ANALYSIS OF RESPONSE IN THE FREQUENCY DOMAIN

It can be demonstrated (see following pages) that the ***unit impulse response function*** $h(t)$ (***Συνάρτηση Απόκρισης σε Μοναδιαία Ωστική Δύναμη***) and **the *complex frequency response function*** $H_u(\Omega)$ (***Μιγαδική Συνάρτηση Συχνοτικής Απόκρισης***) **form a Fourier Transform pair**, *i.e.*

$$\boxed{h(t) \leftrightarrow H_u(\Omega)}$$

where:

$$h(t) = \frac{1}{m\omega_d} e^{-\xi\omega t} \sin(\omega_d t)$$

$$H_u(\Omega) = \frac{\left(\frac{1}{k}\right)}{(1-\beta^2) + i(2\xi\beta)} \qquad \left(\beta = \frac{\Omega}{\omega}\right)$$

Recall that the response of a SDOF system, starting from rest and subjected to a general dynamic load $p(t)$, is expressed by **Duhamel's (convolution) integral (*ολοκλήρωμα συνέλιξης*)**:

$$u(t) = p(t) * h(t) \stackrel{\text{def}}{=} \int_0^t p(\tau) h(t-\tau)\, d\tau$$

Let:

$$p(t) \leftrightarrow P(\Omega)$$
$$u(t) \leftrightarrow U(\Omega)$$

Then applying the Convolution Theorem of Fourier Transform Theory, we obtain:

$$\left.\begin{aligned} u(t) = p(t) * h(t) \quad &\Longrightarrow \quad U(\Omega) = P(\Omega) H_u(\Omega) \\ u(t) \leftrightarrow U(\Omega) \quad &\Longrightarrow \quad u(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} U(\Omega) e^{i\Omega t} \, d\Omega \end{aligned}\right\} \Longrightarrow$$

$$\Longrightarrow \quad \boxed{u(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} P(\Omega) H_u(\Omega) e^{i\Omega t} \, d\Omega}$$

Thus, **the essential steps in the frequency domain analysis** can be summarized as follows:

1) Compute the Fourier Transforms of the excitation function and the unit impulse response function.
2) Take the product of the two transforms computed in step (1).
3) Take the inverse Fourier Transform of the product to obtain the desired response.

Ordinarily, the computations involved in obtaining the ***Discrete Fourier Transform*** **(*Διακριτός Μετασχηματισμός Fourier*)** of the functions being convolved, taking the product of these transforms and then evaluating the ***Discrete Inverse Fourier Transform*** **(*Αντίστροφος Διακριτός Μετασχηματισμός Fourier*)**, are no less than these in a direct evaluation of the convolution.

However, the development of a **special algorithm** called ***Fast Fourier Transform (FFT)*** **(*Ταχύς Μετασχηματισμός Fourier*)** has completely altered this position. **The FFT algorithm** (which derives its efficiency from exploiting the harmonic property of a discrete transform) **cuts down the computations by several orders of magnitude, and makes frequency-domain analysis highly efficient**.

TIME DOMAIN		***FREQUENCY DOMAIN***
$\underbrace{p(t)}$ ↓	⟶ $P(\Omega) = \int_{-\infty}^{+\infty} p(t) e^{-i\Omega t}\, dt$ ⟶	$\underbrace{P(\Omega)}$ ↓
$u(t) = \int_0^t p(\tau) h(t-\tau)\, d\tau$		$U(\Omega) = H_u(\Omega) P(\Omega)$
↓ $\overbrace{u(t)}$	⟵ $u(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} U(\Omega) e^{i\Omega t}\, d\Omega$ ⟵	↓ $\overbrace{U(\Omega)}$

Indirect proof of $\boldsymbol{h(t) \leftrightarrow H_u(\Omega)}$ *using the Convolution Theorem*

We know that the response of the SDOF system to $\boldsymbol{p(t) = \delta(t)}$ is $\boldsymbol{u(t) = h(t)}$, *i.e.*

$$p(t) = \delta(t) \quad \longrightarrow \boxed{SDOF} \longrightarrow \quad u(t) = h(t)$$

Furthermore, it is straightforward to demonstrate, **using the sifting property of the Dirac (delta) function**, that the Fourier Transform, $\Delta(\boldsymbol{\Omega})$**,** of $\boldsymbol{\delta(t)}$ is equal to **1**, *i.e.*

$$\boxed{\Delta(\Omega) = \int_{-\infty}^{+\infty} \delta(t) e^{-i\Omega t} \, dt = 1}$$

Therefore, using the Convolution Theorem, we may express the response $h(t)$ of the SDOF system to a loading $p(t) = \delta(t)$ as follows:

$$\begin{aligned} u(t) \quad = \quad h(t) \quad &= \quad \frac{1}{2\pi} \int_{-\infty}^{+\infty} U(\Omega) e^{i\Omega t} \, d\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Delta(\Omega) \cdot H_u(\Omega) e^{i\Omega t} \, d\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} 1 \cdot H_u(\Omega) e^{i\Omega t} \, d\Omega \end{aligned}$$

Which implies that the **inverse Fourier Transform** of $\boldsymbol{H_u(\Omega)}$ is $\boldsymbol{h(t)}$.

Therefore, $\boldsymbol{h(t) \leftrightarrow H_u(\Omega)}$.

Proof of $\boldsymbol{h(t) \leftrightarrow H_u(\Omega)}$ by direct evaluation of the inverse Fourier Transform Integral using Jordan's Lemma.

Using **Jordan's Lemma** (see below):

$$\int_{-\infty}^{+\infty} H_u(\Omega) e^{i\Omega t}\, d\Omega = 2\pi i \sum \text{Res}\{H_u(\text{z})e^{izt}; \alpha_k\} \quad (t \geq 0)$$

$$H_u(\Omega) = \frac{\left(\frac{1}{k}\right)}{(1-\beta^2)+i(2\xi\beta)} = \frac{\left(\frac{1}{m}\right)}{(\omega^2-\Omega^2)+i(2\xi\omega\Omega)}$$

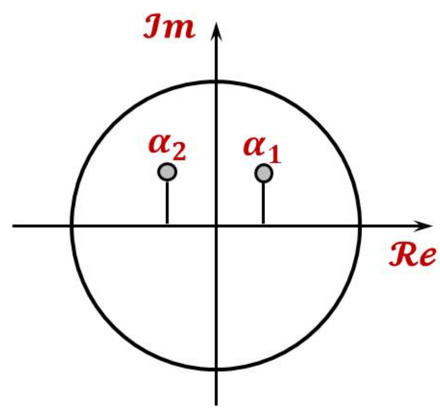
$$= \frac{-\left(\frac{1}{m}\right)}{\left[\Omega - \underbrace{\omega\left(i\xi + \sqrt{1-\xi^2}\right)}_{\alpha_1}\right]\left[\Omega - \underbrace{\omega\left(i\xi - \sqrt{1-\xi^2}\right)}_{\alpha_2}\right]}$$

$$\text{Res}(\alpha_1) = \frac{-\exp\left[i\omega\left(i\xi+\sqrt{1-\xi^2}\right)t\right]}{m2\omega\sqrt{1-\xi^2}} = \frac{-\exp[-\xi\omega t]\cdot\exp[i\omega_d t]}{m2\omega_d}$$

$$\text{Res}(\alpha_2) = \frac{-\exp\left[i\omega\left(i\xi-\sqrt{1-\xi^2}\right)t\right]}{-m2\omega\sqrt{1-\xi^2}} = \frac{\exp[-\xi\omega t]\cdot\exp[-i\omega_d t]}{m2\omega_d}$$

Therefore:

$$Res(\alpha_1) + Res(\alpha_2) = \frac{-\exp[-\xi\omega t]}{m\omega_d}\cdot i\, sin(\omega_d t) \quad (t \geq 0)$$



Closing the contour in the upper half-space for $(t \geq 0)$ (so that the integral converges), it follows that:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} H_u(\Omega)e^{i\Omega t}\, d\Omega = \frac{1}{2\pi}2\pi i \frac{-\exp[-\xi\omega t]}{m\omega_d}\cdot i\, sin(\omega_d t)$$

$$= \frac{\exp[-\xi\omega t]}{m\omega_d}\cdot sin(\omega_d t)$$

$$= h(t)$$

For $(t < 0)$, the contour is closed **'down'** (*i.e.*, in the lower half plane) and yields zero (no residues).

Therefore:

$$\boxed{h(t) \leftrightarrow H_u(\Omega)}$$

JORDAN'S LEMMA:

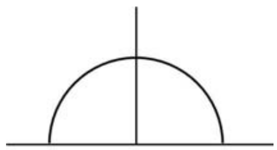
If, along a circular arc C_r of radius r, we have $|f(z)| \leq M_r$, where M_r is a bound depending only on r and hence independent of angular position on C_r, and if $M_r \to 0$ as $r \to \infty$, then we will say that $f(z)$ tends to zero ***uniformly*** on C_r as $r \to \infty$.

It is not difficult to show, in particular, that **any *rational function* (ratio of polynomials) whose denominator is of higher degree than the numerator tends uniformly to zero on any** C_r as $r \to \infty$.

THEOREM:

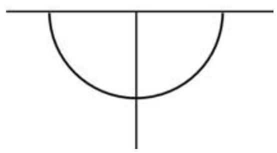
Suppose that, in a circular arc C_r with radius R and center at the origin, $f(z) \to 0$ uniformly as $R \to \infty$.

Then:



$$\lim_{R\to\infty} \int_{C_R} e^{imz} f(z)\, dz \quad (m > 0)$$

and



$$\lim_{R\to\infty} \int_{C_R} e^{-imz} f(z)\, dz \quad (m > 0)$$

If $f(z)$ is finite for all **real** values of z, then:

$$\int_{-\infty}^{+\infty} e^{\pm imx} f(x)\, dx = 2\pi i \sum \mathrm{Res}\{e^{\pm imz} f(\mathrm{z}); \alpha_k\} \quad (m \geq 0)$$

where: α_k are the poles of $f(z)$ in the $\begin{Bmatrix} (+)\ \textit{upper} \\ (-)\ \textit{lower} \end{Bmatrix}$ half-plane.

STATE-SPACE METHOD

Equation of Motion:

$$m\frac{d^2u}{dt^2}+c\frac{du}{dt}+ku=p(t) \quad \Longrightarrow \quad \frac{d^2u}{dt^2}+2\xi\omega\frac{du}{dt}+\omega^2u=\frac{1}{m}p(t)$$

Introducing the transformation:

$$\begin{aligned} u(t) &= z_1(t) \\ \frac{du(t)}{dt} &= \frac{dz_1(t)}{dt} = z_2(t) \\ \frac{d^2u(t)}{dt^2} &= \frac{dz_2(t)}{dt} \end{aligned}$$

we write the desired ***state equations*** as follows:

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) \\ \dot{z}_2(t) &= (-\omega^2)z_1(t)+(-2\xi\omega)z_2(t)+\left(\frac{1}{m}\right)p(t) \end{aligned}$$

or, in matrix form:

$$\underbrace{\begin{Bmatrix}\dot{z}_1(t)\\ \dot{z}_2(t)\end{Bmatrix}}_{\dot{\boldsymbol{Z}}(t)} = \underbrace{\begin{pmatrix}0 & 1\\ -\omega^2 & -2\xi\omega\end{pmatrix}}_{\boldsymbol{A}}\underbrace{\begin{Bmatrix}z_1(t)\\ z_2(t)\end{Bmatrix}}_{\boldsymbol{Z}(t)} + \underbrace{\begin{Bmatrix}0\\ \left(\frac{1}{m}\right)p(t)\end{Bmatrix}}_{\boldsymbol{f}(t)}$$

i.e.,

$$\dot{\boldsymbol{Z}}(t)=\boldsymbol{A}\boldsymbol{Z}(t)+\boldsymbol{f}(t) \qquad \textit{State – Space Equation of Motion}$$

This is a **system of linear 1st order ODE's**. For such a system we have a closed form solution.

Evaluation of the response (*i.e.*, solution* of the ***state-space equation of motion***) for $\boldsymbol{t \geq t_0}$, given the initial condition: $\boldsymbol{Z_0} \stackrel{\text{def}}{=} \boldsymbol{Z}(\boldsymbol{t_0}) = [\boldsymbol{u}(\boldsymbol{t_0}) \quad \dot{\boldsymbol{u}}(\boldsymbol{t_0})]^{\boldsymbol{T}}$:

$$\boldsymbol{Z}(t) = \underbrace{e^{A(t-t_0)}\boldsymbol{Z}(t_0)}_{\textit{homogeneous solution}} + \underbrace{\int_{t_0}^{t} e^{\boldsymbol{A}(t-\tau)}\boldsymbol{f}(\tau)\, d\tau}_{\textit{particular solution}}$$

where:

$$e^{\boldsymbol{A}t} \stackrel{\text{def}}{=} \boldsymbol{I} + \boldsymbol{A}t + \frac{1}{2!}(\boldsymbol{A}t)^2 + \frac{1}{3!}(\boldsymbol{A}t)^3 + \cdots \qquad \textit{State Transition Matrix}$$

- The **homogeneous solution** represents the response of the system to **the given initial condition** and **zero forcing function**.
- The **particular solution** represents the response of the system to **the given forcing function** and starting with **zero initial conditions**.

*__NOTE__: For a review of matrix analysis of differential equations see:

FRANKLIN, J.N. (1968). *Matrix Theory*, DOVER Publications, Inc.

STRANG, G. (1976). *Linear Algebra and its Applications*, Academic Press.

We proceed to **diagonalize** $\boldsymbol{A}$, and express $e^{\boldsymbol{A}t}$ in terms of the **eigenvalues** & **eigenvectors** of $\boldsymbol{A}$.

Let: λ_1, λ_2 = **eigenvalues** of $\boldsymbol{A}$

$\boldsymbol{\psi}_1, \boldsymbol{\psi}_2$ = corresponding **eigenvectors** of $\boldsymbol{A}$

i.e., $\boldsymbol{A}\boldsymbol{\psi}_1 = \lambda_1 \boldsymbol{\psi}_1$ & $\boldsymbol{A}\boldsymbol{\psi}_2 = \lambda_2 \boldsymbol{\psi}_2$

Let:

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \textit{\textbf{eigenvalue matrix of A}}$$

$$\boldsymbol{S} = \begin{pmatrix} \downarrow & \downarrow \\ \boldsymbol{\psi}_1 & \boldsymbol{\psi}_2 \\ \downarrow & \downarrow \end{pmatrix} \quad \textit{\textbf{eigenvector matrix of A}}$$

Then:

$$\boldsymbol{AS} = \boldsymbol{A}[\boldsymbol{\psi}_1 \quad \boldsymbol{\psi}_2] = [\lambda_1\boldsymbol{\psi}_1 \quad \lambda_2\boldsymbol{\psi}_2] = [\boldsymbol{\psi}_1 \quad \boldsymbol{\psi}_2]\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \boldsymbol{S\Lambda}$$

$$i.e. \quad \boldsymbol{AS} = \boldsymbol{S\Lambda} \quad \Longrightarrow \quad \boldsymbol{ASS}^{-1} = \boldsymbol{S\Lambda S}^{-1}$$

Therefore:

$$\boldsymbol{A} = \boldsymbol{S\Lambda S}^{-1} \quad (\textit{\textbf{canonical form of matrix A}})$$

[**NOTE**: The above development presupposes that $\boldsymbol{A}$ **possesses two <u>linearly independent</u> eigenvectors**, which is **always the case for an under-damped oscillator**.]

Observing that the powers of $\boldsymbol{A} = \boldsymbol{S\Lambda S}^{-1}$ telescope into:

$$\boldsymbol{A}^k = (\boldsymbol{S\Lambda S}^{-1}) \cdots (\boldsymbol{S\Lambda S}^{-1}) = \boldsymbol{S\Lambda}^k \boldsymbol{S}^{-1}$$

the infinite series of the exponential become:

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{I} + \boldsymbol{S\Lambda S}^{-1} + \frac{\boldsymbol{S\Lambda}^2\boldsymbol{S}^{-1}}{2!}t^2 + \frac{\boldsymbol{S\Lambda}^3\boldsymbol{S}^{-1}}{3!}t^3 + \cdots \\ &= \underbrace{\boldsymbol{S}\left\{\boldsymbol{I} + \boldsymbol{\Lambda} + \frac{(\boldsymbol{\Lambda}t)^2}{2!} + \frac{(\boldsymbol{\Lambda}t)^3}{3!} + \cdots\right\}\boldsymbol{S}^{-1}}_{e^{\boldsymbol{\Lambda}t}} \\ &= \boldsymbol{S}e^{\boldsymbol{\Lambda}t}\boldsymbol{S}^{-1} \end{aligned}$$

Thus, we have demonstrated that:

$$\boxed{e^{\boldsymbol{A}t} = \boldsymbol{S}e^{\boldsymbol{\Lambda}t}\boldsymbol{S}^{-1}}$$

i.e., that the state transition matrix $e^{\boldsymbol{A}t}$ of a SDOF system may be calculated by finding first all the eigenvalues & eigenvectors of $\boldsymbol{A}$.

Specifically, solving the eigenvalue problem we obtain the eigenvalues:

$$\det[\boldsymbol{A} - \lambda \boldsymbol{I}] = \begin{vmatrix} -\lambda & 1 \\ -\omega^2 & -2\xi\omega - \lambda \end{vmatrix} = \lambda^2 + 2\xi\omega\lambda + \omega^2 = 0$$

$$\Longrightarrow \quad \begin{matrix} \lambda_1 = -\xi\omega + i\omega_d \\ \lambda_2 = -\xi\omega - i\omega_d \end{matrix} \quad \left(\omega_d = \omega\sqrt{1-\xi^2}\right)$$

with **corresponding eigenvectors**:

$$\boldsymbol{\psi}_1 = \begin{pmatrix} 1 \\ -\xi\omega + i\omega_d \end{pmatrix} \qquad \boldsymbol{\psi}_2 = \begin{pmatrix} 1 \\ -\xi\omega - i\omega_d \end{pmatrix}$$

Therefore:

$$\boldsymbol{\Lambda} = \begin{pmatrix} -\xi\omega + i\omega_d & 0 \\ 0 & -\xi\omega - i\omega_d \end{pmatrix} \qquad \boldsymbol{S} = \begin{pmatrix} 1 & 1 \\ -\xi\omega + i\omega_d & -\xi\omega - i\omega_d \end{pmatrix}$$

Inverting ***S***, we obtain:

$$\boldsymbol{S}^{-1} = -\frac{1}{i2\omega_d} \begin{pmatrix} -\xi\omega - i\omega_d & -1 \\ \xi\omega - i\omega_d & 1 \end{pmatrix}$$

It is straightforward to verify that:

$$e^{\boldsymbol{\Lambda} t} = \exp\begin{pmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

Therefore:

$$\begin{aligned} e^{\boldsymbol{A}t} &= \boldsymbol{S}e^{\boldsymbol{\Lambda}t}\boldsymbol{S}^{-1} \\ &= e^{-\xi\omega t}\begin{pmatrix} \cos(\omega_d t) + \left(\frac{\xi\omega}{\omega_d}\right)\sin(\omega_d t) & \frac{1}{\omega_d}\sin(\omega_d t) \\ -\left(\frac{\omega^2}{\omega_d}\right)\sin(\omega_d t) & \cos(\omega_d t) - \left(\frac{\xi\omega}{\omega_d}\right)\sin(\omega_d t) \end{pmatrix} \end{aligned}$$

Therefore, the response is given by:

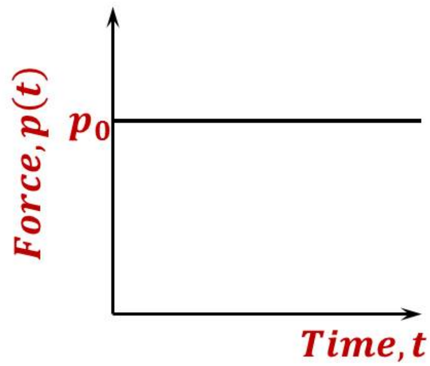
$$\begin{aligned} \begin{Bmatrix} u(t) \\ \dot{u}(t) \end{Bmatrix} &= e^{-\xi\omega t}\begin{pmatrix} \cos(\omega_d t) + \left(\frac{\xi\omega}{\omega_d}\right)\sin(\omega_d t) & \frac{1}{\omega_d}\sin(\omega_d t) \\ -\left(\frac{\omega^2}{\omega_d}\right)\sin(\omega_d t) & \cos(\omega_d t) - \left(\frac{\xi\omega}{\omega_d}\right)\sin(\omega_d t) \end{pmatrix}\begin{Bmatrix} u(t_0) \\ \dot{u}(t_0) \end{Bmatrix} \\ &\quad + \int_{t_0}^{t} \frac{p(\tau)}{m} e^{-\xi\omega(t-\tau)} \begin{pmatrix} \frac{1}{\omega_d}\sin[\omega_d(t-\tau)] \\ \cos[\omega_d(t-\tau)] - \left(\frac{\xi\omega}{\omega_d}\right)\sin[\omega_d(t-\tau)] \end{pmatrix} d\tau \end{aligned}$$

Therefore:

$$\begin{aligned} u(t) &= e^{-\xi\omega t}\left[u(t_0)\cos(\omega_d t) + \frac{\xi\omega u(t_0) + \dot{u}(t_0)}{\omega_d}\sin(\omega_d t)\right] \\ &\quad + \int_{t_0}^{t} p(\tau)h(t-\tau)\,d\tau \end{aligned}$$

which is identical to the expression for the displacement response that we obtained previously by other means.

RESPONSE TO A STEP FUNCTION LOAD (ΒΑΘΜΙΔΩΤΗ ΔΥΝΑΜΗ)



Equation of Motion: $m\ddot{u} + c\dot{u} + ku = p_0$

Solution: $u(t) = e^{-\xi\omega t}[A\cos(\omega_d t) + B\sin(\omega_d t)] + \frac{p_0}{k}$

Initial Conditions: $u_0 \stackrel{\text{def}}{=} u(0) = 0 \quad \& \quad \dot{u}_0 \stackrel{\text{def}}{=} \dot{u}(0) = 0$

Then $A = -\left(\frac{p_0}{k}\right) \quad \& \quad B = -\left(\frac{p_0}{k}\right)\left(\frac{\xi\omega}{\omega_d}\right) = -\left(\frac{p_0}{k}\right)\frac{\xi}{\sqrt{1-\xi^2}}$

Therefore, the solution becomes:

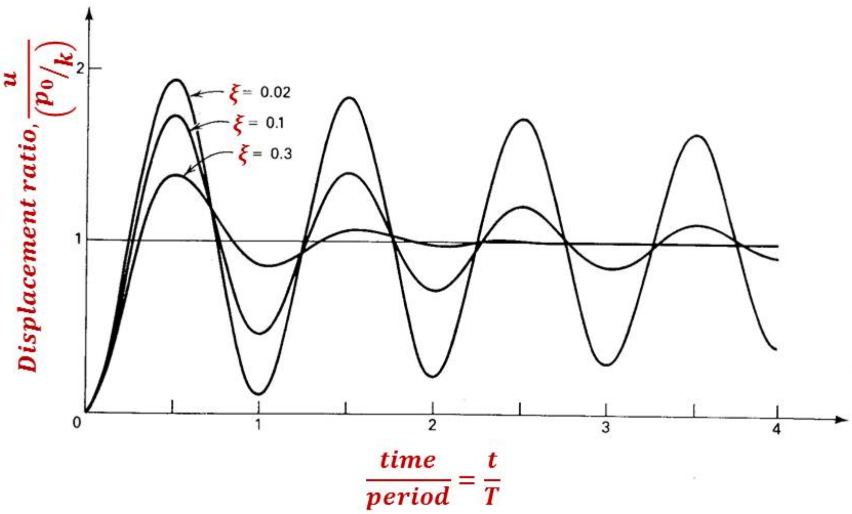
$$\boxed{u(t) = \left(\frac{p_0}{k}\right)\left[1 - e^{-\xi\omega t}\left[\cos(\omega_d t) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_d t)\right]\right]}$$

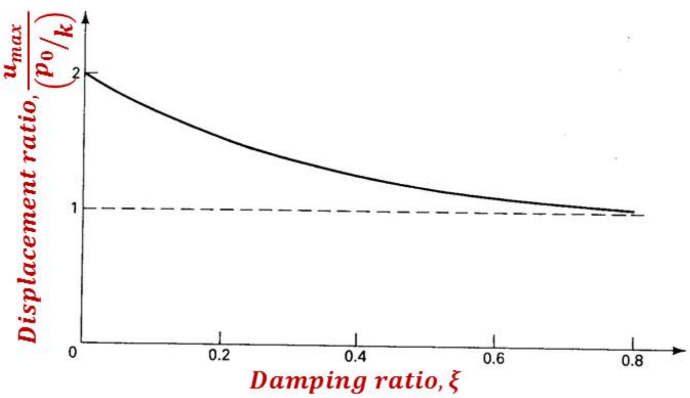
For $\xi = 0$ (*i.e.*, **undamped system**):

$$u(t) = \left(\frac{p_0}{k}\right)[1 - \cos(\omega t)]$$

Therefore, for $\xi = 0$:

$$R_d = \frac{\max_t u(t)}{\left(\frac{p_0}{k}\right)} = 2$$





The time at which the max occurs can be obtained by setting $\dot{u}(t) = 0$, *i.e.*

$$\left(\frac{p_0}{k}\right) e^{-\xi\omega t_p} \left\{\frac{(\omega\xi)^2}{\omega_d} + \omega_d\right\} \sin(\omega_d t_p) = 0$$

$$\Longrightarrow \quad t_p = \frac{n\pi}{\omega_d} \quad n = 0,1,2,\cdots \quad (t_p = \textit{time at peak value})$$

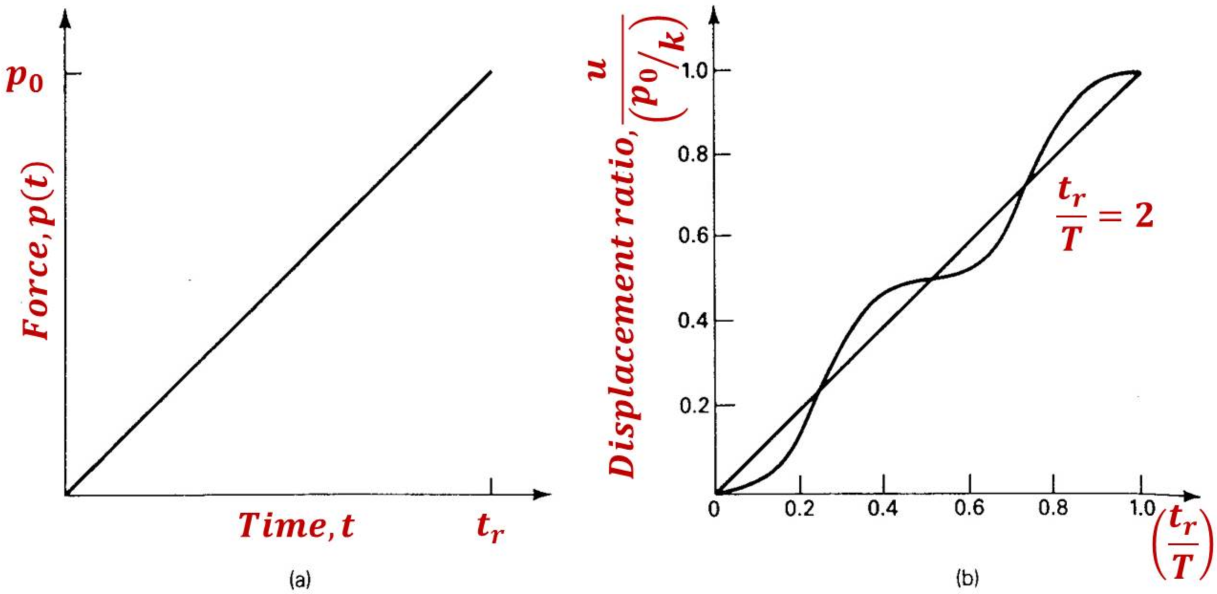
For $n = 0 \Longrightarrow t_p = 0 \;\&\; u = 0$, represents a minimum

For $n = 1 \Longrightarrow t_p = \frac{\pi}{\omega_d}$

$$\boxed{u_{max} = u(t_p) = \left(\frac{p_0}{k}\right)\left(1 + e^{-\left(\frac{\pi\xi}{\sqrt{1-\xi^2}}\right)}\right)}$$

The **normalized maximum displacement** $\left(\frac{u_{max}}{\left(\frac{p_0}{k}\right)}\right)$ is shown above.

RESPONSE TO A RAMP FUNCTION LOAD (ΓΡΑΜΜΙΚΑ ΑΥΞΑΝΟΜΕΝΗ ΔΥΝΑΜΗ)



Loading:

$$p(t) = p_0\left(\frac{t}{t_r}\right)$$

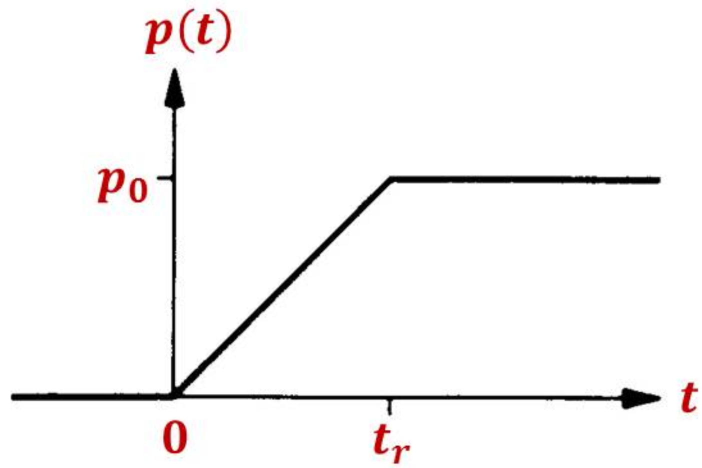
Response of **undamped system** is obtained using Duhamel's integral:

$$\begin{aligned} u(t) &= \frac{1}{m\omega}\int_0^t \frac{p_0\tau}{t_r}\sin[\omega(t-\tau)]\ d\tau \\ &= \left(\frac{p_0}{k}\right)\left\{\frac{t}{t_r} - \frac{\sin(\omega t)}{\omega t_r}\right\} \\ &= \left(\frac{p_0}{k}\right)\left\{\frac{\left(\frac{t}{T}\right)}{\left(\frac{t_r}{T}\right)} - \frac{\sin\left(2\pi\frac{t}{T}\right)}{2\pi\frac{t_r}{T}}\right\} \end{aligned}$$

Notice that **the response depends only on the ratio (t_r/T), not separately on t_r and T**.

The system **oscillates at its natural period $T = (2\pi/\omega)$ about the static solution $p(t)/k$** (*i.e.*, response of the system in the absence of inertia/mass).

RESPONSE TO A STEP FUNCTION LOAD WITH FINITE RISE TIME (ΒΑΘΜΙΔΩΤΗ ΔΥΝΑΜΗ ΜΕ ΠΕΠΕΡΑΣΜΕΝΟ ΧΡΟΝΟ ΑΝΑΠΤΥΞΗΣ)

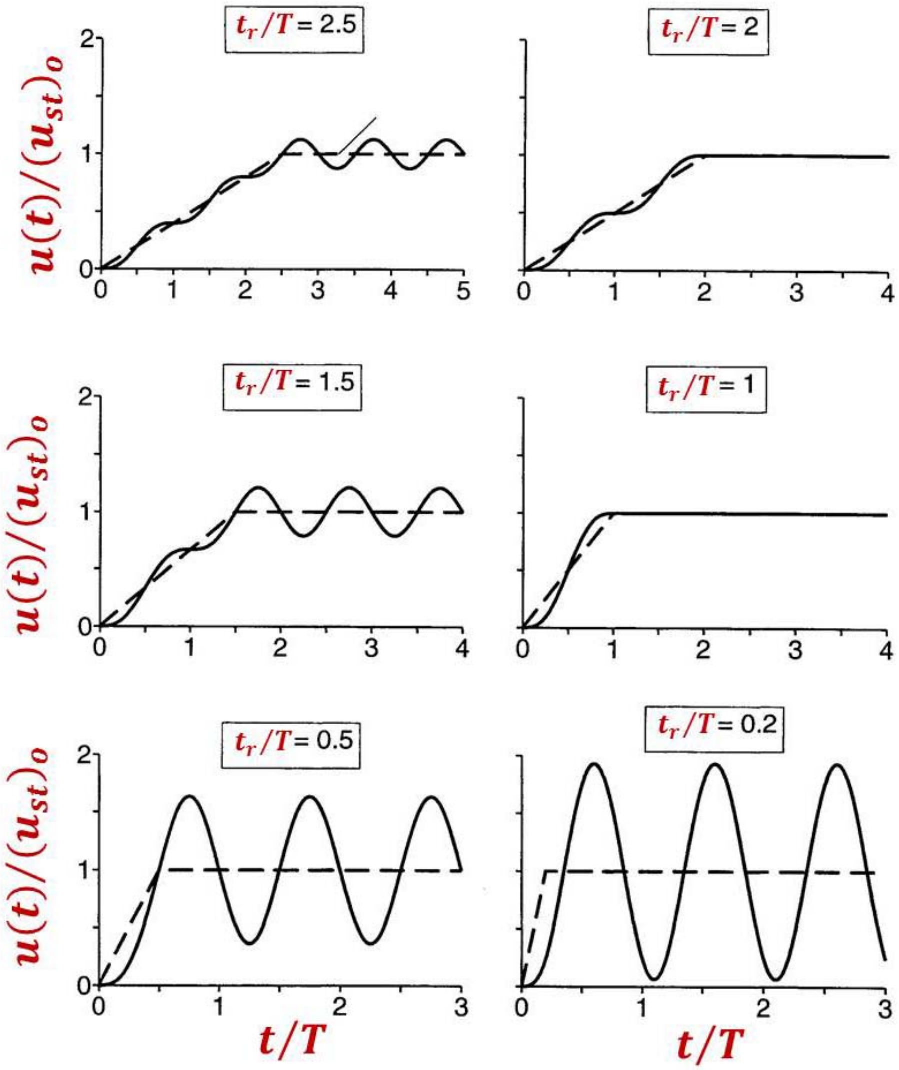


Undamped System ($\xi = 0$):

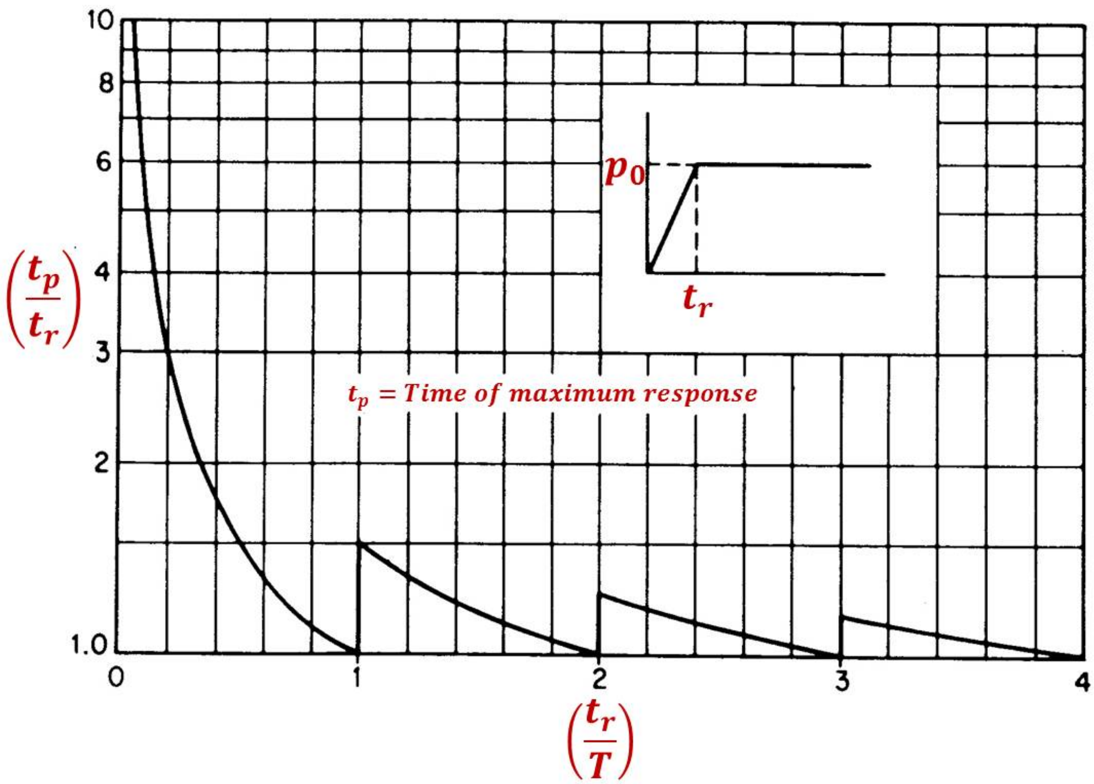
$$u(t) = \begin{cases} \left(\frac{p_0}{k}\right)\left\{\frac{t}{t_r} - \frac{\sin(\omega t)}{\omega t_r}\right\} & t \leq t_r \\ \left(\frac{p_0}{k}\right)\left\{1 - \frac{1}{\omega t_r}[\sin(\omega t) - \sin[\omega(t - t_r)]]\right\} & t \geq t_r \end{cases}$$

The response $u(t)$ depends only on the ratio (t_r/T) because $\omega t_r = 2\pi(t_r/T)$, **not separately on t_r & T**.

Response to step force with rise time



1) **The system oscillates at the natural period T about the static solution** $u_{st}(t) \stackrel{\text{def}}{=} p(t)/k$ **&** $[(u_{st})_0 = p_0/k]$
2) **If the velocity $\dot{u}(t_r)$ is zero at the end of the ramp, the system does not vibrate during the constant-force phase**.
3) **For smaller values of (t_r/T)** (*i.e.*, **relatively short rise time**), the **response is similar to a sudden step force**.
4) **For larger values of (t_r/T)**, the dynamic displacement oscillates close to the static solution (*i.e.*, **dynamic effects are small**).



The deformation attains its maximum value during the constant-force phase of the response (why?).

The time, t_p, at which the peak response occurs is obtained by setting $\dot{u}(t) = 0$. Thus:

$$\frac{p_0}{kt_r}\{-\cos(\omega t) + \cos[\omega(t - t_r)]\} = 0$$

$$\Rightarrow \qquad \tan(\omega t) = \tan\left(\frac{\omega t_r}{2}\right)$$

$$\Rightarrow \qquad \boxed{t_p = \frac{n\pi}{\omega} + \frac{t_r}{2} \quad n = 0,1,2,\cdots}$$

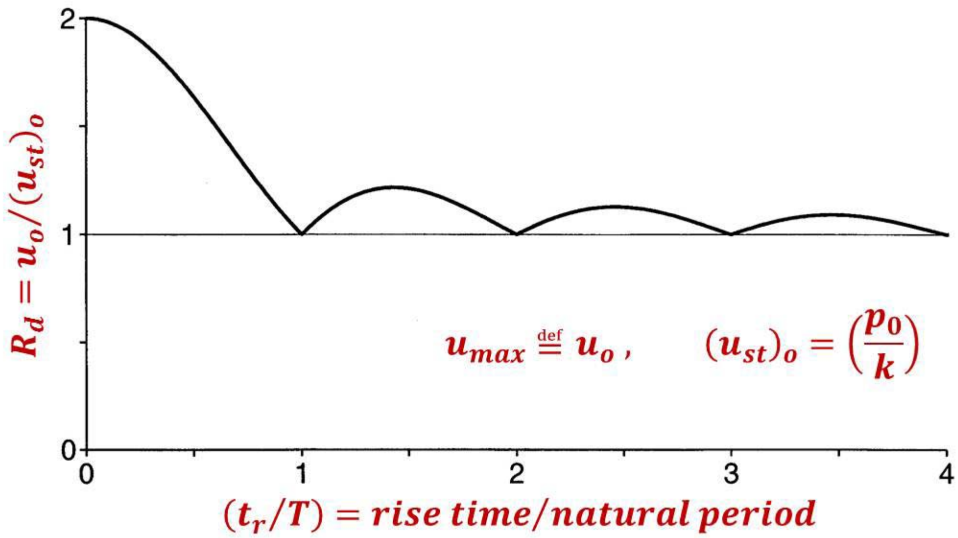
The value of n should be chosen so that $t_p > t_r$.

Also:

$$\left(\frac{t_p}{t_r}\right) = \frac{n\pi}{\omega t_r} + \frac{1}{2} = \frac{n\pi}{2\pi\left(\frac{t_r}{T}\right)} + \frac{1}{2} = \frac{1}{2}\left[\frac{n}{\left(\frac{t_r}{T}\right)} + 1\right]$$

[See plot of $\left(\frac{t_p}{t_r}\right)$ vs. $\left(\frac{t_r}{T}\right)$, above.]

Response spectrum for step force with rise time



Substitution of $t = t_p \stackrel{\text{def}}{=} \frac{n\pi}{\omega} + \frac{t_r}{2}$ in the expression $u(t) = \left(\frac{p_0}{k}\right)\left\{1 - \frac{1}{\omega t_r}[\sin(\omega t) - \sin[\omega(t - t_r)]]\right\}$ $(t > t_r)$ gives the following value for the maximum response:

$$u_{max} \stackrel{\text{def}}{=} u_o = \left(\frac{p_0}{k}\right)\left\{1 + \frac{2\sin\left(n\pi - \frac{\omega t_r}{2}\right)}{\omega t_r}\right\}$$

The true maximum will be obtained by selecting a value of n such that the second term within the braces in the above equation is positive, so that:

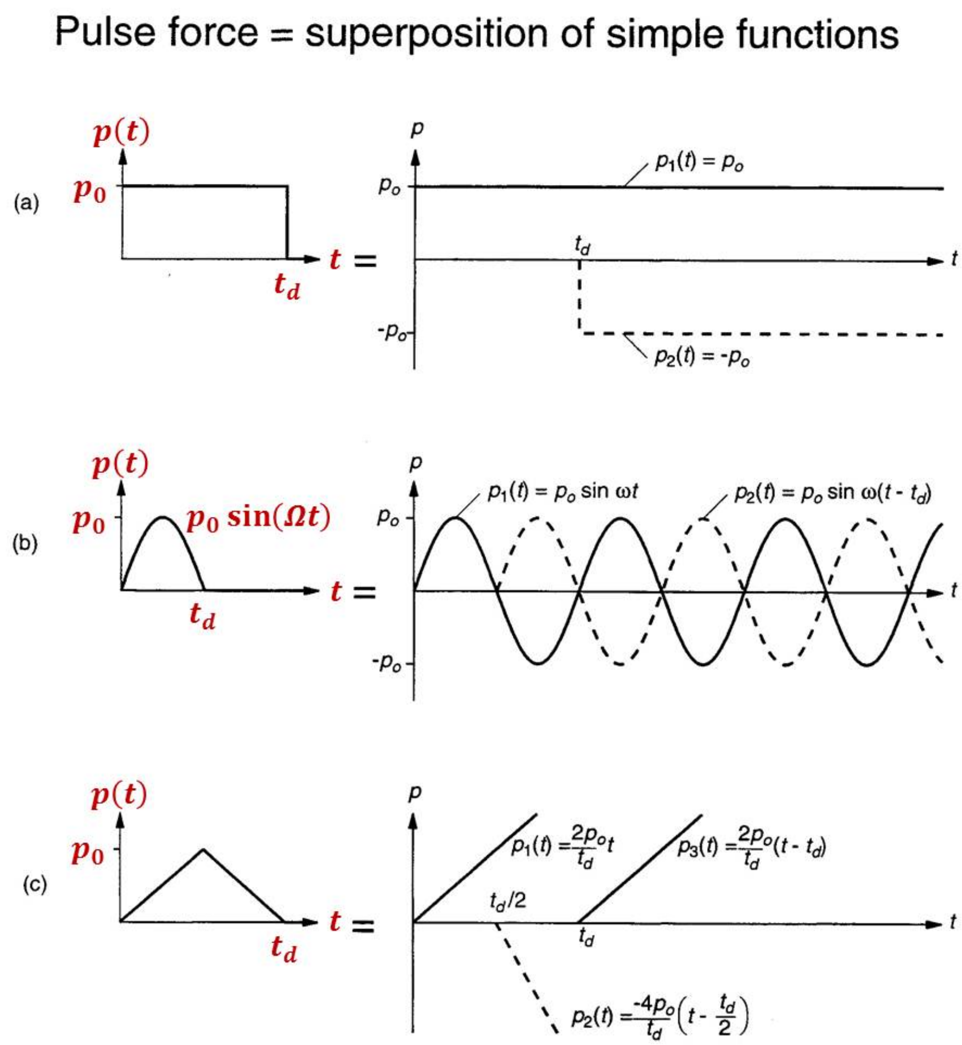
$$u_{max} \stackrel{\text{def}}{=} u_o = \left(\frac{p_0}{k}\right)\left\{1 + \frac{\left|\sin\left(\frac{\omega t_r}{2}\right)\right|}{\left(\frac{\omega t_r}{2}\right)}\right\}$$

Observations:

1) If $t_r < \frac{T}{4}$ then $u_{max} \cong 2\left(\frac{p_0}{k}\right)$
2) If $t_r > 3T$ then $u_{max} \cong \left(\frac{p_0}{k}\right)$ (*i.e.*, **the excitation affects the structure like a static force**)
3) If $(t_r/T) = 1, 2, 3, \cdots$, then $u_{max} \cong \left(\frac{p_0}{k}\right)$, because $\dot{u}(t_r) = 0$ at the end of the force-rise phase, and **the system does not oscillate during the constant-force phase**.

NOTE: Plots such as the above, which **show the relationship between the maximum value of a response parameter and a characteristic of the system** (*e.g.*, T) are called ***response spectra***.

RESPONSE TO PULSE RXCITATION: SOLUTION METHODS

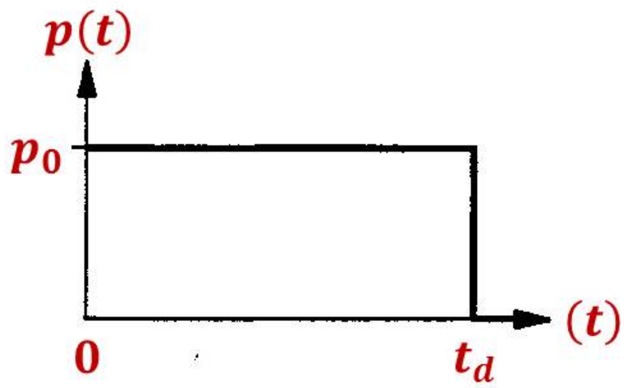


Analytical methods:

1) The **classical method** of solving differential equations;
2) Evaluating **Duhamel's Integral**;
3) Expressing the pulse as **a superposition of two or more simpler functions** for which response solutions are already available or easier to determine.

NOTE: Response to pulse excitations **concerns systems without damping because damping has little influence** on response to pulse excitations.

RECTANGULAR PULSE FORCE (ΟΡΘΟΓΩΝΙΚΟΣ ΠΑΛΜΟΣ)



Equation of Motion: $m\ddot{u} + ku = p(t) = \begin{cases} p_0 & t \le t_d \\ 0 & t \ge t_d \end{cases}$

Initial Conditions: $u(0) = \dot{u}(0) = 0$

Forced vibration phase:

$$\frac{u(t)}{\underbrace{(u_{st})_o}_{(p_0/k)}} = 1 - \cos(\omega t) = 1 - \cos\left(\frac{2\pi t}{T}\right) \quad (t \le t_d)$$

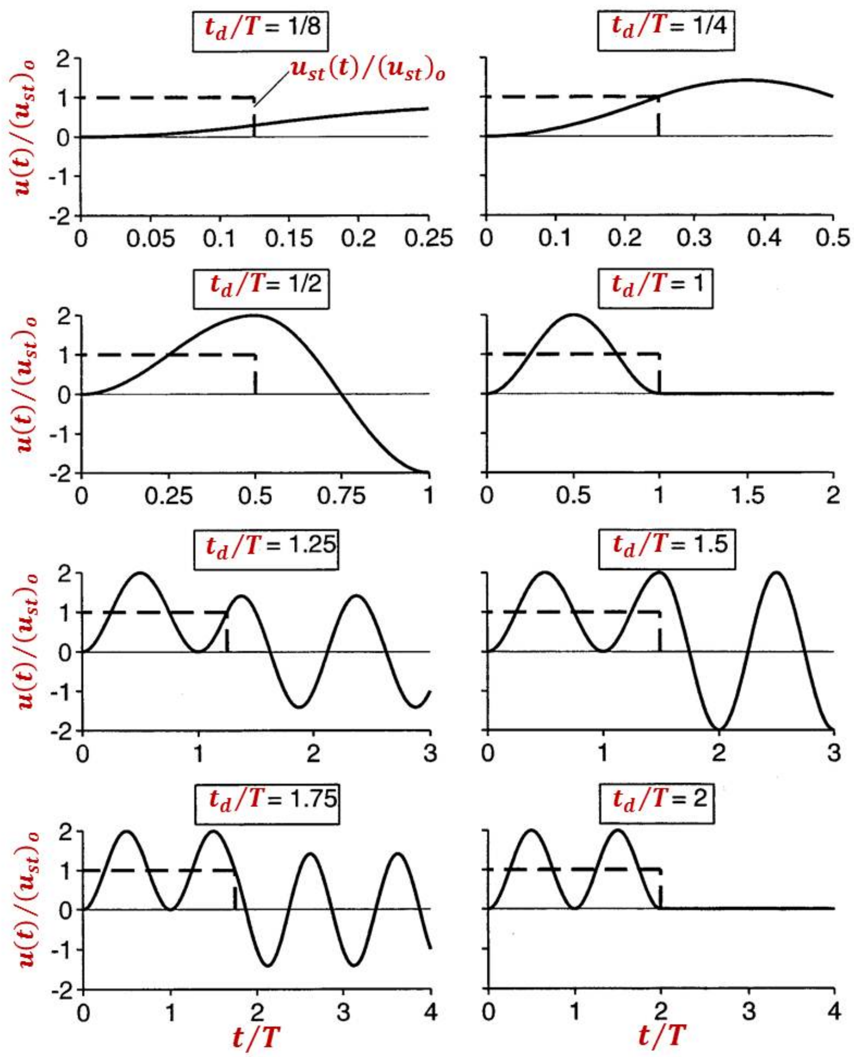
Free vibration phase:

$$\left.\begin{array}{l} u(t) = u(t_d)\cos[\omega(t - t_d)] + \dfrac{\dot{u}(t_d)}{\omega}\sin[\omega(t - t_d)] \\ \text{where:} \quad u(t_d) = (u_{st})_o[1 - \cos(\omega t_d)] \quad \& \quad \dot{u}(t_d) = (u_{st})_o \omega \sin(\omega t_d) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \quad \frac{u(t)}{(u_{st})_o} = \cos[\omega(t - t_d)] - \cos(\omega t) \quad (t \ge t_d)$$

$$\Rightarrow \quad \boxed{\frac{u(t)}{(u_{st})_o} = \left(2\sin\frac{\pi t_d}{T}\right)\sin\left[2\pi\left(\frac{t}{T} - \frac{1}{2}\left(\frac{t_d}{T}\right)\right)\right] \quad (t \ge t_d)}$$

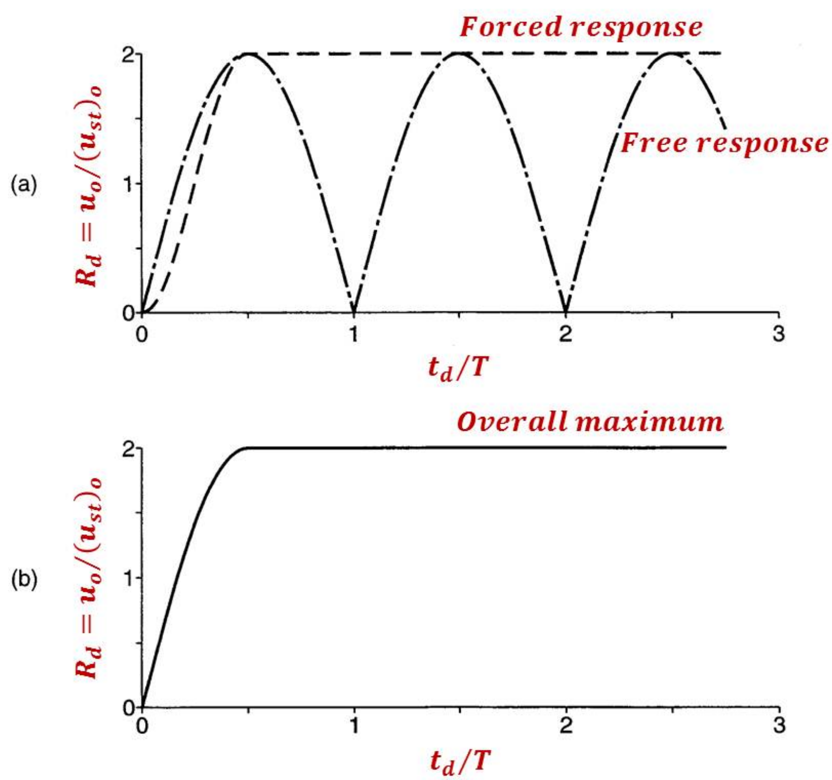
Response to rectangular pulse forces



Response History:

- $u(t)/(u_{st})_o$ **depends only on** (t_d/T) , not separately on t_d or T.
- While the force is applied on the structure, the system oscillates about the shifted position, $(u_{st})_o = p_0/k$, at its own natural period T.
- If $(t_d/T) = 1, 2, 3, \cdots$, **the system stays still at its original undeformed configuration because** $u(t_d) = \dot{u}(t_d) = 0$.
- **It must be** $t_d \geq T/2$ **for at least one peak to develop during the forced vibration phase**; the longer the pulse duration, more such peaks occur, the first peak occurs at $t_o = T/2$ **with the deformation** $u_o = 2(u_{st})_o$.
- **If** $t_d < T/2$ **, no peak will develop during the free vibration phase**.

Shock spectrum for rectangular pulse force



Maximum deformation during the forced vibration phase:

$$R_d = \frac{u_o}{(u_{st})_o} = \begin{cases} 1 - \cos\left(\frac{2\pi t_d}{T}\right) & \left(\frac{t_d}{T}\right) \leq \frac{1}{2} \\ 2 & \left(\frac{t_d}{T}\right) > \frac{1}{2} \end{cases}$$

[Indicated as **'forced response'** in the FIGURE above]

Maximum deformation during the free vibration phase:

$$\left.\begin{aligned} u_o &= \sqrt{[u(t_d)]^2 + \left[\frac{\dot{u}(t_d)}{\omega}\right]^2} \\ \text{where:}\quad u(t_d) &= (u_{st})_o[1 - \cos(\omega t_d)] \quad \& \quad \dot{u}(t_d) = (u_{st})_o \omega \sin(\omega t_d) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \boxed{R_d \overset{\text{def}}{=} \frac{u_o}{(u_{st})_o} = 2\left|\sin\left(\frac{\pi t_d}{T}\right)\right|}$$

[R_d depends only on (t_d/T) and is shown as **'free response'**.]

If $t_d/T < 1/2$ the overall maximum is the peak(s) in $\boldsymbol{u(t)}$ that develops during the **free vibration phase**.

If $t_d/T = 1/2$ the overall maximum in $\boldsymbol{u(t)}$ is given by **either** the **forced-response maximum or** the **free-response maximum** because the two are equal.

If $t_d/T > 1/2$ the overall maximum is the peak(s) in $\boldsymbol{u(t)}$ that develops during the **forced vibration phase**.

In summary:

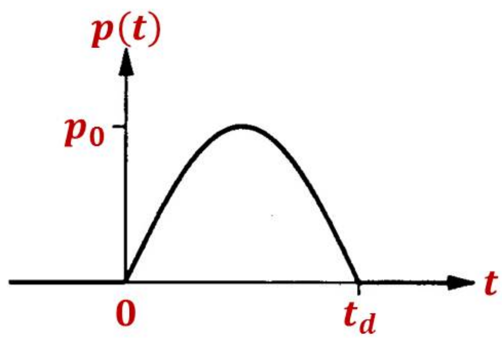
$$R_d \stackrel{\text{def}}{=} \frac{u_o}{(u_{st})_o} = \begin{cases} 2\sin\left(\dfrac{\pi t_d}{T}\right) & \left(\dfrac{t_d}{T}\right) \le \dfrac{1}{2} \\ 2 & \left(\dfrac{t_d}{T}\right) \ge \dfrac{1}{2} \end{cases}$$

A plot of $\boldsymbol{u_o}$ (= **maximum deformation of the system**) vs. $\boldsymbol{T}$ (or **related parameter**), is called a '***response spectrum***'.

When the excitation is a **single pulse**, the terminology '***shock spectrum***' is also used for the response spectrum.

The '***shock/response spectrum***' characterizes the response completely.

HALF-CYCLE SINE PULSE FORCE (ΠΑΛΜΟΣ ΜΙΣΟΥ ΚΥΚΛΟΥ ΗΜΙΤΟΝΟΥ)



Equation of Motion: $m\ddot{u} + ku = p(t) = \begin{cases} p_0 \sin(\pi t/t_d) & t \le t_d \\ 0 & t \ge t_d \end{cases}$

Initial Conditions: $u(0) = \dot{u}(0) = 0$

CASE 1: $(t_d/T) \neq 1/2$

Forced Vibration Phase:

Recall that the response to the harmonic force $p(t) = p_0 \sin(\pi t/t_d)$ is:

$$u(t) = \left(\frac{p_0}{k}\right) \frac{1}{1 - \left(\frac{\Omega}{\omega}\right)^2} \left\{ \sin(\Omega t) - \left(\frac{\Omega}{\omega}\right) \sin(\omega t) \right\}$$

$$\text{where:} \quad \Omega = \left(\frac{\pi}{t_d}\right)$$

Therefore, **the response to the half-sine pulse for $t \le t_d$** is:

$$\frac{u(t)}{(u_{st})_o} = \frac{1}{1 - \left(\frac{T}{2t_d}\right)^2} \left\{ \sin\left(\pi \frac{t}{t_d}\right) - \left(\frac{T}{2t_d}\right) \sin\left(2\pi \frac{t}{T}\right) \right\} \quad t \le t_d$$

$$\text{where:} \quad (u_{st})_o = \left(\frac{p_0}{k}\right)$$

Free Vibration Phase:

We determine $u(t_d)$ & $\dot{u}(t_d)$ from the forced response above, and we obtain:

$$\frac{u(t)}{(u_{st})_o} = \frac{\left(\frac{T}{t_d}\right) \cos\left(\frac{\pi t_d}{T}\right)}{\left(\frac{T}{2t_d}\right)^2 - 1} \sin\left[2\pi\left(\frac{t}{T} - \frac{1}{2}\frac{t_d}{T}\right)\right] \quad t \ge t_d$$

CASE 2: $(t_d/T) = 1/2$

Forced Vibration Phase:

Recall that the response to the harmonic loading $p(t) = p_0 \sin(\omega t)$ (*i.e.*, **phase resonance**) is:

$$u(t) = \frac{1}{2}\left(\frac{p_0}{k}\right)\{\omega t \cos(\omega t) - \sin(\omega t)\}$$

Therefore, the response to the half-sine pulse for $t \leq t_d$ is:

$$\boxed{\begin{array}{c} \dfrac{u(t)}{(u_{st})_o} = \dfrac{1}{2}\left\{\sin\left(\dfrac{2\pi t}{T}\right) - \left(\dfrac{2\pi t}{T}\right)\cos\left(\dfrac{2\pi t}{T}\right)\right\} \quad t \leq t_d \\ \text{where:} \quad (u_{st})_o = \left(\dfrac{p_0}{k}\right) \end{array}}$$

Free Vibration Phase:

We determine $u(t_d)$ **&** $\dot{u}(t_d)$ from the forced response above:

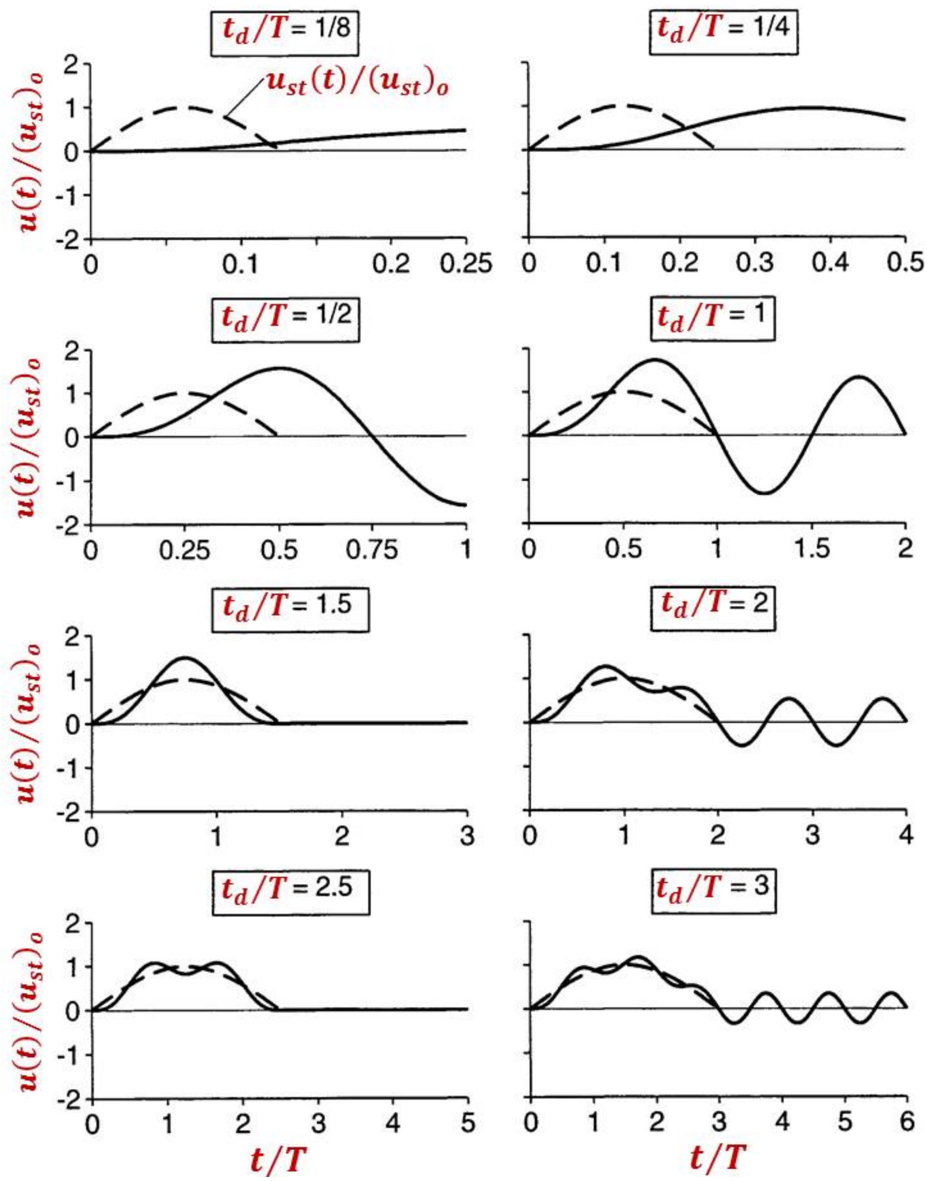
$$\frac{u(t_d)}{(u_{st})_o} = \frac{\pi}{2} \quad \& \quad \dot{u}(t_d) = 0$$

Equation $\dot{u}(t_d) = 0$ implies that the displacement in the forced vibration phase reaches its maximum at the end of this phase.

Therefore, **the response in the free-vibration phase** is:

$$\boxed{\frac{u(t)}{(u_{st})_o} = \frac{\pi}{2}\cos\left[2\pi\left(\frac{t}{T} - \frac{1}{2}\right)\right] \quad t \geq t_d}$$

Response to half-cycle sine pulse forces



Response History:

- **As** (t_d/T) **increases** (*i.e.*, as the variation of the time becomes increasingly slower relative to the natural period T of the system) **the dynamic effects become smaller**.
- **The forced-vibration response** contains both frequencies: $\Omega(=\pi/t_d)$ and ω and it **is positive throughout**.
- If $(t_d/T) = 1.5, 2.5, \cdots$, the mass stays still after the force pulse ends because $u(0) = \dot{u}(0) = 0$.

Maximum Response:

Forced Vibration Phase:

To determine t_o = time instants when the peaks occur, we set $\dot{u}(t_o) = 0$:

$$\dot{u}(t_o) = 0 \quad \Longrightarrow \quad \cos\left(\frac{\pi t_o}{t_d}\right) = \cos\left(\frac{2\pi t_o}{T}\right) \qquad \textbf{\textit{transcedental equation}}$$

$$\Longrightarrow \quad (t_o)_\ell = \frac{2\ell}{1 \pm 2\left(\frac{t_d}{T}\right)} t_d \qquad \ell = 1,2,3,\cdots$$

where: **negative** signs $\leftrightarrow$ local **minima**

positive signs $\leftrightarrow$ local **maxima**

Therefore:

$$\boxed{(t_o)_\ell = \frac{2\ell}{1 + 2\left(\frac{t_d}{T}\right)} t_d \qquad \ell = 1,2,3,\cdots}$$

We consider only $(t_o)_\ell \le t_d$. Substituting t in the expression u of forced response we obtain:

$$\boxed{R_d \overset{\text{def}}{=} \frac{u_o}{(u_{st})_o} = \frac{1}{1 - \left(\frac{T}{2t_d}\right)^2} \left\{ \sin\left(\frac{2\pi\ell}{1 + \frac{2t_d}{T}}\right) - \left(\frac{T}{2t_d}\right) \sin\left(\frac{2\pi\ell}{1 + \frac{T}{2t_d}}\right) \right\}}$$

where: u_o = **peak value**

Free Vibration Phase:

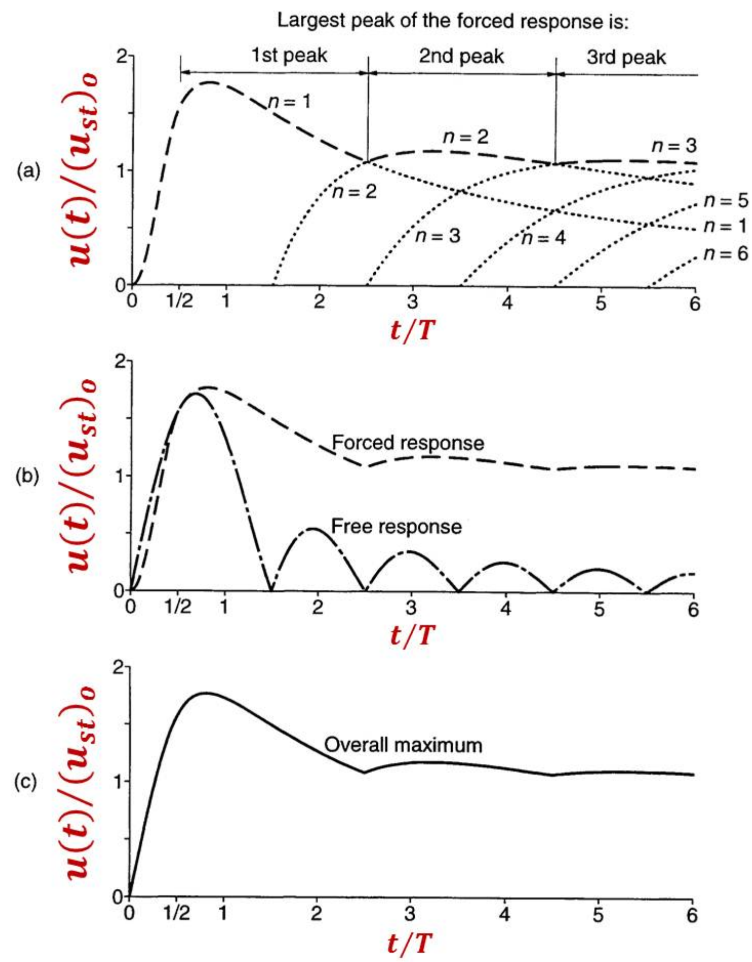
$$\boxed{R_d \overset{\text{def}}{=} \frac{u_o}{(u_{st})_o} = \frac{\left(\frac{T}{t_d}\right)\cos\left(\frac{\pi t_d}{T}\right)}{\left(\frac{T}{2t_d}\right)^2 - 1}}$$

[plotted as **'free response'**]

For the special case of $(t_d/T) = 1/2$, **the maximum of the forced and free vibration phases are the same**:

$$\boxed{R_d \overset{\text{def}}{=} \frac{u_o}{(u_{st})_o} = \frac{\pi}{2}}$$

Shock spectrum for half-cycle sine pulse force



$$R_d \overset{\text{def}}{=} \frac{u_o}{(u_{st})_o} = \frac{1}{1-\left(\frac{T}{2t_d}\right)^2}\left\{\sin\left(\frac{2\pi\ell}{1+\frac{2t_d}{T}}\right) - \left(\frac{T}{2t_d}\right)\sin\left(\frac{2\pi\ell}{1+\frac{T}{2t_d}}\right)\right\}$$

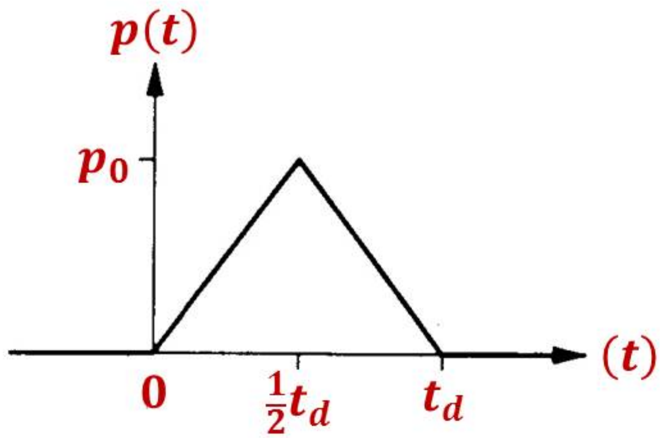
If $\mathbf{0.5 \le (t_d/T) \le 1.5}$ **only one peak, $\ell = 1$, occurs during the forced response**.

If $\mathbf{1.5 < (t_d/T)}$ a second peak develops, but is smaller than the first peak if $\mathbf{1.5 < (t_d/T) < 2.5}$

If $\mathbf{(t_d/T) < 0.5}$ **no peak occurs during the forced vibration phase**
(*i.e.*, for $\ell = 1 \Rightarrow t_o = \frac{2}{1+2\left(\frac{t_d}{T}\right)} t_d \Rightarrow t_o > t_d$)

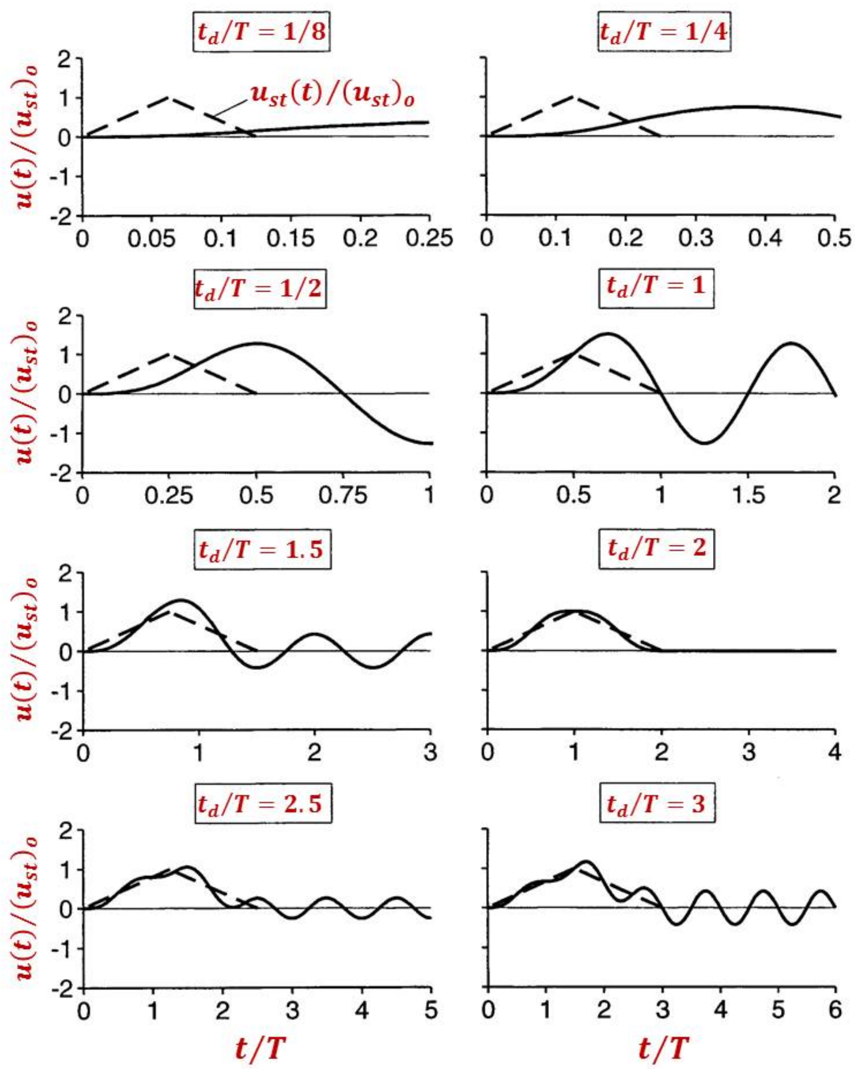
Therefore, **the peak forced response** is: $\frac{u(t_d)}{(u_{st})_o} = \frac{\left(\frac{T}{2t_d}\right)}{\left(\frac{T}{2t_d}\right)^2 - 1}\sin\left(2\pi\frac{t_d}{T}\right)$

SYMMETRICAL TRIANGULAR PULSE FORCE (ΣΥΜΜΕΤΡΙΚΟΣ ΤΡΙΓΩΝΙΚΟΣ ΠΑΛΜΟΣ)



$$\frac{u(t)}{(u_{st})_o} = \begin{cases} 2\left\{\dfrac{t}{t_d} - \dfrac{T}{2\pi t_d}\sin\left(2\pi\dfrac{t}{T}\right)\right\} & 0 \le t \le \dfrac{t_d}{2} \\ 2\left\{1 - \dfrac{t}{t_d} + \dfrac{T}{2\pi t_d}\left[2\sin\left[\dfrac{2\pi}{T}\left(t - \dfrac{1}{2}t_d\right)\right] - \sin\left(2\pi\dfrac{t}{T}\right)\right]\right\} & \dfrac{t_d}{2} \le t \le t_d \\ 2\left\{\dfrac{T}{2\pi t_d}\left\{2\sin\left[\dfrac{2\pi}{T}\left(t - \dfrac{1}{2}t_d\right)\right] - \sin\left[\dfrac{2\pi}{T}(t - t_d)\right] - \sin\left(2\pi\dfrac{t}{T}\right)\right\}\right\} & t_d \le t \end{cases}$$

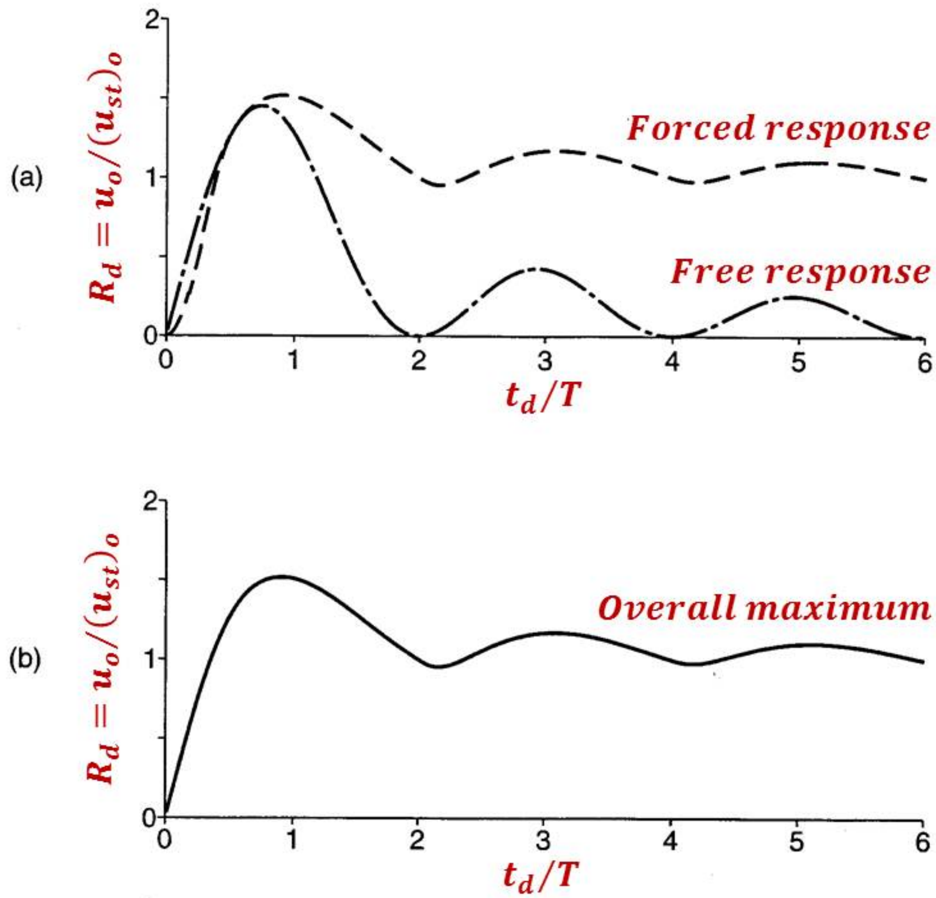
Response to triangular pulse forces



- **The dynamic effects are seen to decrease as the pulse duration t_d increases beyond $2T$.**
- **The first peak develops**

	right at the end of the pulse	if $t_d = T/2$;
	during the pulse	if $t_d > T/2$;
and	**after the pulse**	if $t_d < T/2$

Shock spectrum for triangular pulse force



(ΦΑΣΜΑ ΠΛΗΓΜΑΤΟΣ)

The above FIGURE shows that:

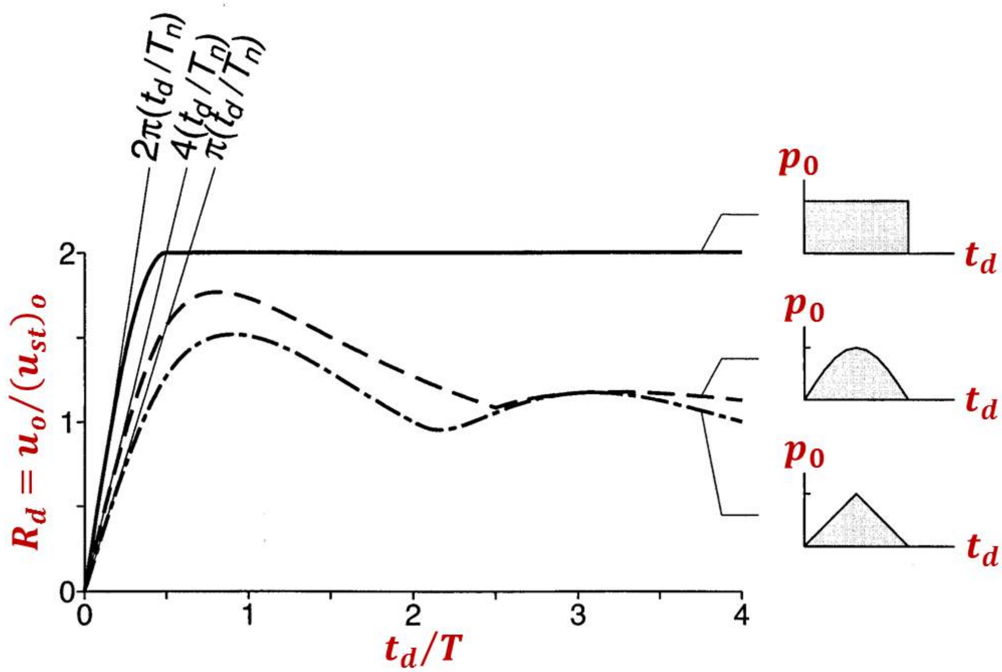
If $t_d > T/2$ **the overall maximum is the largest peak that develops during the force pulse**;

If $t_d < T/2$ **the overall maximum is the peak response during the free vibration phase**;

If $t_d = T/2$ **the forced and free response maxima are equal**

EFFECTS OF PULSE SHAPE AND APPROXIMATE ANALYSIS FOR SHORT PULSES

Shock spectra for force pulses of equal amplitude

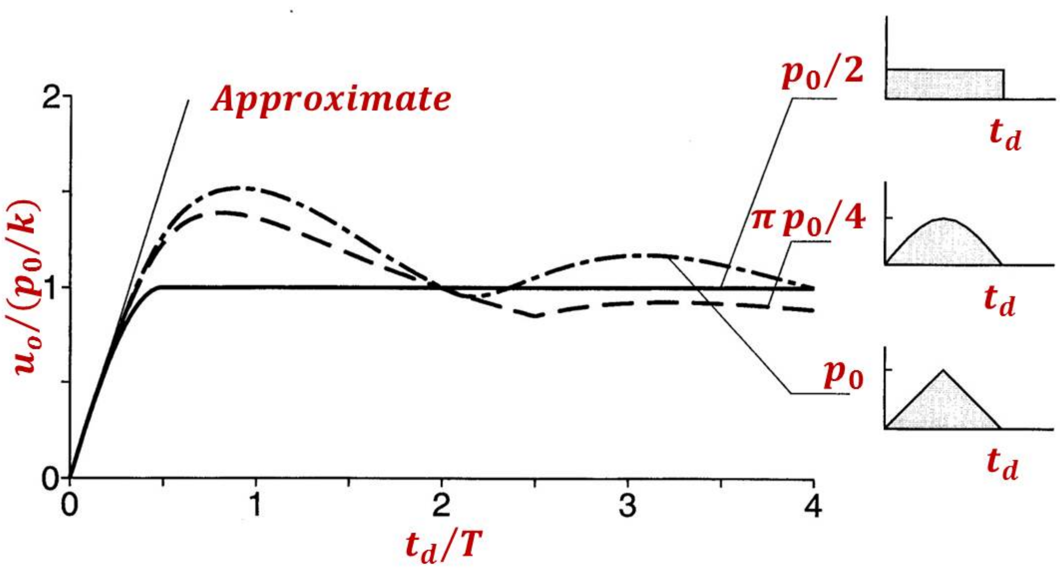


- As shown in the previous analyses, **if** $t_d > T/2$, the overall maximum deformation occurs during the pulse. Then **the pulse shape is of great significance**.
- For **the larger values of** t_d/T, the **overall maximum** is influenced by the **rapidity of loading**.
- If $t_d < T/2$, the **overall maximum** response of the system occurs during its free vibration phase and is **controlled by the time integral of the pulse**

$$I = \int_0^{t_d} p(t)\,dt \quad (= \textit{Impulse})$$

- **This can be demonstrated by considering the limiting case as** $(t_d/T) \to 0$.

Shock spectra for force pulses of equal area



Recall that the response of a SDOF system to an impulse of intensity ***I*** is:

$$u(t) = \frac{I}{m\omega}\sin(\omega t)$$

Thus, the **maximum deformation** is:

$$u_o = \frac{I}{m\omega} = \left(\frac{I}{k}\right)\left(\frac{2\pi}{T}\right)$$

Over the range $(t_d/T) < (1/4)$, the pure impulse solution is close to the exact response, *i.e.* **the maximum deformation is essentially controlled by the pulse area, independent of its shape**.

EFFECTS OF VISCOUS DAMPING

Shock spectra for a half-cycle sine pulse force

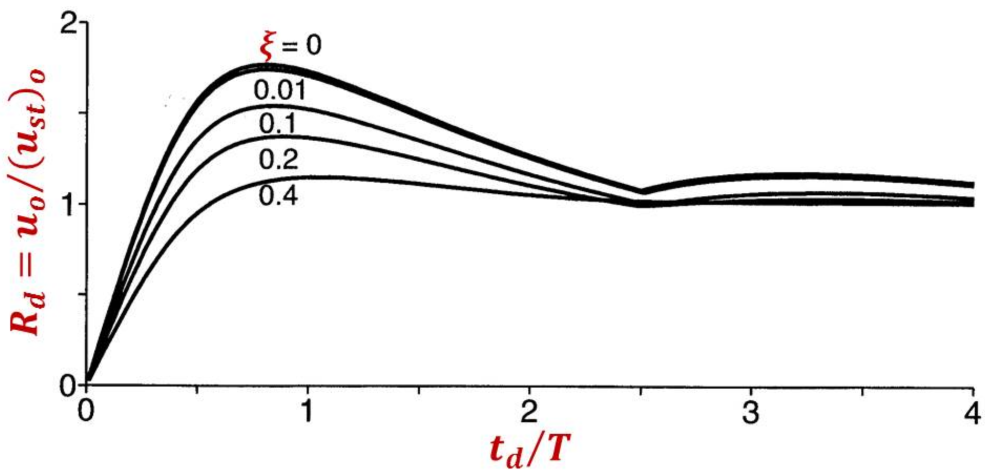


FIGURE: **Shock spectra** for a half-cycle sine pulse force for five damping values

If the excitation is a sine pulse, the effect of damping on the maximum response is usually not important unless the system is highly damped.

Thus a conservative but not overly conservative estimate of the response of many practical structures with damping to pulse-type excitations may be obtained by neglecting damping and using the results for undamped systems.