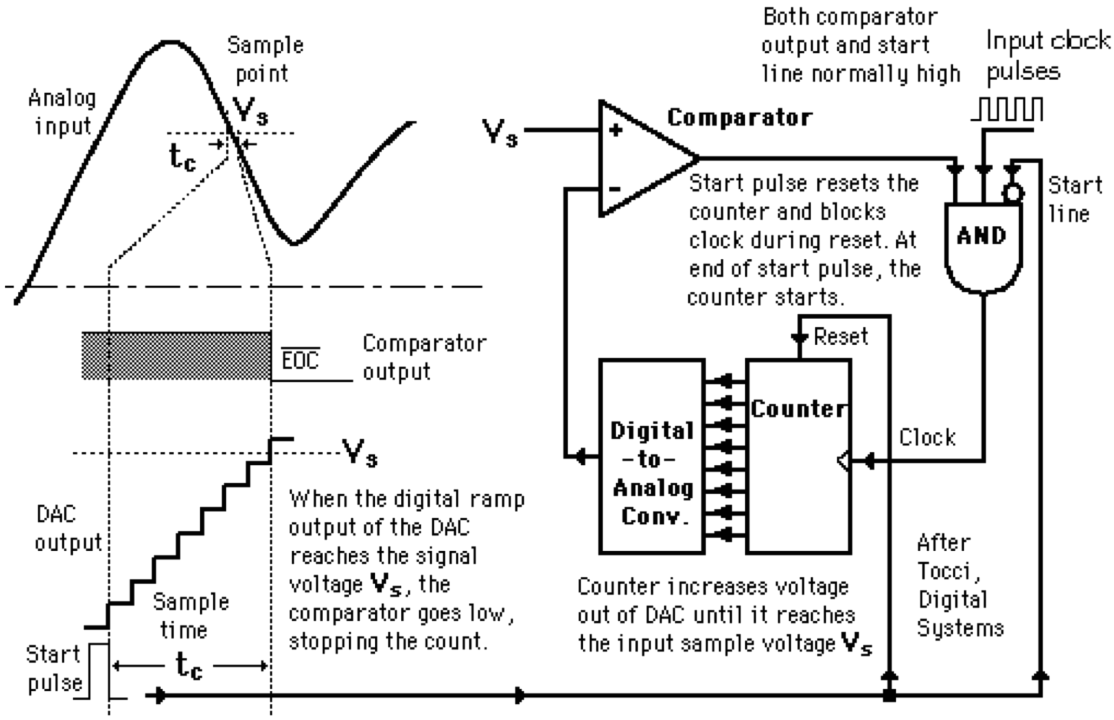
b. Even if the signal is “naturally” band-limited, additive noise has a much broader spectrum than the signal.



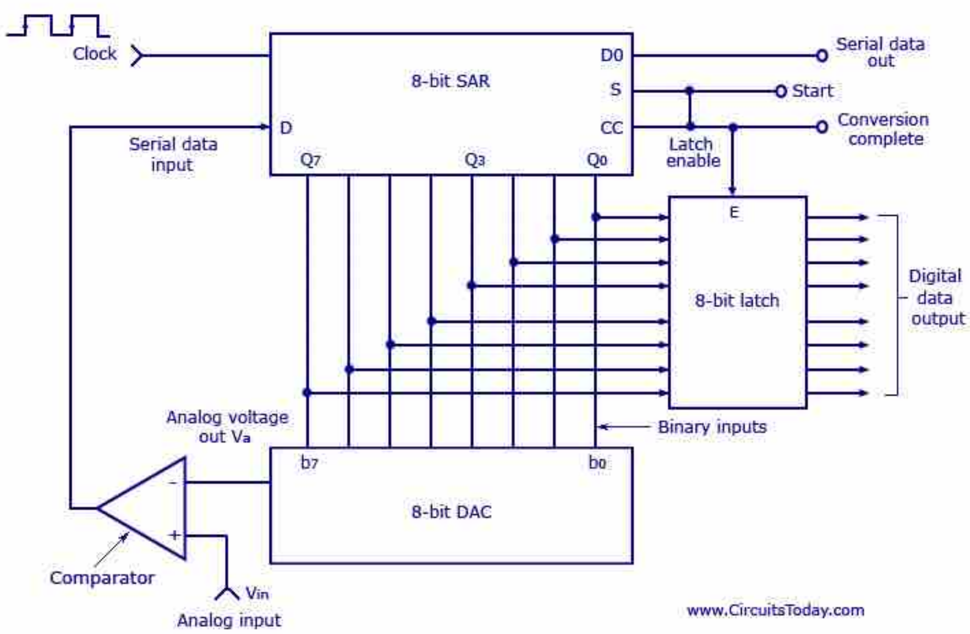
◆ Digital to Analog Converter (DAC or D/A)



