VISCOUSLY DAMPED 1-DOF SYSTEM

Equation of Motion:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = P(t)$$

Initial Conditions:

$$u(t=0) = u_0 \quad , \quad \dot{u}(t=0) = \dot{u}_0$$

Complete Solution:

$$u(t) = u_H(t) + u_P(t)$$

where: $u_H(t) =$ **homogeneous** or **complementary** solution

$u_P(t) =$ **particular** solution

REFERENCE:

BOYCE, W.E. and R.C. Di PRIMA (1970). ***Introduction to Differential Equations***, John Wiley & Sons, Inc.

UNDAMPED FREE VIBRATION

Equation of Motion:	$m\ddot{u}(t) + ku(t) = 0$
Initial Conditions:	$u(t=0) = u_0 \quad , \quad \dot{u}(t=0) = \dot{u}_0$

Potentially, a solution is of the form: $u(t) = Ge^{\lambda t}$

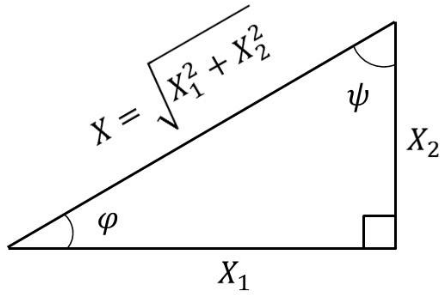
[$G \neq 0$ for **non-trivial** solution, *i.e.*, for non-zero motion; G is in general complex-valued.]

Substituting in the equation of motion:

$$mG\lambda^2 e^{\lambda t} + kGe^{\lambda t} = 0 \quad \Leftrightarrow \quad Ge^{\lambda t}(m\lambda^2 + k) = 0$$

$$\underbrace{m\lambda^2 + k = 0}_{\textit{characteristic equation}} \quad \Leftrightarrow \quad \lambda = \pm i\omega$$

where: $\boxed{\omega \overset{\text{def}}{=} \sqrt{\dfrac{k}{m}}}$



∴ **General solution of homogeneous equation**:

$$u(t) = G_1 e^{i\omega t} + G_2 e^{-i\omega t}$$

or (see NOTE below)

$$u(t) = A\cos(\omega t) + B\sin(\omega t)$$

NOTE: Recall ***Euler's formula***:

$$e^{\pm ix} = \cos x \pm i \sin x \quad \Longrightarrow \quad \cos x = \frac{e^{+ix} + e^{-ix}}{2} \qquad \sin x = \frac{e^{+ix} - e^{-ix}}{2i}$$

Introducing the **initial conditions**:

$$u(t) \quad = \quad u_0\cos(\omega t) + \frac{\dot{u}_0}{\omega}\sin(\omega t) \quad = \quad \begin{cases} \rho\sin(\omega t + \varphi) \\ \text{or} \\ \rho\cos(\omega t - \psi) \end{cases}$$

where: $\rho = \sqrt{u_0^2 + \left(\dfrac{\dot{u}_0}{\omega}\right)^2}$ & $\begin{cases} \tan\varphi = \dfrac{u_0}{(\dot{u}_0/\omega)} \\ \tan\psi = \dfrac{(\dot{u}_0/\omega)}{u_0} \end{cases}$

Clearly, an **oscillatory/harmonic** response with:

Period	$T = \dfrac{2\pi}{\omega}$ [sec]
Natural (circular) frequency	$\omega \quad \left[\dfrac{\text{rad}}{\text{sec}}\right]$
Natural (cyclic) frequency	$f = \dfrac{\omega}{2\pi} = \dfrac{1}{T} \quad \left[\text{Hz} = \dfrac{\text{cycles}}{\text{sec}}\right]$

NOTE:

THEOREM:
Consider the second-order linear ordinary differential equation,

$$\mathcal{L}[y] \quad = \quad y'' + p(t)y' + q(t)y \quad = \quad 0$$

where $p(t)$ and $q(t)$ are continuous real-valued functions. **If** $y(t) = u(t) + iv(t)$ is **a complex-valued solution** of the differential equation, then the **real part** $u(t)$ and the **imaginary part** $v(t)$ **are also solutions** of this equation.

