NUMERICAL EVALUATION OF DYNAMIC RESPONSE

Αριθμητικός Υπολογισμός Δυναμικής Απόκρισης

TIME-STEPPING (or DIFFERENCE) METHODS:

Μέθοδοι Χρονικών Βημάτων

Equation of Motion of an inelastic system:

$$m\ddot{u} + c\dot{u} + f_S(u, \dot{u}) = p(t)$$
$$\text{Initial Conditions (I.C.'s):} \quad u_0 = u(0) \quad \& \quad \dot{u}_0 = \dot{u}(0)$$

Important requirements for a numerical procedure:

(1) **Convergence**: **Σύγκλιση**
As the time step decreases, **the numerical solution should approach the exact solution**.

(2) **Stability**: **Ευστάθεια**
The numerical solution **should be stable in the presence of numerical round-off errors**.

(3) **Accuracy**: **Ακρίβεια**
The numerical procedure **should provide results that are close enough to the exact solution**.

Types of time-stepping procedures:

(1) Methods based on **interpolation of excitation function**. **Μέθοδοι βασισμένοι στην παρεμβολή της διέγερσης**

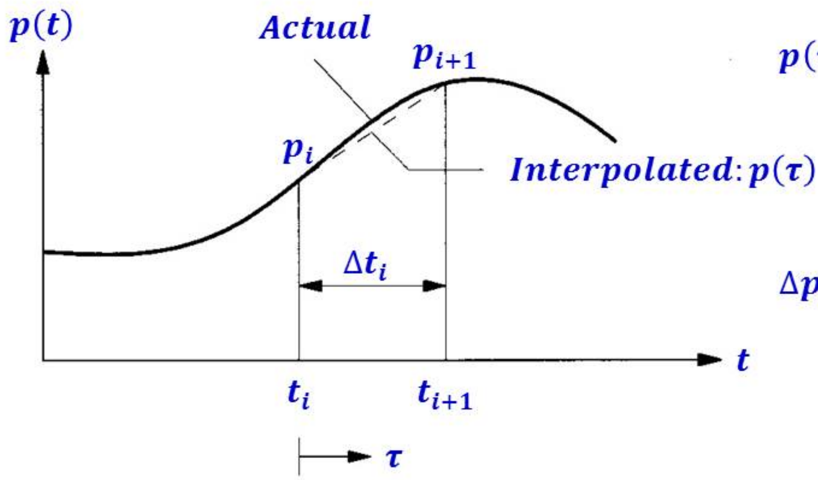
(2) Methods based on **finite difference expressions of velocity & acceleration**. **Μέθοδος κεντρικής διαφοράς**

(3) Methods based on **assumed variation of acceleration**. **Μέθοδοι βασισμένοι στην υπόθεση μεταβολής της επιταχύνσεως**

METHODS BASED ON INTERPOLATION OF EXCITATION (LINEAR SYSTEMS)

Interpolation of the excitation over each time interval

For sufficiently short time intervals, **linear interpolation**:



$$p(\tau) = p_i + \frac{\Delta p_i}{\Delta t_i}\tau$$

$$\Delta p_i = p_{i+1} - p_i$$

Response $u(\tau)$ for $0 \le \tau \le \Delta t_i$:

(1) Free vibration due to **I.C.'s** u_i & $\dot{u}_i$ at $\tau = 0$

+

(2) Response to **step force** p_i with **zero initial conditions**

+

(3) Response to **ramp force** $\left(\frac{\Delta p_i}{\Delta t_i}\tau\right)$ with **zero initial conditions**

Each of the above responses is available in closed form for linear systems. Therefore:

$$\begin{pmatrix} u \\ \dot{u} \end{pmatrix}_{i+1} = \begin{pmatrix} A & B \\ A' & B' \end{pmatrix}\begin{pmatrix} u \\ \dot{u} \end{pmatrix}_i + \begin{pmatrix} C & D \\ C' & D' \end{pmatrix}\begin{pmatrix} p_i \\ p_{i+1} \end{pmatrix}$$

Expressions for $A, B, A', B', C, D, C', D'$ are given in TABLE 5.2.1 of CHOPRA (1995).

The above described procedure is used to compute **elastic response spectra**.

