FORMULATION OF THE MDOF EQUATIONS OF MOTION

ΕΞΙΣΩΣΕΙΣ ΚΙΝΗΣΗΣ, ΔΙΑΤΥΠΩΣΗ ΤΟΥ ΠΡΟΒΛΗΜΑΤΟΣ ΚΑΙ ΜΕΘΟΔΟΙ ΕΠΙΛΥΣΗΣ

For an accurate description of its displaced configuration, a structural system subjected to dynamic disturbances may require the specification of displacements along more than one **coordinate direction**.

Such a system is known as a **Multi-Degree-Of-Freedom (MDOF)** system.

The number of displacement components which must be considered in order **to represent the effects of all significant inertia forces** of a structure may be termed the ***number of dynamic degrees of freedom* of the structure**.

The **degrees of freedom (DOF)** in a ***discrete-parameter system*** may be taken as:

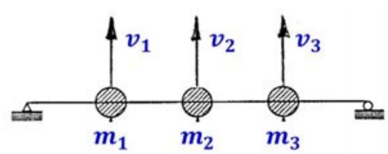
- The **displacement amplitudes of certain selected points** in the structure (FIGURE a)
  or
- They may be **generalized coordinates** representing the amplitudes of a specified set of **displacement patterns** (FIGURE b).

$$v(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

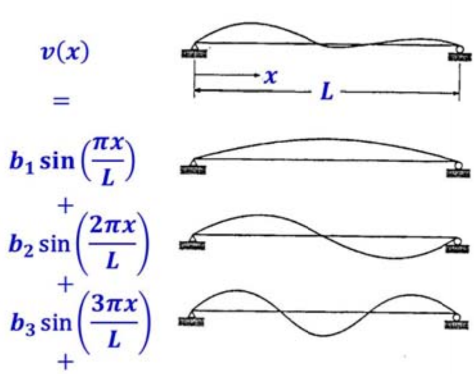
or

$$v(x) = \sum_{n=1}^{\infty} Z_n \psi_n(x) \qquad Z_n = \textbf{generalized coordinates}$$

Ιδιομορφικές ή Κανονικές Συντεταγμένες

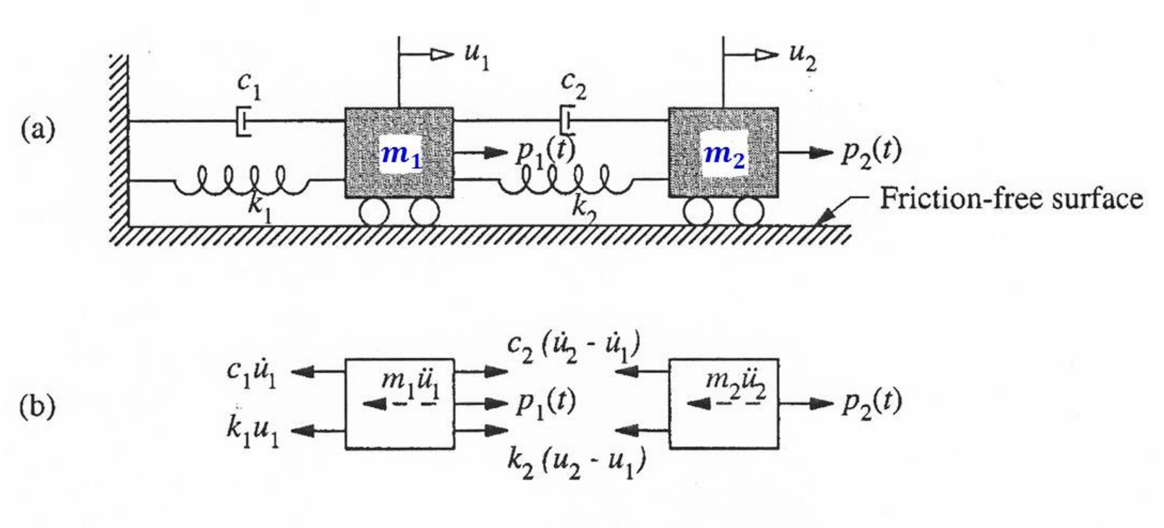


(a)

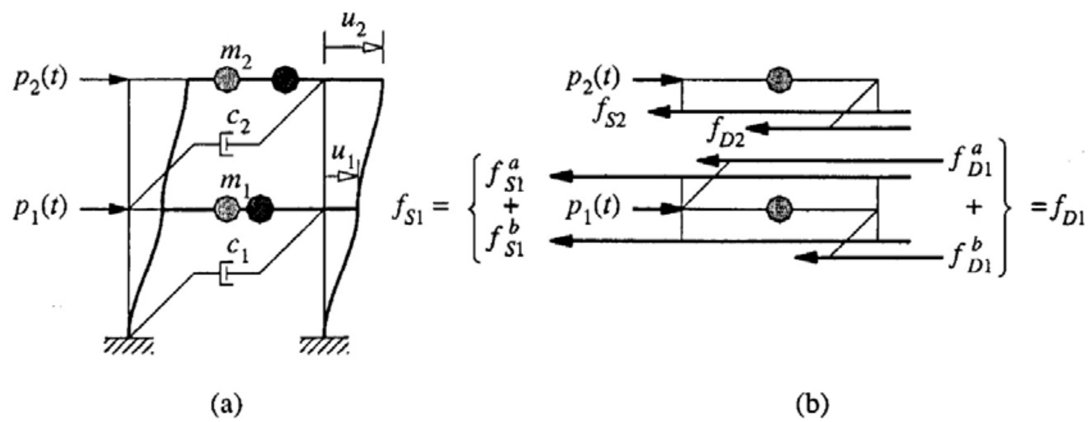


(b)

EXAMPLE: **Mass-Spring-Damper System**



EXAMPLE: Two-Story Shear Frame



Assumption: Mass concentrated at floor levels (generally appropriate for multi-story buildings).

Newton's 2nd Law of Motion for **each** mass:

mass j: $$p_j - f_{Sj} - f_{Dj} = m_j\ddot{u}_j \quad \Leftrightarrow \quad m_j\ddot{u}_j + f_{Dj} + f_{Sj} = p_j(t) \quad (j = 1,2)$$

In matrix form:

$$\underbrace{\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}}_{\underbrace{\mathbf{m}}_{\textit{mass matrix}}} \underbrace{\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix}}_{\ddot{\mathbf{u}}} + \underbrace{\begin{pmatrix} f_{D1} \\ f_{D2} \end{pmatrix}}_{\mathbf{f}_D} + \underbrace{\begin{pmatrix} f_{S1} \\ f_{S2} \end{pmatrix}}_{\mathbf{f}_S} = \underbrace{\begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix}}_{\mathbf{p}(t)}$$

i.e.,

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{f}_D + \mathbf{f}_S = \mathbf{p}(t)$$

where:

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Resisting (elastic or inelastic) force vector $\boldsymbol{f_S}$:

Story shear:

$$V_j = k_j \underbrace{(u_j - u_{j-1})}_{\substack{\Delta_j \\ \textit{story} \\ \textit{deformation} \\ \textit{/drift}}}$$