PROOF:

We substitute $u(t) + iv(t)$ for y in $\mathcal{L}[y]$, obtaining

$$\begin{aligned} \mathcal{L}[y] &= \left(u(t) + iv(t)\right)'' + p(t)\left(u(t) + iv(t)\right)' + q(t)\left(u(t) + iv(t)\right) \\ &= u''(t) + iv''(t) + p(t)\left(u'(t) + iv'(t)\right)q(t)\left(u(t) + iv(t)\right) \\ &= [u'' + p(t)u' + q(t)u] + i[v'' + p(t)v' + q(t)v] \\ &= \mathcal{L}[u] + i\mathcal{L}[v] \end{aligned}$$

Recall that a complex number is zero if and only if its real and imaginary parts are both zero. Therefore,

$$\mathcal{L}[y] = \mathcal{L}[u + iv] = \mathcal{L}[u] + i\mathcal{L}[v] = 0 \quad \Leftrightarrow \quad \begin{cases} \mathcal{L}[u] = 0 \\ \& \\ \mathcal{L}[v] = 0 \end{cases}$$

NOTE:

One approach to demonstrate the equivalence of $[u_0 \cos(\omega t) + (\dot{u}_0/\omega)\sin(\omega t)]$ with either $\rho \sin(\omega t + \varphi)$ or $\rho \cos(\omega t - \psi)$ is the following:

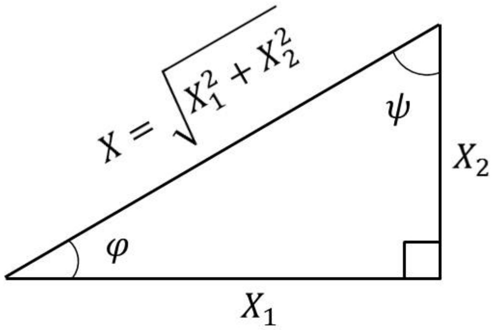
Let:

$$\begin{aligned} x(t) &= X_1 \sin(\omega t) + X_2 \cos(\omega t) \\ &= X\left[\frac{X_1}{X}\sin(\omega t) + \frac{X_2}{X}\cos(\omega t)\right] \\ &= X[\cos\varphi \sin(\omega t) + \sin\varphi \cos(\omega t)] \\ &= X \sin(\omega t + \varphi) \end{aligned}$$

where:

$$X = \sqrt{X_1^2 + X_2^2} \quad \& \quad \varphi = \tan^{-1}\left(\frac{X_2}{X_1}\right)$$

Notice that $|X_1/X| \leq 1$ & $|X_2/X| \leq 1$ and $X^2 = X_1^2 + X_2^2$. Therefore (X_1/X) & (X_2/X) qualify to be equal to the $\cos(\cdot)$ & $\sin(\cdot)$, respectively, of the same angle φ.



Alternatively:

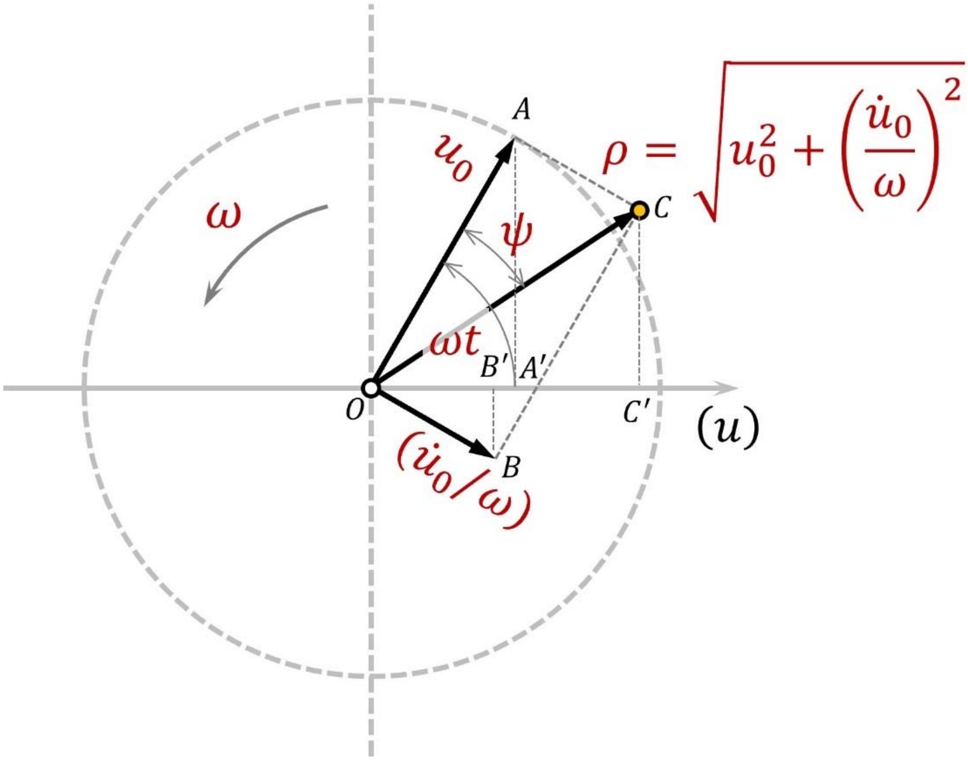
$$\begin{aligned} x(t) &= X_1 \sin(\omega t) + X_2 \cos(\omega t) \\ &= X\left[\frac{X_1}{X}\sin(\omega t) + \frac{X_2}{X}\cos(\omega t)\right] \\ &= X[\sin\psi \sin(\omega t) + \cos\psi \cos(\omega t)] \\ &= X \cos(\omega t - \psi) \end{aligned}$$

where:

$$X = \sqrt{X_1^2 + X_2^2} \quad \& \quad \psi = \tan^{-1}\left(\frac{X_1}{X_2}\right)$$

Notice that $|X_1/X| \leq 1$ & $|X_2/X| \leq 1$ and $X^2 = X_1^2 + X_2^2$. Therefore (X_1/X) & (X_2/X) qualify to be equal to the $\sin(\cdot)$ & $\cos(\cdot)$, respectively, of the same angle ψ.

