Multi-Degree of Freedom Systems – Synopsis

Classification of Problems in Structural Dynamics

By the number of degrees of freedom:

- **Single DOF**
- **Multiple DOFs**
  - lumped mass (discrete) system (finite DOF)
  - continuous systems (infinitely many DOF)

Discrete systems are characterized by systems of **ordinary differential equations** (ODEs), while **continuous systems** are described by systems of **partial differential equations** (PDEs).

By the linearity of the governing equations:

- **Linear systems** – linear elasticity, small motions assumption
- **Nonlinear systems**
  - conservative (elastic) systems
  - nonconservative (inelastic) systems

By the type of excitation:

- **Free vibrations**
- **Forced vibrations**
  - structural loads / seismic loads
    - periodic
      - harmonic
      - nonharmonic
    - transient
      - deterministic excitation
      - random excitation
        - stationary
        - non-stationary

By the type of mathematical problem:

- **Static** – Boundary Value Problems (BVPs)
- **Dynamic**
  - eigenvalue problems (free vibrations)
  - initial value problem, propagation problem (waves)

By the presence of energy dissipating mechanisms:

- **Undamped vibrations**
- **Damped vibrations**
  - viscous damping
  - hysteretic damping
  - Coulomb damping
  - etc.

KAUSEL, E. (2017). Advanced Structural Dynamics. CAMBRIDGE

Modeling – Discretization

Although real physical structures are continuous in nature (i.e. their mass and stiffness are continuously distributed along their structural members), they cannot be analyzed as such. First, they must be cast in the form of discrete systems with a finite number of DOF. There are two major approaches to transform a **continuous system** into a **discrete one**:

- **Heuristic approach** (based on physical approximations): Use common sense (intuition) to **lump** masses, then basic methods to obtain the required stiffness.

- **Mathematical methods**: There are two classes of mathematical discretization schemes based on series of functions expansions, namely,
  - **Rayleigh-Ritz type methods** (such as the **assumed-modes method**, and the well-known **Finite Element Method**) and
  - **Weighted Residual methods** (such as the well-known **Galerkin method**).

Rayleigh-Ritz type methods are **based on a given variational principle** (examples of variational principles are: *Hamilton's Principle*; *Virtual Work*; *The Method of Total Potential Energy*; *Complementary Virtual Work*; *Principle of Total Complementary Energy*; *Reissner's Principle*). By contrast, **weighted residual methods** are more general in scope and they **do not require a variational principle**.

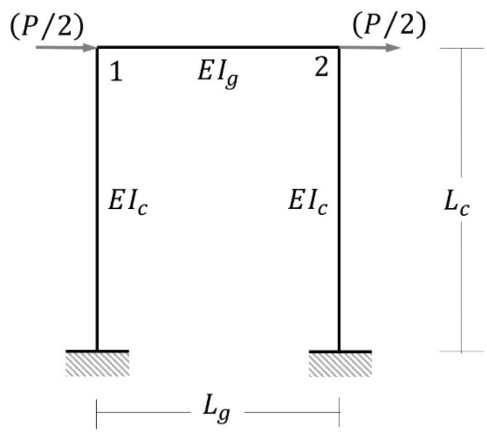
In this course we use exclusively the "heuristic approach" and develop **discrete-parameter models**.

For example, in modeling a structural frame for dynamic analysis we neglect vertical inertia forces and rotational inertias. Note carefully that **this does not imply that the vertical motions or rotations vanish**. Instead, these become static degrees of freedom, and thus **depend linearly on the lateral translations**, that is, they become **slave** DOF to the lateral translations, which are the **master** DOF. The process of reducing the number of DOF as a result of neglecting rotational and translational inertias can formally be achieved by matrix manipulations referred to as **static condensation** (see **EXAMPLES 9.8 & 9.9** in the textbook). We introduce further simplification by assuming that that the beams are axially rigid and thus neglect axial deformations. This introduces a kinematic constraint between the axial components of motion at the two ends of a beam. The formal process by which this is accomplished through matrix manipulations is referred to as **kinematic condensation**.

EXERCISE:

Demonstrate that the **lateral stiffness** $\hat{k}$ of a one-bay, one-story, frame (portal frame) is

$$\hat{k} = \frac{24EI_c}{L_c^3}\left\{\frac{1+\frac{1}{6}\frac{I_c}{I_g}\frac{L_g}{L_c}+4\frac{I_c}{L_c I_g^2}}{1+\frac{2}{3}\frac{I_c}{I_g}\frac{L_g}{L_c}+16\frac{I_c}{L_c I_g^2}}\right\}$$



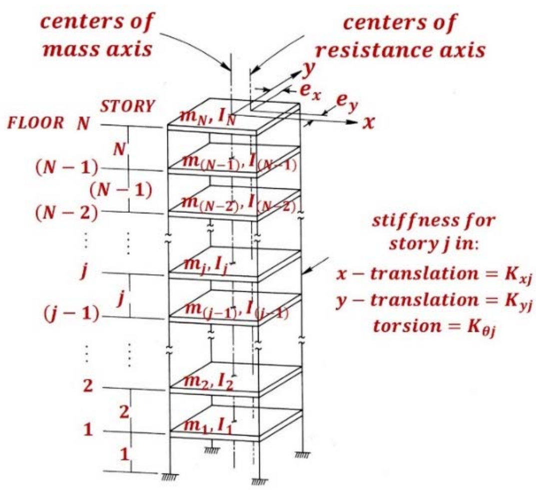
The above expression includes axial deformation of the columns. The girder is axially rigid. All members are made of the same material with modulus of elasticity E.

HINT:

1. Write the stiffness matrix and equilibrium equations for the free joints (*i.e.* joints 1 & 2).
2. Neglecting the axial deformation of the girder and assuming antisymmetric behavior, so that $u_1 = u_2 = u$, $v_1 = -v_2 = v$, $\varphi_1 = \varphi_2 = \varphi$, condense the stiffness matrix to a matrix 3×3 and the load vector to a vector with 3 components. (Write 3 equilibrium equations. What does each one of these equations represent physically?)
3. Eliminating the rotation φ and the axial deformation v of the columns (static condensation) obtain a relation of the form $P = \hat{k}u$ (obtain $\hat{k}$). The parameter $\hat{k}$ is the **lateral stiffness** of the frame.

NOTE: Accounting for axial deformation of the columns may not be important for a one-story frame. However, it becomes a crucial consideration for multi-story structures (say 10 stories and higher).

As another example, for a **multi-story structure**, we lump masses at the level of each floor and we assume that the floor slabs act as **disks** (i.e. rigid in their own planes or, equivalently, they do not deform under shear forces; act as **diaphragms**).



Thus each floor has three degrees of freedom (DOF): two DOFs corresponding to translational displacements along orthogonal axes that coincide with the principal axes of the plan of a typical floor, and a third DOF corresponding to rotation about a vertical axis.

As a side note we remark that for low-rise buildings (less than ten floors), the columns may be assumed inextensible. However, for high-rise buildings (more than ten stories), the axial extensibility of columns must be considered.

Formulation of the Equations of Motion

There are four types of forces involved in the **dynamic equilibrium** of a structure:

$\mathbf{p}(t)$: The **external** applied forces

$\mathbf{f}_I(t)$: The **inertia** forces
(involve accelerations measured w.r.t. an **inertial frame of reference**)

$\mathbf{f}_D(t)$: The **damping** forces
(involve velocities that describe **rate of deformation**)

$\mathbf{f}_S(t)$: The **restoring** (elastic or inelastic) forces
(involve displacements that describe **deformation**)

One way of looking at the problem is to visualize the external forces $\mathbf{p}(t)$ as being distributed among the **three forces $\mathbf{f}_I(t)$**, $\mathbf{f}_D(t)$, and $\mathbf{f}_S(t)$, **all of them resisting motion**, that is

$$\mathbf{f}_I(t) + \mathbf{f}_D(t) + \mathbf{f}_S(t) \quad = \quad \mathbf{p}(t)$$

Another way of looking at the problem is by applying **NEWTON's 2nd law of motion**

$$\mathbf{f}_I(t) \quad = \quad \mathbf{p}(t) - \mathbf{f}_D(t) - \mathbf{f}_S(t)$$

In the above equation the **restoring forces $\mathbf{f}_S(t)$** and the **damping forces $\mathbf{f}_D(t)$** appear with negative sign because these internal forces resist motion.

Both approaches lead to the same equation of motion, as expected.

If the **structure is elastic**, the restoring forces may be expressed as $\mathbf{f}_S(t) = \mathbf{k}\mathbf{u}(t)$, where $\mathbf{k}$ is the **stiffness matrix** of the structure.

If we <u>**assume**</u> that damping in the structure may be described by **linear viscous damping**, then $\mathbf{f}_D(t) = \mathbf{c}\dot{\mathbf{u}}(t)$, where $\mathbf{c}$ is the **damping matrix** of the structure.

Finally, the inertia forces may be expressed as $\mathbf{f}_I(t) = \mathbf{m}\ddot{\mathbf{u}}(t)$, where $\mathbf{m}$ is the **mass matrix** of the structure and the accelerations **$\ddot{\mathbf{u}}(t)$ must be measured w.r.t. an inertial frame of reference**.

In view of the above, the <u>**equation of dynamic equilibrium**</u> may be written as

$$\mathbf{m}\ddot{\mathbf{u}}(t) + \mathbf{c}\dot{\mathbf{u}}(t) + \mathbf{k}\mathbf{u}(t) \quad = \quad \mathbf{p}(t)$$

The above matrix equation is the <u>**equation of motion**</u> of the discretized structure.

As a corollary of **BETTI's law**, we demonstrated that, the **stiffness matrix $\mathbf{k}$** as well as the **flexibility matrix $\tilde{\mathbf{f}} = \mathbf{k}^{-1}$** of a stable structure, are both <u>**symmetric**</u>.

<u>**NOTE**</u>: The symmetry of the flexibility matrix, $\tilde{\mathbf{f}}^T = \tilde{\mathbf{f}}$, is known as <u>**MAXWELL's Law of Reciprocal Deflections**</u>.

Furthermore, the matrices $\mathbf{m}$ & $\mathbf{k}$ are **<u>positive definite</u>** as the **kinetic energy** $T = (1/2)\dot{\mathbf{u}}^T\mathbf{m}\dot{\mathbf{u}}$ and the **strain energy** $U = (1/2)\mathbf{u}^T\mathbf{k}\mathbf{u}$ are **positive definite functions** of velocities and displacements, respectively.

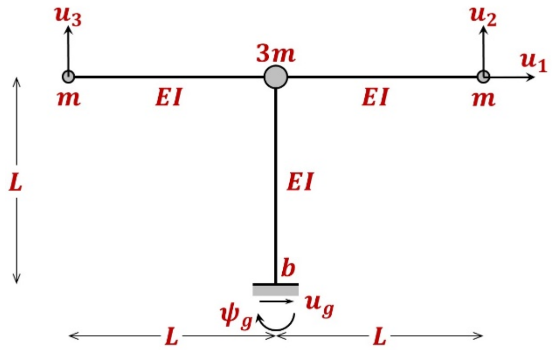
<u>NOTE</u>: For stable civil engineering structures, $\mathbf{k}$ is always positive definite because civil engineering structures are normally supported at fixed points of support and, consequently, **rigid body modes** of motion are not possible (that is, the structure is *restrained* and motion of the structure cannot exist without deformation of the structure). On the other hand, for structures like an airplane, when they are flying (that is, when they are *unrestrained*), rigid body modes exist and, consequently, there exists motion without deformation of the structure; in this case, the stiffness matrix $\mathbf{k}$ is **positive semi-definite**.