METHODS BASED ON DIFFERENCE EXPRESSIONS

Central Difference Method Μέθοδος Κεντρικής Διαφοράς

Approximate $\dot{u}$ & $\ddot{u}$ by difference equations:

$$\dot{u}_i = \frac{u_{i+1} - u_{i-1}}{2(\Delta t)} \quad ; \quad \ddot{u}_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2}$$

One way of obtaining these formulae is by ***Taylor expansion***:

$$u(t + \Delta t) = u(t) + (\Delta t)\dot{u}(t) + \frac{(\Delta t)^2}{2!}\ddot{u}(t) + \frac{(\Delta t)^3}{3!}\dddot{u}(\xi_1) \qquad (1)$$

$$u(t - \Delta t) = u(t) - (\Delta t)\dot{u}(t) + \frac{(\Delta t)^2}{2!}\ddot{u}(t) - \frac{(\Delta t)^3}{3!}\dddot{u}(\xi_2) \qquad (2)$$

Subtracting (2) from (1), we obtain:

$$u(t + \Delta t) - u(t - \Delta t) = 2(\Delta t)\dot{u}(t) + \frac{(\Delta t)^3}{3!}\underbrace{[\dddot{u}(\xi_1) + \dddot{u}(\xi_2)]}_{=2\dddot{u}(\xi)}$$

$$\Rightarrow \quad \dot{u}(t) = \frac{u(t + \Delta t) - u(t - \Delta t)}{2(\Delta t)} - \frac{(\Delta t)^2}{6}\dddot{u}(\xi)$$

Add (2) to (1) after expanding up to 4th power:

$$u(t + \Delta t) + u(t - \Delta t) = 2u(t) + (\Delta t)^2\ddot{u}(t) + \frac{(\Delta t)^4}{4!}\underbrace{\left[\frac{d^4u(\xi_1)}{dt^4} + \frac{d^4u(\xi_2)}{dt^4}\right]}_{=2\frac{d^4u(\xi)}{dt^4}}$$

$$\Rightarrow \quad \ddot{u}(t) = \frac{u(t + \Delta t) - 2u(t) + u(t - \Delta t)}{(\Delta t)^2} - \frac{(\Delta t)^2}{12}\left(\frac{d^4u(\xi)}{dt^4}\right)$$

Equation of Motion at step i :

$$m\ddot{u}_i + c\dot{u}_i + (f_S)_i = p_i$$

Substituting the approximate expression for $\dot{u}_i$ & $\ddot{u}_i$:

$$m\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2} + c\frac{u_{i+1} - u_{i-1}}{2(\Delta t)} + (f_S)_i = p_i$$

Transferring the unknown quantities to the right side, we obtain:

$$\hat{k}u_{i+1} = \hat{p}_i \quad \Leftrightarrow \quad u_{i+1} = \frac{\hat{p}}{\hat{k}}$$

where:
$$\begin{cases} \hat{k} = \dfrac{m}{(\Delta t)^2} + \dfrac{c}{2(\Delta t)} \\ \hat{p} = p_i - \left[\dfrac{m}{(\Delta t)^2} - \dfrac{c}{2(\Delta t)}\right]u_{i-1} - (f_S)_i + \dfrac{2m}{(\Delta t)^2}u_i \end{cases}$$

(Explicit Method)

(Ρητή Μέθοδος)

To initialize the process:

$$\dot{u}_0 = \frac{u_1 - u_{-1}}{2(\Delta t)} \quad ; \quad \ddot{u}_0 = \frac{u_1 - 2u_0 + u_{-1}}{(\Delta t)^2}$$

After eliminating $\boldsymbol{u_1}$, we solve for $\boldsymbol{u_{-1}}$: $\quad u_{-1} = u_0 - (\Delta t)\dot{u}_0 + \frac{(\Delta t)^2}{2}\ddot{u}_0$

$\boldsymbol{u_0}$ & $\boldsymbol{\dot{u}_0}$ are given initial conditions

Equation of Motion: $\quad m\ddot{u}_0 + c\dot{u}_0 + (f_S)_0 = p_0 \Longrightarrow \ddot{u}_0 = \frac{1}{m}[p_0 - c\dot{u}_0 - (f_S)_0]$

Stability requirement:

$$\boxed{\frac{(\Delta t)}{T} < \frac{1}{\pi}}$$

Typically, $\boldsymbol{\frac{(\Delta t)}{T} \leq 0.1}$**, (**$\boldsymbol{T}$ **= natural period of the SDOF system) to define the response adequately**.