Σχετική μετατόπιση ορόφου

Story Stiffness: (for ***shear-building*** $I_g = \infty$) **Διατμητικό κτιριο**

$$k_j = \sum_{columns} \left(\frac{12EI_c}{h^3} \right)$$

mass 1:

$$\begin{aligned} f_{S1} &= \overbrace{f_{S1}^{(b)}}^{\substack{\textit{from} \\ \textit{story} \\ \textit{below}}} + \overbrace{f_{S2}^{(a)}}^{\substack{\textit{from} \\ \textit{story} \\ \textit{above}}} \\ &= k_1\Delta_1 + (-k_2\Delta_2) \\ &= k_1(u_1 - 0) + k_2(u_1 - u_2) \end{aligned}$$

mass 2:

$$\begin{aligned} f_{S2} &= k_2\Delta_2 \\ &= k_2(u_2 - u_1) \end{aligned}$$

Notice that $f_{S1}^{(a)} = -f_{S2}$ (***story shear***) **Τέμνουσα ορόφου**

Therefore:

$$\underbrace{\begin{pmatrix} f_{S1} \\ f_{S2} \end{pmatrix}}_{\mathbf{f_S}} = \underbrace{\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix}}_{\substack{\underbrace{\mathbf{k}} \\ \textit{stiffness matrix of} \\ \textit{two-storey shear bldg.}}} \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{\mathbf{u}} \Rightarrow \boxed{\mathbf{f_S = ku}}$$

Damping force vector $\boldsymbol{f_D}$:

We **assume** that the story shear due to damping is expressed as follows:

$$V_j = c_j \dot{\Delta}_j \quad (j = 1,2)$$

mass 1:

$$f_{D1} = f_{D1}^{(b)} + f_{D1}^{(a)} = c_1 \dot{u}_1 + c_2(\dot{u}_1 - \dot{u}_2)$$

mass 2:

$$f_{D2} = c_2(\dot{u}_2 - \dot{u}_1)$$

Therefore:

$$\underbrace{\begin{pmatrix} f_{D1} \\ f_{D2} \end{pmatrix}}_{\mathbf{f}_D} = \underbrace{\begin{pmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{pmatrix}}_{\underbrace{\mathbf{c}}_{\substack{\text{damping} \\ \text{matrix}}}} \underbrace{\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix}}_{\dot{\mathbf{u}}} \Longrightarrow \boxed{\mathbf{f}_D = \mathbf{c}\dot{\mathbf{u}}}$$

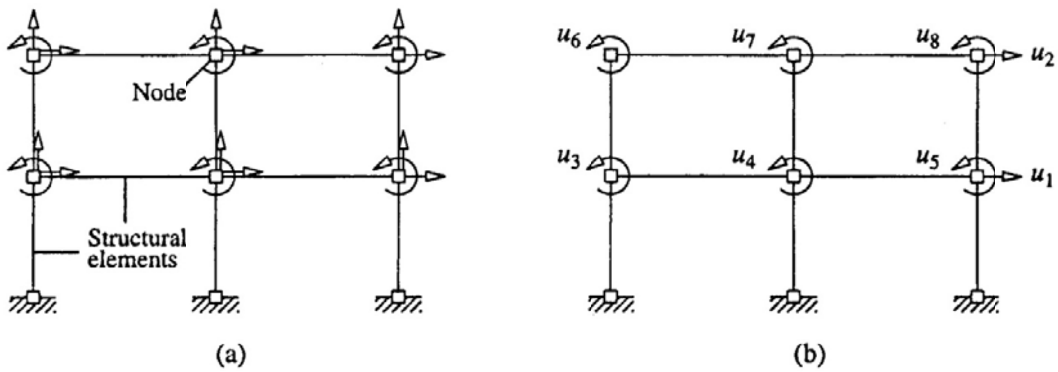
Therefore:

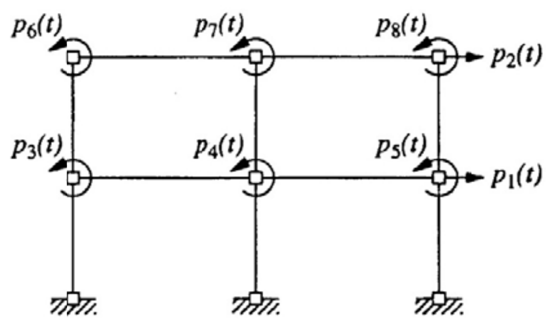
$$\left.\begin{matrix} \mathbf{m}\ddot{\mathbf{u}} + \mathbf{f}_D + \mathbf{f}_S = \mathbf{p}(t) \\ \mathbf{f}_S = \mathbf{k}\mathbf{u} \\ \mathbf{f}_D = \mathbf{c}\dot{\mathbf{u}} \end{matrix}\right\} \Longrightarrow$$

$$\Longrightarrow \boxed{\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t)} \quad \textbf{\textit{Equation of Motion}}$$

GENERAL APPROACH FOR LINEAR SYSTEMS

Discretization: **Διακριτοποίηση**





A **frame structure** can be idealized as an ***assemblage*** **of elements** – beams, columns, walls – interconnected at nodal points or ***nodes***.

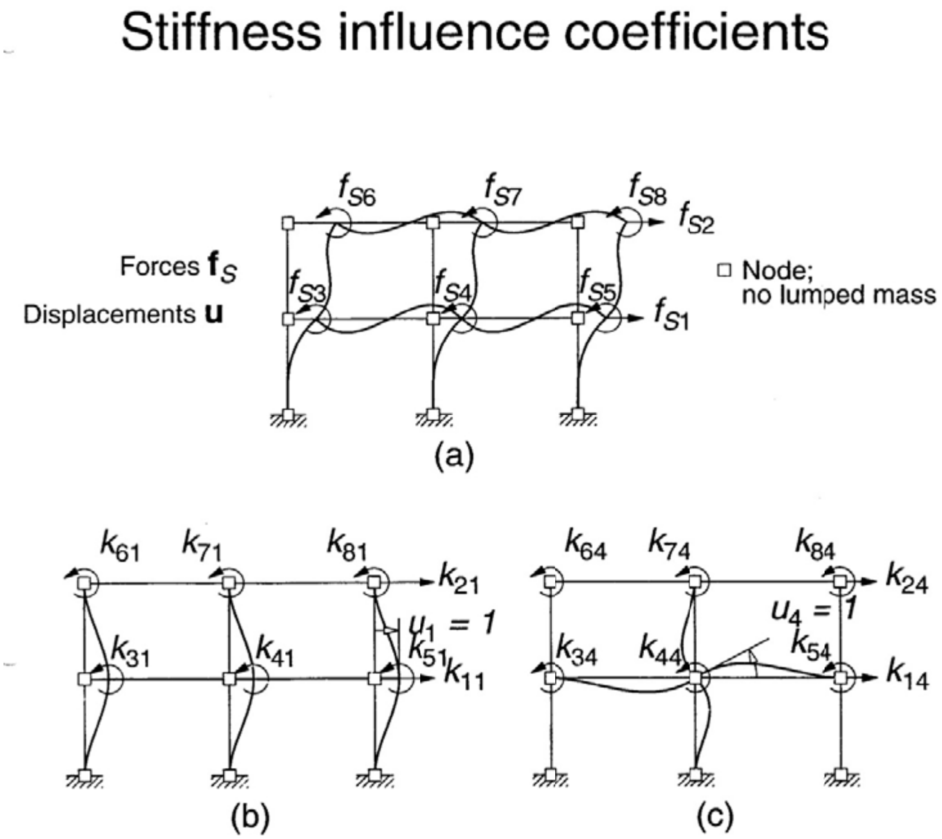
The displacements of the nodes are the ***degrees of freedom (DOF)***.