NOTE:
Another approach of representing vibrations is by means of **rotating vectors**. Imagine a vector $\overline{OA}$ (see FIGURE) of magnitude u_0 rotating counterclockwise with a constant *angular velocity* ω around a fixed point O. This angular velocity is what we call *circular frequency* of vibration. If at time $t = 0$ the vector coincides with the u-axis, the angle which it makes with the same axis at any instant t is equal to ωt. The projection OA' of the vector on the u-axis is equal to $u_0 \cos(\omega t)$ and represents the first term of the expression $[u_0 \cos(\omega t) + (\dot{u}_0/\omega)\sin(\omega t)]$. To represent the second term of the abovementioned expression, i.e., $(\dot{u}_0/\omega)\sin(\omega t) = (\dot{u}_0/\omega)\cos[(\pi/2) - \omega t] = (\dot{u}_0/\omega)\cos[\omega t - (\pi/2)]$, we take another vector $\overrightarrow{OB}$ of magnitude $(\dot{u}_0/\omega)$ and perpendicular to vector $\overrightarrow{OA}$, as shown in the FIGURE. Evidently, the projection OB' of vector $\overrightarrow{OB}$ on the u-axis, gives the second term of the above expression. The total displacement given by the sum of the two terms of the above expression is obtained by adding the two projections, i.e., $OA' + OB'$. The same result may be obtained by adding vectorially the vectors $\overrightarrow{OA}$ & $\overrightarrow{OB}$ to obtain vector $\overrightarrow{OC}$ and taking its projection on the u-axis. Clearly, the magnitude of vector $\overrightarrow{OC}$ is $\rho = \sqrt{u_0^2 + (\dot{u}_0/\omega)^2}$, and its projection OC' on the u-axis is $\rho\cos(\omega t - \psi)$. By now it should be evident that **the result of the addition of two simple harmonic vibrations of the same circular frequency** ω, one proportional to $\cos(\omega t)$ and the other proportional to $\sin(\omega t)$, **is a simple harmonic vibration proportional to** $\cos(\omega t - \psi)$. The maximum displacement of the vibrating mass is given by $\rho = \sqrt{u_0^2 + (\dot{u}_0/\omega)^2}$ and is referred as *amplitude of vibration*. The angle ψ is referred to as *phase difference*.

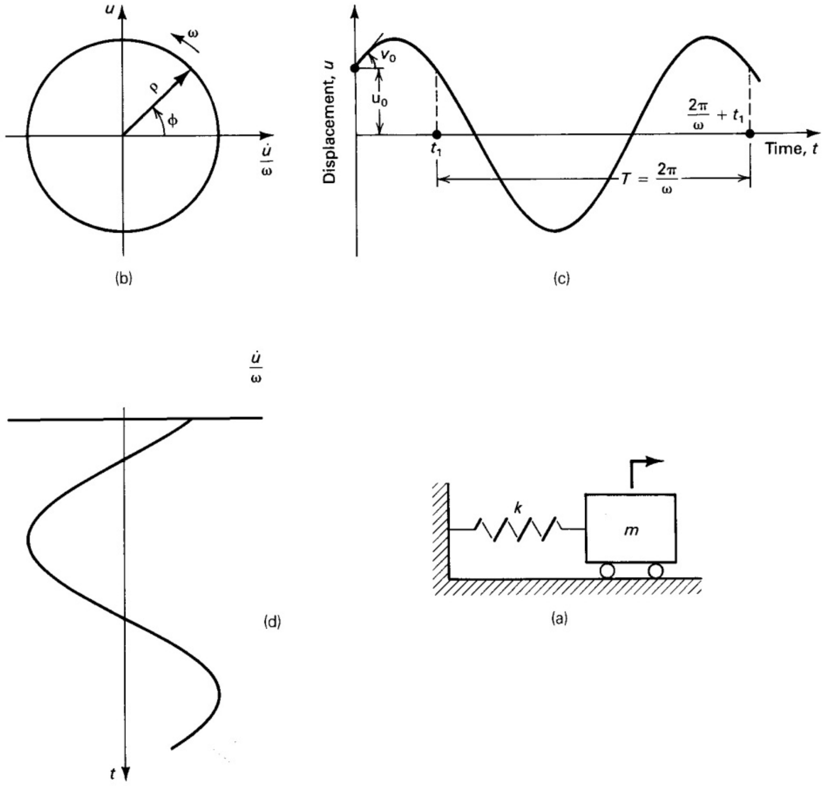


Phase Plane Diagram (or *Poincaré* Phase Plane)