NEWMARK's β METHOD ΜΕΘΟΔΟΣ NEWMARK

'***Newmark's Method***' comprises a family of time-stepping methods based on the following equations:

$$\begin{aligned} \dot{u}_{i+1} &= \dot{u}_i + [(1-\gamma)\Delta t]\ddot{u}_i + (\gamma \Delta t)\ddot{u}_{i+1} \\ u_{i+1} &= u_i + (\Delta t)\dot{u}_i + \left[\left(\tfrac{1}{2}-\beta\right)(\Delta t)^2\right]\ddot{u}_i + [\beta(\Delta t)^2]\ddot{u}_{i+1} \end{aligned}$$

The parameters $\boldsymbol{\beta}$ & $\boldsymbol{\gamma}$:

- define **the variation of acceleration** over $(\Delta \boldsymbol{t})$
- determine the **stability** & **accuracy** characteristics

Method is **implicit** $\Longrightarrow$ iteration

For linear systems, the method may be modified to become **explicit**.

Typical selection of parameters:

$$\gamma = \frac{1}{2} \qquad \frac{1}{6} \le \beta \le \frac{1}{4}$$

for satisfactory performance, including accuracy.

Special Cases:

(1) $\gamma = 0 \quad \beta = 0$: **Constant acceleration method**

(2) $\gamma = \frac{1}{2} \quad \beta = \frac{1}{4}$: **Average acceleration method**

(3) $\gamma = \frac{1}{2} \quad \beta = \frac{1}{6}$: **Linear acceleration method**

Newmark's method is **stable if**:

$$\left(\frac{\Delta t}{T}\right) \le \frac{1}{\pi\sqrt{2}} \frac{1}{\sqrt{\gamma - 2\beta}}$$

For: $\gamma = \frac{1}{2} \qquad \beta = \frac{1}{4} \quad \Longrightarrow \quad \frac{\Delta t}{T} < \infty$

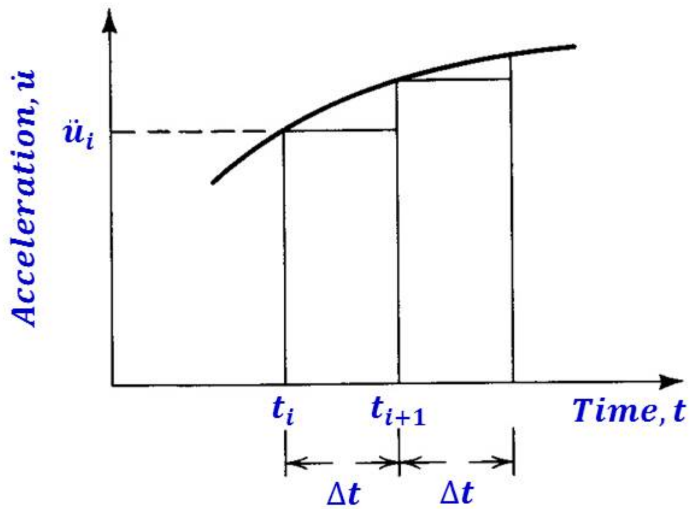
For: $\gamma = \frac{1}{2} \qquad \beta = \frac{1}{6} \quad \Longrightarrow \quad \frac{\Delta t}{T} \le 0.551$

Constant-Acceleration Method: $(\boldsymbol{\gamma = 0 \quad \beta = 0})$

$$\ddot{u}(\tau) = \ddot{u}$$

$$\dot{u}(\tau) = \dot{u}_i + \tau\ddot{u}_i \quad \Rightarrow \quad \boxed{\dot{u}_{i+1} = \dot{u}_i + (\Delta t)\ddot{u}_i}$$

$$u(\tau) = u_i + \tau\dot{u}_i + \frac{\tau^2}{2}\ddot{u}_i \quad \Rightarrow \quad \boxed{u_{i+1} = u_i + (\Delta t)\dot{u}_i + \frac{(\Delta t)^2}{2}\ddot{u}_i}$$



The above two equations provide two of the three equations for time integration, *i.e.* to obtain $\boldsymbol{u_{i+1}, \dot{u}_{i+1}, \ddot{u}_{i+1}}$ from $\boldsymbol{u_i, \dot{u}_i, \ddot{u}_i}$.