◆ Analog to Digital Converter (ADC or A/D)

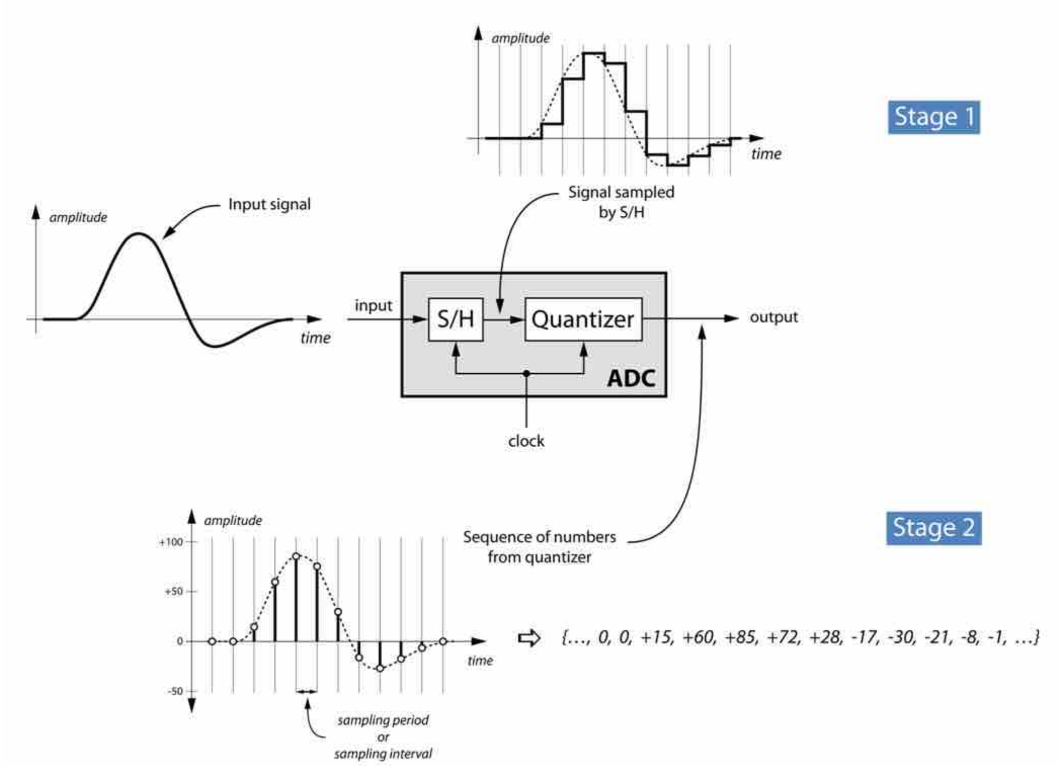


◆ Analog to Digital Converter (ADC or A/D)