Displacement: $u(t) = \rho \sin(\omega t + \varphi)$

Velocity: $\dot{u}(t) = \rho\omega \cos(\omega t + \varphi) \quad \Longrightarrow \quad \frac{\dot{u}(t)}{\omega} = \rho \cos(\omega t + \varphi)$

[Notice that $\left(\frac{\dot{u}(t)}{\omega}\right)$ has **dimensions of displacement**.]



Potential Energy:
$$\begin{aligned} V(t) &= \frac{1}{2}ku^2 \\ &= \frac{1}{2}k\rho^2 \sin^2(\omega t + \varphi) \end{aligned}$$

Kinetic Energy:
$$\begin{aligned} T(t) &= \frac{1}{2}m\dot{u}^2 \\ &= \frac{1}{2}m\omega^2\rho^2 \cos^2(\omega t + \varphi) \\ &\underbrace{=}_{\omega^2 = \frac{k}{m}} \frac{1}{2}k\rho^2 \cos^2(\omega t + \varphi) \end{aligned}$$

Evidently: $\boxed{V(t) + T(t) = \frac{1}{2}k\rho^2}$

∴ **No energy is dissipated** in a system undergoing free vibrations.

Circles (i.e., closed curves) in a phase plane diagram thus represent ***'constant-energy'* states**.

VISCOUSLY DAMPED FREE VIBRATIONS

Equation of Motion:	$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = 0$
Initial Conditions:	$u(t=0) = u_0 \quad , \quad \dot{u}(t=0) = \dot{u}_0$

Possible solution is of the form: $u(t) = Ge^{\lambda t}\ ,\ (G \neq 0)$

Substituting in the equation of motion:

$$\underbrace{m\lambda^2 + c\lambda + k = 0}_{\textit{characteristic equation}} \quad \Longrightarrow \quad \begin{cases} \lambda_1 = -\dfrac{c}{2m} + \sqrt{\left(\dfrac{c}{2m}\right)^2 - \dfrac{k}{m}} \\ \lambda_2 = -\dfrac{c}{2m} - \sqrt{\left(\dfrac{c}{2m}\right)^2 - \dfrac{k}{m}} \end{cases}$$

Critically Damped System: ($\xi = 1$)

When the **discriminant** $\Delta = \left[\left(\frac{c}{2m}\right)^2 - \frac{k}{m}\right]$ is zero:

Then $\Delta = 0 \quad \Leftrightarrow \quad \boxed{c = c_{cr} = 2\sqrt{km} = 2m\omega}$

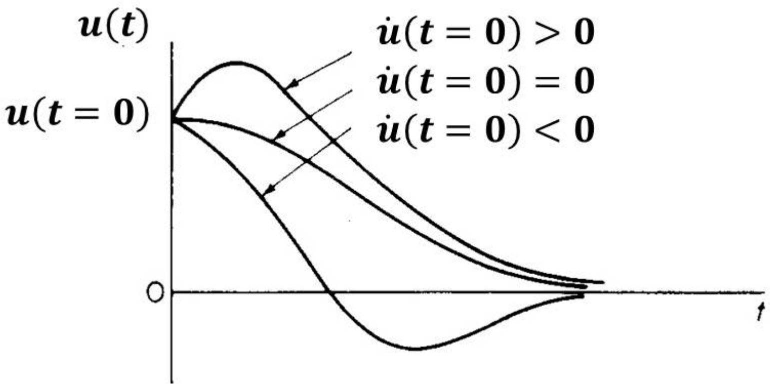
$\lambda_{1,2} = -\frac{c}{2m} = -\omega$ (The characteristic equation has a **double root**)

$\therefore$ General solution: $u(t) = (G_1 + G_2 t)e^{-\omega t}$

Introducing the Initial Conditions:

$$\boxed{u(t) = [u_0 + (\dot{u}_0 + \omega u_0)t]e^{-\omega t}}$$

The above solution represents **non-oscillatory motion**.



NOTE: Mechanical Systems for which it is required that the system return to a zero-displacement position **in the least amount of time** are designed to have **critical damping** (*e.g.*, recoiling gun, weighing scale).