The third equation is the **Equation of Motion**:

$$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + ku_{i+1} = p_{i+1}$$

Then:

$$\ddot{u}_{i+1} = \frac{1}{m}\left\{p_{i+1} - ku_i - [c + k(\Delta t)]\dot{u}_i - \left[c(\Delta t) + k\frac{(\Delta t)^2}{2}\right]\ddot{u}_i\right\}$$

To begin the time integration, we need to know the values of $\boldsymbol{u_0}$, $\boldsymbol{\dot{u}_0}$ & $\boldsymbol{\ddot{u}_0}$, *i.e.* the values of displacement, velocity and acceleration at $\boldsymbol{t = 0}$.

Two of them must be specified; the third is obtained by using the Equation of Motion at $\boldsymbol{t = 0}$.

Average Acceleration Method: $\left(\boldsymbol{\gamma = \frac{1}{2} \quad \beta = \frac{1}{4}}\right)$

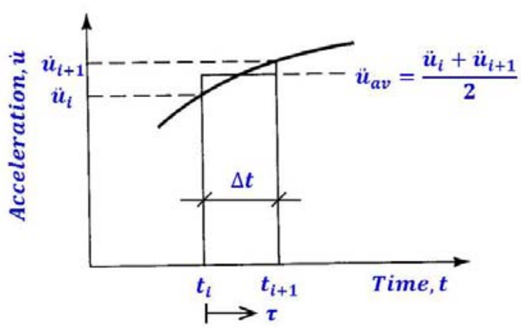
$$\ddot{u}(\tau) = \frac{1}{2}(\ddot{u}_{i+1} + \ddot{u}_i)$$

$$\dot{u}(\tau) = \dot{u}_i + \tau\frac{(\ddot{u}_{i+1} + \ddot{u}_i)}{2}$$

$$\Rightarrow \quad \boxed{\dot{u}_{i+1} = \dot{u}_i + (\Delta t)\frac{(\ddot{u}_{i+1} + \ddot{u}_i)}{2}}$$

$$u(\tau) = u_i + \tau\dot{u}_i + \left(\frac{\tau^2}{2}\right)\frac{(\ddot{u}_{i+1} + \ddot{u}_i)}{2}$$

$$\Rightarrow \quad \boxed{u_{i+1} = u_i + (\Delta t)\dot{u}_i + \frac{(\Delta t)^2}{4}(\ddot{u}_{i+1} + \ddot{u}_i)}$$



Linear Acceleration: $\left(\boldsymbol{\gamma = \frac{1}{2} \quad \beta = \frac{1}{6}}\right)$

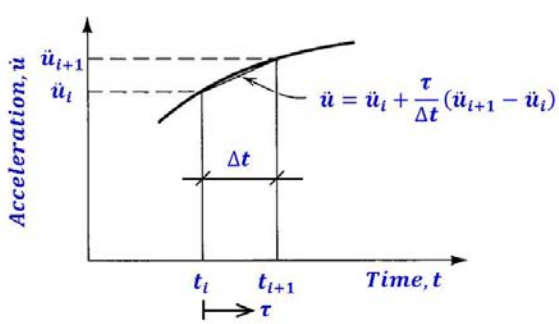
$$\ddot{u}(\tau) = \ddot{u}_i + \frac{\tau}{(\Delta t)}(\ddot{u}_{i+1} - \ddot{u}_i)$$

$$\dot{u}(\tau) = \dot{u}_i + \tau\ddot{u}_i + \frac{\tau^2}{2(\Delta t)}(\ddot{u}_{i+1} - \ddot{u}_i)$$

$$\Rightarrow \quad \boxed{\dot{u}_{i+1} = \dot{u}_i + (\Delta t)\frac{(\ddot{u}_{i+1} + \ddot{u}_i)}{2}}$$

$$u(\tau) = u_i + \tau\dot{u}_i + \left(\frac{\tau^2}{2}\right)\ddot{u}_i + \left(\frac{\tau^3}{6(\Delta t)}\right)(\ddot{u}_{i+1} - \ddot{u}_i)$$

$$\Rightarrow \quad \boxed{u_{i+1} = u_i + (\Delta t)\dot{u}_i + (\Delta t)^2\left(\frac{1}{6}\ddot{u}_{i+1} + \frac{1}{3}\ddot{u}_i\right)}$$



