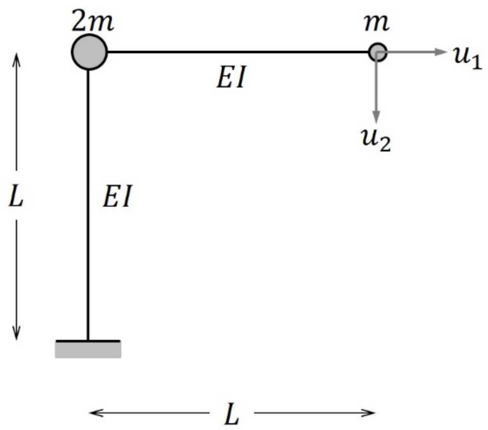
**<u>ΑΣΚΗΣΗ</u>: (<u>**10** μονάδες</u>)**

Για τον εικονιζόμενο φορέα, δίδονται:

$$\mathbf{\Phi} = \begin{bmatrix} 1 & 1 \\ 2.097 & -1.431 \end{bmatrix} \quad , \quad \mathbf{\Omega} = \left(\sqrt{\frac{EI}{mL^3}}\right) \cdot \begin{bmatrix} 0.6987 & \\ & 1.874 \end{bmatrix}$$

Θεωρήσατε τον φορέα άνευ απόσβεσης. Επισης θεωρήσατε οτι παραμορφώσεις λόγω αξονικών και τεμνουσών δυνάμεων ειναι αμελητέες.




(1) Υπολογίσατε την απόκριση, $\mathbf{u}(t)$, του φορέα στην αρχική μετατόπιση του φορέα:

$$\mathbf{u}(t=0) = u_o \cdot \begin{Bmatrix} -1 \\ 1.431 \end{Bmatrix} \quad , \quad \dot{\mathbf{u}}(t=0) = \mathbf{0}$$

Τι παρατηρείτε και γιατί?

(2) Δίδονται οι ακόλουθες δυνάμεις διέγερσης του φορέα:

$$\mathbf{p}(t) = \begin{Bmatrix} 0 \\ \delta(t) \end{Bmatrix}$$

Υπολογίσατε την ιδιομορφική ανάπτυξη (της χωρικής κατανομής) του διανύσματος διέγερσης των ανωτέρω δυνάμεων.

(3) Υπολογίσατε την απόκριση, $\mathbf{u}(t)$, του φορέα στις ανωτέρω δυνάμεις. Θεωρούμε οτι ο φορέας εκκινεί απο την κατάσταση ηρεμίας. Υπολογίσατε την ροπή ανατροπής, $\mathcal{M}_b(t)$, στο σημείο στήριξης του φορέα στο έδαφος, που προκαλείται από τις ανωτέρω ωστικές δυνάμεις.

(4) Δίδονται οι ακόλουθες δυνάμεις διέγερσης του φορέα:

$$\mathbf{p}(t) = \begin{Bmatrix} 0 \\ e^{i\Omega t} \end{Bmatrix}$$

Υπολογίσατε την μόνιμη (steady state) αρμονική απόκριση, $\mathbf{u}(t)$, του φορέα στις ανωτέρω αρμονικές δυνάμεις.
Υπολογίσατε την αντίστοιχη ροπή ανατροπής, $\mathcal{M}_b(t)$, στο σημείο στήριξης του φορέα στο έδαφος, που προκαλείται από τις ανωτέρω αρμονικές δυνάμεις.

**SOLUTION**:

Equation of motion for the structure (with no damping) due to initial conditions (without any external forces):

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{0} \quad , \quad \left(\mathbf{u}(t=0) = \mathbf{u}_o = u_o \cdot \begin{pmatrix} -1 \\ 1.431 \end{pmatrix} \quad , \quad \dot{\mathbf{u}}(t=0) = \dot{\mathbf{u}}_o = \mathbf{0}\right)$$

We notice that:

$$\mathbf{u}(t=0) \quad = \quad \mathbf{u}_o \quad = \quad (-u_o) \cdot \boldsymbol{\phi}_2$$