Axial deformation of beams can be **neglected** in analyzing most buildings, and **axial deformation of columns** need not be considered **for low-rise buildings**.

The external dynamic forces are applied at the nodes. The external moments $p_3(t)$ to $p_8(t)$ are zero in most practical cases.

Elastic Forces:

(We assume that the structural behavior is linear, so that the ***principle of superposition*** **applies**). **Αρχή της επαλληλίας**



We apply a unit displacement along DOF j, **holding all other displacements to zero as shown**.

To maintain these displacements, forces must be applied generally along all DOFs.

> **The stiffness influence coefficient** k_{ij} is the **force** required along DOF i due to unit displacement at DOF j.

The force f_{Si} at DOF i associated with displacements $u_1, u_2, \cdots, u_N$ is obtained by **superposition**:

$$f_{Si} = k_{i1}u_1 + k_{i2}u_2 + \cdots + k_{ij}u_j + \cdots + k_{iN}u_N \quad (i = 1,2,\cdots,N)$$

In matrix form:

$$\begin{pmatrix} f_{S1} \\ f_{S2} \\ \vdots \\ f_{Si} \\ \vdots \\ f_{SN} \end{pmatrix} = \underbrace{\begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1j} & \cdots & k_{1N} \\ k_{21} & k_{22} & \cdots & k_{2j} & \cdots & k_{2N} \\ \vdots & \vdots & \ddots & & & \vdots \\ k_{i1} & k_{i2} & \cdots & k_{ij} & \cdots & k_{iN} \\ \vdots & \vdots & & & \ddots & \vdots \\ k_{N1} & k_{N2} & \cdots & k_{Nj} & \cdots & k_{NN} \end{pmatrix}}_{\substack{\mathbf{k} \\ stiffness\ matrix}} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_j \\ \vdots \\ u_N \end{pmatrix}$$

or

$$\boxed{\mathbf{f}_S = \mathbf{k}\mathbf{u}}$$

The stiffness matrix **k** for a discretized system can be determined by any one of several methods.

The most commonly used method is the ***direct stiffness method***.

$$\mathbf{f}_S = \mathbf{k}\mathbf{u} \quad \Leftrightarrow \quad \mathbf{u} = \mathbf{k}^{-1}\mathbf{f}_S = \tilde{\mathbf{f}}\mathbf{f}_S$$

$$\boxed{\tilde{\mathbf{f}} = \left[\tilde{f}_{ij}\right] \quad \boldsymbol{flexibility\ matrix} \quad \tilde{\mathbf{f}} = \mathbf{k}^{-1}}$$

The flexibility influence coefficient $\tilde{f}_{ij}$ is the **deflection** of DOF ***i*** due to unit load applied to DOF ***j*** (while no loads are applied to all other DOFs).

The evaluation of flexibility coefficients for any given system is a **standard problem of structural analysis**.

Any desired method of analysis may be used to compute these deflections resulting from the applied unit loads.

Inertia Forces:

We apply a **unit acceleration** along DOF j, while the **acceleration in all other DOFs are kept zero**.

According to ***d'Alembert's principle***, the **fictitious inertia forces oppose these accelerations**.

Therefore, **external forces will be necessary to equilibrate these inertia forces**.

The mass influence coefficient m_{ij} is the external **force** in DOF i due to unit acceleration along DOF j.

The force f_{Ii} at DOF i associated with accelerations $\ddot{u}_1, \ddot{u}_2, \cdots, \ddot{u}_N$ is obtained by **superposition**:

$$f_{Ii} = m_{i1}\ddot{u}_1 + m_{i2}\ddot{u}_2 + \cdots + m_{ij}\ddot{u}_j + \cdots + m_{iN}\ddot{u}_N \quad (i = 1,2,\cdots,N)$$

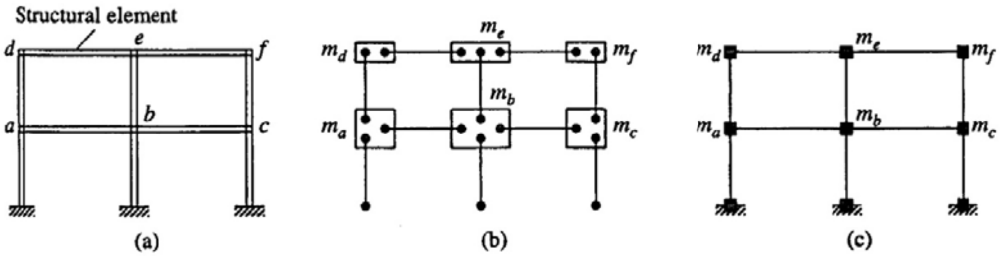
In matrix form:

$$\begin{pmatrix} f_{I1} \\ f_{I2} \\ \vdots \\ f_{Ii} \\ \vdots \\ f_{IN} \end{pmatrix} = \underbrace{\begin{pmatrix} m_{11} & m_{12} & \cdots & m_{1j} & \cdots & m_{1N} \\ m_{21} & m_{22} & \cdots & m_{2j} & \cdots & m_{2N} \\ \vdots & \vdots & \ddots & & & \vdots \\ m_{i1} & m_{i2} & \cdots & m_{ij} & \cdots & m_{iN} \\ \vdots & \vdots & & & \ddots & \vdots \\ m_{N1} & m_{N2} & \cdots & m_{Nj} & \cdots & m_{NN} \end{pmatrix}}_{\substack{\mathbf{m} \\ \textit{mass matrix}}} \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \vdots \\ \ddot{u}_j \\ \vdots \\ \ddot{u}_N \end{pmatrix}$$

or

$$\boxed{\mathbf{f}_I = \mathbf{m}\ddot{\mathbf{u}}}$$

Inertia Forces (continued): *Lumped-Mass Matrix*



The simplest procedure for defining the mass properties of any structure is to assume that the entire mass is **lumped / concentrated** at the **points/nodes** at which the translational displacements are defined.

