A Simple Model for the Calculation of the Fluid Discharge from a Small Orifice

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A Simple Model for the Calculation of the Fluid Discharge from a Small Orifice

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One of the easily accessible results in elementary fluid mechanics is the so-called Torricelli's theorem (or law), which states that the velocity U_{th} of the fluid exiting from an orifice at depth h from the free surface of a container filled with fluid, is the same as the velocity of a free-falling body from rest over a height h. This simple result can be verified experimentally by several methods, e.g., by measuring the parabolic path of the free jet exiting from the container and back-calculating its exit velocity.[1] It turns out that the ratio of the experimental velocity, U_{exp}, to the theoretical velocity, U_{th}, called the velocity coefficient C_u, varies from 0.95 to 0.99.[1] Yet, if one was to calculate the outflow volumetric flowrate (or discharge or rate of efflux) from the container by multiplying the Torricelli velocity by the cross-section area of the orifice, and compare it to the flowrate derived by dividing the volume of the bucket into which the jet flowed by the time it took to fill the bucket, one would get a large discrepancy, which, in view of the value of C_u, cannot be attributed to the deviation of the experimental velocity from the Torricelli value. This result seems paradoxical. It offers, therefore, an excellent opportunity to clarify important details in the application of simple fluid mechanics equations, and in addition to introduce at an early stage the concept of producing an approximate result through flow modeling, as we shall see below.

Torricelli's theorem and the trajectory of a jet that issues from an orifice in the side of a container have a long history in introductory physics and fluid mechanics.[2,3] This theorem can be derived from Bernoulli's equation along a streamline, under the assumption of steady, incompressible frictionless flow,[4] and can be written as

$$U_{th} = \sqrt{2gh}, \tag{1}$$

where g is the measure of the acceleration of gravity and h the distance of the orifice from the free surface (Fig. 1). For the flow to be steady, the container should be considered infinite in dimensions; alternatively, its cross section could be considered to be very large with respect to the cross section of the orifice, so that the flow be quasi-steady. To quantify the above mentioned discrepancy, let us define the coefficient of discharge[5] C_d as the ratio of the experimental value of the volumetric discharge (referred to henceforth as simply discharge) Q_e to the theoretical value of the discharge defined as $Q_{th} = U_{th}\,A$, where A is the area of the orifice:

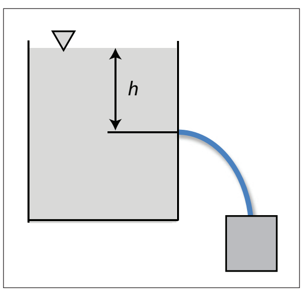

Fig. 1. Outflow of a fluid jet issuing from an orifice on the side of a container filled with fluid. The jet is collected into a bucket.

$$C_d = Q_e/Q_{th}. \tag{2}$$

Based on experimental measurements, the value of C_d has been found to be about 0.61,[6] that is, the measured discharge is only about 60% of the theoretical value.

The source of the discrepancy is very lucidly described in Lamb[7]: "its motion [i.e., the motion of the fluid in the container approaching the orifice] is not, therefore, throughout the area of the orifice, everywhere perpendicular to this area [i.e., the area of the orifice] but becomes more and more oblique as we pass from the center to the sides." As a result, although the *measure* of the velocity is constant and equal to the Torricelli value across the entire area of the orifice (provided the opening of the orifice is small with respect to the height h), its *direction* is not perpendicular to this area. To put it in terms of the equations of introductory fluid mechanics, in the equation of continuity:

$$Q = \bar{U} A \tag{3}$$

The mean velocity $\bar{U}$ is defined to be perpendicular to the area A. Thus, the mean value of velocity in the cross section of orifice cannot be equal to the Torricelli value because, although the velocity at each point of the cross section is constant in measure and equal to the Torricelli value, it is not perpendicular to it, except near its center.

The problem of calculating analytically the exact value of the inviscid discharge exiting from a slot at the side of a container has been solved by Kirchhoff[8,9] using the technique of conformal mapping. Since, however, this technique presupposes complex analysis, which is outside the reach of incoming college students, we need a simpler model, be it approximate, in order to materialize in mathematical terms the above qualitative explanation. This we may attempt in the spirit of the early approximate methods in fluid mechanics termed integral methods. The best example of such methods is the approximate boundary layer calculations using the Karman-Pohlhausen integral.[10] The essence of the method is to assume a "reasonable" shape of the velocity profile within the boundary layer, i.e., a profile satisfying at least the appropriate boundary conditions, and subsequently to substitute this profile into the pertinent equations and extract by integration the required information, such as the boundary layer growth, the shear stress at the boundary, etc.

What we need here then is, extending the scope of the integral methods, an assumption about how the angle between the velocity vector and the horizontal direction varies along the cross section of the slot. The simplest reasonable requirement, depicted in Fig. 2, is that the velocity be symmetrical around its center, perpendicular to the cross section at the center of the slot, parallel to it at its edge, and vary linearly in between.