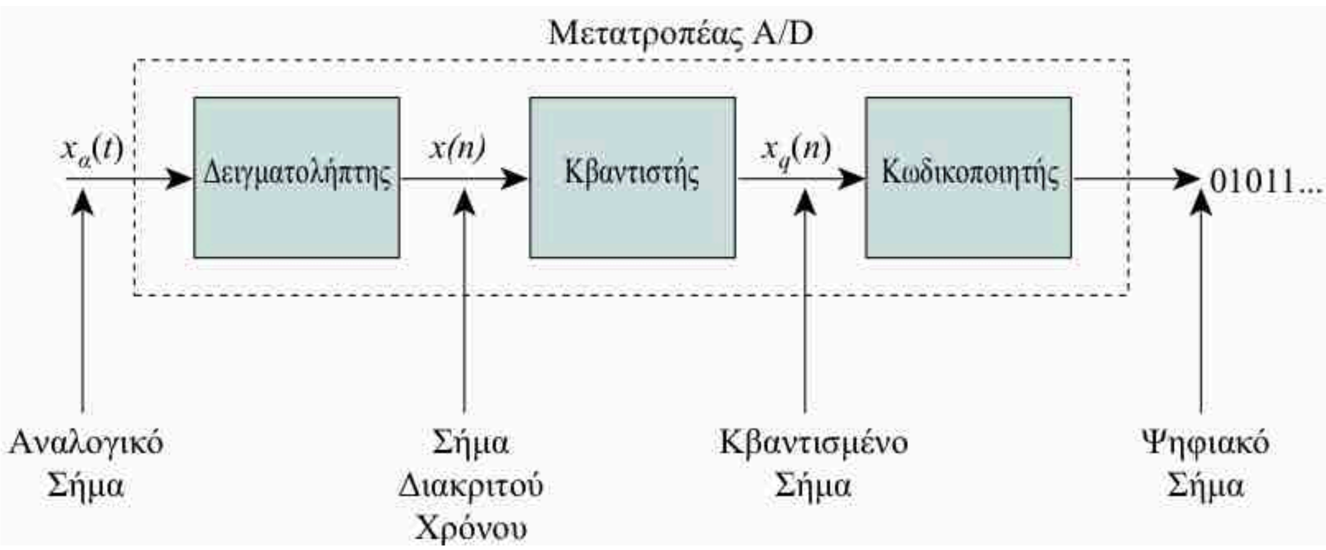


Successive Approximation Type Analog to Digital Converter

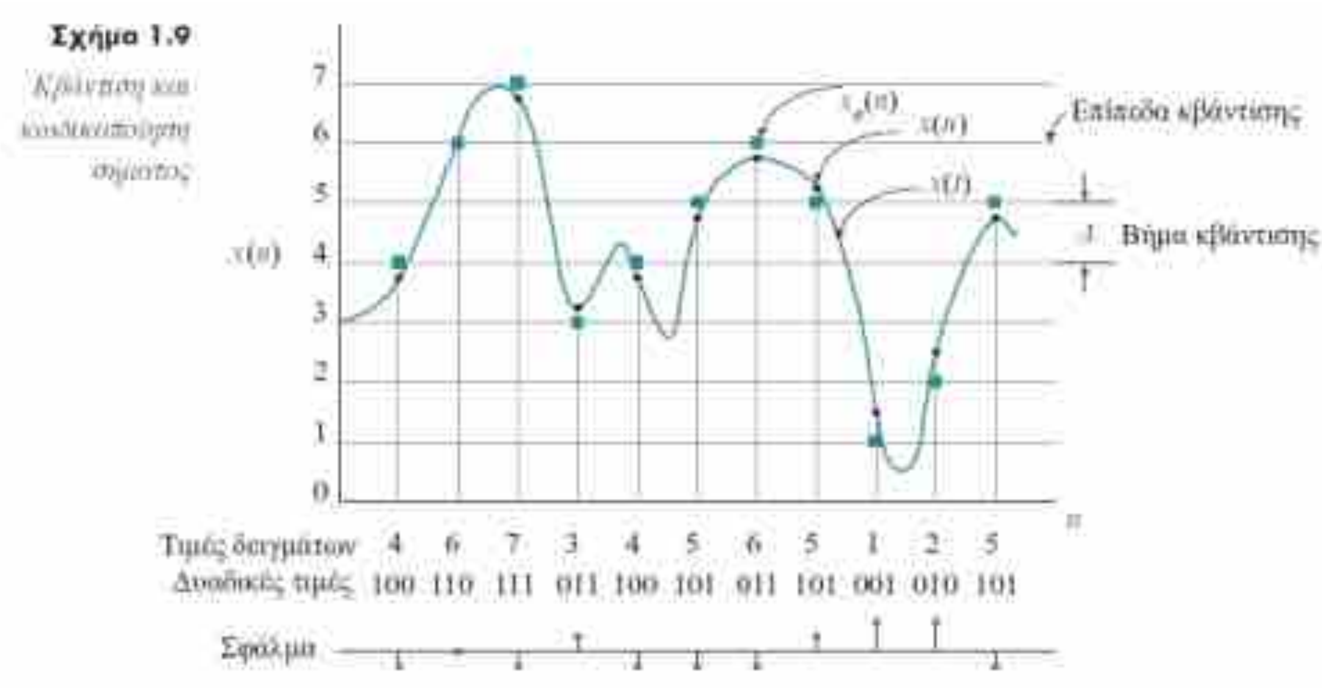
◆ Analog to Digital Conversion Stages



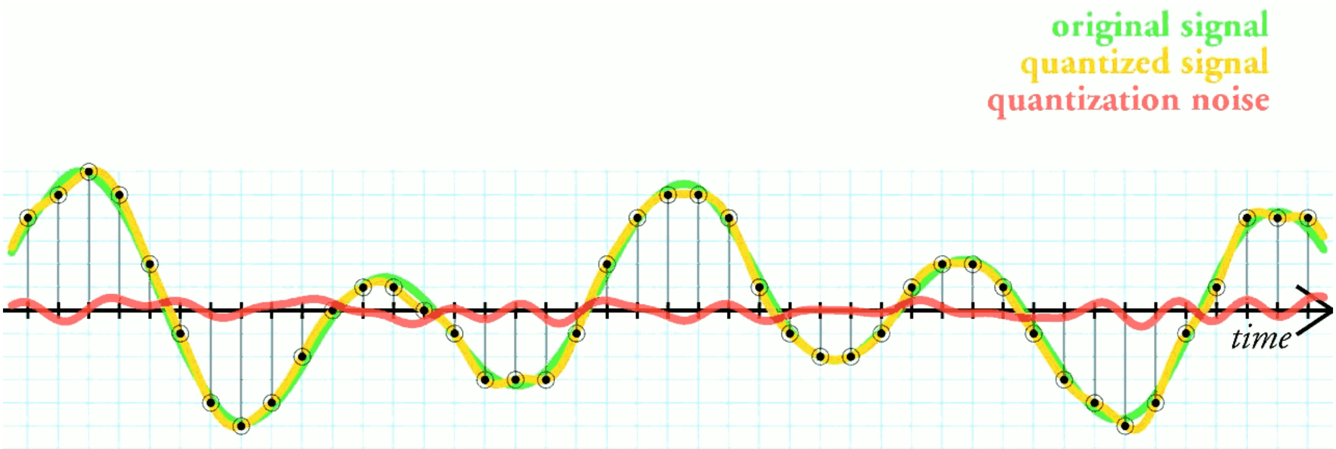
Quantization & Coding

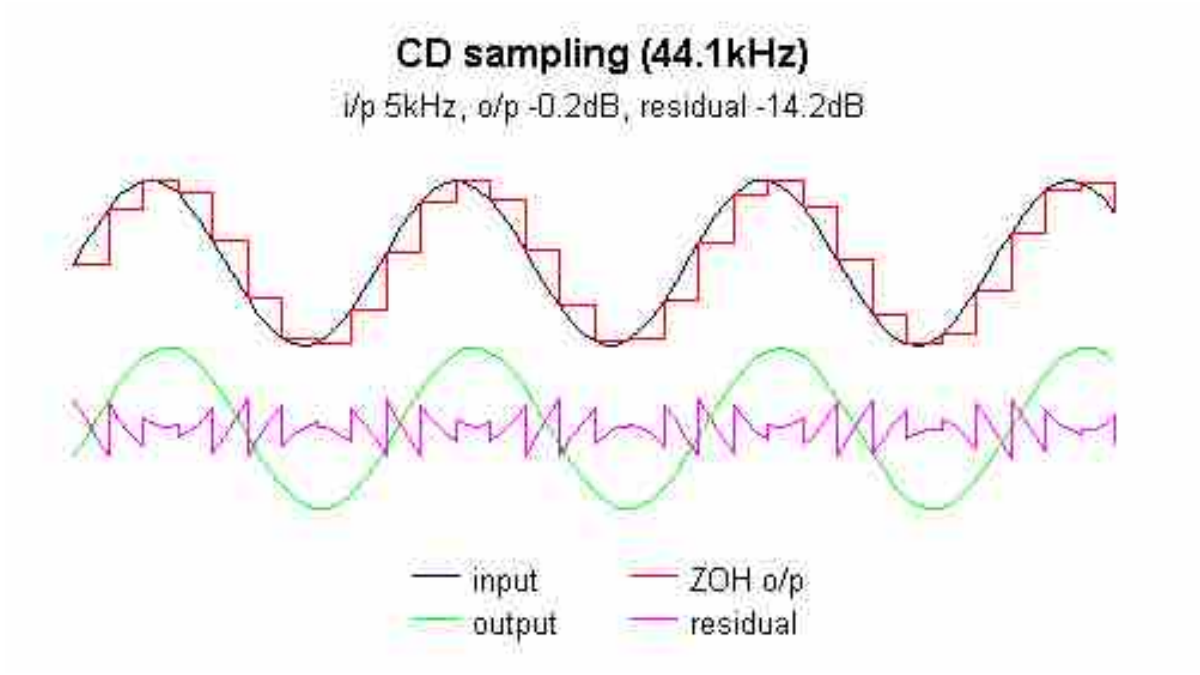


Quantization & Coding



Quantization & Coding





Quantization introduces an error which cannot be removed !

The level of the error is a function of the number of bits ADC, being approx. equal to ½ of an LSB.

Example: A 12-bit ADC with an input voltage range of $\pm 10V$ will have a LSB of $20/2^{12}$V, of 4.9mV and a quantization error of 2.45mV

For an ADC with ***b*** bits the number of quantization levels is 2^b, and the interval between the levels, that is the quantization step size ***q*** is

$$q = \frac{V_{fs}}{2^b - 1} \approx \frac{V_{fs}}{2^b}$$

where V_{fs} is the **f**ull-**s**cale range of the ADC with bipolar signal inputs.

Maximum quantization error for rounding is $\pm q/2$

For a sine wave input of amplitude A, the quantization step size becomes

$$q = \frac{2A}{2^b}$$

The quantization error for each sample **e** is assumed to be **random** and **uniformly** distributed in the interval $\pm q/2$ with **zero mean**. Thus, the quantization noise power of variance is given by

$$\sigma^2_e = \frac{q^2}{12}$$

For a sine wave input, the average signal power is $A^2/2$. The signal-to-quantization noise power ratio (SQNR), in decibels, is

$$SQNR = 10\log_{10}\left(\frac{\text{signal power}}{\text{noise power}}\right) =$$

$$= 10\log_{10}\left(\frac{A^2/2}{q^2/12}\right) =$$

$$= 6.02\text{b} + 1.76 \qquad \text{dB}$$

that is, SQNR increases by **6dB** for **each additional bit** in the wordlength.