The lumped mass at a node is determined from **the portion of the weight that can reasonably be assigned to the node.**

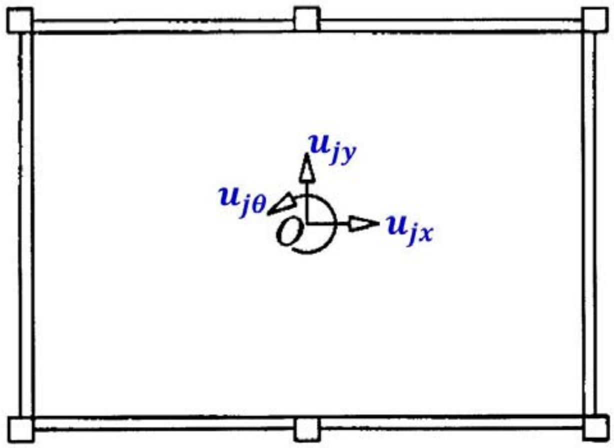
The off-diagonal terms m_{ij} ($i \neq j$) of the '*lumped-mass*' (συγκεντρωμένη μάζα) matrix vanish because an acceleration of any mass point produces an inertial force at that point only, *i.e.*

$$m_{ij} = 0 \quad \text{for} \quad i \neq j$$

Thus **the lumped-mass matrix is a diagonal matrix** which will include **zero diagonal elements for rotational DOFs**, *i.e.*

$$m_{jj} = \begin{cases} m_j & \\ 0 & \text{(for rotational DOFs)} \end{cases}$$

The rotational DOFs are eliminated by '*static condensation*' (explained later) (στατική συμπύκνωση).



The mass representation can be simplified for multistory buildings because of the constraining effects of the floor slabs or **floor diaphragms**.

Each floor diaphragm is usually assumed to be **rigid in its own plane** but is **flexible in bending in the vertical direction**, which is reasonable representation of the true behavior of several types of floor systems (*e.g.*, cast-in-place concrete).

Both horizontal (x & y) DOFs of all the nodes at a floor level are related to the three rigid-body DOFs of the floor-diaphragm in its own plane.

For the jth floor diaphragm these three DOFs, defined at the center of mass, are **translations** u_{jx} & u_{jy} and **rotation** $u_{j\theta}$ **about a vertical axis**.

Damping Forces:

(assuming that damping is of ***viscous type***, *i.e.* **depends on velocity**)

$$\begin{pmatrix} f_{D1} \\ f_{D2} \\ \vdots \\ f_{Di} \\ \vdots \\ f_{DN} \end{pmatrix} = \underbrace{\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1N} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{iN} \\ \vdots & \vdots & & & \ddots & \vdots \\ c_{N1} & c_{N2} & \cdots & c_{Nj} & \cdots & c_{NN} \end{pmatrix}}_{\substack{\mathbf{c} \\ \textit{damping matrix}}} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \vdots \\ \dot{u}_j \\ \vdots \\ \dot{u}_N \end{pmatrix}$$

The damping influence coefficient c_{ij} is the **force** acting along DOF i due to a unit velocity of DOF j.

In matrix form:

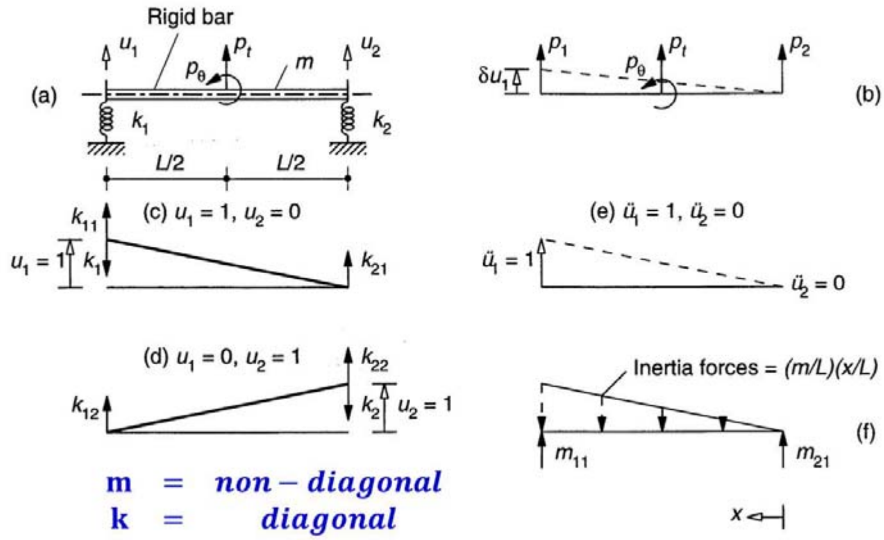
$$\boxed{\mathbf{f}_D = \mathbf{c}\dot{\mathbf{u}}}$$

However, it is **impractical / impossible** to compute the coefficients c_{ij} of the damping matrix directly from the dimensions of the structure and the sizes of the structural elements.

Damping for MDOF systems is generally specified by numerical values for the damping ratios, as for SDOF systems, **based on experimental data for similar structures**.

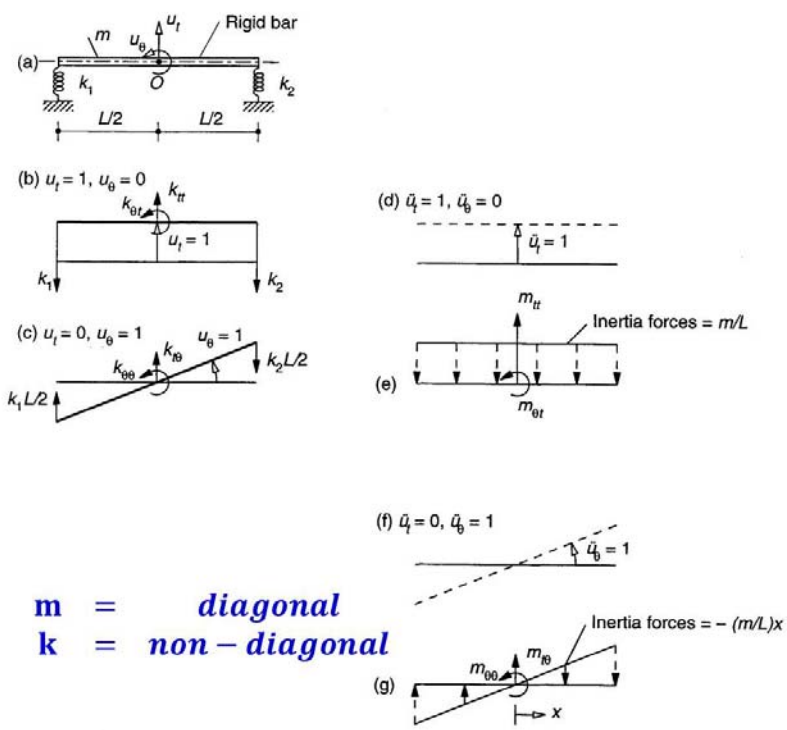
Methods are available to construct the damping matrix from known damping ratios.

