ΣΣΕ ΙΙ — 9 Δεκεμβρίου 2011 ①

Αυτοσυσχέτιση / Συνδιακύμανση

$X(t) = A\cos(\omega t + \phi)$

$E[X(t)] = 0$

$$R_{XX}(t_1, t_2) = \frac{A^2}{2}\cos\omega(t_1 - t_2)$$

$R_{XX}(0) = E[X^2(t)] \leftarrow$ μέση ισχύ σε σήματα ισχύος

$R_{XX}(\tau) = \mu^2$ για $\tau \to$ μεγάλο

$\boxed{\text{~~Φίλτρα~~}}$

$$C_{XX}(t_1, t_2) = E\{[X(t_1) - \mu_X(t_1)][X(t_2) - \mu_X(t_2)]\}$$

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2)$$

$$\rho(t_1, t_2) = \frac{C(t_1, t_2)}{\sqrt{C(t_1, t_1)\,C(t_2, t_2)}}$$

$\boxed{\text{Φίλτρα ΣΔ}}$ || $\boxed{\text{ΔΣΡΣ ΙΙ}}$

Φίλτρα → $\boxed{\text{Διάσπορες}}$

Δύο κατηγορίες

ΜΙ

- στατικές
- δυναμικές

σελίς 11

α) Αυστηρώς ΣΔ

b) στατικές με την ευρεία έννοια

σελ. 13

$R_X(t_1,t_2)=R_X(t_2-t_1)$

Αυτοσυνδιακύμανση

$C_X(t_1,t_2)=E[(X(t_1)-m_X)(X(t_2)-m_X)]$

$= R_X(t_2-t_1)-m_X^2$

Ιδιότητες συνάρτησης $R_X(t_1,t_2)$ σελίς 15

σελίς 289

+ Normal _ Gaussian

κανονικές ΣΔ σελίς 291

Εργοδικές ΣΔ - κεφ 10

σελίδες 16 →

Autocorrelation Function of Stationary Random Processes

If a random process is wide sense stationary, then by the definition of stationarity

$$R_{XX}(t_1 + \tau, t_1) = R_{XX}(t_1 + t + \tau, t_1 + t) = R_{XX}(t_2 + \tau, t_2). \quad \forall t_1, t_2. \tag{11-17}$$

Thus R_{XX} for stationary processes depends only on τ, the time difference, and we will write $R_{XX}(t_1 + \tau, t_1)$ as $R_{XX}(\tau)$. We now show some of the properties of $R_{XX}(\tau)$.

$$R_{XX}(\tau) = R_{XX}(-\tau). \tag{11-18}$$

This follows from letting $t_2 = t_1 - \tau$ in (11-17) and recalling that $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = E[X(t_2)X(t_1)] = R_{XX}(t_2, t_1)$.

From (11-17) with $\tau = 0$

$$R_{XX}(0) = E[X^2(t)]. \tag{11-19}$$

Thus
$$R_{XX}(0) \geq 0.$$

Also
$$R_{XX}(0) \geq |R_{XX}(\tau)|. \tag{11-20}$$

This last result follows from the same argument made earlier (Chapter V) to show that the correlation coefficient was bounded by ± 1.

If $X(t_1)$ and $X(t_1 + \tau)$ are independent

$$R_{XX}(\tau) = \mu^2. \tag{11-21}$$

In many practical cases $X(t_1)$ and $X(t_1 + \tau)$ will be independent for large values of τ.

EXAMPLE 11-11

Why are the functions shown in Figure 11-11 not autocorrelation functions?

(a) Is not symmetrical and thus violates (11-18).
(b) Violates (11-20).

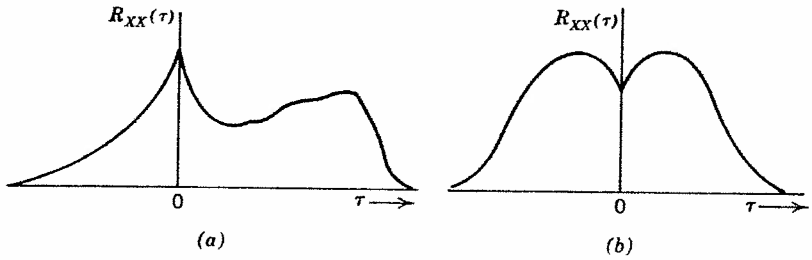


Figure 11-11

Now $E[Y_k Y_m] = E[Y_k]E[Y_m]$ if $k \neq m$ due to independence. Thus

$$\begin{aligned} R_{XX}(i,j) &= iE[Y_k^2] + (i^2 - i)[E(Y_k)]^2 + iE[Y_k](j-i)E[Y_m] \\ &= i + (i^2 - i)(p-q)^2 + i(j-i)(p-q)^2 \\ &= i + (ij - i)(p-q)^2, \qquad i \leq j. \end{aligned}$$

If $j \leq i$ then

$$R_{XX}(i,j) = j + (ij - j)(p-q)^2, \qquad j \leq i.$$

If $p = q$ (fair coin) then

$$R_{XX}(i,j) = i, \qquad i \leq j.$$

The autocorrelation function for the case where t_i and/or t_j are not integers is easily derived from the above results.

EXAMPLE 11-8

Find the mean and autocorrelation of the random process given in Example 11-3. The mean $\mu(t)$ is

$$\begin{aligned} \mu(t) &= E[A\cos(\omega t + \phi)] \\ &= \int_{-\pi}^{\pi} A\cos(\omega t + \theta)\frac{1}{2\pi}\,d\theta \\ &= \frac{A}{2\pi}[\sin(\omega t + \theta)]\Big|_{-\pi}^{\pi} \\ &= 0. \end{aligned}$$

The autocorrelation function is

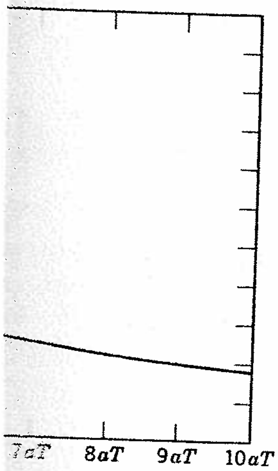
$$\begin{aligned} R_{XX}(t_1, t_2) &= E[A\cos(\omega t_1 + \phi)A\cos(\omega t_2 + \phi)] \\ &= \int_{-\pi}^{\pi} A^2\cos(\omega t_1 + \theta)\cos(\omega t_2 + \theta)\frac{1}{2\pi}\,d\theta \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi}\{\tfrac{1}{2}\cos(\omega t_1 + \omega t_2 + 2\theta) + \tfrac{1}{2}\cos[\omega(t_1 - t_2)]\}\,d\theta \\ &= 0 + \frac{A^2}{2}\cos\omega(t_1 - t_2). \end{aligned}$$

EXAMPLE 11-9

Find the mean and autocorrelation function for the Poisson process. The expected value of $X(t)$ is

$$E[X(t)] = \mu t.$$

g produces the same mean



Ergodicity

The basic idea of ergodicity is that expected values can be replaced by time averages. This concept seems natural when "noise" is considered. In this case randomness is present not only in the selection of the sample function, but also in how the function behaves in time. The thermal noise of a resistor is random not because of the choice of a resistor, but because of the uncertainty associated with the collisions of the "free" electrons in the metal. The randomness of shot noise similarly is due to the random behavior of the electrons in the filament and one might expect that time averages of the output noise of one tube would be the same as averaging over the instantaneous output of many tubes of the same type operating under equivalent conditions.

Such ideas have led to a loose definition of ergodicity. A random process is ergodic if (with probability one)

$$E\{g[X(t)]\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} g[X(t)]\,dt \tag{11-33}$$

for all functions g.

For our purposes the most interesting functions g are

$$g[X(t)] = X(t)$$

and

$$g[X(t)] = X(t)X(\tau + t).$$

We now examine what is necessary in order for the expected value μ to equal the time average η.

$$\overline{X(t)} \qquad \boxed{\eta = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\,dt.}$$

Note that if the time ~~average η exists, this~~ average must not be a function of time. Also note that in general η will depend on which sample function is chosen.

Thus if the expected value μ is to equal η then μ must be constant, that is, $X(t)$ must be stationary in the mean. Moreover μ is not a function of the sample space, that is, we have averaged over the sample space. Thus η must not depend on the sample space, that is, η must be the same (with probability one) for all choices of sample functions.

If ~~$\mu = \eta$ then we say that $X(t)$ is~~ ergodic in the mean? We have seen that stationarity of the mean is necessary for ergodicity in the mean. We have also seen that η must be a constant ~~over the~~ sample space. Thus if η is a constant this will imply that $\boxed{\sigma_\eta^2 = 0}$. Similarly if $\sigma_\eta^2 = 0$, then by Tchebycheff's inequality η is a constant. Thus a necessary and sufficient condition for stationarity of the mean is that $\sigma_\eta^2 = 0$.

~~The same ideas~~ are useful in determining if a random process is ergodic in the autocorrelation function. The process must be wide sense stationary. In addition the average over the sample space $R_{XX}(\tau)$ must equal the time average autocorrelation function $\mathscr{R}_{XX}(\tau)$ where

$$\mathscr{R}_{XX}(\tau) = \lim_{T\to\infty} \mathscr{R}_T(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)\,dt.$$

In order for $R_{XX}(\tau)$ to be the same as $\mathscr{R}_{XX}(\tau)$ it is, in addition to wide sense stationarity, necessary that (with probability one) $\mathscr{R}_{XX}(\tau)$ does not depend on which sample function is chosen.

In practice it is usually impossible to decide on the basis of data if a random process is ergodic. One has to decide based on reasoning about the situation. In order to be ergodic a random process must be stationary, and "randomness" must be evident in the time variation as well as in the selection of a sample function. In addition the time average must not depend on which sample function is selected.

EXAMPLE 11-16

Is the process of Examples 11-3 and 11-8 ergodic in the mean and in the autocorrelation function?

We see from Example 11-14 that $\lim_{T\to\infty} \eta(0, T) = 0$. Thus the mean (see Example 11-8) and the time average agree. We now find the time average of $X(t)X(t+\tau)$.

$$\begin{aligned}\mathscr{R}_T(\tau) &= \frac{1}{2T}\int_{-T}^{T} A\cos(\omega t+\phi)A\cos(\omega t+\omega\tau+\phi)\,dt\\ &= \frac{A^2}{4T}\int_{-T}^{T}[\cos(2\omega t+\tau+2\phi)+\cos(\omega\tau)]\,dt\\ &= \frac{A^2}{4T}\left[\frac{1}{2\omega}\sin(2\omega T+\tau+2\phi)\right.\\ &\qquad \left.-\frac{1}{2\omega}\sin(-2\omega T+\tau+2\phi)\right]+\frac{A^2}{2}\cos\omega\tau.\end{aligned}$$

Now $\lim_{T\to\infty} \mathscr{R}_T(\tau) = (A^2/2)\cos\omega\tau$. Note that as $T\to\infty$, the time average does not depend on the sample function selected, that is, ϕ is not in the final answer, and the time average autocorrelation function agrees with $R_{XX}(\tau)$. Thus this example is ergodic in mean and in autocorrelation.

11-7 POWER DENSITY SPECTRUM

If the autocorrelation function $R(\tau)$ of a stationary random process is such that

$$\int_{-\infty}^{\infty} |R(\tau)|\,d\tau$$