<u>REMINDER</u>: If the real **quadratic form** $\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}$, associated with a **real symmetric** matrix $\mathbf{A}$, is **nonnegative** for all **real** $\boldsymbol{\xi}$, and is **zero** only if $\boldsymbol{\xi} = \mathbf{0}$, then the quadratic form is said to be **<u>positive definite</u>**. Then, by convention, we say that that the **matrix A** is **<u>positive definite</u>**. On the other hand, **a quadratic form** $\boldsymbol{\xi}^T\mathbf{A}\boldsymbol{\xi}$, associated with a **real symmetric** matrix $\mathbf{A}$, is said to be **<u>positive semi-definite</u>** when it takes on only nonnegative values for all real $\boldsymbol{\xi}$, but vanishes for some $\boldsymbol{\xi} \neq \mathbf{0}$. In this case we say that that the **matrix A** is **<u>positive semi-definite</u>**.

<u>EXAMPLE</u> [Problem 9.13 of the textbook]:

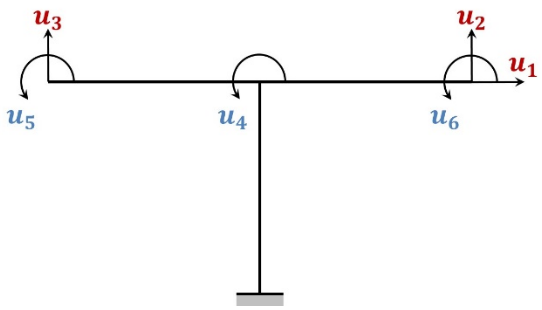
An umbrella structure has been idealized as an assemblage of three flexural elements with lumped masses at the nodes as shown in FIGURE.
(a) Identify the DOFs to represent the elastic properties and determine the stiffness matrix. Neglect axial deformations in all members.
(b) Identify the DOFs to represent the inertial properties and determine the mass matrix.
(c) Formulate the equations of motion governing the DOFs in part (b) when the excitation is (*i*) horizontal ground motion, (*ii*) vertical ground motion, (*iii*) ground motion in direction $b - d$, (*iv*) ground motion in direction $b - c$, and (*v*) rocking ground motion in the plane of the structure.

<u>SOLUTION</u>:



In view of the fact that we neglect axial deformation for all members we have only one horizontal DOF for all nodes (and consequently for all the corresponding concentrated masses m, $3m$ & m) at the level of the beams. Thus, we end up with a total number of six (6) DOFs for the entire structure (see FIGURE). Now, the DOFs u_4, u_5 & u_6, are associated with rotations of the corresponding nodes (and of the corresponding concentrated nitrated and thus are associated with zero rotational inertias. Thus,

$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_t \\ \mathbf{u}_0 \end{Bmatrix} \quad \mathbf{u}_t = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \quad \mathbf{u}_0 = \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$



Recall that the DOFs $\mathbf{u}_t$ are associated with **significant inertias** and are called the **dynamic DOFs**, while $\mathbf{u}_0$ are associated with **insignificant** (i.e. **zero**) **inertias**, are referred to as **static DOFs** and will be eliminated by **static condensation**.

The **stiffness matrix** can formally be derived by the **direct stiffness matrix method** (i.e. the stiffness matrix of each individual member is formulated and rotated to the selected global reference system; the element matrices are assembled to form the global matrix; the boundary conditions are imposed and the equations not involving reactions are retained).

For a simple structure (with few DOFs) like the one we are analyzing, we can obtain the stiffness matrix by imposing a unit displacement at each one of the DOFs sequentially, while '**locking**' (i.e. setting equal to zero) all other DOFs. Thus, by setting $u_1 = 1$ we obtain, using simple statics, the elements of the 1st column: k_{11}, k_{21}, k_{31}, k_{41}, k_{51}, and k_{61}. Proceeding with all the other DOFs in a similar way, we obtain

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 0 & 0 & 6L & 0 & 0 \\ 0 & 12 & 0 & -6L & 0 & -6L \\ 0 & 0 & 12 & 6L & 6L & 0 \\ 6L & -6L & 6L & 12L^2 & 2L^2 & 2L^2 \\ 0 & 0 & 6L & 2L^2 & 4L^2 & 0 \\ 0 & -6L & 0 & 2L^2 & 0 & 4L^2 \end{bmatrix}$$

The partitioned stiffness matrix may be written as

$$\mathbf{k} = \begin{bmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t0} \\ \mathbf{k}_{0t} & \mathbf{k}_{00} \end{bmatrix}$$

$$\mathbf{k}_{tt} = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} \quad \mathbf{k}_{t0} = \mathbf{k}_{0t}^T = \begin{bmatrix} 6L & 0 & 0 \\ -6L & 0 & -6L \\ 6L & 6L & 0 \end{bmatrix} \quad \mathbf{k}_{00} = \begin{bmatrix} 12L^2 & 2L^2 & 2L^2 \\ 2L^2 & 4L^2 & 0 \\ 2L^2 & 0 & 4L^2 \end{bmatrix}$$

In view of the fact that all masses are concentrated / lumped, the mass matrix $\mathbf{m}$ is a diagonal matrix

$$\mathbf{m} = m\begin{bmatrix} (3+1+1) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = m\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The **condensed stiffness matrix** (related only to the translational DOFs) is

$$\hat{\mathbf{k}}_{tt} = \mathbf{k}_{tt} - \mathbf{k}_{t0}\mathbf{k}_{00}^{-1}\mathbf{k}_{0t} = \frac{3EI}{10L^3}\begin{bmatrix} 28 & 6 & -6 \\ 6 & 7 & 3 \\ -6 & 3 & 7 \end{bmatrix}$$

The equations of dynamic equilibrium of the structure, for base excitation, are

$$\mathbf{m}\ddot{\mathbf{u}}^t + \hat{\mathbf{k}}_{tt}\mathbf{u} = \mathbf{0} \quad \Rightarrow \quad m\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1^t \\ \ddot{u}_2^t \\ \ddot{u}_3^t \end{Bmatrix} + \frac{3EI}{10L^3}\begin{bmatrix} 28 & 6 & -6 \\ 6 & 7 & 3 \\ -6 & 3 & 7 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For **horizontal base excitation** $u_{gx}(t)$

$$\mathbf{u}^t = \mathbf{u} + \boldsymbol{\iota}_x u_{gx} \quad \Rightarrow \quad \begin{Bmatrix} u_1^t \\ u_2^t \\ u_3^t \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} u_{gx} \quad \Rightarrow \quad \begin{cases} \mathbf{m}\ddot{\mathbf{u}} + \hat{\mathbf{k}}_{tt}\mathbf{u} = \mathbf{p}_{eff}(t) \\ \mathbf{p}_{eff}(t) = -\begin{Bmatrix} 5m \\ 0 \\ 0 \end{Bmatrix}\ddot{u}_{gx} \end{cases}$$

for **vertical base excitation** $u_{gy}(t)$

$$\mathbf{u}^t = \mathbf{u} + \boldsymbol{\iota}_y u_{gy} \quad \Rightarrow \quad \begin{Bmatrix} u_1^t \\ u_2^t \\ u_3^t \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} u_{gy} \quad \Rightarrow \quad \begin{cases} \mathbf{m}\ddot{\mathbf{u}} + \hat{\mathbf{k}}_{tt}\mathbf{u} = \mathbf{p}_{eff}(t) \\ \mathbf{p}_{eff}(t) = -\begin{Bmatrix} 0 \\ m \\ m \end{Bmatrix}\ddot{u}_{gy} \end{cases}$$

for **base excitation** $u_{gbd}(t)$ in the $b-d$ direction

$$\mathbf{u}^t = \mathbf{u} + \boldsymbol{\iota}_{bd} u_{gbd} \quad \Rightarrow \quad \begin{Bmatrix} u_1^t \\ u_2^t \\ u_3^t \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{Bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix} u_{gbd} \quad \Rightarrow \quad \begin{cases} \mathbf{m}\ddot{\mathbf{u}} + \hat{\mathbf{k}}_{tt}\mathbf{u} = \mathbf{p}_{eff}(t) \\ \mathbf{p}_{eff}(t) = -\begin{Bmatrix} 5m/\sqrt{2} \\ m/\sqrt{2} \\ m/\sqrt{2} \end{Bmatrix}\ddot{u}_{gbd} \end{cases}$$

for **base excitation** $u_{gbc}(t)$ in the $b-c$ direction

$$\mathbf{u}^t = \mathbf{u} + \boldsymbol{\iota}_{bc} u_{gbc} \quad \Rightarrow \quad \begin{Bmatrix} u_1^t \\ u_2^t \\ u_3^t \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{Bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{Bmatrix} u_{gbc} \quad \Rightarrow \quad \begin{cases} \mathbf{m}\ddot{\mathbf{u}} + \hat{\mathbf{k}}_{tt}\mathbf{u} = \mathbf{p}_{eff}(t) \\ \mathbf{p}_{eff}(t) = -\begin{Bmatrix} -5m/\sqrt{2} \\ m/\sqrt{2} \\ m/\sqrt{2} \end{Bmatrix}\ddot{u}_{gbc} \end{cases}$$

for **rocking base excitation** $u_{g\theta}(t)$ (i.e. rotation of base about a horizontal axis normal to the plane of the structure)

$$\mathbf{u}^t = \mathbf{u} + \boldsymbol{\iota}_\theta u_{g\theta} \quad \Rightarrow \quad \begin{Bmatrix} u_1^t \\ u_2^t \\ u_3^t \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} + \begin{Bmatrix} L \\ -L \\ L \end{Bmatrix} u_{gy} \quad \Rightarrow \quad \begin{cases} \mathbf{m}\ddot{\mathbf{u}} + \hat{\mathbf{k}}_{tt}\mathbf{u} = \mathbf{p}_{eff}(t) \\ \mathbf{p}_{eff}(t) = -\begin{Bmatrix} 5mL \\ -mL \\ mL \end{Bmatrix}\ddot{u}_{g\theta} \end{cases}$$

Solution of the Equations of Motion – Modal Superposition Method

Free Vibration of Systems without Damping

We start by considering the free-vibrations of an undamped system. Indeed, we are looking for the response of the system performing **synchronous motion** (i.e. the system vibrates **maintaining the overall shape** and changing only the amplitude by a time-dependent proportionality factor).