*Non-iterative Formulation of Newmark's β Method (**Linear** Systems)*:

Μη επαναληπτική (ή μη θαμιστική) διατύπωση

Let:

$$\Delta u_i \stackrel{\text{def}}{=} u_{i+1} - u_i$$
$$\Delta \dot{u}_i \stackrel{\text{def}}{=} \dot{u}_{i+1} - \dot{u}_i$$
$$\Delta \ddot{u}_i \stackrel{\text{def}}{=} \ddot{u}_{i+1} - \ddot{u}_i \qquad \Delta p_i = p_{i+1} - p_i$$

Newmark's ***time-stepping* equations**: **Εξισώσεις χρονικών βημάτων**

$$\Delta \dot{u}_i = (\Delta t)\ddot{u}_i + \gamma(\Delta t)\Delta \ddot{u}_i \tag{1a}$$

$$\Delta u_i = (\Delta t)\dot{u}_i + \frac{(\Delta t)^2}{2}\ddot{u}_i + \beta(\Delta t)^2\Delta \ddot{u}_i \tag{1b}$$

$$(1b) \Longrightarrow \Delta \ddot{u}_i = \frac{1}{\beta(\Delta t)^2}\Delta u_i - \frac{1}{\beta(\Delta t)}\dot{u}_i - \frac{1}{2\beta}\ddot{u}_i \tag{2}$$

$$(2) + (1a) \Longrightarrow \Delta \dot{u}_i = \frac{\gamma}{\beta(\Delta t)}\Delta u_i - \frac{\gamma}{\beta}\dot{u}_i + (\Delta t)\left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}_i \tag{3}$$

Incremental Equation of Motion: **Αυξητική εξίσωση κίνησης**

$$m\Delta \ddot{u}_i + c\Delta \dot{u}_i + k\Delta u_i = \Delta p_i \tag{4}$$

$$(2) + (3) + (4) \quad \Longrightarrow \quad \hat{k}\Delta u_i = \Delta \hat{p}_i$$

$$\text{where:} \quad \hat{k} = k + \frac{\gamma}{\beta(\Delta t)}c + \frac{1}{\beta(\Delta t)^2}m$$

$$\text{and} \quad \Delta \hat{p}_i = \Delta p_i + \left(\frac{1}{\beta(\Delta t)}m + \frac{\gamma}{\beta}c\right)\dot{u}_i + \left[\frac{1}{2\beta}m + (\Delta t)\left(\frac{\gamma}{2\beta} - 1\right)c\right]\ddot{u}_i$$

Therefore: $\left.\begin{array}{c}\Delta u_i = \frac{\Delta \hat{p}_i}{\hat{k}} \\ \text{Equation (3)}\end{array}\right\} \Longrightarrow \Delta \dot{u}_i \Longrightarrow \begin{cases} u_{i+1} \\ \dot{u}_{i+1}\end{cases}$

$$\boxed{\ddot{u}_{i+1} = \frac{p_{i+1} - c\dot{u}_{i+1} - ku_{i+1}}{m}} \quad \Longrightarrow \quad \ddot{u}_{i+1}$$

Implicit Method (Πεπλεγμένη Μέθοδος)

ERRORS INVOLVED IN NUMERICAL INTEGRATION

Types of errors:

(1) ***Round-off errors*** due to precision of the floating point arithmetic of computers. **Αριθμητικό σφάλμα στρογγυλοποίησης**

(2) ***Truncation errors*** involved in representing u_{i+1} or $\dot{u}_{i+1}$ by a finite number of terms in the Taylor series expansion. This error is represented analytically by the term $C(\Delta t)^p \left(\frac{d^p u(\xi)}{dt^p}\right)$. **Αριθμητικό σφάλμα περικοπής**