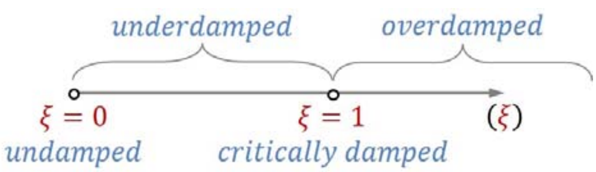
Definition of **Damping Ratio**:

$$\boxed{\xi = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}}}$$

Then: $\frac{c}{2m} = \frac{\xi c_{cr}}{2m} = \xi\omega$

Therefore: $\lambda_{1,2} = -\xi\omega \pm \omega\sqrt{\xi^2 - 1}$

For the **critically damped case**: $\xi = 1$



Over-damped System: ($\xi > 1$)

Clearly:
$$\lambda_1 = -\xi\omega + \omega\sqrt{\xi^2 - 1}$$
$$\lambda_2 = -\xi\omega - \omega\sqrt{\xi^2 - 1}$$

Let: $$\boxed{\bar{\omega} = \omega\sqrt{\xi^2 - 1}}$$

Then: $$\boxed{u(t) = e^{-\xi\omega t}\left(G_1 e^{\bar{\omega} t} + G_2 e^{-\bar{\omega} t}\right)}$$

Introducing the **Initial Conditions**, we obtain:

$$\boxed{\begin{aligned} G_1 &= \frac{\dot{u}_0 + (\xi\omega + \bar{\omega})u_0}{2\bar{\omega}} \\ G_2 &= \frac{-\dot{u}_0 - (\xi\omega - \bar{\omega})u_0}{2\bar{\omega}} \end{aligned}}$$

Alternatively, the response may be expressed as:

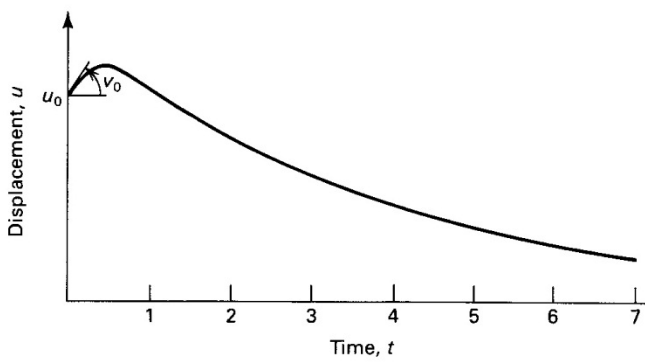
$$\boxed{u(t) = e^{-\xi\omega t}\left[u_0 \cosh(\bar{\omega} t) + \frac{\dot{u}_0 + \xi\omega u_0}{\bar{\omega}} \sinh(\bar{\omega} t)\right]}$$

The above solution represents **non-oscillatory motion**.

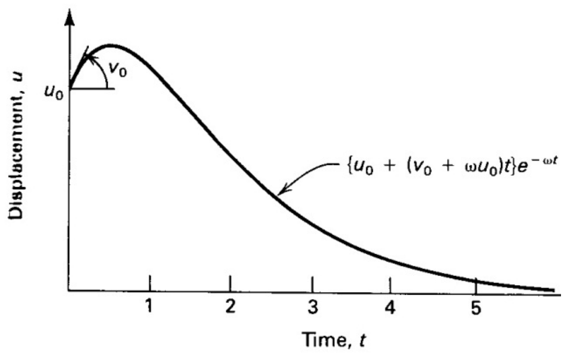
NOTE: Recall the definition of **hyperbolic functions** in terms of the exponential function:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

Over-damped



Critically Damped



NOTE: Certain recoil mechanisms (*e.g.*, an automatic door closer) are designed to have over-damping.

Under-damped System: $(\xi < 1)$

Roots of the characteristic equation:

$$\lambda_1 = -\xi\omega + i\omega\sqrt{1-\xi^2}$$
$$\lambda_2 = -\xi\omega - i\omega\sqrt{1-\xi^2}$$

Definition of **Damped Circular Frequency**:

$$\boxed{\omega_d \stackrel{\text{def}}{=} \omega\sqrt{1-\xi^2}}$$

The general solution may be written as:

$$u(t) = e^{-\xi\omega t}\left(G_1 e^{i\omega_d t} + G_2 e^{-i\omega_d t}\right)$$

or

$$u(t) = e^{-\xi\omega t}[A\cos(\omega_d t) + B\sin(\omega_d t)]$$

Introducing the Initial Conditions, we obtain:

$$u(t) = e^{-\xi\omega t}\left[u_0\cos(\omega_d t) + \frac{\dot{u}_0 + \xi\omega u_0}{\omega_d}\sin(\omega_d t)\right]$$

Therefore:

