DAMPED FREE VIBRATION WITH COULOMB DAMPING

Equation of Motion:

$\dot{u} > 0$: (i.e., **moving to the right**) $m\ddot{u} + ku = -\mu N$

$\dot{u} < 0$: (i.e., **moving to the left**) $m\ddot{u} + ku = +\mu N$

Solution:

For $\dot{u} > 0$: $u(t) = A\cos(\omega t) + B\sin(\omega t) - \frac{\mu N}{k}$

For $\dot{u} < 0$: $u(t) = C\cos(\omega t) + D\sin(\omega t) + \frac{\mu N}{k}$

Initial Conditions: $u(t=0) = u_0$ & $\dot{u}(t=0) = 0$

Let us assume that $u_0 > 0$. Then, assuming that $u_0 > \left(\frac{\mu N}{k}\right)$ (i.e., the spring force is sufficiently large to overcome the frictional force):

Block starts moving **to the left**:

Then, solution for $0 \le t \le \frac{\pi}{\omega}$:

$$\boxed{u(t) = \left(u_0 - \frac{\mu N}{k}\right)\cos(\omega t) + \frac{\mu N}{k}}$$

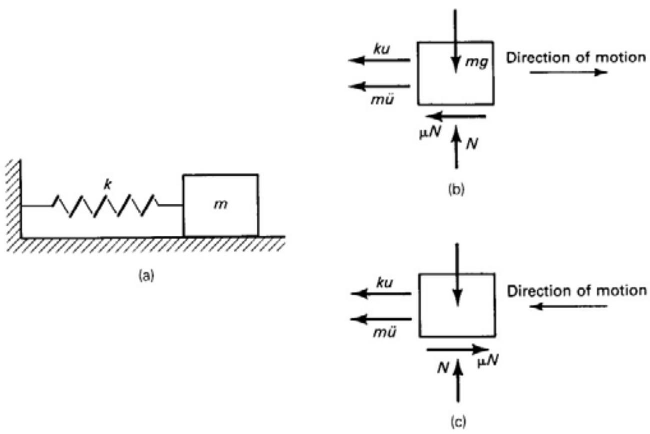
$$\therefore \begin{cases} u\left(t = \frac{\pi}{\omega}\right) & = & -\left(u_0 - \frac{2\mu N}{k}\right) \\ \dot{u}\left(t = \frac{\pi}{\omega}\right) & = & 0 \end{cases}$$

Block starts moving **to the right**:

Then, solution for $\frac{\pi}{\omega} \le t \le \frac{2\pi}{\omega}$:

$$\boxed{u(t) = \left(u_0 - \frac{3\mu N}{k}\right)\cos(\omega t) - \frac{\mu N}{k}}$$

$$\therefore \begin{cases} u\left(t = \frac{2\pi}{\omega}\right) & = & \left(u_0 - \frac{4\mu N}{k}\right) \\ \dot{u}\left(t = \frac{2\pi}{\omega}\right) & = & 0 \end{cases}$$



- The block has **completed a cycle of motion**.

- The period of motion (*i.e.*, the time it takes to complete one cycle) is $\left(\frac{2\pi}{\omega}\right)$, *i.e.*, **Coulomb friction does not change the period of vibration**.

- The amplitude reduces by $\left(\frac{4\mu N}{k}\right)$ over one cycle of motion.

- At any instant when $\dot{u} = 0$, *i.e.*, **when the block is at its extreme left or extreme right position**,

$$if \quad u \leq \frac{\mu N}{k} \quad then \quad \underbrace{ku}_{\substack{spring\\force}} \leq \underbrace{\mu N}_{\substack{friction\\force}}$$

  and **the block will cease to move**.

  Thus, the block will come to rest at the end of a half-cycle in a position which is displaced from its original position of rest.

- In the equation of motion, we must use μ_d = **dynamic coefficient of friction**.
  In checking/determining the position that the block will come to rest, we use μ_s = **static coefficient of friction**.

- The amplitude decay per cycle due to Coulomb friction is $\left(\frac{4\mu N}{k}\right)$. Thus, **the envelopes of u vs. t curves are straight lines**.

  In the cases of **viscous** or **hysteretic damping**, amplitude **decay is exponential** and **theoretically the system never comes to rest**.

- Because the damping force (i.e., frictional force) changes abruptly at times $t = \frac{\pi}{\omega}, \frac{2\pi}{\omega}, \cdots$, (i.e., **at the end of each half-cycle**), **the slope of the velocity** curve is **discontinuous** at these times.

Phase Plane Representation of Vibrations Under Coulomb Damping

First half-cycle of motion:

$$u(t) - \frac{\mu N}{k} = \rho_1 \sin(\omega t + \varphi_1) \quad (0 \le t \le \frac{\pi}{\omega})$$

where:

$$\rho_1 = \left(u_0 - \frac{\mu N}{k}\right) \quad \& \quad \varphi_1 = \left(\frac{\pi}{2}\right)$$

Also:

$$\frac{\dot{u}(t)}{\omega} = \rho_1 \cos(\omega t + \varphi_1) \quad (0 \le t \le \frac{\pi}{\omega})$$

Thus, the **first half-cycle** is represented by **circular arc AB** centered at point $O_1 = \left(0, \frac{\mu N}{k}\right)$.

Second half-cycle of motion:

$$u(t) + \frac{\mu N}{k} = \rho_2 \sin(\omega t + \varphi_2) \quad (\frac{\pi}{\omega} \le t \le \frac{2\pi}{\omega})$$

where:

$$\rho_2 = \left(u_0 - \frac{3\mu N}{k}\right) \quad \& \quad \varphi_2 = \left(\frac{\pi}{2}\right)$$

Also:

$$\frac{\dot{u}(t)}{\omega} = \rho_2 \cos(\omega t + \varphi_2) \quad (\frac{\pi}{\omega} \le t \le \frac{2\pi}{\omega})$$

Thus, the **second half-cycle** is represented by **circular arc BC** centered at point $O_2 = \left(0, -\frac{\mu N}{k}\right)$