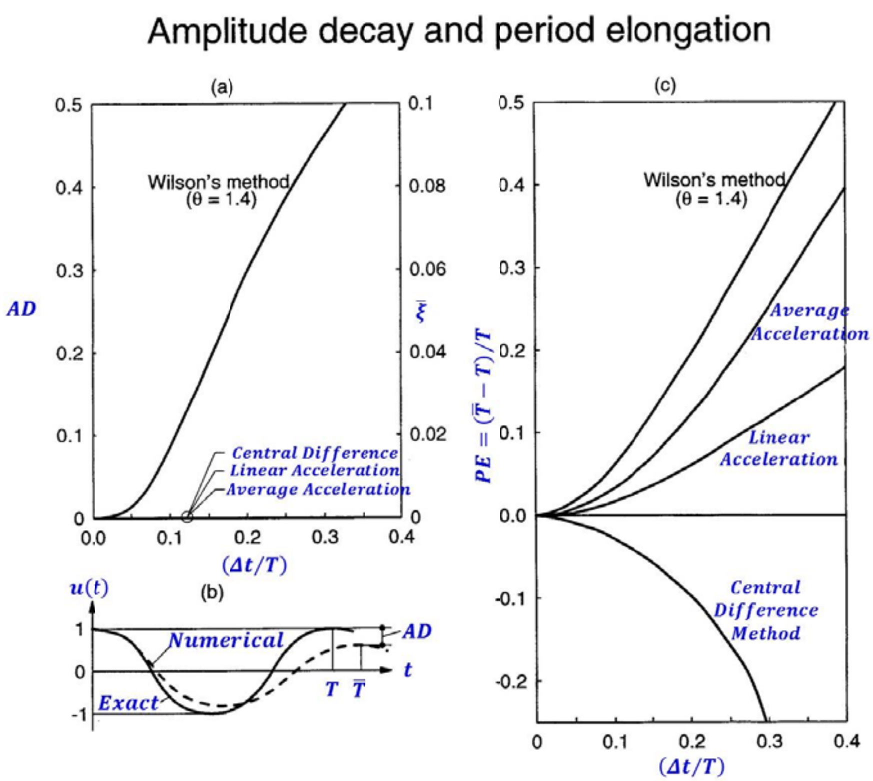
A very important aspect in the error analysis of numerical methods is the **growth** or **accumulation of errors** as computation progresses (***error propagation***).

This consideration is related to ***the stability*** of the numerical scheme.

The performance of a numerical scheme is evaluated by examining two important characteristics:

(1) **Amplitude decay (AD)** as a function of $\left(\frac{\Delta t}{T}\right)$ **Εξασθένιση Εύρους Ταλάντωσης**

(2) **Period Elongation (PE)** as a function of $\left(\frac{\Delta t}{T}\right)$ **Επιμήκυνση Περιόδου**



The ***Central Difference***, ***Linear Acceleration*** & ***Average Acceleration*** **Methods** introduce **no artificial damping**, *i.e.* $AD = 0$.

ANALYSIS OF NONLINEAR RESPONSE

ΑΝΑΛΥΣΗ ΜΗ ΓΡΑΜΜΙΚΗΣ ΑΠΟΚΡΙΣΗΣ

Equation of Motion: $m\ddot{u} + c\dot{u} + f_S(u, \dot{u}) = p(t)$

$t = t_i$: $m\ddot{u}_i + c\dot{u}_i + (f_S)_i = p_i$

$t = t_{i+1}$: $m\ddot{u}_{i+1} + c\dot{u}_{i+1} + (f_S)_{i+1} = p_{i+1}$

$$m(\ddot{u}_{i+1} - \ddot{u}_i) + c(\dot{u}_{i+1} - \dot{u}_i) + [(f_S)_{i+1} - (f_S)_i] = (p_{i+1} - p_i)$$

or $\boxed{m\Delta\ddot{u}_i + c\Delta\dot{u}_i + (\Delta f_S)_i = \Delta p_i}$

The **incremental resisting force**:

$$(\Delta f_S)_i = (k_i)_{sec} \cdot \Delta u_i$$

where: $(k_i)_{sec}$ = ***secant stiffness*** (not known) **τέμνουσα δυσκαμψία**

or approximately:

$$(\Delta f_S)_i \cong (k_i)_T \cdot \Delta u_i$$

where: $(k_i)_T$ = ***tangent stiffness*** **εφαπτομενική δυσκαμψία**

Application of **Newmark's Method**, which is the most popular method because of its accuracy:

$$\hat{k}_i \cdot \Delta u_i = \Delta \hat{p}_i$$

where:

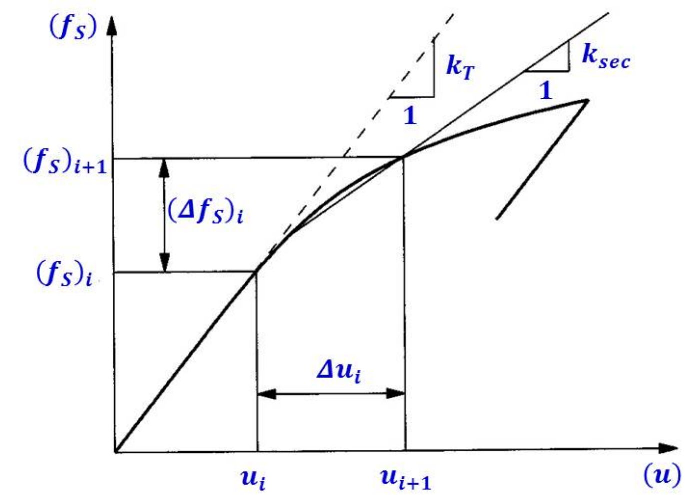
$$\begin{cases} \hat{k}_i = (k_i)_T + \dfrac{\gamma}{\beta(\Delta t)} c + \dfrac{1}{\beta(\Delta t)^2} m \\ \Delta\hat{p}_i = \Delta p_i + \left(\dfrac{1}{\beta(\Delta t)} m + \dfrac{\gamma}{\beta} c\right)\dot{u}_i + \left[\dfrac{1}{2\beta} m + (\Delta t)\left(\dfrac{\gamma}{2\beta} - 1\right) c\right]\ddot{u}_i \end{cases}$$

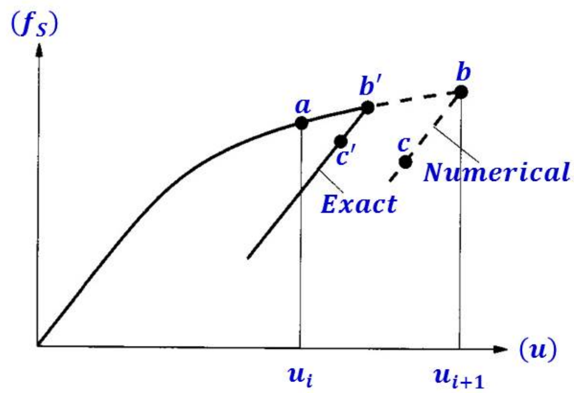
i.e., **same formulae as in the linear case** except that $\boldsymbol{k}$ has been replaced by $\boldsymbol{(k_i)_T}$.

Significant error arises for **two** reasons:

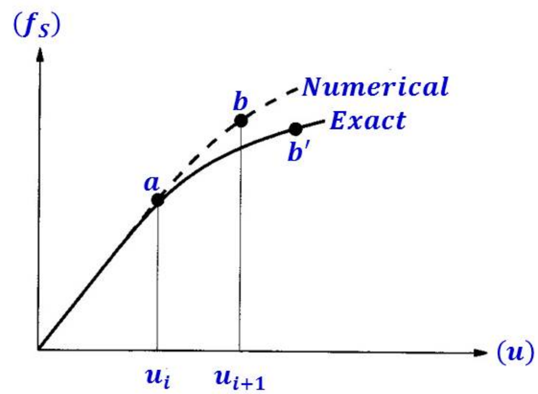
(1) **The tangent stiffness is used instead of the secant stiffness**;

(2) **Use of a constant time step delays detection of the transition in the force deformation relation**.