We seek solutions (of the equations of motion of the undamped system) of the form $\mathbf{u}(t) = e^{st}\boldsymbol{\phi}$, where $\boldsymbol{\phi}$ is the shape that the system maintains as it vibrates freely and e^{st} is the time-dependent proportionality factor:

$$\left.\begin{array}{r}\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{0} \\ \mathbf{u}(t) = e^{st}\boldsymbol{\phi}\end{array}\right\} \quad \Rightarrow \quad e^{st}(s^2\mathbf{m}\boldsymbol{\phi} + \mathbf{k}\boldsymbol{\phi}) = \mathbf{0}$$

$$\Rightarrow \quad \mathbf{k}\boldsymbol{\phi} = -s^2\mathbf{m}\boldsymbol{\phi}$$

Therefore,

$$\boxed{\mathbf{k}\boldsymbol{\phi} = \lambda\mathbf{m}\boldsymbol{\phi} \quad , \quad \lambda = -s^2}$$

The above problem is known as the **algebraic** (or **matrix**) **eigenvalue problem** (or **characteristic-value problem**).

Facts that we learn from **Linear Algebra** regarding the matrix eigenvalue problem:

(1) **The eigenvalues** of an algebraic eigenvalue problem, in which **k** & **m** are **both symmetric**, and at **least one positive definite**, are **all real**.
[**NOTE**: For stable civil engineering structures both matrices **k** & **m** are positive definite]

(2) **When matrices k** & **m** are **both positive definite**, **the eigenvalues are all positive**.

(3) When **k** is **singular, at least one** of the eigenvalues must be **zero**.
[**NOTE**: An example of a stable (but unconstrained) structure which has a singular **k** matrix is an airborne airplane.]
When **m** is **singular, at least one** of the eigenvalues must be **infinite**.
[**NOTE**: For a discrete model of a civil engineering structure that we develop and analyze in this course, **m** is always non-singular as we have eliminated (by static condensation) any degrees associated with insignificant inertia.]

(4) The **eigenvectors, corresponding to different eigenvalues, are orthogonal to each other with respect** to both **k** & **m**. Therefore

$$\boldsymbol{\phi}_n^T \mathbf{m} \boldsymbol{\phi}_m = \begin{cases} 0 & n \neq m \\ M_n & n = m \end{cases}$$
$$\boldsymbol{\phi}_n^T \mathbf{k} \boldsymbol{\phi}_m = \begin{cases} 0 & n \neq m \\ K_n & n = m \end{cases}$$

Evidently,

$$\mathbf{k}\boldsymbol{\phi}_n = \omega_n^2 \mathbf{m}\boldsymbol{\phi}_n \quad \Rightarrow \quad \boldsymbol{\phi}_n^T \mathbf{k}\boldsymbol{\phi}_n = \omega_n^2 \boldsymbol{\phi}_n^T \mathbf{m}\boldsymbol{\phi}_n \quad \Rightarrow \quad \omega_n^2 = \frac{\boldsymbol{\phi}_n^T \mathbf{k}\boldsymbol{\phi}_n}{\boldsymbol{\phi}_n^T \mathbf{m}\boldsymbol{\phi}_n} = \frac{K_n}{M_n}$$

(5) To an eigenvalue of **multiplicity** m there correspond m **linearly independent** eigenvectors. These m linearly independent eigenvectors can be made orthogonal (w.r.t. $\mathbf{k}$ & $\mathbf{m}$) to each other using the **Gram-Schmidt orthogonalization procedure**. These eigenvectors are already orthogonal to the rest $(n - m)$ eigenvectors
[**NOTE**: It is rather unlikely for a civil engineering structure to have eigenvalues with multiplicity higher than 1. However, it is not inconceivable.]

(6) **Any arbitrary N-vector $\mathbf{u}$** can **always** be expressed as a linear combination of the **linearly independent** eigenvectors of the algebraic eigenvalue problem:

$$\mathbf{u} = \sum_{r=1}^{N} \boldsymbol{\phi}_r q_r = \boldsymbol{\Phi}\mathbf{q}$$

where: $$\boldsymbol{\Phi} = \underbrace{\begin{pmatrix} \downarrow & \downarrow & & \downarrow \\ \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \cdots & \boldsymbol{\phi}_N \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}}_{\textit{Modal Matrix}}$$

where q_r $(r = 1,2,\cdots,N)$ are scalar multipliers called **modal coordinates** or **normal coordinates** and $\mathbf{q} = [q_1 \quad q_2 \quad \cdots \quad q_N]^T$; N = **no. of Degrees of Freedom (DOF) of the structure**.
[**NOTE**: The above statement, known as the "**eigenvector expansion theorem**", guaranties that whatever shape the structure takes as it vibrates, that shape can **always** be expressed as a **linear combination** of the modal shapes of the structure. This theorem forms the basis of the "**modal superposition method**" of expressing the response of the structure.]

Returning to the solution of the matrix eigenvalue problem, now we know that $\lambda > 0$ (always true for civil engineering structures). Therefore, we can set $\lambda = \omega^2$. This implies that $s = \pm\sqrt{-\lambda} = \pm i\omega$. It follows that the general solution of $\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{0}$ is $\mathbf{u}(t) = C_1 e^{i\omega t}\boldsymbol{\phi} + C_2 e^{-i\omega t}\boldsymbol{\phi} = \left(C_1 e^{i\omega t} + C_2 e^{-i\omega t}\right)\boldsymbol{\phi}$; **for this expression to represent actual vibrations it must be a real expression**; therefore $C_2 = \bar{C}_1 = A' + iB'$ (assuming that $C_1 = A' - iB'$). Then,

$$\begin{aligned}\mathbf{u}(t) &= 2\mathcal{R}e\left(C_1 e^{i\omega t}\right)\boldsymbol{\phi} \\ &= 2[A'\cos(\omega t) + B'\sin(\omega t)]\boldsymbol{\phi} \\ &= [A\cos(\omega t) + B\sin(\omega t)]\boldsymbol{\phi} \\ &= \rho\sin(\omega t + \theta)\,\boldsymbol{\phi} \qquad \begin{cases}\rho = \sqrt{A^2 + B^2} \\ \theta = \tan^{-1}\left(\dfrac{A}{B}\right)\end{cases}\end{aligned}$$

This demonstrates that the system/structure **performs synchronous harmonic motion maintaining all the time the same shape**.

Now, if we give an arbitrary **initial displacement** $\mathbf{u}(t=0) = \mathbf{u}_0$ and arbitrary **initial velocity** $\dot{\mathbf{u}}(t=0) = \dot{\mathbf{u}}_0$ to the structure, the response may be expressed by invoking the **eigenvector expansion theorem**. Specifically

$$\mathbf{u}(t) = \sum_{n=1}^{N} q_n(t)\boldsymbol{\phi}_n$$

Evidently, the step that precedes the above one is to solve the matrix eigenvalue problem $\mathbf{k}\boldsymbol{\phi} = \lambda\mathbf{m}\boldsymbol{\phi}$, and we know that we are going to obtain N **positive values** $\lambda_n = \omega_n^2$, $(n = 1,2,\cdots,N)$ and corresponding N **real eigenvectors** $\boldsymbol{\phi}_n$, $(n = 1,2,\cdots,N)$.

In order to decouple the equations of motion, we substitute the above expression for the response / solution $\mathbf{u}(t)$, we pre-multiply by $\boldsymbol{\phi}_j^T$ both sides of the equation of motion and we invoke the orthogonality theorem of the eigenvectors. As a result we obtain N uncoupled modal equations

$$\ddot{q}_j(t) + \omega_j^2 q_j(t) = 0 \quad , \quad (j = 1,2,\cdots,N)$$

subject to the following initial conditions

$$q_j(0) = \frac{\boldsymbol{\phi}_j^T\mathbf{m}\mathbf{u}_0}{M_n} \quad , \quad \dot{q}_j(0) = \frac{\boldsymbol{\phi}_j^T\mathbf{m}\dot{\mathbf{u}}_0}{M_n}$$

The solution / response of the above set of uncoupled modal equations is

$$q_j(t) = q_j(0)\cos(\omega_j t) + \frac{\dot{q}_j(0)}{\omega_j}\sin(\omega_j t) \quad , \quad (j = 1,2,\cdots,N)$$

[**NOTE**: The modal equations are mathematically equivalent to the equation of motion of a SDOF system. Clearly, any method that is appropriate to use to solve the equation of motion of a SDOF system is also appropriate to solve also the modal equations.]

We observe that in the general case, (that is **for arbitrary initial displacement and velocity**), **all modes participate in the response**. If we would like to excite only one mode, say the m-th mode, **both** the $\mathbf{u}_0$ and $\dot{\mathbf{u}}_0$ would have to be **proportional to** $\boldsymbol{\phi}_m$, i.e. $\mathbf{u}_0 = \alpha\boldsymbol{\phi}_m$ & $\dot{\mathbf{u}}_0 = \beta\boldsymbol{\phi}_m$.

EXAMPLE [Problem 10.23 of the textbook]:

(a) For the umbrella structure, determine the natural vibration frequencies and modes. Express the frequencies in terms of m, EI, & L, and sketch the modes.
(b) The structure is pulled through a lateral displacement $u_1(0) = 1$ and released. Determine the free vibration response.

SOLUTION:

We recall that

$$\mathbf{m} = m\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \hat{\mathbf{k}}_{tt} = \frac{3EI}{10L^3}\begin{bmatrix} 28 & 6 & -6 \\ 6 & 7 & 3 \\ -6 & 3 & 7 \end{bmatrix}$$

We form the eigenvalue problem:

$$\hat{\mathbf{k}}_{tt} - \omega^2\mathbf{m} = \frac{3EI}{10L^3}\begin{bmatrix} 28-5\lambda & 6 & -6 \\ 6 & 7-\lambda & 3 \\ -6 & 3 & 7-\lambda \end{bmatrix} \quad , \quad \lambda = \frac{10mL^3}{3EI}\omega^2$$

$$\begin{vmatrix} 28-5\lambda & 6 & -6 \\ 6 & 7-\lambda & 3 \\ -6 & 3 & 7-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad 5\lambda^3 - 98\lambda^2 + 520\lambda - 400 = 0$$

The **cubic equation** (like the **quadratic** & **quartic** equations **but not higher order equations**) can be solved algebraically; for a detailed discussion see https://en.wikipedia.org/wiki/Cubic_equation).