Quantization & Coding

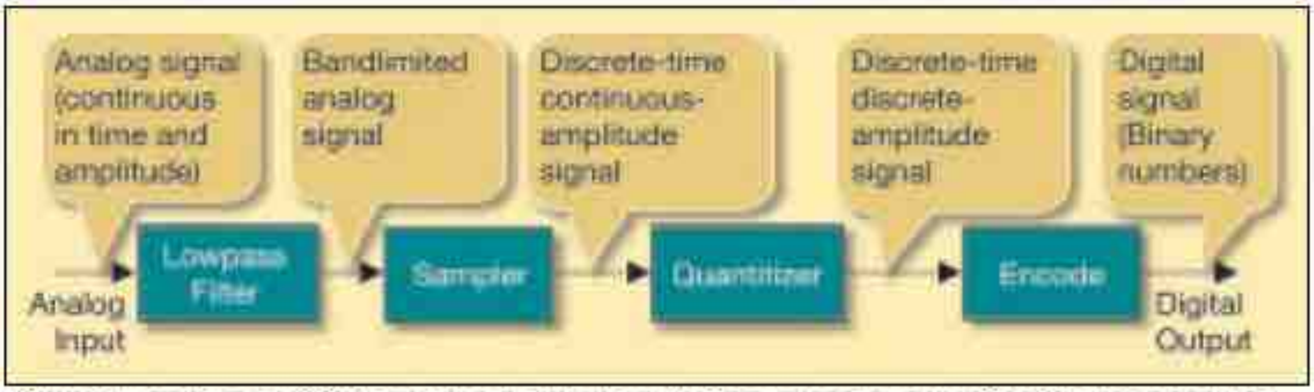


Figure 2. Analog to digital conversion involves filtering, sampling, quantization and encoding.

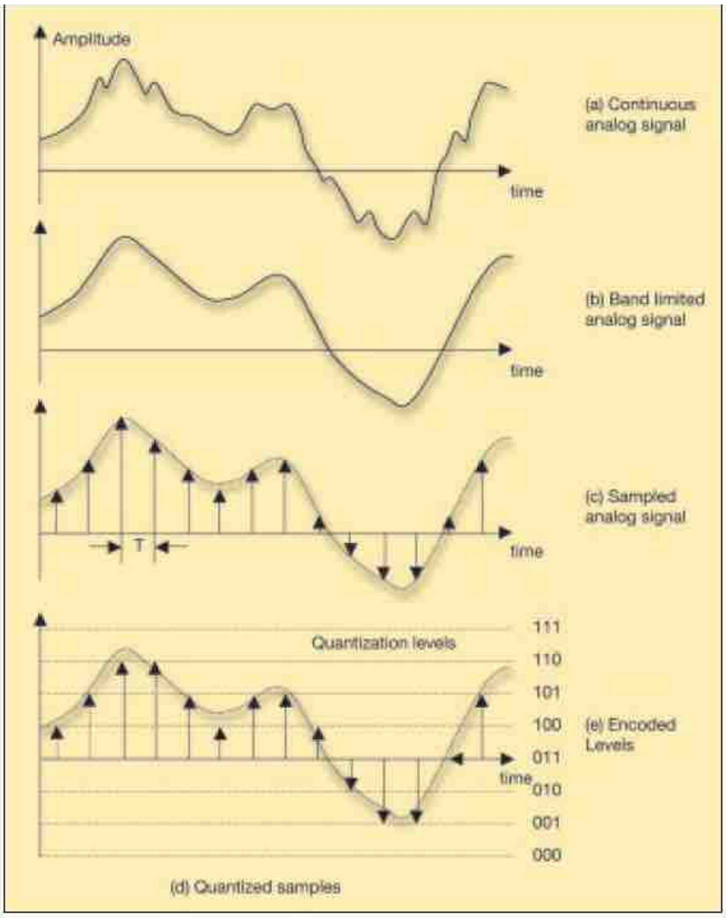


Figure 3. Signal waveforms at different stages of conversion.