Modal expansion of the response:

$$\mathbf{u}(t) \quad = \quad \sum_{i=1}^{N} \boldsymbol{\phi}_i q_i(t)$$

Modal equations:

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = 0 \quad , \quad (i = 1,2,\cdots,N)$$

with initial conditions:

$$q_1(t=0) \quad = \quad \frac{\boldsymbol{\phi}_1^T \mathbf{m} \mathbf{u}_o}{\boldsymbol{\phi}_1^T \mathbf{m} \boldsymbol{\phi}_1} \quad = \quad (-u_o) \cdot \frac{\boldsymbol{\phi}_1^T \mathbf{m} \boldsymbol{\phi}_2}{\boldsymbol{\phi}_1^T \mathbf{m} \boldsymbol{\phi}_1} \quad = \quad 0$$

$$q_2(t=0) \quad = \quad \frac{\boldsymbol{\phi}_2^T \mathbf{m} \mathbf{u}_o}{\boldsymbol{\phi}_2^T \mathbf{m} \boldsymbol{\phi}_2} \quad = \quad (-u_o) \cdot \frac{\boldsymbol{\phi}_2^T \mathbf{m} \boldsymbol{\phi}_2}{\boldsymbol{\phi}_2^T \mathbf{m} \boldsymbol{\phi}_2} \quad = \quad (-u_o)$$

Obviously:

$$\dot{q}_1(t=0) \quad = \quad \frac{\boldsymbol{\phi}_1^T \mathbf{m} \dot{\mathbf{u}}_o}{\boldsymbol{\phi}_1^T \mathbf{m} \boldsymbol{\phi}_1} \quad = \quad 0$$

$$\dot{q}_2(t=0) \quad = \quad \frac{\boldsymbol{\phi}_2^T \mathbf{m} \dot{\mathbf{u}}_o}{\boldsymbol{\phi}_2^T \mathbf{m} \boldsymbol{\phi}_2} \quad = \quad 0$$

Therefore:

$$\left.\begin{array}{r} \ddot{q}_1(t) + \omega_1^2 q_1(t) = 0 \\ q_1(t=0) = 0 \\ \dot{q}_1(t=0) = 0 \end{array}\right\} \quad \rightarrow \quad q_1(t) = 0$$

Clearly, there is no response component in the first (fundamental) mode because the original displacement is proportional to the second mode shape.

$$\left.\begin{array}{r} \ddot{q}_2(t) + \omega_2^2 q_2(t) = 0 \\ q_2(t=0) = (-u_o) \\ \dot{q}_2(t=0) = 0 \end{array}\right\} \quad \rightarrow \quad q_2(t) = q_2(t=0) \cdot \cos(\omega_2 t) + \frac{\dot{q}_2(t=0)}{\omega_2} \cdot \sin(\omega_2 t) = (-u_o) \cdot \cos(\omega_2 t)$$

Therefore:

$$\boxed{\mathbf{u}(t) \quad = \quad \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} \quad = \quad (-u_o) \cdot \cos(\omega_2 t) \cdot \begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix}}$$

For the given forces:

$$\mathbf{p}(t) = \begin{Bmatrix} 0 \\ \delta(t) \end{Bmatrix} = \underbrace{\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}}_{\mathbf{s}} \cdot \delta(t)$$

Therefore:

$$\mathbf{s} = \sum_{r=1}^{N} \mathbf{s}_r = \sum_{r=1}^{N} \Gamma_r \mathbf{m} \boldsymbol{\phi}_r = \sum_{r=1}^{N} \left( \frac{\boldsymbol{\phi}_r^T \mathbf{s}}{M_r} \right) \mathbf{m} \boldsymbol{\phi}_r$$

$$L_1 = \boldsymbol{\phi}_1^T \mathbf{s} = \begin{bmatrix} 1 & 2.097 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = 2.097$$

$$M_1 = \boldsymbol{\phi}_1^T \mathbf{m} \boldsymbol{\phi}_1 = \begin{bmatrix} 1 & 2.097 \end{bmatrix} \begin{bmatrix} 3m & \\ & m \end{bmatrix} \begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} = 7.397m$$