The roots of the above **characteristic equation** are:

$$\lambda_1 = 0.9219 \qquad \lambda_2 = 8.6780 \qquad \lambda_3 = 10$$

$$\omega_1 = 0.5259\sqrt{\frac{EI}{mL^3}} \quad \omega_2 = 1.6135\sqrt{\frac{EI}{mL^3}} \quad \omega_3 = 1.7321\sqrt{\frac{EI}{mL^3}}$$

Then, from the equation $\left[\hat{\mathbf{k}}_{tt} - \omega_n^2\mathbf{m}\right]\boldsymbol{\phi}_n = \mathbf{0}$, and after setting one of the elements of $\boldsymbol{\phi}_n$ equal to an arbitrary value (see **EXAMPLE** 10.1), we obtain

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} 1 \\ -1.9492 \\ 1.9492 \end{Bmatrix} \quad \boldsymbol{\phi}_2 = \begin{Bmatrix} 1 \\ 1.2826 \\ -1.2826 \end{Bmatrix} \quad \boldsymbol{\phi}_3 = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

Initial conditions:

$$\mathbf{u}(0) = \mathbf{u}_0 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$q_1(0) = \frac{\boldsymbol{\phi}_1^T \mathbf{m} \mathbf{u}_0}{\boldsymbol{\phi}_1^T \mathbf{m} \boldsymbol{\phi}_1} = 0.3969 \quad \dot{q}_1(0) = 0$$
$$q_2(0) = \frac{\boldsymbol{\phi}_2^T \mathbf{m} \mathbf{u}_0}{\boldsymbol{\phi}_2^T \mathbf{m} \boldsymbol{\phi}_2} = 0.6031 \quad \dot{q}_2(0) = 0$$
$$q_3(0) = \frac{\boldsymbol{\phi}_3^T \mathbf{m} \mathbf{u}_0}{\boldsymbol{\phi}_3^T \mathbf{m} \boldsymbol{\phi}_3} = 0 \qquad \dot{q}_3(0) = 0$$

The free vibration response to the given initial displacement is

$$\mathbf{u}(t) = \sum_{n=1}^{3} \boldsymbol{\phi}_n q_n(t) = \sum_{n=1}^{3} \boldsymbol{\phi}_n \left[q_n(0)\cos(\omega_n t) + \frac{\dot{q}_n(0)}{\omega_n}\sin(\omega_n t) \right]$$

or

$$\mathbf{u}(t) = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = \begin{Bmatrix} 0.3969 \\ -0.7736 \\ 0.7736 \end{Bmatrix} \cos(\omega_1 t) + \begin{Bmatrix} 0.6031 \\ 0.7736 \\ -0.7736 \end{Bmatrix} \cos(\omega_2 t)$$

Notice that the 3rd mode does not contribute to the response because the initial conditions do not contain a component in that mode (recall that $q_3(0) = 0$ & $\dot{q}_3(0) = 0$).

Free Vibration of Systems with Classical Damping

In order to simulate the **damping mechanisms** present in our physical structure, we introduce in the equations of motion the term $\mathbf{c}\dot{\mathbf{u}}$, where the **damping matrix** $\mathbf{c}$ is **<u>assumed</u> to satisfy the same properties of orthogonality as the matrices** $\mathbf{k}$ & $\mathbf{m}$. Specifically, the eigenvectors (modal shapes) $\mathbf{\Phi} = [\boldsymbol{\phi}_1 \quad \boldsymbol{\phi}_2 \quad \cdots \quad \boldsymbol{\phi}_n \quad \cdots \quad \boldsymbol{\phi}_N]$ that we obtain by solving the matrix eigenvalue problem $\mathbf{k}\boldsymbol{\phi} = \omega^2\mathbf{m}\boldsymbol{\phi}$ are <u>**assumed**</u> to be orthogonal w.r.t. the damping matrix $\mathbf{c}$ as well, i.e.

$$\boldsymbol{\phi}_n^T\mathbf{c}\boldsymbol{\phi}_m \quad = \quad \begin{cases} 0 & n \neq m \\ 2\xi_n\omega_n M_n & n = m \end{cases}$$

or in terms of the **modal matrix** $\mathbf{\Phi}$

$$\mathbf{\Phi}^T\mathbf{c}\mathbf{\Phi} \quad = \quad \underbrace{\begin{bmatrix} \ddots & & \\ & \underbrace{2\xi_n\omega_n M_n}_{C_n} & \\ & & \ddots \end{bmatrix}}_{diagonal\ matrix}$$

The parameters $\xi_n \,.\, (n = 1,2,\cdots,N)$ specify the damping ratios of the modes of the structure and are referred to as **modal damping ratios**.

[<u>**NOTE**</u>: It is **important to point out** that the matrix eigenvalue problem $\mathbf{k}\boldsymbol{\phi} = \omega^2\mathbf{m}\boldsymbol{\phi}$, solution of which produces the modal shapes $\boldsymbol{\phi}_n$ and characteristic circular frequencies ω_n, <u>**does not involve the matrix**</u> $\mathbf{c}$.]

The above kind of damping is referred to as <u>**classical damping**</u>. This model of damping is appropriate for all common/conventional civil engineering structures. However, if a structure is equipped with modern protective (aseismic) systems, such as base isolation and/or energy absorbing dashpots, then the damping model of classical damping is not satisfactory and one has to make use of the <u>**non-classical damping model**</u>. This model is not discussed in this course.

Let us consider the free vibrations of **a structure with classical damping**, subjected to **initial displacement** $\mathbf{u}(t = 0) = \mathbf{u}_0$ and **initial velocity** $\dot{\mathbf{u}}(t = 0) = \dot{\mathbf{u}}_0$. Therefore

$$\text{Eqn of Motion:} \qquad \mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{0}$$
$$\text{Initial Conditions:} \quad \mathbf{u}(t = 0) = \mathbf{u}_0 \quad , \quad \dot{\mathbf{u}}(t = 0) = \dot{\mathbf{u}}_0$$

Invoking again the **eigenvector expansion theorem**, we express the response / solution of the structure as a linear combination of the modal shapes:

$$\mathbf{u}(t) \quad = \quad \sum_{n=1}^{N} q_n(t)\boldsymbol{\phi}_n$$

Substituting the above expansion in the equations of motion, pre-multiplying by $\boldsymbol{\phi}_j^T$ and, invoking the **orthogonality property of the modal shapes** w.r.t. $\mathbf{k}$, $\mathbf{m}$ & $\mathbf{c}$, we obtain

N uncoupled **modal equations** in terms of the **normal co-ordinates** $q_n(t)$, $(j = 1,2,\cdots,N)$:

$$\ddot{q}_j(t) + 2\xi_j\omega_j\dot{q}_j(t) + \omega_j^2 q_j(t) = 0 \quad , \quad (j = 1,2,\cdots,N)$$

subject to the following initial conditions

$$q_j(0) = \frac{\boldsymbol{\phi}_j^T \mathbf{m}\mathbf{u}_0}{M_n} \quad , \quad \dot{q}_j(0) = \frac{\boldsymbol{\phi}_j^T \mathbf{m}\dot{\mathbf{u}}_0}{M_n}$$

The solution / response of the above set of uncoupled modal equations is

$$q_j(t) = e^{-\xi_j\omega_j t}\left\{q_j(0)\cos(\omega_{Dj}t) + \frac{\dot{q}_j(0) + \xi_j\omega_j q_j(0)}{\omega_{Dj}}\sin(\omega_{Dj}t)\right\} \quad , \quad (j = 1,2,\cdots,N)$$

$$\omega_{Dj} = \omega_j\sqrt{1-\xi_j^2}$$

We observe that in the general case, (that is **for arbitrary initial displacement and velocity**), **all modes participate in the response**. If we would like to excite only one mode, say the m-th mode, both the $\mathbf{u}_0$ and $\dot{\mathbf{u}}_0$ would have to be **proportional to $\boldsymbol{\phi}_m$**, i.e. $\mathbf{u}_0 = \alpha\boldsymbol{\phi}_m$ & $\dot{\mathbf{u}}_0 = \beta\boldsymbol{\phi}_m$.

NOTE: For the analysis of **conventional structures** using discrete models developed and discussed in this course, it is not necessary to form the damping matrix $\mathbf{c}$. The only information that the analyst needs is the damping ratios ξ_j , $(j = 1,2,\cdots,N)$ of the modes of the structure. The damping ratios ξ_j are used in the modal equations.

However, there are problems of dynamic analysis of structures that require the formation of a damping matrix $\mathbf{c}$. For example, a Finite Element Model (FEM) of the structure including part of the soil supporting the structure would be such a problem. The reason is that the nature and amount of damping in the soil is very different from that in the structure. Therefore, an explicit damping matrix needs to be formed for the soil part of the finite element mesh and a separate damping matrix for the mesh modeling the structure. Eventually, the two submatrices are combined to form a damping matrix for the complete soil-structure system.

One way to form a damping matrix that provides classical damping is using the so called **CAUGHEY damping series**. Specifically

$$\left.\begin{aligned} \mathbf{c} &= \mathbf{m}\sum_{\ell} \alpha_\ell[\mathbf{m}^{-1}\mathbf{k}]^\ell \\ \xi_j &= \frac{1}{2\omega_j}\sum_{\ell} \alpha_\ell \omega_j^{2\ell} \end{aligned}\right\} \qquad \ell = \cdots,-2,-1,0,+1,+2,\cdots$$

The damping ratios ξ_j for as many modes need to be provided. However, for effective use of the CAUGHEY damping series, an **<u>even number of terms</u> must be used in the series expression**.

Dynamic Analysis of Structural Systems with Classical Damping

Let us consider a N-DOF system, having **classical damping** and subjected to the general loading $\mathbf{p}(t)$ and having **initial displacement** $\mathbf{u}(t=0)=\mathbf{u}_0$ and **initial velocity** $\dot{\mathbf{u}}(t=0)=\dot{\mathbf{u}}_0$. [**NOTE**: Normally, in most cases, civil engineering structures start vibrating from rest.] Therefore

$$\text{Eqn of Motion:} \qquad \mathbf{m}\ddot{\mathbf{u}}+\mathbf{c}\dot{\mathbf{u}}+\mathbf{k}\mathbf{u}=\mathbf{p}(t)$$
$$\text{Initial Conditions:} \quad \mathbf{u}(t=0)=\mathbf{u}_0 \quad , \quad \dot{\mathbf{u}}(t=0)=\dot{\mathbf{u}}_0$$

Invoking again the **eigenvector expansion theorem**, we express the response / solution of the structure as a linear combination of the modal shapes:

$$\mathbf{u}(t)=\sum_{n=1}^{N}\mathbf{u}_n(t)=\sum_{n=1}^{N}q_n(t)\boldsymbol{\phi}_n$$

Substituting the above expansion in the equations of motion, pre-multiplying by $\boldsymbol{\phi}_j^T$ and invoking the **orthogonality property of the modal shapes** w.r.t. $\mathbf{k}$, $\mathbf{m}$ & $\mathbf{c}$ we obtain N uncoupled **modal equations** in terms of the **normal co-ordinates** $q_j(t)\,,(j=1,2,\cdots,N)$:

$$M_j\ddot{q}_j(t)+C_j\dot{q}_j(t)+K_jq_j(t)=P_j(t) \quad , \quad (j=1,2,\cdots,N)$$
$$\Rightarrow \qquad \ddot{q}_j(t)+\frac{C_j}{M_j}\dot{q}_j(t)+\frac{K_j}{M_j}q_j(t)=\frac{P_j(t)}{M_j}$$
$$\Rightarrow \qquad \ddot{q}_j(t)+2\xi_j\omega_j\dot{q}_j(t)+\omega_j^2q_j(t)=\frac{P_j(t)}{M_j}$$
$$\text{where} \qquad P_j(t)=\boldsymbol{\phi}_j^T\mathbf{p}(t)$$

subject to the following initial conditions

$$q_j(0)=\frac{\boldsymbol{\phi}_j^T\mathbf{m}\mathbf{u}_0}{M_n} \quad , \quad \dot{q}_j(0)=\frac{\boldsymbol{\phi}_j^T\mathbf{m}\dot{\mathbf{u}}_0}{M_n}$$

The above equations governing the response of the **normal co-ordinates** $q_j(t)\,,(j=1,2,\cdots,N)$ is mathematically identical to the equation of motion of a SDOF system. Any one of the techniques that we developed in obtaining the response of the SDOF system, evidently may be applied here as well.