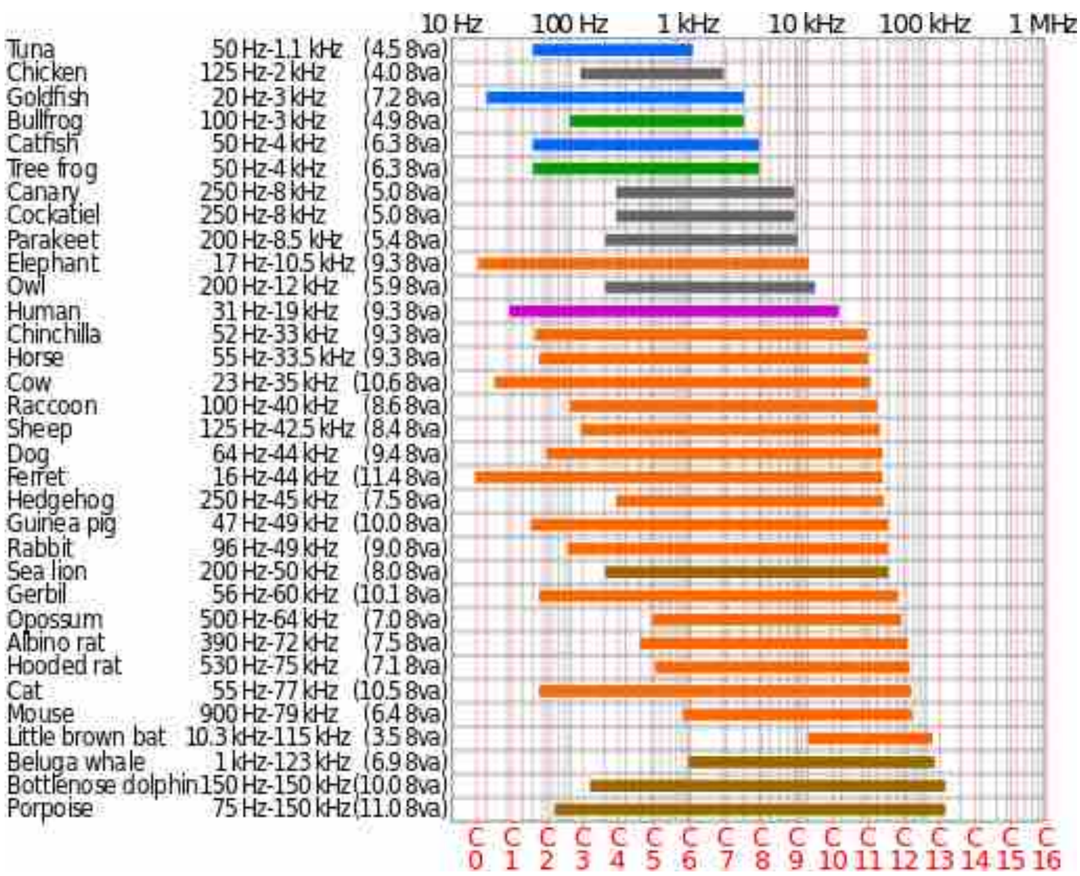
Appendix

FREQUENCY RANCES OF SOME BIOLOGICAL SIGNALS

Type of Signal	Frequency Range (Hz)
Electrororetinogram[a]	0-20
Electronystagmogram[b]	0-20
Pneumogram[c]	0-40
Electrocardiogram (ECG)	0-100
Electroencephalogram (EEG)	0-100
Electromyogram[d]	10-200
Sphygmomanogram[e]	0-200
Speech	100-4000

[a] A graphic recording of retina characteristics.
[b] A graphic recording of involuntary movement of the eyes.
[c] A graphic recording of respiratory activity.
[d] A graphic recording of muscular action, such as muscular contraction.
[e] A recording of blood pressure.