Example 9.2



$$\mathbf{m} = non-diagonal$$
$$\mathbf{k} = diagonal$$

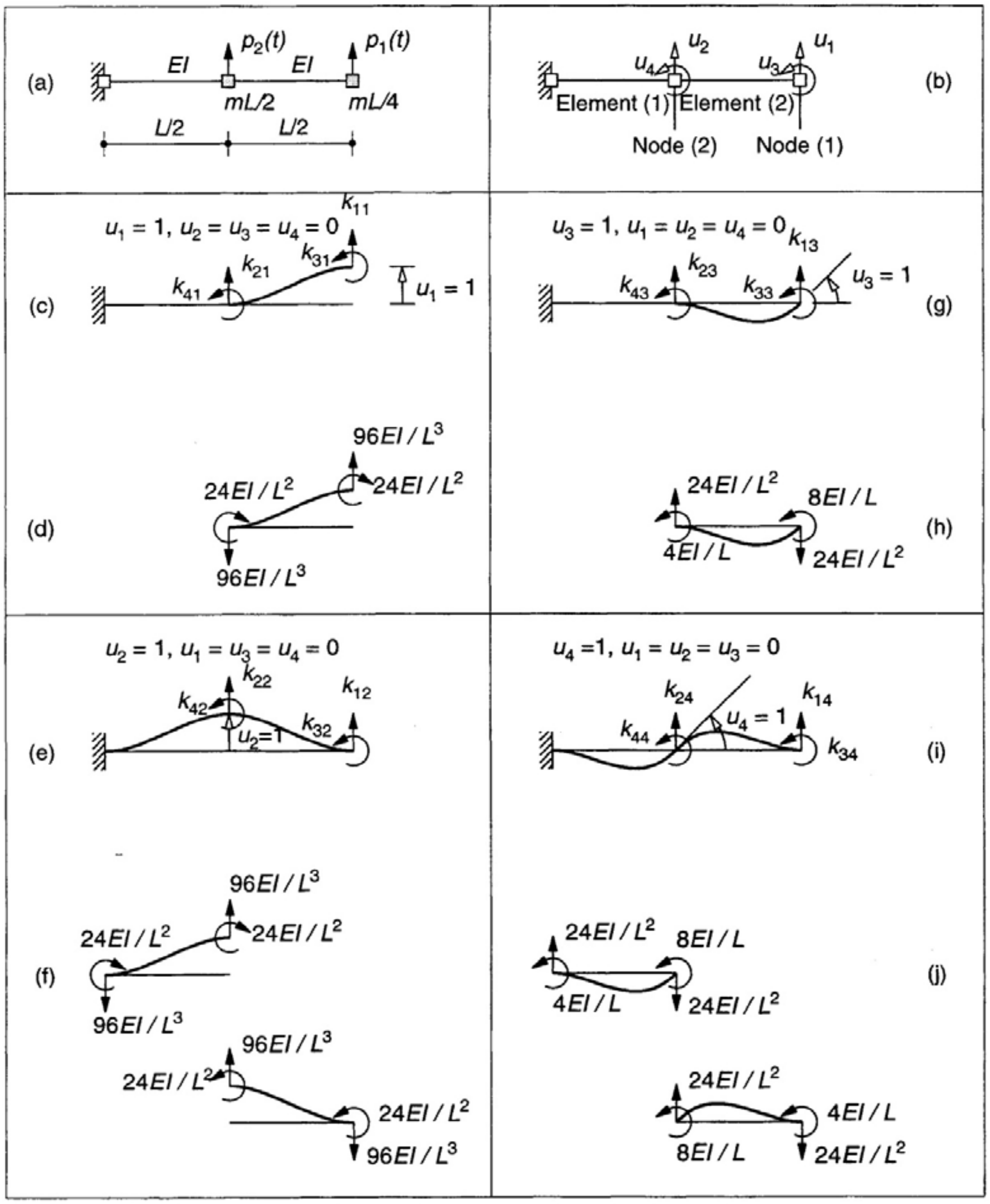
Example 9.3



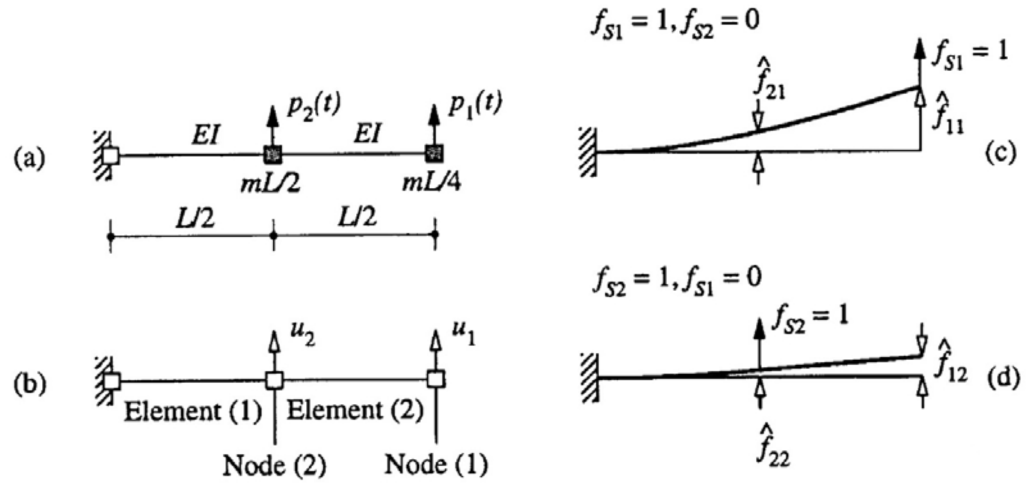
$$\mathbf{m} = diagonal$$
$$\mathbf{k} = non-diagonal$$

Direct Stiffness Method

Example 9.4

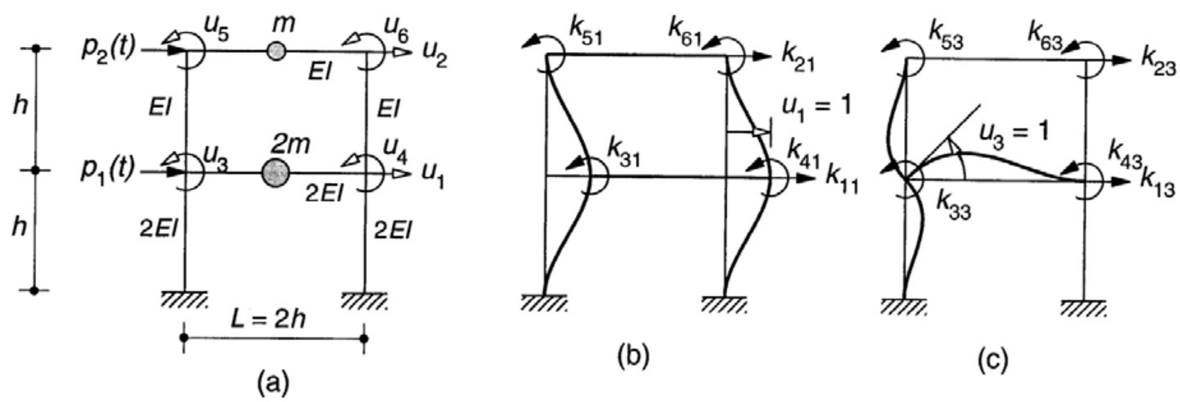


Flexibility Method



Direct Stiffness Method

Example 9.7



EQUATIONS OF MOTION: EXTERNAL FORCES

The equations of motion of a structural system can be formulated by expressing the (dynamic) equilibrium of the effective forces associated with each of its DOFs.