Modal Analysis for $\mathbf{p}(t)=\mathbf{s}p(t)$

One particular type of loading $\mathbf{p}(t)$ is of interest to us. This is the case of $\mathbf{p}(t)=\mathbf{s}p(t)$, where all the applied loads have **common time variation** $p(t)$, while the (time-independent) vector $\mathbf{s}$ describes the **spatial distribution** of the load. This type of load describes various practical cases, including the **earthquake load** (i.e. the load induced by support motion) to be considered later.

We **resolve** vector $\mathbf{s}$ into its **modal components**:

$$\mathbf{s} = \sum_{r=1}^{N} \mathbf{s}_r = \sum_{r=1}^{N} \Gamma_r \mathbf{m} \boldsymbol{\phi}_r$$

$$\Gamma_r = \frac{\boldsymbol{\phi}_r^T \mathbf{s}}{M_r} \qquad \textbf{\textit{Modal Participation Factor}}$$

It is evident that Γ_r is **not independent** of how the modal shape is normalized. However, the modal component $\mathbf{s}_r = \Gamma_r \mathbf{m} \boldsymbol{\phi}_r$ **is independent** of how the modal shape is normalized.

The above expansion of the $\mathbf{s}$ vector has two useful properties:

(1) The force vector $\mathbf{s}_n p(t)$ **produces response only in the** n^{th} **mode** but **no response in any other mode**.
(2) The dynamic response of **the** nth **mode is entirely due to the partial force vector** $\mathbf{s}_n p(t)$.

It should be noted that **the spatial distribution of the inertia forces** $(\mathbf{f}_I)_n$ associated with the n^{th} mode, **is the same as that of** $\mathbf{s}_n = \Gamma_n \mathbf{m} \boldsymbol{\phi}_n$:

$$(\mathbf{f}_I)_n = \mathbf{m}\ddot{\mathbf{u}}_n(t) = \mathbf{m}\boldsymbol{\phi}_n \ddot{q}_n(t)$$

The uncoupled modal equations in this case are:

$$\ddot{q}_n(t) + 2\xi_n \omega_n \dot{q}_n(t) + \omega_n^2 q_n(t) = \frac{P_n(t)}{M_n} \quad (n = 1,2,\cdots,N)$$

$$\text{where:} \quad P_n(t) = \boldsymbol{\phi}_n^T \mathbf{p}(t) = \boldsymbol{\phi}_n^T \mathbf{s} p(t) = \Gamma_n M_n p(t)$$

Therefore:

$$\ddot{q}_n(t) + 2\xi_n \omega_n \dot{q}_n(t) + \omega_n^2 q_n(t) = \Gamma_n p(t) \quad (n = 1,2,\cdots,N)$$

Introducing the new variable: $D_n(t) = q_n(t)/\Gamma_n \Leftrightarrow q_n(t) = \Gamma_n D_n(t)$, we obtain

$$\ddot{D}_n(t) + 2\xi_n \omega_n \dot{D}_n(t) + \omega_n^2 D_n(t) = p(t) \quad (n = 1,2,\cdots,N)$$

The reason that we express the modal equations in terms of the variable $D_n(t)$ (a seemingly trivial substitution) is because $D_{no} \stackrel{\text{def}}{=} \max_t |D_n(t)|$ **may be read directly from the response spectrum of** $p(t)$.

In order to find **element forces** (i.e. actions such as moments and shear forces of the various structural members of a structure subjected to dynamic analysis) we implement the **equivalent static force method**. Specifically, as the structure vibrates it deforms and the equivalent static forces that would cause the deformed shape $\mathbf{u}(t)$ at any instant in time are the **elastic forces** $\mathbf{f}_S(t) = \mathbf{k}\mathbf{u}(t)$ corresponding to that time instant. It is evident that

$$\mathbf{u}(t) = \sum_{n=1}^{N} \mathbf{u}_n(t) = \sum_{n=1}^{N} q_n(t)\boldsymbol{\phi}_n = \sum_{n=1}^{N} \Gamma_n D_n(t)\boldsymbol{\phi}_n$$

and

$$\mathbf{f}_S(t) = \mathbf{k}\mathbf{u}(t) = \mathbf{k}\sum_{n=1}^{N} \mathbf{u}_n(t) = \sum_{n=1}^{N} \mathbf{k}\mathbf{u}_n(t) = \sum_{n=1}^{N} \mathbf{f}_{Sn}(t)$$

<u>We work with each mode separately</u>. Therefore, **the equivalent static forces $\mathbf{f}_{Sn}(t)$ in the nth mode are**

$$\mathbf{f}_{Sn}(t) = \mathbf{k}\mathbf{u}_n(t) = \Gamma_n \underbrace{\mathbf{k}\boldsymbol{\phi}_n}_{\omega_n^2 \mathbf{m}\boldsymbol{\phi}_n} D_n(t) = \underbrace{\Gamma_n \mathbf{m}\boldsymbol{\phi}_n}_{\mathbf{s}_n} [\omega_n^2 D_n(t)]$$

$$\Rightarrow \boxed{\mathbf{f}_{Sn}(t) = \mathbf{s}_n[\omega_n^2 D_n(t)]}$$

In words, the above result makes evident that the elastic forces in the nth mode consist of **the modal component $\mathbf{s}_n$** (which describes the spatial distribution of the elastic forces in the nth mode and is time independent) **scaled by the time-varying coefficient** $[\omega_n^2 D_n(t)]$. This suggests that in order to compute **any <u>response quantity</u>** $r(t)$ [such as **actions** (*e.g.* moment, shear force) acting on members of the structure, or **displacements** at any of the nodes of the structure], we need to perform N **static analyses**. The results of these static analyses, r_n^{st} , $(n = 1,2, \cdots, N)$, will be scaled by the respective time-varying coefficients, $[\omega_n^2 D_n(t)]$, $(n = 1,2, \cdots, N)$, to obtain the corresponding modal contribution, $r_n(t)$, $(n = 1,2, \cdots, N)$, to the response quantity $r(t)$.

$$
\begin{array}{ccccccc}
\mathbf{s}_1 & \rightarrow \boxed{MDOF\ System} \rightarrow & r_1^{st} & \cdot & \omega_1^2 D_1(t) & = & r_1(t) \\
\mathbf{s}_2 & \rightarrow \boxed{MDOF\ System} \rightarrow & r_2^{st} & \cdot & \omega_2^2 D_2(t) & = & r_2(t) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{s}_N & \rightarrow \boxed{MDOF\ System} \rightarrow & r_N^{st} & \cdot & \omega_N^2 D_N(t) & = & r_N(t)
\end{array}
$$

$$\mathbf{s} = \sum_N \mathbf{s}_n \quad \rightarrow \boxed{MDOF\ System} \rightarrow \quad r^{st} = \sum_N r_n^{st} \qquad\qquad r(t) = \sum_N r_n(t)$$

<u>**NOTE**</u>: From the relation $r^{st} = \sum_N r_n^{st}$ we obtain $1 = \sum_N \left(\frac{r_n^{st}}{r^{st}}\right) = \sum_N \bar{r}_n$. Evidently $\bar{r}_n = \left(\frac{r_n^{st}}{r^{st}}\right)$ **expresses the contribution of the nth mode to the response quantity** $r(t)$ and that is why it is referred to as <u>**the nth modal contribution factor**</u> to the response quantity $r(t)$. The advantages of these factors over the <u>nth</u> modal participation factors Γ_n are: (1) they are **dimensionless**; (2) they are **independent of how modes are normalized**; and (3) the sum of the modal contribution factors over all modes is unity, that is $\sum_N \bar{r}_n = 1$.

Now, let us consider the **peak response of the system**. Let us introduce the following definitions:

$$D_{no} \stackrel{\text{def}}{=} \max_t |D_n(t)|$$

$$\left(D_{n,st}\right)_o = \max_t \underbrace{|D_{n,st}(t)|}_{\substack{static\\response}} = \left(\frac{\max_t |p(t)|}{\omega_n^2}\right) = \left(\frac{p_o}{\omega_n^2}\right) \quad \underbrace{R_{dn} = \frac{D_{no}}{\left(D_{n,st}\right)_o}}_{\substack{Dynamic\\Response\ Factor}}$$

NOTE: The **static response** $D_{n,st}(t)$ is obtained from $\ddot{D}_n(t) + 2\xi_n\omega_n\dot{D}_n(t) + \omega_n^2 D_n(t) = p(t)$, **by dropping the** $\ddot{D}_n$ **&** $\dot{D}_n$ **terms**.

It follows that:

$$r_{no} = \underbrace{r^{st}\bar{r}_n}_{r_n^{st}}\,\omega_n^2 D_{no} = r^{st}\bar{r}_n p_o R_{dn}$$

NOTE: The algebraic sign of r_{no} is the same as that of $r_n^{st} \stackrel{\text{def}}{=} r^{st}\bar{r}_n$ because R_{dn} is positive by definition.

- r^{st} & $\bar{r}_n$: **depend on** the **spatial distribution s** of the applied forces, but are **independent of** the **time variation** $p(t)$ of the applied forces.
- R_{dn} : **depends on** $p(t)$, but is **independent of s.**

Earthquake Analysis of Linear Systems

In the case of **base excitation** (**earthquake problem**) there are **no external forces acting on the structure**, i.e. $\mathbf{p}(t) = \mathbf{0}$. Therefore

$$\mathbf{f}_I(t) + \mathbf{f}_D(t) + \mathbf{f}_S(t) = \mathbf{0}$$

It should be emphasized that the inertia forces vector depends on **absolute accelerations** (i.e. **accelerations measured w.r.t. an inertial / Newtonian frame of reference**). The **absolute / total displacement vector** $\mathbf{u}^t(t)$ (the superscript 't' stands for 'total') may be resolved as follows

$$\mathbf{u}^t(t) = \mathbf{u}(t) + \boldsymbol{\iota} u_g(t)$$

where: $\boldsymbol{\iota} u_g(t)$ is the part of displacements that describes **rigid body motion** of the structure as it undergoes support motion $u_g(t)$, (i.e. if the structure were massless and were subjected to support motion $u_g(t)$); $\boldsymbol{\iota}$ is the **influence vector**; $\mathbf{u}(t)$ are the **additional displacements / deformations** that the structure (with its mass) experiences **due to the inertia forces that are induced** as the structure accelerates due to support movement.