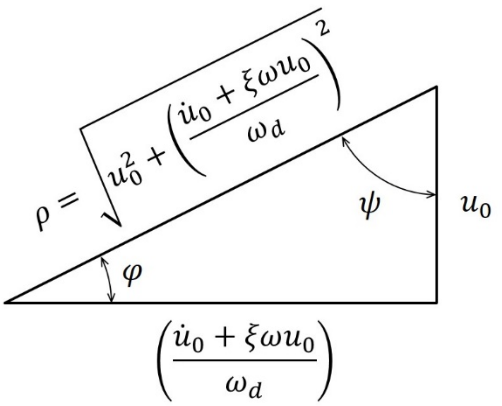
$$u(t) = e^{-\xi\omega t}\left[u_0\cos(\omega_d t) + \frac{\dot{u}_0 + \xi\omega u_0}{\omega_d}\sin(\omega_d t)\right] = \begin{cases} \rho e^{-\xi\omega t}\sin(\omega_d t + \varphi) \\ \& \\ \rho e^{-\xi\omega t}\cos(\omega_d t - \psi) \end{cases}$$

where

$$\begin{cases} \rho = \sqrt{u_0^2 + \left(\dfrac{\dot{u}_0 + \xi\omega u_0}{\omega_d}\right)^2} \\ \& \\ \tan\varphi = \dfrac{u_0\omega_d}{\dot{u}_0 + \xi\omega u_0} \end{cases}$$

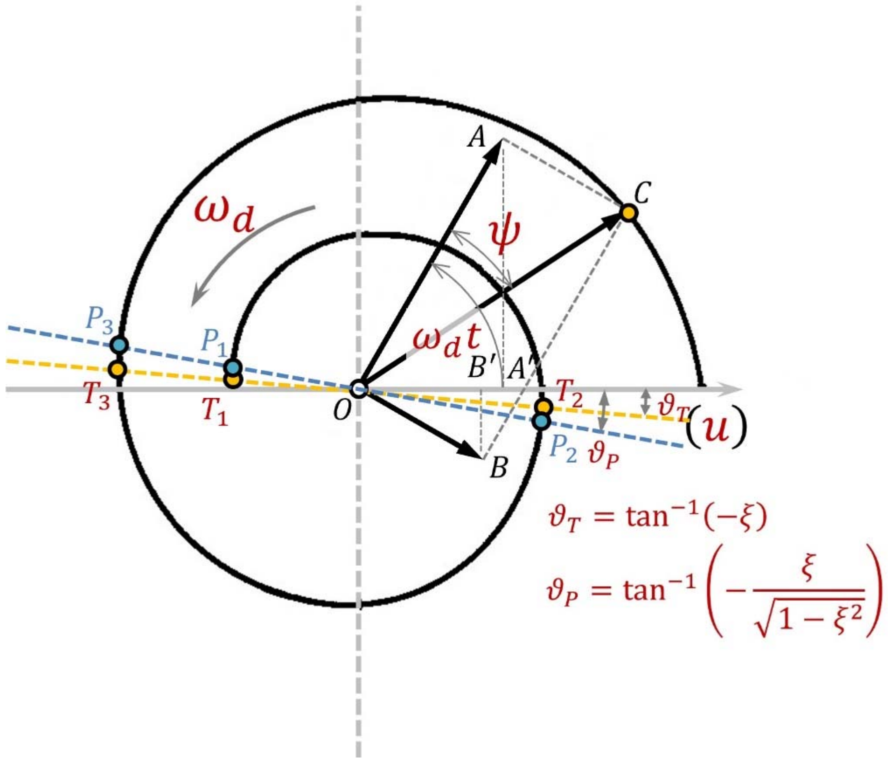
The above expression represents **decaying oscillatory motion**.

NOTE: For $\xi = 0$ (*i.e.*, $c = 0$) the solution reduces to undamped free vibrations.



NOTE:

We have seen that we could describe undamped vibrations by making use of a **rotating vector**. Such a vector **can be used also in the case of damped vibrations**. Imagine a vector $\overrightarrow{OA}$ of time-varying magnitude $u_0 e^{-\xi\omega t}$ rotating with a constant angular velocity ω_d (see FIGURE). Measuring the angle of rotation counterclockwise starting from the positive u-axis, the projection OA' of the vector $\overrightarrow{OA}$ is equal to $u_0 e^{-\xi\omega t}\cos(\omega_d t)$ and represents the first term of the response expression $e^{-\xi\omega t}\left[u_0\cos(\omega_d t)+\frac{\dot{u}_0+\xi\omega u_0}{\omega_d}\sin(\omega_d t)\right]$. In the same manner, by taking a vector $\overrightarrow{OB}$ of magnitude $\frac{\dot{u}_0+\xi\omega u_0}{\omega_d}e^{-\xi\omega t}$ and perpendicular to vector $\overrightarrow{OA}$ and projecting it on the u-axis we get the projection OB' which is equal to $\frac{\dot{u}_0+\xi\omega u_0}{\omega_d}e^{-\xi\omega t}\cos[\omega_d t-(\pi/2)]=\frac{\dot{u}_0+\xi\omega u_0}{\omega_d}e^{-\xi\omega t}\cos[(\pi/2)-\omega_d t]=\frac{\dot{u}_0+\xi\omega u_0}{\omega_d}e^{-\xi\omega t}\sin(\omega_d t)$ and represents the second term of the response expression. Thus, the sum of the two projections, i.e., $OA'+OB'$, represents the complete response. Therefore, the complete response is represented by the projection of vector $\overrightarrow{OC}$ which results by summing the vectors $\overrightarrow{OA}$ & $\overrightarrow{OB}$. Clearly, the magnitude of vector $\overrightarrow{OC}$ is $\rho e^{-\xi\omega t}=\sqrt{u_0^2+\left(\frac{\dot{u}_0+\xi\omega u_0}{\omega_d}\right)^2}\,e^{-\xi\omega t}$ and the response of the underdamped oscillator may be written as: $u(t)=\rho e^{-\xi\omega t}\cos(\omega_d t-\psi)$, where $\tan\psi=\frac{\dot{u}_0+\xi\omega u_0}{\omega_d u_0}$.