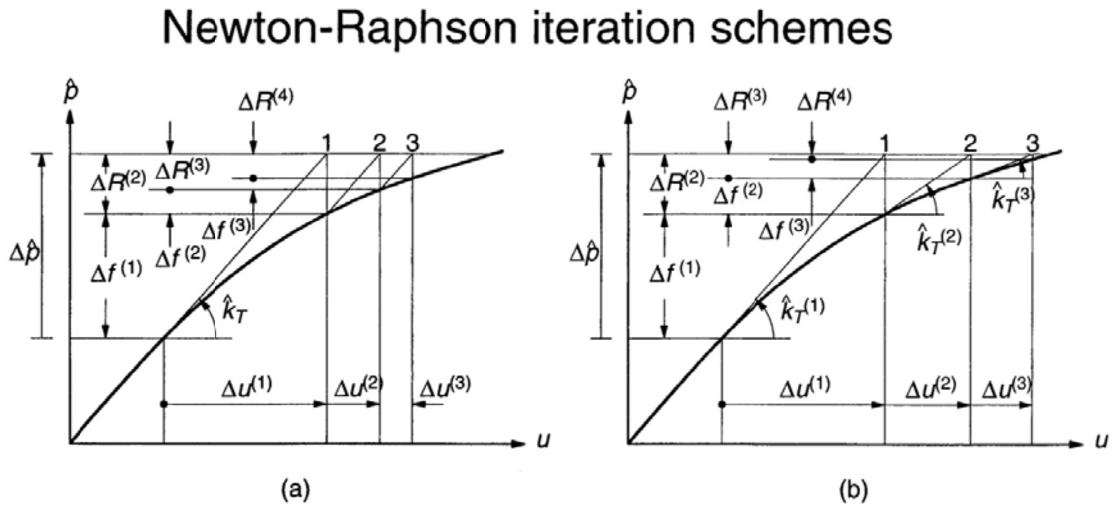
***Error* (2)**



***Error* (1)**

Error (2) is eliminated by using an **iterative process** in which integration is resumed from t_i with a smaller step whose size is **progressively adjusted** so that **at the end of such an adjusted time step, the velocity is close to zero**.

Error (1) is eliminated by an **iteration process** (**θαμιστική μέθοδος**) that is known as ***Newton-Raphson Method***.



Initialize data:

$$u_{i+1}^{(0)} = u_i \quad f_S^{(0)} = (f_S)_i \quad \Delta R^{(1)} = \Delta \hat{p}_i$$

Calculation **for each iteration**: $j = 1,2,3,\cdots$

$$\hat{k}_i \Delta u^{(j)} = \Delta R^{(j)} \Longrightarrow \Delta u^{(j)}$$

$$u_{i+1}^{(j)} = u_{i+1}^{(j-1)} + \Delta u^{(j)}$$

$$\Delta f^{(j)} = f_S^{(j)} - f_S^{(j-1)} + \left[\frac{\gamma}{\beta(\Delta t)} c + \frac{1}{\beta(\Delta t)^2} m\right] \Delta u^{(j)}$$

$$\Delta R^{(j+1)} = \Delta R^{(j)} - \Delta f^{(j)}$$

LINEAR MULTISTEP (LMS) METHODS

LMS Methods for 1st-order Equations:

Consider a system of 1st-order Ordinary Differential Equations (ODE's):

$$\boxed{\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, t)} \tag{1}$$

A ***k-step linear multistep method*** for equation (1) is defined by the following expression:

$$\sum_{i=0}^{k} \{\alpha_i \boldsymbol{y}_{n+1-i} + (\Delta t)\beta_i \boldsymbol{f}(\boldsymbol{y}_{n+1-i}, t_{n+1-i})\} = \boldsymbol{0} \tag{2}$$

The $\boldsymbol{\alpha_i}$ **&** $\boldsymbol{\beta_i}$ are parameters that define the method.
Note that the word ***'linear'***, in ***'linear multistep method'***, has nothing to do with the linearity of equation (1). Indeed, equation (2) is perfectly well defined for a nonlinear function $\boldsymbol{f}(\boldsymbol{y}, t)$.

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Excellent references for **LMS methods** of this type are:

GEAR, C.W. (1971), *Numerical Initial Value Problems in Ordinary Differential Equations*, Prentice-Hall, Englewood Cliffs, NJ.

ISERLES, A. (1996), *A First Course in the Numerical Analysis of Differential Equations*, Cambridge University Press.

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EXAMPLE:

The equation of motion of the linear, viscously damped, SDOF system can be cast in 1st-order form:

$$m\ddot{u} + c\dot{u} + ku = p(t) \quad \Longrightarrow \quad \ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = \left(\frac{1}{m}\right)p(t)$$

Let:

$$y_1 = u$$
$$y_2 = \dot{u}$$

Then: $\quad$ *Equation of Motion* $\Longrightarrow \quad \dot{y}_2 = \left(\frac{1}{m}\right)p(t) - 2\xi\omega y_2 - \omega^2 y_1$

Therefore:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\xi\omega \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \left(\frac{1}{m}\right)p(t) \end{pmatrix} = \begin{pmatrix} f_1(\boldsymbol{y}, t) \\ f_2(\boldsymbol{y}, t) \end{pmatrix}$$

i.e. $$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, t)$$

Multistep Methods:

Runge-Kutta (one step method)

Adams-Bashford

Adams-Multon

Predictor-Corrector Methods