Evidently, the elastic forces $\mathbf{f}_S(t)$ are associated with $\mathbf{u}(t)$ (i.e. the part of displacements associated with the deformations of the structure), i.e. $\mathbf{f}_S(t) = \mathbf{k}\mathbf{u}(t)$, while the damping forces $\mathbf{f}_D(t)$ are associated with the rate of deformations $\dot{\mathbf{u}}(t)$, i.e. $\mathbf{f}_D(t) = \mathbf{c}\dot{\mathbf{u}}(t)$. Therefore, the equation of dynamic equilibrium transforms to

$$\begin{aligned} \mathbf{m}\left[\ddot{\mathbf{u}}(t) + \boldsymbol{\iota}\ddot{u}_g(t)\right] + \mathbf{c}\dot{\mathbf{u}}(t) + \mathbf{k}\mathbf{u}(t) &= \mathbf{0} \\ \mathbf{m}\ddot{\mathbf{u}}(t) + \mathbf{c}\dot{\mathbf{u}}(t) + \mathbf{k}\mathbf{u}(t) &= -\mathbf{m}\boldsymbol{\iota}\ddot{u}_g(t) \end{aligned}$$

The right-hand side term $\mathbf{p}_{eff}(t) = -\mathbf{m}\boldsymbol{\iota}\ddot{u}_g(t)$ are the **effective earthquake forces**. Therefore, $\mathbf{p}_{eff}(t)$ are of the form $\mathbf{s}p(t)$ where $\mathbf{s} = \mathbf{m}\boldsymbol{\iota}$ and $p(t) = \left(-\ddot{u}_g(t)\right)$. Thus, all the development presented above regarding the response / solution for $\mathbf{p}(t) = \mathbf{s}p(t)$ applies also for $\mathbf{p}_{eff}(t) = -\mathbf{m}\boldsymbol{\iota}\ddot{u}_g(t)$.

For instance, the expansion of the vector $\mathbf{s} = \mathbf{m}\boldsymbol{\iota}$ of the effective earthquake forces is

$$\mathbf{s} = \mathbf{m}\boldsymbol{\iota} = \sum_{r=1}^{N} \mathbf{s}_r = \sum_{r=1}^{N} \Gamma_r \mathbf{m}\boldsymbol{\phi}_r \quad , \quad \Gamma_r = \frac{\boldsymbol{\phi}_r^T \mathbf{s}}{M_r} = \frac{\boldsymbol{\phi}_r^T \mathbf{m}\boldsymbol{\iota}}{M_r}$$

The modal equations are

$$\ddot{D}_n(t) + 2\xi_n\omega_n\dot{D}_n(t) + \omega_n^2 D_n(t) = -\ddot{u}_g(t) \quad (n = 1,2,\cdots,N)$$

The above equation is identical to the equation that is used to compute **earthquake response spectra** which display the peak (absolute) value of the response $D_n(t)$, i.e. $D_{no} \overset{\text{def}}{=} \max_t |D_n(t)|$, by sweeping the circular frequency axis and for selected values of the damping ratio. **Response spectra** may display the same information in two alternative but equivalent forms:

- in terms of the **pseudo-velocity** $V_{no} = \max_t |V_n(t)| = \max_t |\omega_n D_n(t)| = \omega_n D_{no}$; and,
- in terms of **pseudo-acceleration** $A_{no} = \max_t |A_n(t)| = \max_t |\omega_n^2 D_n(t)| = \omega_n^2 D_{no}$

EXAMPLE [Problem 13.17 of the textbook]:

For the umbrella structure of the FIGURE excited by **horizontal** ground motion $\ddot{u}_{gx}(t)$, determine (a) the modal expansion of effective earthquake forces, (b) the displacement response in terms of $D_n(t)$, and (c) the bending moments at the base of the column and at location a of the beam in terms of $A_n(t)$.

SOLUTION:

We recall that for the given structure we have:

$$\mathbf{m} = m\begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad \hat{\mathbf{k}}_{tt} = \frac{3EI}{10L^3}\begin{bmatrix} 28 & 6 & -6 \\ 6 & 7 & 3 \\ -6 & 3 & 7 \end{bmatrix}$$

$$\omega_1 = 0.5259\sqrt{\frac{EI}{mL^3}} \quad \omega_2 = 1.6135\sqrt{\frac{EI}{mL^3}} \quad \omega_3 = 1.7321\sqrt{\frac{EI}{mL^3}}$$

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} 1 \\ -1.9492 \\ 1.9492 \end{Bmatrix} \quad \boldsymbol{\phi}_2 = \begin{Bmatrix} 1 \\ 1.2826 \\ -1.2826 \end{Bmatrix} \quad \boldsymbol{\phi}_3 = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

For the given excitation, the **influence vector** $\boldsymbol{\iota}$ is

$$\boldsymbol{\iota} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

The **effective earthquake forces** are:

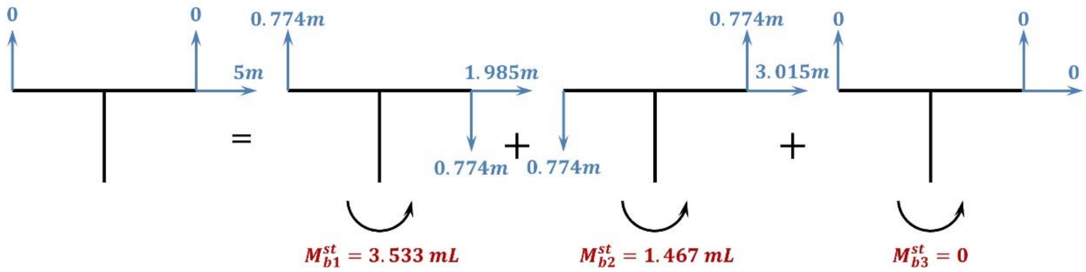
$$\mathbf{s} = \sum_{n=1}^{3} \mathbf{s}_n = \sum_{n=1}^{3} \Gamma_n \mathbf{m} \boldsymbol{\phi}_n$$

$$L_1 = \boldsymbol{\phi}_1^T \mathbf{m} \boldsymbol{\iota} = 5m \quad L_2 = \boldsymbol{\phi}_2^T \mathbf{m} \boldsymbol{\iota} = 5m \quad L_3 = \boldsymbol{\phi}_3^T \mathbf{m} \boldsymbol{\iota} = 0$$

$$M_1 = \boldsymbol{\phi}_1^T \mathbf{m} \boldsymbol{\phi}_1 = 12.597m \quad M_2 = \boldsymbol{\phi}_2^T \mathbf{m} \boldsymbol{\phi}_2 = 8.292m \quad M_3 = \boldsymbol{\phi}_3^T \mathbf{m} \boldsymbol{\phi}_3 = 2m$$

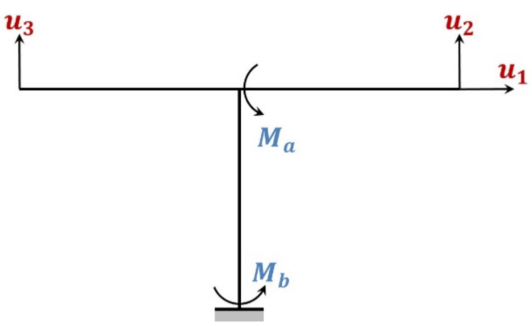
$$\Gamma_1 = \frac{L_1}{M_1} = 0.397 \quad \Gamma_2 = \frac{L_2}{M_2} = 0.603 \quad \Gamma_3 = \frac{L_3}{M_3} = 0$$

$$\mathbf{s}_1 = \Gamma_1 \mathbf{m} \boldsymbol{\phi}_1 = \begin{Bmatrix} 1.985m \\ -0.774m \\ 0.774m \end{Bmatrix} \quad \mathbf{s}_2 = \Gamma_2 \mathbf{m} \boldsymbol{\phi}_2 = \begin{Bmatrix} 3.015m \\ 0.774m \\ -0.774m \end{Bmatrix} \quad \mathbf{s}_3 = \Gamma_3 \mathbf{m} \boldsymbol{\phi}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$



The displacement response in terms of $D_n(t)$ is

$$\mathbf{u}(t) = \sum_{n=1}^{3} \Gamma_n \boldsymbol{\phi}_n D_n(t) = \begin{Bmatrix} 0.397 \\ -0.774 \\ 0.774 \end{Bmatrix} D_1(t) + \begin{Bmatrix} 0.603 \\ 0.774 \\ -0.774 \end{Bmatrix} D_2(t) + \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} D_3(t)$$



The bending moment at the base of the column (point b) in terms of its modal contributions is

$$
\begin{aligned}
M_b(t) &= \sum_{n=1}^{3} M_{bn}(t) \\
&= \sum_{n=1}^{3} M_{bn}^{st} A_n(t) \\
&= 3.533 \cdot mL \cdot A_1(t) + 1.467 \cdot mL \cdot A_2(t)
\end{aligned}
$$

The bending moment at location a of the beam in terms of its modal contributions is

$$
\begin{aligned}
M_a(t) &= \sum_{n=1}^{3} M_{an}(t) \\
&= \sum_{n=1}^{3} M_{an}^{st} A_n(t) \\
&= -0.774 \cdot mL \cdot A_1(t) + 0.774 \cdot mL \cdot A_2(t)
\end{aligned}
$$

In **Earthquake Engineering** we have two types of analysis:

- **Response History Analysis (RHA)**
- **Response Spectrum Analysis (RSA)**

Response History Analysis (RHA) is feasible with the capacity and capabilities of present day personal computers. It is an "exact" analysis in that it is based on the exact (analytical or numerical) integration of the equations of motion that govern the response of the mathematical model that we have adopted. Thus, for a given ground acceleration, we can compute the time history of any response quantity $r(t) = \sum_N r_n(t)$.

Structural design is usually based on **peak values of forces** and **deformations** over the duration of the earthquake-induced response. Estimation / calculation of such peak values may be determined directly from the **response spectrum**. Such an approach is referred to as **Response Spectrum Analysis (RSA)**. We can use the response spectrum to **<u>predict exactly</u>** the response of a **SDOF system**, and **<u>estimate approximately</u>** the response of **MDOF systems**. The latter estimate is **accurate enough for structural design applications**.

Peak Modal Response of any **response quantity** $r(t)$ is given by

$$r_{no} = r_n^{st} A_{no}$$

where, we recall that $A_{no} = \max_t |A_n(t)| = A(T_n, \xi_n)$; the value $A(T_n, \xi_n)$ (which is always non-negative) is the **peak (absolute) value** of the **pseudo-acceleration** and is **read from the response spectrum**. Evidently, **all response quantities** $r(t)$

associated with a particular mode, say the nth mode, **reach their peak values at the same time instant as $A_n(t)$ reaches its peak**.

[**NOTE**: The textbook is using the following notation: $A_n = \max_t |A_n(t)|$]

The **basic question** is the following: **How do we combine the peak modal responses** $r_{no} (n = 1,2,\cdots,N)$ to determine the **peak value** $r_o \overset{\text{def}}{=} \max_t |r(t)|$ **of the total response**? It will **not** be possible to determine the exact value of r_o from r_{no} **because**, in general, **the modal responses $r_n(t)$ attain their peaks at different time instants and the combined $r(t)$ attains its peak at yet a different instant**.

Modal Combination Rules

- The **Absolute Sum (ABSSUM)** modal combination rule:

$$r_o \leq \sum_{n=1}^{N} |r_{no}|$$

This **upper-bound value** is usually **too conservative**.