In general, four types of forces will be involved at any point $\boldsymbol{i}$:

- The **external applied load** $\boldsymbol{p_i(t)}$
- The **inertia force** $\boldsymbol{f_{Ii}(t)}$
- The **damping force** $\boldsymbol{f_{Di}(t)}$
- The **restoring (elastic** or **inelastic) force** $\boldsymbol{f_{Si}(t)}$

Forces resulting from motion: $\boldsymbol{f_{Ii}(t), f_{Di}(t), f_{Si}(t)}$

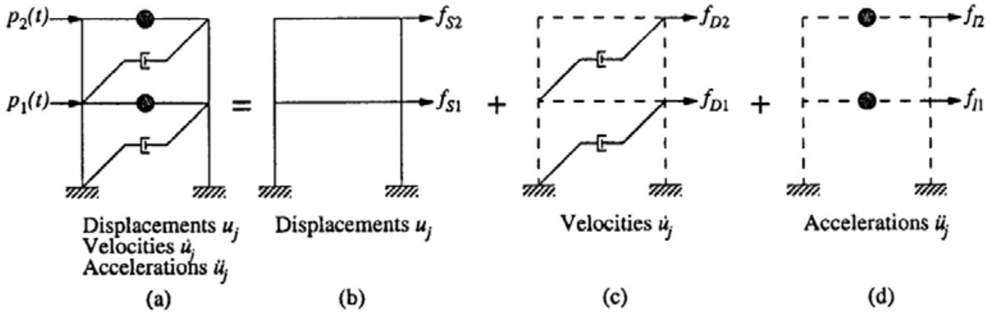
$$\begin{aligned} f_{I1} &= p_1(t) - f_{S1} - f_{D1} \\ f_{I2} &= p_2(t) - f_{S2} - f_{D2} \\ &\vdots \\ &\vdots \end{aligned}$$

The <u>elastic</u> and <u>damping forces</u> are shown acting in the opposite direction because they are <u>internal forces resisting the motion</u>.

The above set of scalar equations may be organized in matrix form:

$$\boxed{\mathbf{f}_I + \mathbf{f}_D + \mathbf{f}_S \quad = \quad \mathbf{p}(t)}$$

Equations of Motion (continued): *Alternative approach*



We visualize the **structural system as** the **combination of three pure components**:

1) **Stiffness component**: the frame **without damping or mass**

2) **Damping component**: the frame with its damping property but **no stiffness** or **mass**

3) **Mass component**: the floor masses **without the stiffness** or **damping** of the frame

The external forces $\mathbf{p}(t)$ may be **visualized as distributed among the above three components** of the structural system:

- $\mathbf{f}_S(t)$ to the stiffness component (related to displacements)
- $\mathbf{f}_D(t)$ to the damping component (related to velocities)
- $\mathbf{f}_I(t)$ to the mass component (related to accelerations)

Therefore:

$$\mathbf{f}_I(t) + \mathbf{f}_D(t) + \mathbf{f}_S(t) = \mathbf{p}(t)$$

or

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{p}(t)$$

BASIC STRUCTURAL CONCEPTS

Strain Energy:

The **strain energy *U*** of a structural system **is equal to the work done in deforming the system**:

$$\boxed{U = \frac{1}{2}\sum_{i=1}^{N} p_i U_i = \frac{1}{2}\mathbf{p}^T\mathbf{u}}$$

where the factor $(\mathbf{1/2})$ results from **the forces which increase linearly with the displacements**.

Clearly:

$$\left.\begin{array}{r} U = \dfrac{1}{2}\mathbf{p}^T\mathbf{u} \\ \mathbf{u} = \tilde{\mathbf{f}}\mathbf{p} \end{array}\right\} \Longrightarrow \boxed{U = \frac{1}{2}\mathbf{p}^T\tilde{\mathbf{f}}\mathbf{p}}$$

and

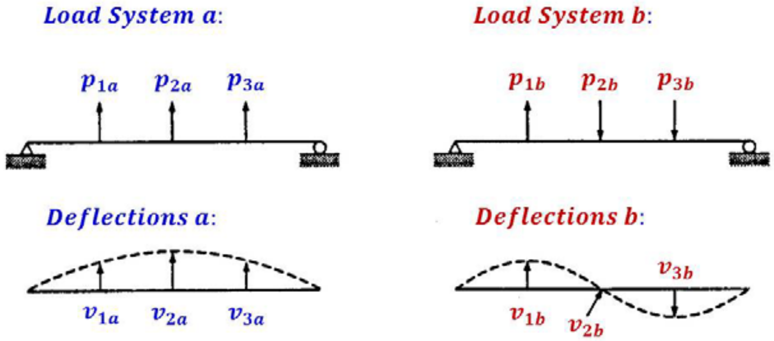
$$\left.\begin{array}{r} U = \dfrac{1}{2}\mathbf{u}^T\mathbf{p} \\ \mathbf{p} = \mathbf{k}\mathbf{u} \end{array}\right\} \Rightarrow \boxed{U = \frac{1}{2}\mathbf{u}^T\mathbf{k}\mathbf{u}}$$

Noting that the strain energy stored in a **stable** structure during any distortion/deformation must always be positive, it is evident that:

$$\boxed{\mathbf{u}^T\mathbf{k}\mathbf{u} > 0 \quad \text{and} \quad \mathbf{p}^T\tilde{\mathbf{f}}\mathbf{p} > 0}$$

Matrices which satisfy this condition, where $\mathbf{u}$ or $\mathbf{p}$ is **any arbitrary nonzero vector**, are said to be ***positive definite***; positive definite matrices (and consequently the flexibility & stiffness matrices of a stable structure) **are nonsingular and can be inverted**.

BETTI's Law



Case 1:

Loads a: $W_{aa} = \frac{1}{2}\sum p_{ia}v_{ia} = \frac{1}{2}\mathbf{p}_a^T\mathbf{u}_a$

Loads b: $W_{bb} + W_{ab} = \frac{1}{2}\mathbf{p}_b^T\mathbf{u}_b + \mathbf{p}_a^T\mathbf{u}_b$

Total: $W_1 = W_{aa} + W_{bb} + W_{ab} = \frac{1}{2}\mathbf{p}_a^T\mathbf{u}_a + \frac{1}{2}\mathbf{p}_b^T\mathbf{u}_b + \mathbf{p}_a^T\mathbf{u}_b$

Case 2:

Loads b: $W_{bb} = \frac{1}{2}\mathbf{p}_b^T\mathbf{u}_b$

Loads a: $W_{aa} + W_{ba} = \frac{1}{2}\mathbf{p}_a^T\mathbf{u}_a + \mathbf{p}_b^T\mathbf{u}_a$

Total: $W_2 = W_{bb} + W_{aa} + W_{ba} = \frac{1}{2}\mathbf{p}_b^T\mathbf{u}_b + \frac{1}{2}\mathbf{p}_a^T\mathbf{u}_a + \mathbf{p}_b^T\mathbf{u}_a$

The deformation of an elastic structure is **independent of the loading sequence**.