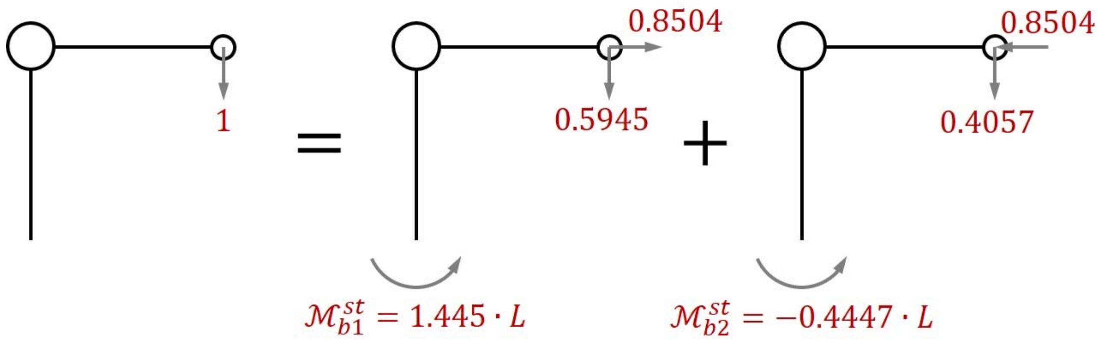
$$\mathbf{s}_1 = \left( \frac{L_1}{M_1} \right) \mathbf{m} \boldsymbol{\phi}_1 = \left( \frac{2.097}{7.397m} \right) \begin{bmatrix} 3m & \\ & m \end{bmatrix} \begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} = \begin{Bmatrix} 0.8505 \\ 0.5945 \end{Bmatrix}$$

and

$$L_2 = \boldsymbol{\phi}_2^T \mathbf{s} = \begin{bmatrix} 1 & -1.431 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = -1.431$$

$$M_2 = \boldsymbol{\phi}_2^T \mathbf{m} \boldsymbol{\phi}_2 = \begin{bmatrix} 1 & -1.431 \end{bmatrix} \begin{bmatrix} 3m & \\ & m \end{bmatrix} \begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix} = 5.048m$$

$$\mathbf{s}_2 = \left( \frac{L_2}{M_2} \right) \mathbf{m} \boldsymbol{\phi}_2 = \left( \frac{-1.431}{5.048m} \right) \begin{bmatrix} 3m & \\ & m \end{bmatrix} \begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix} = \begin{Bmatrix} -0.8504 \\ 0.4057 \end{Bmatrix}$$




It follows that the induced static moment at the base of the column (for each mode separately):

$$\mathcal{M}_{b1}^{st} = 0.8505 \cdot L + 0.5945 \cdot L = 1.445 \cdot L$$
$$\mathcal{M}_{b2}^{st} = -0.8504 \cdot L + 0.4057 \cdot L = -0.4447 \cdot L$$

Displacement response:

$$\mathbf{u}(t) = \sum_{i=1}^{2} \Gamma_i \boldsymbol{\phi}_i D_i(t)$$

Where:

$$\ddot{D}_i(t) + \omega_i^2 D_i(t) = \delta(t) \quad , \quad (i = 1,2) \quad D_i(t = 0) = 0 \quad \& \quad \dot{D}_i(t = 0) = 0$$

It follows that:

$$D_i(t) = \frac{1}{\omega_i} \cdot \sin(\omega_i t)$$

Therefore:

$$
\begin{aligned}
\mathbf{u}(t) &= \sum_{i=1}^{2} \Gamma_i \boldsymbol{\phi}_i D_i(t) \\
&= \left(\frac{L_1}{M_1}\right) \boldsymbol{\phi}_1 D_1(t) + \left(\frac{L_2}{M_2}\right) \boldsymbol{\phi}_2 D_2(t) \\
&= \boxed{0.2835 \cdot \begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} \cdot \frac{1}{m\omega_1} \cdot \sin(\omega_1 t) + (-0.2835) \cdot \begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix} \cdot \frac{1}{m\omega_2} \cdot \sin(\omega_2 t)}
\end{aligned}
$$

