Differences between strict-sense-stationary and wide-sense-stationary random processes

ChatGPT said:

The concepts of **stationary** and **wide-sense stationary (WSS)** random processes are important in signal processing and stochastic processes. Here's a breakdown of their differences:

1. Stationary Random Process (Strict-Sense Stationarity - SSS)

A random process is **strict-sense stationary** if its statistical properties are invariant with respect to time. This means that for any set of time shifts, the joint probability distribution of the process at different time points remains unchanged. Formally, a random process $X(t)$X(t) is stationary if, for any set of time points $t_1, t_2, \ldots, t_n$ and any time shift τ:

$$F_{X(t_1),X(t_2),\ldots,X(t_n)} = F_{X(t_1+\tau),X(t_2+\tau),\ldots,X(t_n+\tau)}$$

where F represents the joint cumulative distribution function (CDF) of the random variables at those times.

In simpler terms, strict-sense stationarity requires **all** statistical moments (mean, variance, higher moments) and the entire distribution of the process to remain constant over time. This is a very strong condition and often difficult to meet in real-world applications.

2. Wide-Sense Stationary (WSS) Random Process

A random process is **wide-sense stationary (WSS)** (also called **second-order stationary** or **weakly stationary**) if only its **mean**, **variance**, and **autocovariance** function are invariant with respect to time. In other words, WSS only requires conditions on the first two moments:

- The **mean** $\mu_X(t)$ is constant:

$$\mathbb{E}[X(t)] = \mu, \text{ for all } t$$

- The **variance** (or autocovariance at zero lag) is also constant:

$$\mathbb{E}[(X(t) - \mu)^2] = \sigma^2, \text{ for all } t$$

- The **autocovariance** function $R_X(t_1, t_2)$ depends only on the time difference $\tau = t_2 - t_1$ (it is a function of lag τ, not absolute time):

$$R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau)$$

WSS is a much weaker and more practical condition than strict-sense stationarity because it only requires the process to have time-invariant first and second moments. Many physical processes (such as most real-world signals) are modeled as WSS even if they are not strictly stationary.

Key Differences

- **Strict-Sense Stationarity (SSS)** requires the joint distribution of the process at any time points to be identical when shifted in time. All moments (mean, variance, higher-order moments) and the distribution must be invariant.

- **Wide-Sense Stationarity (WSS)** only requires the first two moments (mean, variance, and autocovariance) to be time-invariant. This makes WSS a more relaxed and often more applicable condition for real-world processes.

Practical Example

Consider a sinusoidal signal $X(t) = A\cos(\omega t + \phi)$, where ϕ is a random phase uniformly distributed between 0 and 2π:

- This signal is **wide-sense stationary** because its mean and autocovariance are time-invariant.
- However, it is not **strict-sense stationary** because the distribution of the process (including higher moments) changes over time.

In summary, **stationary processes** are more stringent, while **wide-sense stationary processes** are more commonly used in practice since many engineering applications, like in signal processing, focus primarily on mean and covariance rather than higher-order statistics.