- The **Square-Root-of-Sum-of-Squares (SRSS)** rule:

$$r_o \cong \left(\sum_{n=1}^{N} r_{no}^2 \right)^{1/2}$$

This rule provides **excellent response estimates for structures with well-separated natural frequencies**.

- The **Complete Quadratic Combination (CQC)** rule:

$$r_o \cong \left(\sum_{i=1}^{N} \sum_{n=1}^{N} \rho_{in} r_{io} r_{no} \right)^{1/2}$$

where: ρ_{in} = **correlation coefficient** of modes i & n

[$0 \leq \rho_{in} \leq 1$; $\rho_{in} = 1$ for $i = n$]

It can be demonstrated that the **double summation** inside the parentheses is **always positive**.

The expression for the CQC rule may be written as:

$$r_o \cong \left(\sum_{n=1}^{N} r_{no}^2 + \underbrace{\sum_{i=1}^{N} \sum_{n=1}^{N}}_{i \neq n} \rho_{in} r_{io} r_{no} \right)^{1/2}$$

The estimate for r_o, obtained by the CRC rule, maybe **larger** or **smaller** than the estimate provided by the SRSS rule.

The SRSS & CQC rules have been derived based on **RANDOM VIBRATION THEORY** (also referred to as **STOCHASTIC STRUCTURAL DYNAMICS**)

Implications of the assumptions behind the derivations:

The modal combination rules would be **most accurate for**:

- earthquake excitation that contain a **wide band of frequencies** (**white noise** assumption);
- with **long phases of strong shaking** (**stationarity**);
- which (i.e., long phases) are **several times longer than** T_1 (=fundamental period) of the structure (**stationarity**);
- which (i.e., modes) are **not too lightly damped** ($\xi_n > 0.005$).

The modal combination rules become **less accurate** for **short-duration impulsive ground motions** and are **not recommended** for **ground motions that contain many cycles of essentially harmonic excitation**

EXAMPLE [Problem 13.50 of the textbook, but using metric units]:

The umbrella structure of the FIGURE (also of previous EXAMPLES) is made of $150 - mm -$nominal diameter standard steel pipe. Its properties are: $I = 1171.6\ cm^4$, $E = 200{,}000\ MPa$, mass = $28.23\ kg/m$, $m = 680\ kg$, and $L = 3\ m$.

Determine the peak response of this structure to **horizontal** ground motion characterized by the design spectrum of Fig. 6.9.5 (for 5% damping) scaled to $0.20g$ peak ground acceleration. Using the SRSS combination rule, estimate:

(a) displacements u_1, u_2, and u_3, and
(b) the bending moments at the base of the column and at location a of the beam.

SOLUTION:

Weight of the pipe (we consider: $g = 9.81\ m/s^2$; recall that: $1\ N = 1\ kg \times 1(\frac{m}{s^2})$):
$3 \times 3m \times 28.23 \frac{kg}{m} \times g = 2{,}492.43\ N$;

The weight of the concentrated masses is:

$$(3+1+1)mg = 5 \cdot 680\ kg \cdot 9.81 \frac{m}{s^2} = 33{,}354\ N$$

Notice that the weight of the structural members is very small compared to the weight of the concentrated masses; thus we ignore it.

Compute the characteristic periods of the structure ($1\ Pa = 1\ N/m^2$):

$$\omega_1 = 0.5259\sqrt{\frac{EI}{mL^3}} \quad \omega_2 = 1.6135\sqrt{\frac{EI}{mL^3}} \quad \omega_3 = 1.7321\sqrt{\frac{EI}{mL^3}}$$

$$\sqrt{\frac{EI}{mL^3}} = \sqrt{\frac{2\times10^{11}\left(\frac{N}{m^2}\right)\times 1171.6\times10^{-8}m^4}{680\left(\frac{N\cdot s^2}{m}\right)\times(3)^3 m^3}} = 11.2971\left(\frac{rad}{s}\right)$$

$$\omega_1 = 5.927\left(\frac{rad}{s}\right) \quad \omega_2 = 17.951\left(\frac{rad}{s}\right) \quad \omega_3 = 19.634\left(\frac{rad}{s}\right)$$
$$T_1 = 1{,}06\ s \qquad T_2 = 0.35\ s \qquad T_3 = 0.32\ s$$

For the above values of the natural periods of the structure, the design spectrum of FIG. 6.9.5 gives

$$A_1 = 0.2 \times \frac{1.80g}{1.06} = 0.340g \quad \Rightarrow \quad D_1 = 9.5\ cm$$
$$A_2 = 0.2 \times 2.71g = 0.542g \quad \Rightarrow \quad D_2 = 1{,}7\ cm$$
$$A_3 = 0.2 \times 2.71g = 0.542g \quad \Rightarrow \quad D_3 = 1{,}4\ cm$$

Recall that we have previously determined that

$$\mathbf{u}(t) = \begin{Bmatrix} 0.397 \\ -0.774 \\ 0.774 \end{Bmatrix} D_1(t) + \begin{Bmatrix} 0.603 \\ 0.774 \\ -0.774 \end{Bmatrix} D_2(t) + \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} D_3(t)$$

Therefore, the **peak modal responses of displacement** are:

$$\mathbf{u}_1 = \max_t(\mathbf{u}_1(t)) = \begin{Bmatrix} 0.397 \\ -0.774 \\ 0.774 \end{Bmatrix} \times 9.5 = \begin{Bmatrix} 3{,}77 \\ -7{,}35 \\ 7{,}35 \end{Bmatrix} cm$$
$$\mathbf{u}_2 = \max_t(\mathbf{u}_1(t)) = \begin{Bmatrix} 0.603 \\ 0.774 \\ -0.774 \end{Bmatrix} \times 1{,}7 = \begin{Bmatrix} 1.03 \\ 1.32 \\ -1.32 \end{Bmatrix} cm$$

$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	SRSS
$\begin{Bmatrix} 3{,}77 \\ -7{,}35 \\ 7{,}35 \end{Bmatrix} cm$	$\begin{Bmatrix} 1.03 \\ 1.32 \\ -1.32 \end{Bmatrix} cm$	$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} cm$	$\begin{Bmatrix} 3.91 \\ 7.47 \\ 7.47 \end{Bmatrix} cm$

Recall that for **horizontal excitation**, modal components of the effective earthquake forces $\mathbf{s}$ are

$$\mathbf{s}_1 = \begin{Bmatrix} 1.985m \\ -0.774m \\ 0.774m \end{Bmatrix} \quad \mathbf{s}_2 = \begin{Bmatrix} 3.015m \\ 0.774m \\ -0.774m \end{Bmatrix} \quad \mathbf{s}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Peak responses in the nth mode are **induced by the equivalent static forces** $\mathbf{f}_n = \mathbf{s}_n A_n$:

$$\mathbf{f}_1 = \mathbf{s}_1 A_1 = \begin{Bmatrix} 1.985m \\ -0.774m \\ 0.774m \end{Bmatrix} 0.340g = \begin{Bmatrix} 0.6749 \\ -0.2632 \\ 0.2632 \end{Bmatrix} mg = \begin{Bmatrix} 4.502 \\ -1.755 \\ 1.755 \end{Bmatrix} kN$$

$$\mathbf{f}_2 = \mathbf{s}_2 A_2 = \begin{Bmatrix} 3.015m \\ 0.774m \\ -0.774m \end{Bmatrix} 0.542g = \begin{Bmatrix} 1.6341 \\ 0.4195 \\ -0.4195 \end{Bmatrix} mg = \begin{Bmatrix} 10.901 \\ 2.798 \\ -2.798 \end{Bmatrix} kN$$

$$\mathbf{f}_3 = \mathbf{s}_3 A_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} 0.542g = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} kN$$

We subject the structure to forces $\mathbf{f}_n = \mathbf{s}_n A_n$, we perform **static analysis** of the structure and we obtain the **peak values** $(M_a)_{no}$ & $(M_b)_{no}$ of the bending moments **due to each mode**:

	Mode 1	Mode 2	Mode 3	SRSS
$(M_a)_{no}$	5.265	7.829	0	9.435
$(M_b)_{no}$	24.036	15.915	0	28.827

<u>*Comment*</u>

In evaluating the accuracy of the estimates based on the SRSS rule, one obvious choice is to compute the time history response and compare the SRSS estimates with the peak values of the time-history response. The other option is to obtain estimates based on the CQC combination rule and compare them with the SRSS results. We observe that the periods of the 2nd and 3rd modes are very close ($0.35\ s$ vs. $0.32\ s$). As a consequence we would expect the cross-terms involving these two modes to make a significant contribution. All other cross-terms are expected to be significant. However, we have shown that the 3rd mode does not participate in the response. Consequently, the estimates using the CQC rule are expected to be very close to those of the SRSS rule.

<u>EXAMPLE</u>:

The umbrella structure of the previous examples is subjected to an impulsive loading $\delta(t)$ along DOF #1 (the structure starts moving from rest).

(1) Compute the displacement response of the structure to the impulsive loading.

(2) Using the above result compute the response of the structure to a loading acting along DOF #1 with time variation $p(t)$.

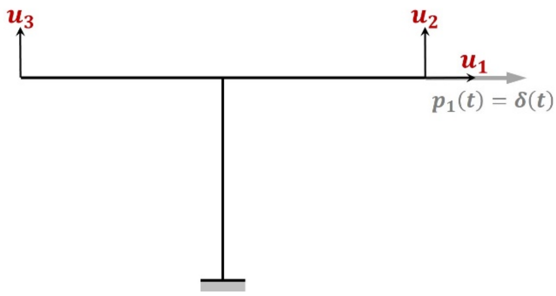
<u>SOLUTION</u>:

For the given structure we know:

$$\boldsymbol{\phi}_1 = \begin{Bmatrix} 1 \\ -1.9492 \\ 1.9492 \end{Bmatrix} \quad \boldsymbol{\phi}_2 = \begin{Bmatrix} 1 \\ 1.2826 \\ -1.2826 \end{Bmatrix} \quad \boldsymbol{\phi}_3 = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} \quad \mathbf{m} = m \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The given impulsive loading may be expressed as follows:

$$\mathbf{p}(t) = \mathbf{s}\delta(t) = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \delta(t)$$



The vector $\mathbf{s}$ is resolved into its modal components:

$$\mathbf{s} = \sum_{n=1}^{3} \mathbf{s}_n = \sum_{n=1}^{3} \Gamma_n \mathbf{m} \boldsymbol{\phi}_n$$

$L_1 = \boldsymbol{\phi}_1^T \mathbf{s} = 1$ $\quad$ $L_2 = \boldsymbol{\phi}_2^T \mathbf{s} = 1$ $\quad$ $L_3 = \boldsymbol{\phi}_3^T \mathbf{s} = 0$

$M_1 = \boldsymbol{\phi}_1^T \mathbf{m} \boldsymbol{\phi}_1 = 12.597m$ $\quad$ $M_2 = \boldsymbol{\phi}_2^T \mathbf{m} \boldsymbol{\phi}_2 = 8.292m$ $\quad$ $M_3 = \boldsymbol{\phi}_3^T \mathbf{m} \boldsymbol{\phi}_3 = 2m$