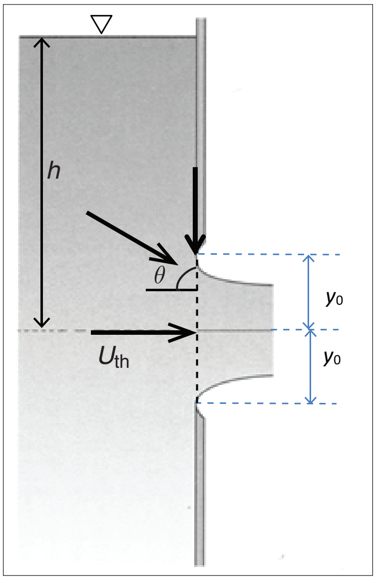

Fig. 2. Angle of the velocity exiting a rectangular slot of height $2y_0$ and width *b*. The slot opening has been highly exaggerated with respect to the height of the container *h*; in fact, the opening $2y_0$ should be very much smaller than *h*.

This means that

$$\theta(y)=\frac{\pi}{2}\frac{y}{y_0}, \quad (4)$$

where the zero of the y-axis is in the middle of the slot and y_0 is its half-width.

With this assumption, the component of the velocity perpendicular to the orifice is $U_{th}\cos(\theta)$, and the elementary discharge emanating from an element of the slot is

$$dQ = U_{th}\cos(\theta)\, b\, dy,$$

where b is the width of the slot. Integrating throughout the orifice cross section,

$$Q=\int_{-y_0}^{y_0} U_{th}\cos(\theta) b\, dy, \quad (5)$$

and evaluating the integral we get $Q = 4\, y_0\ U\, b/\pi$. We can now define the contraction coefficient as the coefficient C_c, which accounts for the deviation of the inviscid flowrate from the theoretical value of the discharge Q_{th}. Thus, $C_d = C_u\, C_c$, but given the values of C_u reported above, we hardly need to make the distinction between C_d and C_c in the present context. Finally, the contraction coefficient can be calculated from Eq. (5) to be equal to $2/\pi \cong 0.64$. This result is to be compared to the exact calculation of Kirchhoff,[8] which is equal to $\pi/(\pi+2)\cong 0.61$ and to the experimental value $C_d = 0.61$, which produces C_c between 0.62 and 0.64 .

The impressive success of this application may be in part misleading. For the integral method is expected to produce qualitatively correct results if it is based on *qualitatively* correct hypotheses, and this is true for the problem already treated. In order, however, to produce results *quantitatively* acceptable (and what is acceptable depends of course on the specific application treated), the basic hypothesis may need refinement by comparison to experimental data, usually. For example, returning to the boundary layer on a flat plate problem, the assumption of a linear velocity profile within the boundary layer, which qualitatively captures the rapid increase of the velocity from zero on the boundary (no-slip condition) to its free-stream value within a thin layer δ, produces approximations to the pertinent dependent variables (boundary layer thickness, shear stress on boundary, etc.), which can be nevertheless greatly improved by choosing a velocity profile closer to experimental data,[11] such as a cubic or a sinusoidal profile.[12]

To dwell on a variation of the problem treated above, it is instructive to adapt the same hypothesis in order to analyze the coefficient of discharge of the circular cross-section aperture. Assuming again that the velocity vectors are symmetrical, but now with respect to the axis vertical at the center of the cross section, and that the angle θ is again distributed linearly, but this time along a radius, it follows that the flowrate Q is given by

$$Q=\int_0^{y_0} U_{th}\cos(\theta) 2\pi\, y\, dy, \quad (6)$$

where y is the distance from the center of the cross section in the radial direction and y_0 is the radius. Evaluating the integral and dividing by $Q_{th} = U_{th}\,\pi\, y_0^2$, we find the contraction coefficient to be equal to $2^2\,(\pi-2)/\pi^2 \cong 0.46$. This is to be compared to the experimental values, which have been found to range between 0.62[13] and 0.65.[14] This result is disappointing, although it still points in the right direction, i.e., the contraction coefficient is less than 1. What is to blame for this moderate failure (alternatively, for this success to be moderate)? Clearly, it is the modeling hypothesis, namely the assumption that the value of the angle θ varies linearly along the radius. Reasonable as it may seem, this or any other assumption can be justified only if it is compatible with experimental data or if it rests on detailed physical reasoning.

In the present case, however, since observing experimentally the angle is rather difficult, recourse must be had to an analytical solution. Such a solution appears in Kennard.[15] From Fig. 7.49 in Johnson,[16] it is apparent that the angle θ varies slowly near the center of the circular cross section, and much more rapidly away from it. Thus, a more realistic assumption would be

$$\theta(y)=\left(\frac{y}{y_0}\right)^2\frac{\pi}{2}. \quad (7)$$

Following the same procedure as above, we calculate the contraction coefficient again and find it to be equal to $2/\pi \cong 0.64$, which is now much closer to the experimental value.

Conclusion

The simple model described above is meant to materialize in mathematical terms the qualitative explanation, which resolves the seeming paradox of the value of the contraction coefficient. It requires minimal analytical effort, well within the abilities of introductory students. Furthermore, it provides a way to introduce students to another kind of approximation, which they have probably not encountered early in their studies. That is, the approximation that is not based on an analytical or numerical approximation of the exact equations, but rather on "modeling" the flow, in the spirit of integral methods. Such approximations have the disadvantage of making it hard to estimate the error involved, but on the other hand they enhance physical insight, which is not always the case with numerical approximations. Furthermore, the modeling assumptions can be refined by comparison to experimental or analytical data, and this process enhances further the physical insight. In this sense the model can be considered to be diagnostic rather than prognostic. Nowadays, integral methods are mostly used for educational purposes, but they have been used extensively in the area of fluid mechanics before the advent of computers[17] and continue to be of use occasionally also in research.[18] Finally, the model is expected to enhance students' understanding of the phenomenon of vena contracta.[19]

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