Appendix



FREQUENCY RANCES OF SOME SEISMIC SIGNALS

Type of Signal	Frequency Range (Hz)
Wind noise	100-1000
Seismic exploration signals	10-100
Earthquake and nuclear explosion signals	0.01-10
Seismic noise	0.1-1

FREQUENCY RANCES OF SOME ELECTROMAGNETIC SIGNALS

Type of Signal	Wavelength (m)	Frequency Range (Hz)
Radio broadcast	10^4-10^2	3×10^4-3×10^6
Shortwave radio signals	10^2-10^{-2}	3×10^6-3×10^{10}
Radar, satellite communications, space communications, common-carrier microwave	1-10^{-2}	3×10^8-3×10^{10}
Infrared	10^{-3}-10^{-6}	3×10^{11}-3×10^{14}
Visible light	3.9×10^{-7}-8.1×10^{-7}	3.7×10^{14}-7.7×10^{14}
Ultraviolet	10^{-7}-10^{-8}	3×10^{15}-3×10^{16}
Gamma rays and x-rays	10^{-9}-10^{-10}	3×10^{17}-3×10^{18}

The need for Decibels

Since one of the major uses of the frequency domain is to resolve small signals in the presence of large ones, let us now address the problem of how we can see both large and small signals on our display simultaneously.

Suppose we wish to measure a distortion component that is 0.1% of the signal. If we set the fundamental to full scale on a four inch (10cm) screen, the harmonic would be only four thousandths of an inch (0.1mm) tall. Obviously, we could barely see such a signal, much less measure it accurately. Yet many analyzers are available with the ability to measure signals even smaller than this.

Since we want to be able to see all the components easily at the same time, the only answer is to change our amplitude scale. A logarithmic scale would compress our large signal amplitude and expand the small ones, allowing all components to be displayed at the same time.

Appendix

Alexander Graham Bell discovered that the human ear responded logarithmically to power difference and invented a unit, the Bel, to help him measure the ability of people to hear. One tenth of a Bel, the deciBel (dB) is the most common unit used in the frequency domain today. A table of the relationship between volts, power and dB is given in Figure A. From the table we can see that our 0.1% distortion component example is 60 dB below the fundamental. If we had an 80 dB display as in Figure B, the distortion component would occupy ¼ of the screen, not 1/1000 as in a linear display.

Appendix

dB	Power Ratio	dB	Voltage Ratio
+ 20	100	+ 40	100
+ 10	10	+ 20	10
+ 3	2	+ 6	2
0	1	0	1
− 3	1 / 2	− 6	1 / 2
− 10	1 / 10	− 20	1 / 10
− 20	1 / 100	− 40	1 / 100

dB=10 log (Power Ratio)=20 log (Voltage Ratio)

Figure A
The relationship between decibels, power and voltage

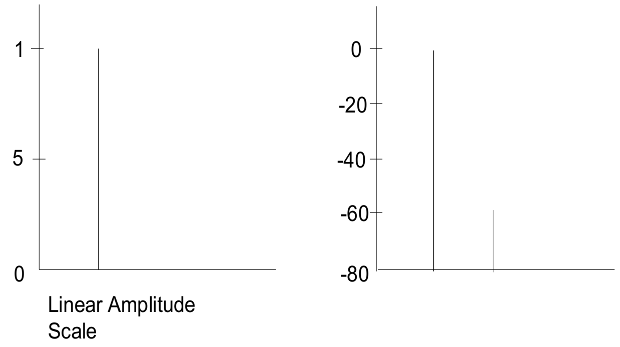


Figure B
Small signals can be measured with a logarithmic amplitude scale