$\Gamma_1 = \dfrac{L_1}{M_1} = 0.079m^{-1}$ $\quad$ $\Gamma_2 = \dfrac{L_2}{M_2} = 0.121m^{-1}$ $\quad$ $\Gamma_3 = \dfrac{L_3}{M_3} = 0$

$\mathbf{s}_1 = \Gamma_1 \mathbf{m} \boldsymbol{\phi}_1 = \begin{Bmatrix} 0.395 \\ -0.154 \\ 0.154 \end{Bmatrix}$ $\quad$ $\mathbf{s}_2 = \Gamma_2 \mathbf{m} \boldsymbol{\phi}_2 = \begin{Bmatrix} 0.605 \\ 0.155 \\ -0.155 \end{Bmatrix}$ $\quad$ $\mathbf{s}_3 = \Gamma_3 \mathbf{m} \boldsymbol{\phi}_3 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

NOTE: Above, we have evaluated the modal components of $\mathbf{s}$ even though this is not necessary for computing the displacement response. The modal components of $\mathbf{s}$ will be necessary to calculate any r_n^{st}(e.g. moments, shears, etc.) that may be needed. Specifically

$$r(t) = \sum_N r_n(t) = \sum_N r_n^{st} A_n(t) = \sum_N r_n^{st} \omega_n^2 D_n(t)$$

Therefore, the displacement response is:

$$\mathbf{h}(t) \stackrel{\text{def}}{=} \mathbf{u}(t)|_{\delta(t)} = \sum_{n=1}^{N} \mathbf{u}_n(t) = \sum_{n=1}^{N} q_n(t) \boldsymbol{\phi}_n = \sum_{n=1}^{N} \Gamma_n D_n(t) \boldsymbol{\phi}_n$$

$$\begin{Bmatrix} 0.079m^{-1} \\ -0.154m^{-1} \\ 0.154m^{-1} \end{Bmatrix} D_1(t) + \begin{Bmatrix} 0.121m^{-1} \\ 0.155m^{-1} \\ -0.155m^{-1} \end{Bmatrix} D_2(t)$$

where $D_n(t)\,, (n = 1,2,3)$ is governed by the modal equations

$$\ddot{D}_n(t) + 2\xi_n \omega_n \dot{D}_n(t) + \omega_n^2 D_n(t) = \delta(t)$$

with initial conditions

$$D_n(0) = \frac{1}{\Gamma_n} q_n(0) = \frac{1}{\Gamma_n} \frac{\boldsymbol{\phi}_n^T \mathbf{m} \mathbf{u}_0}{M_n} = 0 \quad , \quad \dot{D}_n(0) = \frac{1}{\Gamma_n} \dot{q}_n(0) = \frac{1}{\Gamma_n} \frac{\boldsymbol{\phi}_n^T \mathbf{m} \dot{\mathbf{u}}_0}{M_n} = 0$$

The solution is

$$D_n(t) = \frac{1}{\omega_n} e^{-\xi_n \omega_n t} \sin(\omega_{dn} t) \quad , \quad \omega_{dn} = \omega_n \sqrt{1 - \xi_n^2}$$

Notice that the above solution is the Green's function (unit impulse response) $h_n(t)$ of the governing differential equation $\ddot{h}_n(t) + 2\xi_n \omega_n \dot{h}_n(t) + \omega_n^2 h_n(t) = \delta(t)$.

The response to any other kind of loading $p(t)$ may be computed by convolving the above derived solution with $p(t)$, i.e.

$$\mathbf{u}(t)|_{p(t)} = \mathbf{h}(t) * p(t) = \sum_{n=1}^{N} \Gamma_n [h_n(t) * p(t)] \boldsymbol{\phi}_n$$

EXAMPLE:

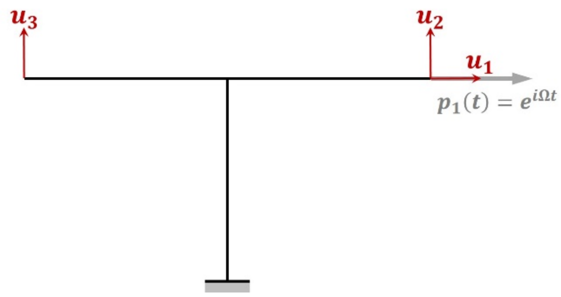
The umbrella structure of the previous examples is subjected to harmonic loading $e^{i\Omega t}$ along DOF #1 (the structure starts moving from rest).

(1) Compute the steady-state displacement response of the structure.
(2) Using the above response results, compute the response of the structure to a loading acting along DOF #1 with time variation $p(t) \leftrightarrow P(\Omega)$.

SOLUTION:

The given impulsive loading may be expressed as follows:

$$\mathbf{p}(t) = \mathbf{s} e^{i\Omega t} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} e^{i\Omega t}$$



The **steady-state** response may be written as follows:

$$\mathbf{u}(t)|_{ss} = \sum_{n=1}^{N} \mathbf{u}_n(t)|_{ss} = \sum_{n=1}^{N} q_n(t)|_{ss} \boldsymbol{\phi}_n = \sum_{n=1}^{N} \Gamma_n D_n(t)|_{ss} \boldsymbol{\phi}_n$$

where $D_n(t)|_{ss}$, $(n = 1,2,3)$ is governed by the modal equations

$$\ddot{D}_n(t)\big|_{ss} + 2\xi_n \omega_n \dot{D}_n(t)\big|_{ss} + \omega_n^2 D_n(t)|_{ss} = e^{i\Omega t}$$

We know that the steady-state solution of the above equation is of the form $D_n(t)|_{ss} = H_n(\Omega) e^{i\Omega t}$. If we substitute the above expression in the modal equation we obtain

$$[(i\Omega)^2 H_n(\Omega) + 2\xi_n \omega_n (i\Omega) H_n(\Omega) + \omega_n^2 H_n(\Omega)] e^{i\Omega t} = e^{i\Omega t}$$

Or

$$H_n(\Omega) = \frac{1}{\omega_n^2 - \Omega^2 + i2\xi_n\omega_n\Omega} = \frac{\left(\frac{1}{\omega_n^2}\right)}{\left(1 - \left(\frac{\Omega}{\omega_n}\right)^2\right) + i2\xi_n\left(\frac{\Omega}{\omega_n}\right)} = \frac{\left(\frac{1}{\omega_n^2}\right)}{(1-\beta^2) + i2\xi_n\beta}$$

where: $\beta = (\Omega/\omega_n)$.

Therefore, the **steady-state response** is

$$D_n(t)|_{ss} = H_n(\Omega)e^{i\Omega t} = \frac{\left(\frac{1}{\omega_n^2}\right)}{(1-\beta^2) + i2\xi_n\beta} e^{i\Omega t}$$

Recall, that the function: $H_n(\Omega) = [\omega_n^2 - \Omega^2 + i2\xi_n\omega_n\Omega]^{-1}$ is referred to as **complex frequency response**.

Now, the modal equations for the loading $\mathbf{p}(t) = \mathbf{s}p(t)$ would be written as

$$\ddot{D}_n(t) + 2\xi_n\omega_n\dot{D}_n(t) + \omega_n^2 D_n(t) = p(t) \quad , \quad (n = 1,2,3)$$

Let us solve the above equations by using the **Fourier Transform** $\mathcal{F}\{\quad\}$. Let the Fourier Transform pairs $p(t) \leftrightarrow P(\Omega)$ & $D_n(t) \leftrightarrow \widetilde{D}_n(\Omega)$. Then

$$\mathcal{F}\{\ddot{D}_n(t) + 2\xi_n\omega_n\dot{D}_n(t) + \omega_n^2 D_n(t)\} = \mathcal{F}\{p(t)\}$$

$$(i\Omega)^2\widetilde{D}_n(\Omega) + 2\xi_n\omega_n(i\Omega)\widetilde{D}_n(\Omega) + \omega_n^2\widetilde{D}_n(\Omega) = P(\Omega)$$

Therefore

$$\widetilde{D}_n(\Omega) = \frac{1}{\omega_n^2 - \Omega^2 + i2\xi_n\omega_n\Omega} P(\Omega)$$

<u>**NOTE**</u>: The Fourier Transform $\widetilde{D}_n(\Omega)$ of $D_n(t)$ is the product of the **complex frequency response function** $H_n(\Omega)$ times the Fourier Transform of the time variation of the loading $p(t) \leftrightarrow P(\Omega)$.

The response $D_n(t)$ in the time domain is obtained by **inverse Fourier Transform** $\mathcal{F}^{-1}\{\quad\}$:

$$D_n(t) = \mathcal{F}^{-1}\{\widetilde{D}_n(\Omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widetilde{D}_n(\Omega)e^{+i\Omega t}\, d\Omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{P(\Omega)e^{+i\Omega t}}{\omega_n^2 - \Omega^2 + i2\xi_n\omega_n\Omega}\, d\Omega$$

In the previous EXAMPLE we derived the response to an arbitrary loading $\mathbf{p}(t) = \mathbf{s}p(t)$:

$$\mathbf{u}(t)|_{p(t)} = \mathbf{h}(t) * p(t) = \sum_{n=1}^{N} \Gamma_n[h_n(t) * p(t)]\boldsymbol{\Phi}_n$$

If we take the Fourier Transform of the above expression, we have

$$\mathcal{F}\left\{\mathbf{u}(t)|_{p(t)}\right\} = \mathcal{F}\{\mathbf{h}(t) * p(t)\} = \mathcal{F}\left\{\sum_{n=1}^{N} \Gamma_n [h_n(t) * p(t)] \boldsymbol{\phi}_n\right\}$$

Or

$$\tilde{\mathbf{u}}(\Omega)|_{P(\Omega)} = \tilde{\mathbf{h}}(\Omega) \cdot P(\Omega) = \sum_{n=1}^{N} \Gamma_n \left[\tilde{h}_n(\Omega) \cdot P(\Omega)\right] \boldsymbol{\phi}_n$$

Evidently

$$\tilde{\mathbf{h}}(\Omega) = \sum_{n=1}^{N} \Gamma_n \tilde{h}_n(\Omega) \boldsymbol{\phi}_n$$

The above result could have been derived by taking the Fourier Transform of the following expression that we have derived in the previous EXAMPLE:

$$\mathbf{h}(t) \stackrel{\text{def}}{=} \mathbf{u}(t)|_{\delta(t)} = \sum_{n=1}^{N} \Gamma_n D_n(t) \boldsymbol{\phi}_n$$

Or

$$\mathcal{F}\{\mathbf{h}(t)\} = \mathcal{F}\left\{\mathbf{u}(t)|_{\delta(t)}\right\} = \mathcal{F}\left\{\sum_{n=1}^{N} \Gamma_n D_n(t) \boldsymbol{\phi}_n\right\}$$

Or

$$\tilde{\mathbf{h}}(\Omega) = \tilde{\mathbf{u}}(\Omega)|_1 = \sum_{n=1}^{N} \Gamma_n \widetilde{D}_n(\Omega) \boldsymbol{\phi}_n$$