It easily follows that:

$$
\begin{aligned}
\mathcal{M}_b(t) &= \mathcal{M}_{b1}^{st} \cdot [\omega_1^2 \cdot D_1(t)] + \mathcal{M}_{b2}^{st} \cdot [\omega_2^2 \cdot D_2(t)] \\
&= \mathcal{M}_{b1}^{st} \cdot \omega_1 \cdot \sin(\omega_1 t) + \mathcal{M}_{b2}^{st} \cdot \omega_2 \cdot \sin(\omega_2 t) \\
&= \left(\sqrt{\frac{EI}{mL}}\right) \cdot [1.445 \cdot 0.6987 \cdot \sin(\omega_1 t) - 0.4447 \cdot 1.874 \cdot \sin(\omega_2 t)] \\
&= \boxed{\left(\sqrt{\frac{EI}{mL}}\right) \cdot [1.0096 \cdot \sin(\omega_1 t) - 0.8334 \cdot \sin(\omega_2 t)]}
\end{aligned}
$$

For the harmonic forces:

$$
\mathbf{p}(t) = \begin{Bmatrix} 0 \\ e^{i\Omega t} \end{Bmatrix} = \underbrace{\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}}_{\mathbf{s}} \cdot e^{i\Omega t}
$$

The steady-state displacement response:

$$
\mathbf{u}(t) = \sum_{i=1}^{2} \Gamma_i \boldsymbol{\phi}_i D_i(t)
$$

Where:

$$
\ddot{D}_i(t) + \omega_i^2 D_i(t) = e^{i\Omega t} \quad , \quad (i = 1,2)
$$

and

$$
D_i(t) = H_i(\Omega) \cdot e^{i\Omega t}
$$

[<u>NOTE</u>: The participation factors are the same as for the previous case as vector **s** remains the same.]

It follows that:

$$
H_i(\Omega) = \frac{1}{\omega_i^2 - \Omega^2}
$$

Therefore:

$$
\begin{aligned}
\mathbf{u}(t) &= \sum_{i=1}^{2} \Gamma_i \boldsymbol{\phi}_i D_i(t) \\
&= \left(\frac{L_1}{M_1}\right) \boldsymbol{\phi}_1 D_1(t) + \left(\frac{L_2}{M_2}\right) \boldsymbol{\phi}_2 D_2(t) \\
&= \boxed{\left(0.2835 \cdot \begin{Bmatrix} 1 \\ 2.097 \end{Bmatrix} \cdot \frac{1}{\omega_1^2 - \Omega^2} + (-0.2835) \cdot \begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix} \cdot \frac{1}{\omega_2^2 - \Omega^2}\right) \cdot e^{i\Omega t}}
\end{aligned}
$$

The corresponding bending moment is expressed as follows:

$$
\begin{aligned}
\mathcal{M}_b(t) &= \mathcal{M}_{b1}^{st} \cdot [\omega_1^2 \cdot D_1(t)] + \mathcal{M}_{b2}^{st} \cdot [\omega_2^2 \cdot D_2(t)] \\
&= \left(\mathcal{M}_{b1}^{st} \cdot \frac{\omega_1^2}{\omega_1^2 - \Omega^2} + \mathcal{M}_{b2}^{st} \cdot \frac{\omega_2^2}{\omega_2^2 - \Omega^2}\right) \cdot e^{i\Omega t} \\
&= \left(\frac{EI}{mL^2}\right) \cdot \left[1.445 \cdot 0.6987^2 \cdot \frac{1}{\omega_1^2 - \Omega^2} - 0.4447 \cdot 1.874^2 \cdot \frac{1}{\omega_2^2 - \Omega^2}\right] \cdot e^{i\Omega t} \\
&= \boxed{\left(\frac{EI}{mL^2}\right) \cdot \left[0.7054 \cdot \frac{1}{\omega_1^2 - \Omega^2} - 1.5617 \cdot \frac{1}{\omega_2^2 - \Omega^2}\right] \cdot e^{i\Omega t}}
\end{aligned}
$$