Therefore: $W_1 = W_2 \quad \Longrightarrow \quad \mathbf{p}_a^T\mathbf{u}_b = \mathbf{p}_b^T\mathbf{u}_a$

Betti's Law: ***The work done by one set of loads on the deflections due to a second set of loads is equal to the work of the second set of loads acting on the deflections due to the first.***

Corollaries of Betti's Law:

$$\begin{array}{lcl} & \mathbf{p}_a^T\mathbf{u}_b = \mathbf{p}_b^T\mathbf{u}_a & \\ \Longrightarrow & \mathbf{p}_a^T\tilde{\mathbf{f}}\mathbf{p}_b = \mathbf{p}_b^T\tilde{\mathbf{f}}\mathbf{p}_a & \\ \Longrightarrow & \tilde{\mathbf{f}} = \tilde{\mathbf{f}}^T & \end{array}$$

the flexibility matrix is symmetric

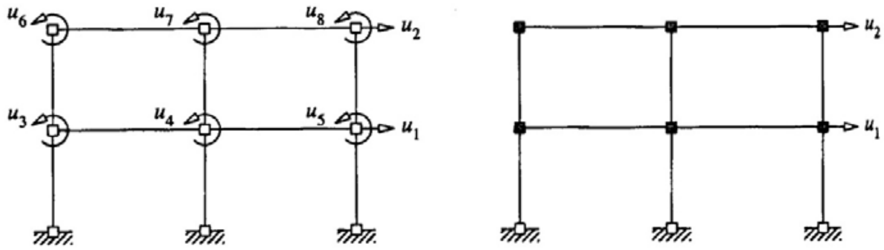
i.e. $f_{ij} = f_{ji}$ ***Maxwell's Law of Reciprocal Deflections***

Similarly:

$$\begin{array}{lcl} & \mathbf{p}_a^T\mathbf{u}_b = \mathbf{p}_b^T\mathbf{u}_a & \\ \Longrightarrow & \mathbf{u}_a^T\mathbf{k}^T\mathbf{u}_b = \mathbf{u}_b^T\mathbf{k}^T\mathbf{u}_a & \\ \Longrightarrow & \mathbf{k} = \mathbf{k}^T & \end{array}$$

the stiffness matrix is symmetric

Static Condensation:



(a) (b)

The **static condensation method** is used to **eliminate** from dynamic analysis **those DOFs** of a structure **to which zero mass is assigned**; however, **all the DOFs are included in the static analysis**.

Equation of Motion:

$$\begin{pmatrix} \mathbf{m}_{tt} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \ddot{\mathbf{u}}_t \\ \ddot{\mathbf{u}}_0 \end{pmatrix} + \begin{pmatrix} \mathbf{k}_{tt} & \mathbf{k}_{t0} \\ \mathbf{k}_{0t} & \mathbf{k}_{00} \end{pmatrix} \begin{pmatrix} \mathbf{u}_t \\ \mathbf{u}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{p}_t(t) \\ \mathbf{0} \end{pmatrix}$$

where: $\mathbf{u}_0$ = DOFs with **zero** mass

$\mathbf{u}_t$ = DOFs with mass (= ***dynamic DOFs***)

$$\Longrightarrow \left.\begin{matrix} \mathbf{m}_{tt}\ddot{\mathbf{u}}_t + \mathbf{k}_{tt}\mathbf{u}_t + \mathbf{k}_{t0}\mathbf{u}_0 = \mathbf{p}_t(t) & (i) \\ \mathbf{k}_{0t}\mathbf{u}_t + \mathbf{k}_{00}\mathbf{u}_0 = \mathbf{0} & (ii) \end{matrix}\right\} \Longrightarrow \left.\mathbf{u}_0 = -\mathbf{k}_{00}^{-1}\mathbf{k}_{0t}\mathbf{u}_t\right\} \Longrightarrow$$

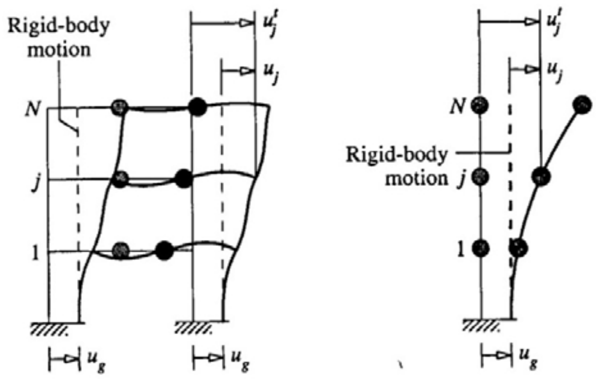
NOTE: Because **no inertia terms** or **external forces** are associated with $\mathbf{u}_0$, Equation (ii) permits a static relationship between $\mathbf{u}_0$ and $\mathbf{u}_t$.

$$\Longrightarrow \quad \mathbf{m}_{tt}\ddot{\mathbf{u}}_t + \hat{\mathbf{k}}_{tt}\mathbf{u}_t = \mathbf{p}_t(t)$$

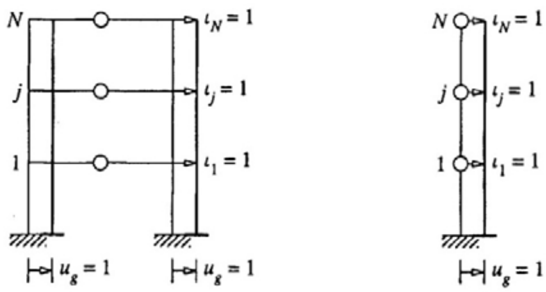
where:

$$\boxed{\hat{\mathbf{k}}_{tt} = \mathbf{k}_{tt} - \mathbf{k}_{0t}^T\mathbf{k}_{00}^{-1}\mathbf{k}_{0t} \quad \textit{condensed stiffness matrix}}$$

Planar Systems: Translational Ground Motion



(a) (b)



(a) (b)

$$\mathbf{u}^t(t) = \underbrace{\mathbf{u}(t)}_{\substack{\text{relative} \\ \text{(to base)} \\ \text{motion}}} + \underbrace{u_g(t)\mathbf{1}}_{\substack{\text{rigid-body} \\ \text{motion}}}$$

$$\text{where:} \quad \mathbf{1} = \underbrace{[1,1,\cdots,1]}_{N}^T$$

Equation of ***Dynamic Equilibrium***: $\mathbf{f}_I + \mathbf{f}_D + \mathbf{f}_S = \mathbf{0}$

Only the relative motions u between the masses and the base due to structural deformations **produce elastic and damping forces** (*i.e.*, **the rigid-body component of the displacement of the structure produces only inertial forces**).