During the rotation of vector $\overrightarrow{OC}$ the point C describes a spiral. The spiral is referred to as ***equiangular spiral*** because the tangent (at point C) to the spiral makes a constant angle $\vartheta_T=\tan^{-1}(-\xi)$, with the perpendicular to the radius vector $\overrightarrow{OC}$ (see NOTE below for a proof). The points at which the envelopes $\rho e^{-\xi\omega t}$ & $-\rho e^{-\xi\omega t}$ touch the response curve $u(t) = \rho e^{-\xi\omega t}\cos(\omega_d t - \psi)$, occur at the instants when $|\cos(\omega_d t - \psi)| = 1$, i.e., when the tangent to the spiral is perpendicular to the u-axis. Consequently, the points of tangency (i.e., points $T_1, T_2, T_3, \cdots$ shown in the FIGURE) lie on a line that passes through the origin and forms an angle $\vartheta_T = \tan^{-1}(-\xi)$ with the u-axis.

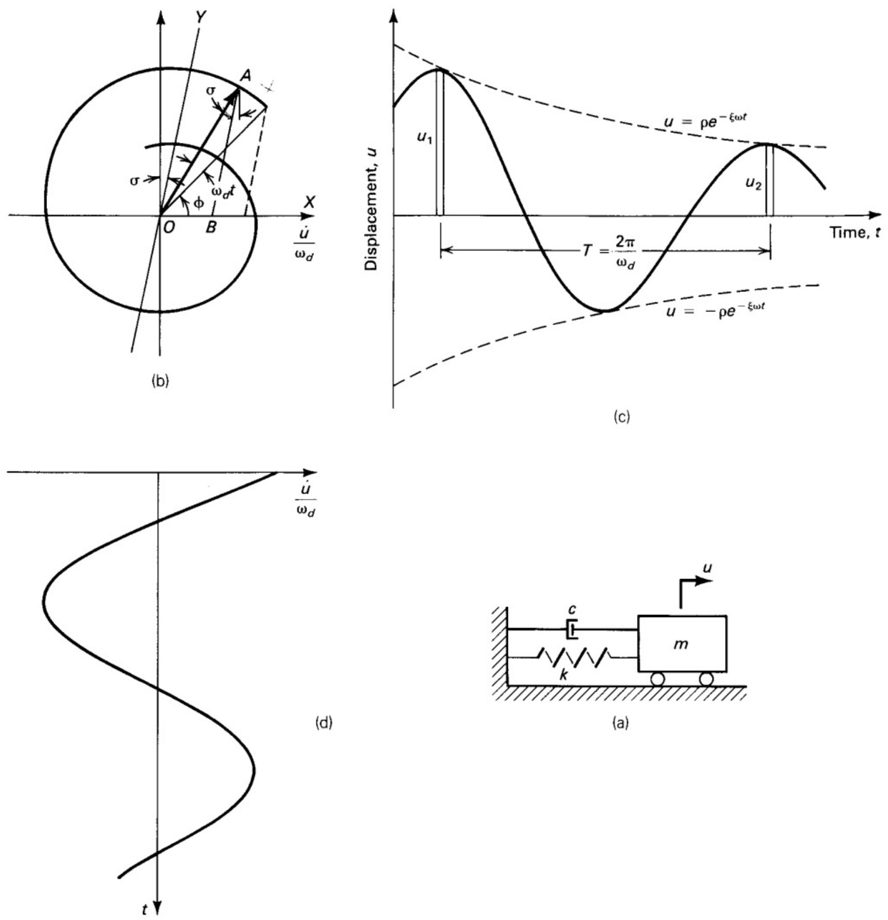
The points of tangency, though, are not the points of local maxima. The points of local maxima (i.e., local peaks of the response displacement; points $P_1, P_2, P_3, \cdots$ shown in the FIGURE), are slightly ahead (in time) of the points of tangency and are located by setting the velocity equal to zero:

$$u(t) = \rho e^{-\xi\omega t}\cos(\omega_d t - \psi) \quad \Rightarrow \quad \dot{u}(t) = -\xi\omega\rho e^{-\xi\omega t}\cos(\omega_d t - \psi) - \omega_d \rho e^{-\xi\omega t}\sin(\omega_d t - \psi)$$

