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A foundation in Digital Communication

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Το προσαρμοσμένο φίλτρο
The matched filter

5.7 Filters

A **filter of impulse response h** is a physical device that when fed the input waveform $\mathbf{x}$ produces the output waveform $\mathbf{h} \star \mathbf{x}$. The impulse response $\mathbf{h}$ is assumed to be a real or complex signal, and it is tacitly assumed that we only feed the device with inputs $\mathbf{x}$ for which the convolution $\mathbf{x} \star \mathbf{h}$ is defined.[2]

Definition 5.7.1 (Stable Filter). *A filter is said to be **stable** if its impulse response is integrable.*

Stable filters are also called **bounded-input/bounded-output stable** or **BIBO stable**, because, as the next proposition shows, if such filters are fed a bounded signal, then their output is also a bounded signal.

Proposition 5.7.2 (BIBO Stability). *If $\mathbf{h}$ is integrable and if $\mathbf{x}$ is a bounded Lebesgue measurable signal, then the signal $\mathbf{x} \star \mathbf{h}$ is also bounded.*

Proof. If the impulse response $\mathbf{h}$ is integrable, and if the input $\mathbf{x}$ is bounded by some constant σ_∞, then (5.8a) and (5.8b) are both satisfied, and the boundedness of the output then follows from (5.8c). □

Definition 5.7.3 (Causal Filter). *A filter of impulse response $\mathbf{h}$ is said to be **causal** or **nonanticipative** if $\mathbf{h}$ is zero at negative times, i.e., if*

$$h(t) = 0, \quad t < 0. \tag{5.13}$$

Causal filters play an important role in engineering because (5.13) guarantees that the present filter output be computable from the past filter inputs. Indeed, the time-t filter output can be expressed in the form

$$\begin{aligned}(\mathbf{x} \star \mathbf{h})(t) &= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, \mathrm{d}\tau \\ &= \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, \mathrm{d}\tau, \quad \mathbf{h} \text{ causal},\end{aligned}$$

where the calculation of the latter integral only requires knowledge of $x(\tau)$ for $\tau < t$. Here the first equality follows from the definition of the convolution (5.2), and the second equality follows from (5.13).

5.8 The Matched Filter

In Digital Communications inner products are often computed using a **matched filter**. In its definition we shall use the notation (5.1).

[2]This definition of a filter is reminiscent of the concept of a "linear time invariant system." Note, however, that since we do not deal with Dirac's Delta in this book, our definition is more restrictive. For example, a device that produces at its output a waveform that is identical to its input is excluded from our discussion here because we do not allow $\mathbf{h}$ to be Dirac's Delta.

Definition 5.8.1 (The Matched Filter). *The* ***matched filter*** *for the signal* $\boldsymbol{\phi}$ *is a filter whose impulse response is* $\overleftarrow{\boldsymbol{\phi}}^*$*, i.e., the mapping*

$$t \mapsto \phi^*(-t). \tag{5.14}$$

The main use of the matched filter is for computing inner products:

Theorem 5.8.2 (Computing Inner Products with a Matched Filter). *The inner product* $\langle \mathbf{u}, \boldsymbol{\phi} \rangle$ *between the energy-limited signals* $\mathbf{u}$ *and* $\boldsymbol{\phi}$ *is given by the output at time* $t = 0$ *of a matched filter for* $\boldsymbol{\phi}$ *that is fed* $\mathbf{u}$*:*

$$\boxed{\langle \mathbf{u}, \boldsymbol{\phi} \rangle = \big(\mathbf{u} \star \overleftarrow{\boldsymbol{\phi}}^*\big)(0), \quad \mathbf{u}, \boldsymbol{\phi} \in \mathcal{L}_2.} \tag{5.15}$$

More generally, if $\mathbf{g}\colon t \mapsto \phi(t - t_0)$*, then* $\langle \mathbf{u}, \mathbf{g} \rangle$ *is the time-*t_0 *output corresponding to feeding the waveform* $\mathbf{u}$ *to the matched filter for* $\boldsymbol{\phi}$*:*

$$\int_{-\infty}^{\infty} u(t)\, \phi^*(t - t_0)\, \mathrm{d}t = \big(\mathbf{u} \star \overleftarrow{\boldsymbol{\phi}}^*\big)(t_0). \tag{5.16}$$

Proof. We shall prove the second part of the theorem, i.e., (5.16); the first follows from the second by setting $t_0 = 0$. We express the time-t_0 output of the matched filter as:

$$\begin{aligned} \big(\mathbf{u} \star \overleftarrow{\boldsymbol{\phi}}^*\big)(t_0) &= \int_{-\infty}^{\infty} u(\tau)\, \overleftarrow{\phi}^*(t_0 - \tau)\, \mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} u(\tau)\, \phi^*(\tau - t_0)\, \mathrm{d}\tau, \end{aligned}$$

where the first equality follows from the definition of convolution (5.2) and the second from the definition of $\overleftarrow{\boldsymbol{\phi}}^*$ as the conjugated mirror image of $\boldsymbol{\phi}$. □

From the above theorem we see that if we wish to compute, say, the three inner products $\langle \mathbf{u}, \mathbf{g}_1 \rangle$, $\langle \mathbf{u}, \mathbf{g}_2 \rangle$, and $\langle \mathbf{u}, \mathbf{g}_3 \rangle$ in the very special case where the functions $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ are all time shifts of the same waveform $\boldsymbol{\phi}$, i.e., when $\mathbf{g}_1\colon t \mapsto \phi(t - t_1)$, $\mathbf{g}_2\colon t \mapsto \phi(t - t_2)$, and $\mathbf{g}_3\colon t \mapsto \phi(t - t_3)$, then we need only one filter, namely, the matched filter for $\boldsymbol{\phi}$. Indeed, we can feed $\mathbf{u}$ to the matched filter for $\boldsymbol{\phi}$ and the inner products $\langle \mathbf{u}, \mathbf{g}_1 \rangle$, $\langle \mathbf{u}, \mathbf{g}_2 \rangle$, and $\langle \mathbf{u}, \mathbf{g}_3 \rangle$ simply correspond to the filter's outputs at times t_1, t_2, and t_3. One circuit computes all three inner products. This is so exciting that it is worth repeating:

Corollary 5.8.3 (Computing Many Inner Products using One Filter). *If the energy-limited signals* $\{\mathbf{g}_j\}_{j=1}^{\mathrm{J}}$ *are all time shifts of the same signal* $\boldsymbol{\phi}$ *in the sense that*

$$\mathbf{g}_j\colon t \mapsto \phi(t - t_j), \quad j = 1, \dots, \mathrm{J},$$

and if $\mathbf{u}$ *is any energy-limited signal, then all* J *inner products*

$$\langle \mathbf{u}, \mathbf{g}_j \rangle, \quad j = 1, \dots, \mathrm{J}$$

can be computed using one filter by feeding $\mathbf{u}$ *to a matched filter for* ϕ *and sampling the output at the appropriate times* $t_1, \ldots, t_J$*:*

$$\langle \mathbf{u}, \mathbf{g}_j \rangle = \big(\mathbf{u} \star \overleftarrow{\phi}^*\big)(t_j), \quad j = 1, \ldots, J. \tag{5.17}$$

5.9 The Ideal Unit-Gain Lowpass Filter

The impulse response of the **ideal unit-gain lowpass filter** of cutoff frequency W_c is denoted by $\mathrm{LPF}_{W_c}(\cdot)$ and is given for every $W_c > 0$ by[3]

$$\mathrm{LPF}_{W_c}(t) \triangleq \begin{cases} 2W_c \frac{\sin(2\pi W_c t)}{2\pi W_c t} & \text{if } t \neq 0, \\ 2W_c & \text{if } t = 0, \end{cases} \quad t \in \mathbb{R}. \tag{5.18}$$

This can be alternatively written as

$$\mathrm{LPF}_{W_c}(t) = 2W_c \operatorname{sinc}(2W_c t), \quad t \in \mathbb{R}, \tag{5.19}$$

where the function $\operatorname{sinc}(\cdot)$ is defined by[4]

$$\operatorname{sinc}(\xi) \triangleq \begin{cases} \frac{\sin(\pi\xi)}{\pi\xi} & \text{if } \xi \neq 0, \\ 1 & \text{if } \xi = 0, \end{cases} \quad \xi \in \mathbb{R}. \tag{5.20}$$

Notice that the definition of $\operatorname{sinc}(0)$ as being 1 makes sense because, for very small (but nonzero) values of ξ the value of $\sin(\xi)/\xi$ is approximately 1. In fact, with this definition at zero the function is not only continuous at zero but also infinitely differentiable there. Indeed, the function from $\mathbb{C}$ to $\mathbb{C}$

$$z \mapsto \begin{cases} \frac{\sin(\pi z)}{\pi z} & \text{if } z \neq 0, \\ 1 & \text{otherwise}, \end{cases}$$

is an entire function, i.e., an analytic function throughout the complex plane.

The importance of the ideal unit-gain lowpass filter will become clearer when we discuss the filter's frequency response in Section 6.3. It is thus named because the Fourier Transform of $\mathrm{LPF}_{W_c}(\cdot)$ is equal to 1 (hence "unit gain"), whenever $|f| \leq W_c$, and is equal to zero, whenever $|f| > W_c$. See (6.38) ahead.

From a mathematical point of view, working with the ideal unit-gain lowpass filter is tricky because the impulse response (5.18) is not an integrable function. (It decays like $1/t$, which does not have a finite integral from $t = 1$ to $t = \infty$.) This filter is thus not a stable filter. We shall revisit this issue in Section 6.4. Note, however, that the impulse response (5.18) is of finite energy. (The square of the impulse response decays like $1/t^2$ which does have a finite integral from one to infinity.) Consequently, the result of feeding an energy-limited signal to the ideal unit-gain lowpass filter is always well-defined.

Note also that the ideal unit-gain lowpass filter is not causal.

[3] For convenience we define the impulse response of the ideal unit-gain lowpass filter of cutoff frequency zero as the all zero signal. This is in agreement with (5.19).

[4] Some texts omit the π's in (5.20) and define the $\operatorname{sinc}(\cdot)$ function as $\sin(\xi)/\xi$ for $\xi \neq 0$.