Therefore:

$$\mathbf{f}_D = \mathbf{c}\dot{\mathbf{u}} \quad \& \quad \mathbf{f}_S = \mathbf{k}\mathbf{u}$$

However, **the inertia forces** $\mathbf{f}_I$ are related to **the total accelerations** $\ddot{\mathbf{u}}^t$ **of the masses**, *i.e.*

$$\mathbf{f}_I = \mathbf{m}\ddot{\mathbf{u}}^t$$

Therefore:

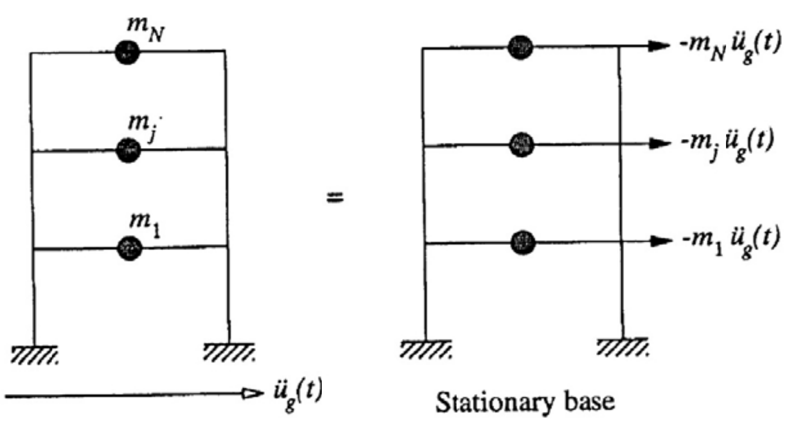
$$\mathbf{f}_I + \mathbf{f}_D + \mathbf{f}_S = \mathbf{0}$$

$$\Rightarrow \left.\begin{array}{c} \mathbf{m}\ddot{\mathbf{u}}^t + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \mathbf{0} \\ \mathbf{u}^t(t) = \mathbf{u}(t) + u_g(t)\mathbf{1} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \boxed{\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = -\mathbf{m}\mathbf{1}\ddot{u}_g(t)}$$

The above equation contains N **(coupled) ODE's** governing the relative displacements $u_j(t)$ of a linearly elastic MDOF system subjected to **ground acceleration** $\ddot{u}_g(t)$.

The **stiffness matrix** $\mathbf{k}$ refers to horizontal displacements and is **obtained by static condensation** in order to eliminate rotational and vertical DOFs of the nodes $\mathbf{k}$ is known as the ***lateral stiffness matrix*** (**μητρώο πλευρικής δυσκαμψίας**).



Equation of Motion:

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \underbrace{-\mathbf{m}\mathbf{1}\ddot{u}_g(t)}_{\substack{\mathbf{p}_{eff}(t) \\ \textit{Effective} \\ \textit{Earthquake} \\ \textit{Forces}}}$$

In general, the total displacement of each mass is expressed as:

$$\textit{its displacement } u_j^S \textit{ due to static application of ground motion}$$
$$+$$
$$\textit{the dynamic displacement } u_j \textit{ relative to the quasi} - \textit{static displacement}$$

i.e., $u_j^t(t) = u_j^S(t) + u_j(t) \quad or \quad \mathbf{u}^t(t) = \mathbf{u}^S(t) + \mathbf{u}(t)$

The ***quasi-static displacements*** (**οιονεί στατικές μετατοπίσεις**) can be expressed as:

$$\mathbf{u}^S(t) = \boldsymbol{i} u_g(t)$$

where: $\boldsymbol{i}$ = ***influence vector*** (**διάνυσμα επιρροής**)

represents the displacements of the masses resulting from <u>static</u> application of a unit ground displacement

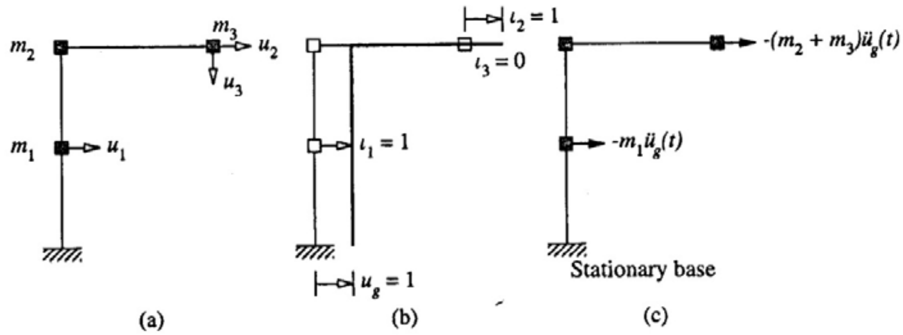
Therefore:

$$\boxed{\mathbf{u}^t(t) = \boldsymbol{i} u_g(t) + \mathbf{u}(t)}$$

<u>*Equation of Motion*</u>:

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{c}\dot{\mathbf{u}} + \mathbf{k}\mathbf{u} = \underbrace{-\mathbf{m}\boldsymbol{i}\ddot{u}_g(t)}_{\mathbf{p}_{eff}(t)}$$

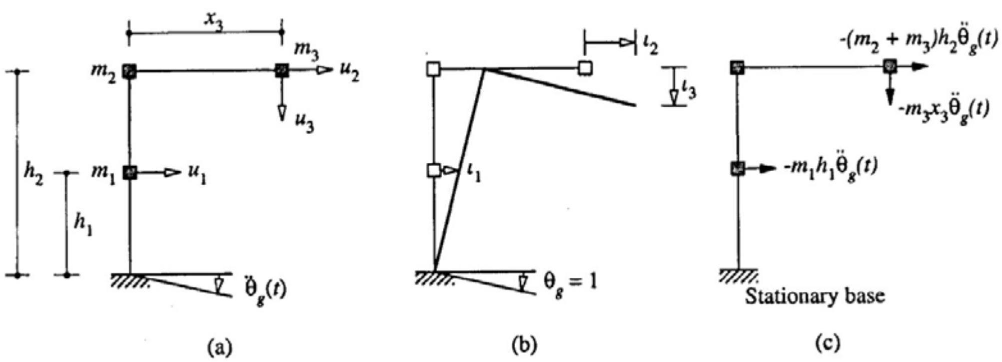
EXAMPLE:



$$\boldsymbol{i} = [u_1 \quad u_2 \quad u_3]^T \underbrace{=}_{\substack{\textbf{static} \\ \textbf{application} \\ \textbf{of } u_g=1}} [1 \quad 1 \quad 0]^T$$

Therefore:

$$\mathbf{p}_{eff}(t) = -\mathbf{m}\boldsymbol{i}\ddot{u}_g(t) = -\ddot{u}_g(t)\begin{pmatrix} m_1 & & \\ & m_2 + m_3 & \\ & & m_3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = - - \ddot{u}_g(t)\begin{pmatrix} m_1 \\ m_2 + m_3 \\ 0 \end{pmatrix}$$



$$\boldsymbol{i} = [u_1 \quad u_2 \quad u_3]^T \underbrace{=}_{\substack{\textbf{static} \\ \textbf{application} \\ \textbf{of } \theta_g=1}} [h_1 \quad h_2 \quad x_3]^T$$

Therefore:

$$\mathbf{p}_{eff}(t) = -\mathbf{m}\boldsymbol{i}\ddot{\theta}_g(t) = -\ddot{\theta}_g(t)\begin{pmatrix} m_1 h_1 \\ (m_2 + m_3)h_2 \\ m_3 x_3 \end{pmatrix}$$