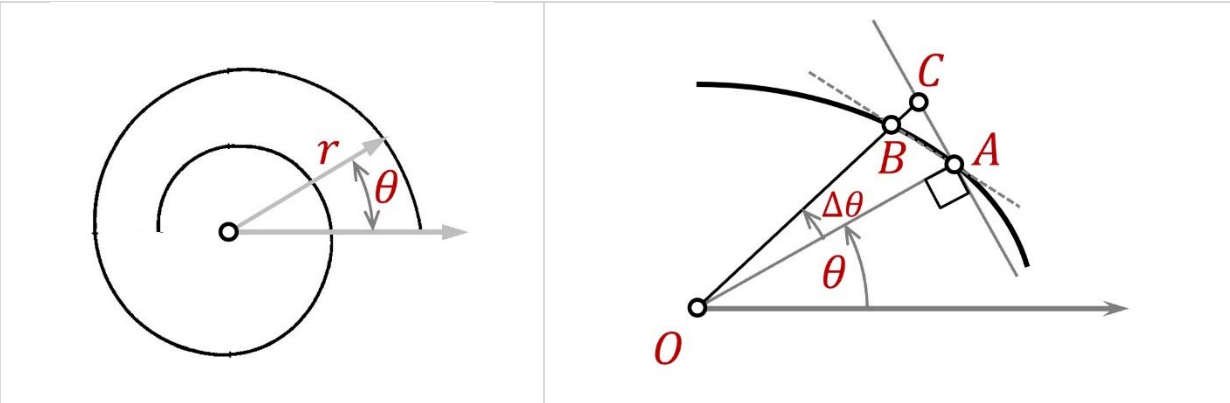
Therefore,

$$\dot{u}(t) = 0 \quad \Rightarrow \quad \tan(\omega_d t - \psi) = -\left(\xi/\sqrt{1-\xi^2}\right)$$

Indeed, this suggests that the local peaks of the response lie on a line passing through the origin and forming an angle $\vartheta_P = \tan^{-1}\left(-\xi/\sqrt{1-\xi^2}\right)$ with the u-axis that (in absolute value) is slightly larger than $|\vartheta_T|$, i.e., $|\vartheta_T| < |\vartheta_P|$.



NOTE: Logarithmic or Exponential or Equiangular Spiral



$$\boxed{\text{Equation of Exponential Spiral:}\quad r = \rho e^{-\xi\omega t} = \rho e^{-\xi\theta}}$$

The shape of the spiral depends on ξ **only**. **For a given value of ξ a spiral has to be drawn only once**.

Using the drawing as a template, the spiral can be transferred to the phase-plane diagram by selecting the required value of r on the spiral.

Let ϑ_T be the angle that is formed by the **tangent at a point** on the spiral and by the **perpendicular to the polar radius r** at that point. Specifically, let us consider a point A on the spiral and let us draw a perpendicular to the polar radius OA at point A. Then, let us consider another point B on the spiral, adjacent to point A. As point B moves towards point A line AB tends to become tangent to the spiral at point A. Thus, at the limit, as point B tends to point A, the angle $\widehat{BAC}$ tends to the aforementioned angle ϑ_T, point C tends to point A, the angle $\widehat{BCA}$ tends to $(\pi/2)$, and the triangle $\triangle\,(BCA)$ tends to become an orthogonal triangle. Therefore,

$$\tan\left(\widehat{BAC}\right) \cong \frac{CB}{AC} = \frac{\Delta r}{r\cdot\Delta\theta} = \frac{(\Delta r/\Delta\theta)}{r} \quad\Rightarrow\quad \tan\vartheta_T = \lim_{B\to A}\left[\tan\left(\widehat{BAC}\right)\right] = \frac{\lim(\Delta r/\Delta\theta)}{r} = \frac{(dr/d\theta)}{r}$$

Therefore:

$$\tan\vartheta_T = \frac{(dr/d\theta)}{r} = \frac{-\xi\rho e^{-\xi\theta}}{\rho e^{-\xi\theta}} = -\xi$$

NOTE: This is why the exponential spiral is called sometimes ***'equiangular'*** spiral.

HISTORICAL NOTE: The investigation of *Spirals* began at least with the ancient Greeks. The famous *Equiangular Spiral* was discovered by **DESCARTES**, its properties of self-reproduction by **James (Jacob) BERNOULLI (1654-1705)** who requested that the curve be engraved upon his tomb with the phrase "*Eadem mutata resurgo*" ("I shall arise the same, though changed").

YATES, R. C. (1952). *Curves and Their Properties*. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS