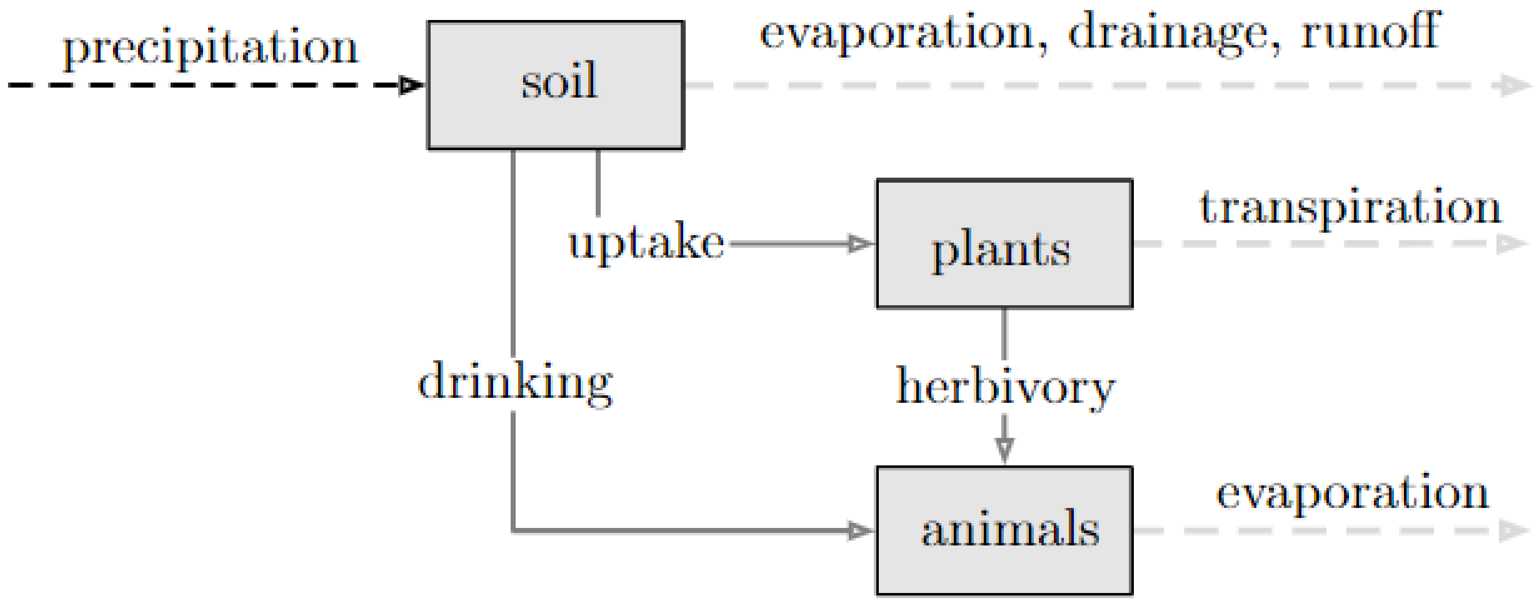


Noy-Meir water flow model for a desert ecosystem

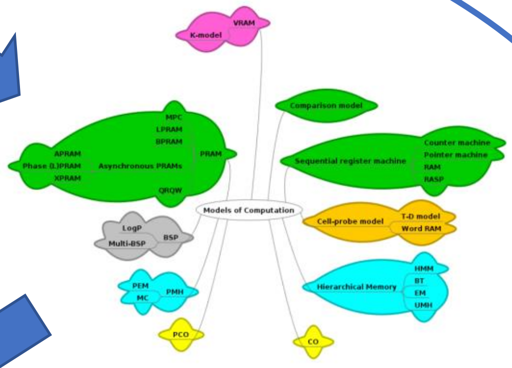
Modeling (Natural) Processes

Up Until Now… & From Now and on…

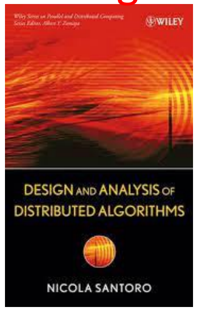
Problem

Feedback Loop?

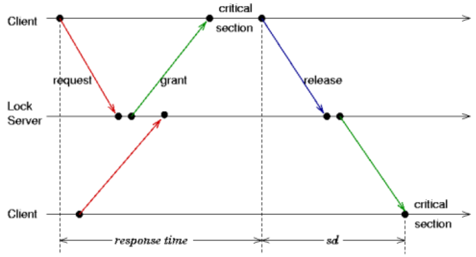
Model



Protocol Design



Execution Results



Phenomenon

Intervention?

Model

Analysis

Natural Processes

We look at certain modeling techniques for natural processes in:

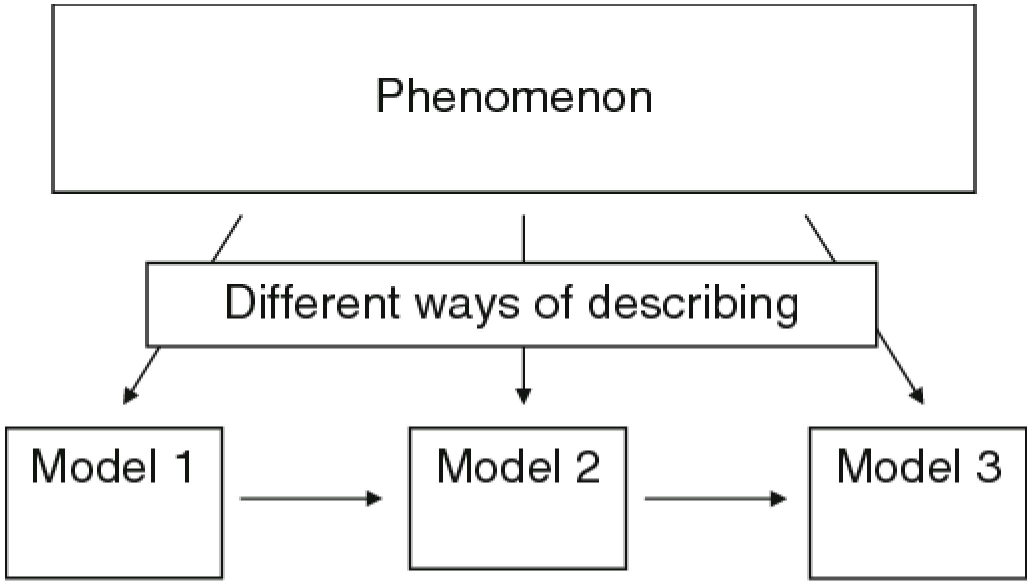
- Physics (e.g., fluid mechanics)
- Environmental Sciences (e.g., river modeling)
- Biology (e.g., tissue growth)
- Ecosystems (e.g,. epidemics, competition, ant behavior)
- Finance, social sciences, traffic, pedestrian movement,…
- …

What is a Model?

This is not a pipe. It is only a depiction of a pipe.

- Simplified abstraction of reality
- Only necessary ingredients are retained
  - These must be related to the questions at hand
- Representation of phenomena in a mathematical or computer language (algorithms?)

Why a Model?



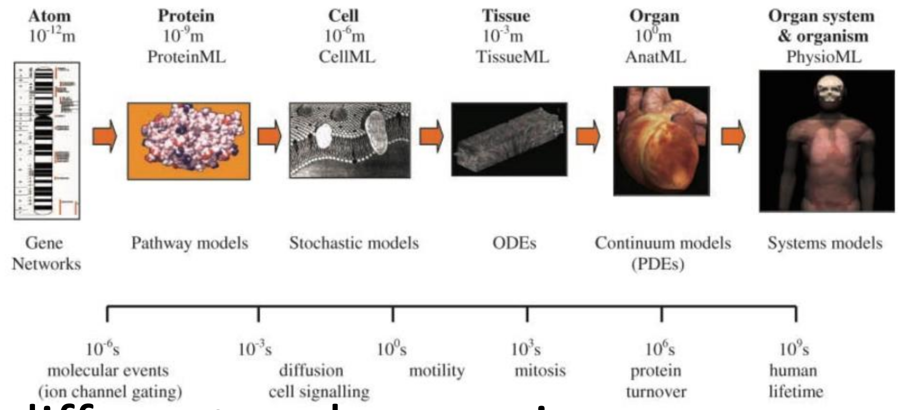
Describe, classify and:

- Understand
- Predict
- Control Phenomena (Make Interventions)

What is a good model?

- It depends on the questions asked. Based on these questions, different aspects of the process are studied.

Scales



The same system can be described at different scales, and different scales require different methods to be applied

- mechanical parts, car, traffic
- cells, tissues, organs, human being, societies
- atoms, molecules, fluid mechanics, pressure fields, climate
- virus propagation within human body, within a family, within a school/workplace, within a town, in a country, globally

1. One has to identify important ingredients and their interactions
2. Often, one defines a model at a finer scale than the scale on which questions are asked.

Several Models – Different Languages

Partial differential equation for a fluid (Navier-Stokes)

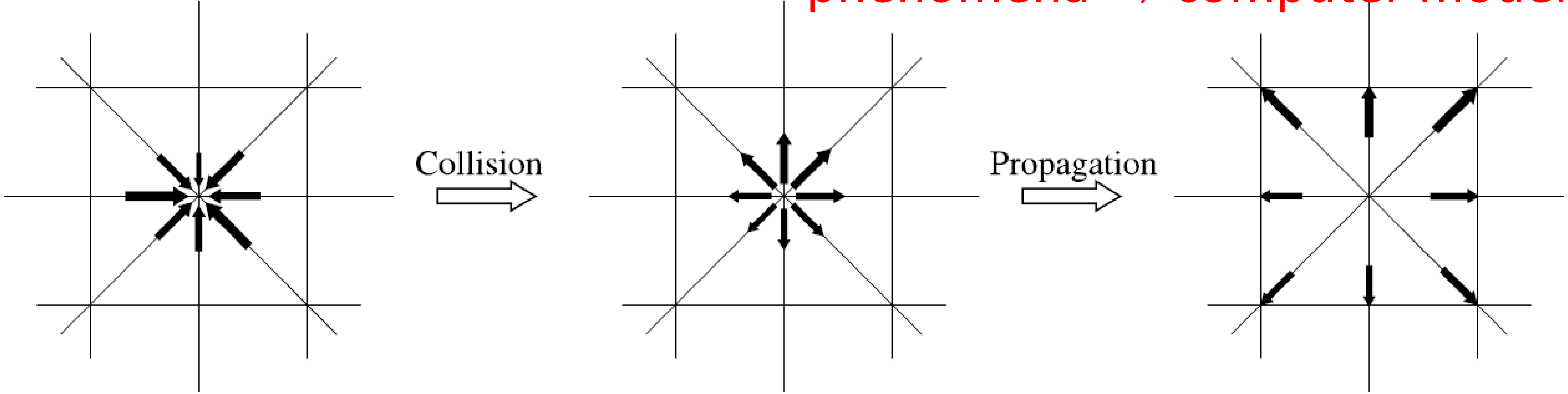
$$\vartheta_t u + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p + v \nabla^2 u$$

phenomena → PDE → discretization → numerical solution

vs

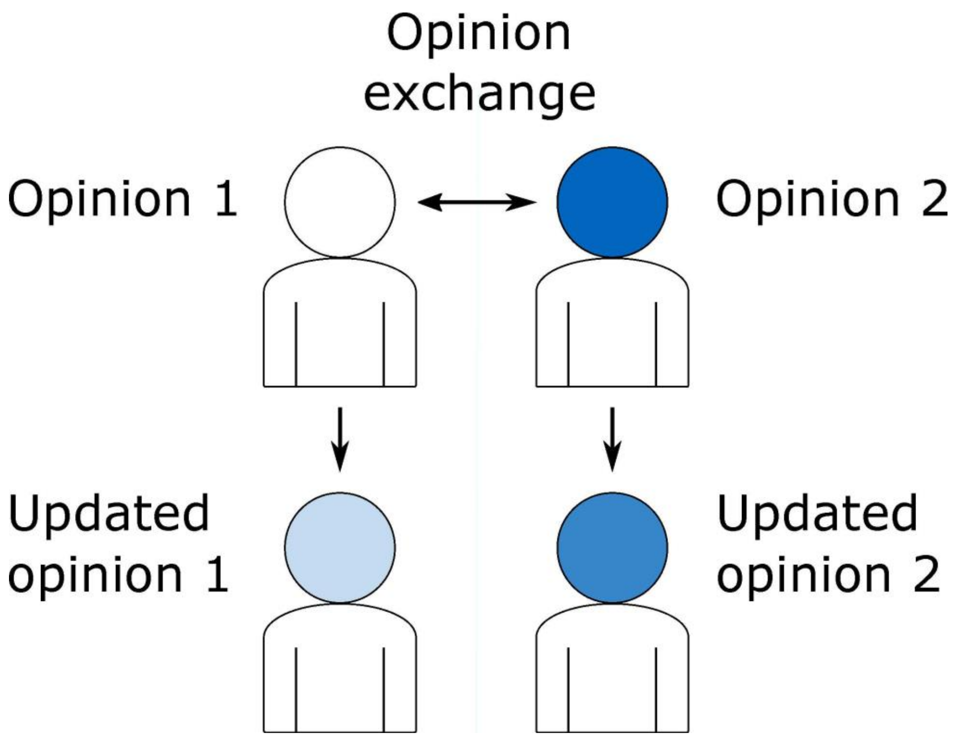
Virtual model of reality. We consider a discrete universe as an abstraction of the real world.

phenomena → computer model



Models…

- Mathematical Equations, ODE, PDE
- Monte-Carlo methods
- Cellular Automata
- Multi-agent Systems
- Complex Networks
- …



Opinion Dynamics - Consensus

Let's start with opinion models with linear dynamics

Common Types of Opinion Dynamics Phenomena and Respective Modesls

Phenomenon	Description	Model
Consensus	The agents achieve an agreement	DeGroot
Stubborness	The agents tend to defend their opinions in a discussion	Friedkin-Johnsen
Polarization	The agents achieve two different, opposite consensus values	Bipartite consensus
Bounded confi-dence	The agents "trust" only other agents having similar opinions. They cluster according to the initial opinions	Hegselmann-Krause
Biased assimilation	Describes how the agents give more impor-tance to their own opinions	Pravan-Goed-Lee
Reflected self-appraisal	Describes the evolution of the "social power" of the agents on a sequence of discussions	DeGroot-Friedkin
Eco chambers	Similar opinions tend to reinforce each other	?
Herding	Leader-follower or "bandwagon" behavior	?
Wisdom of crowds	Crowds tend to reject extremisms	?
⋮		

Seeing from a Learning "Lens"

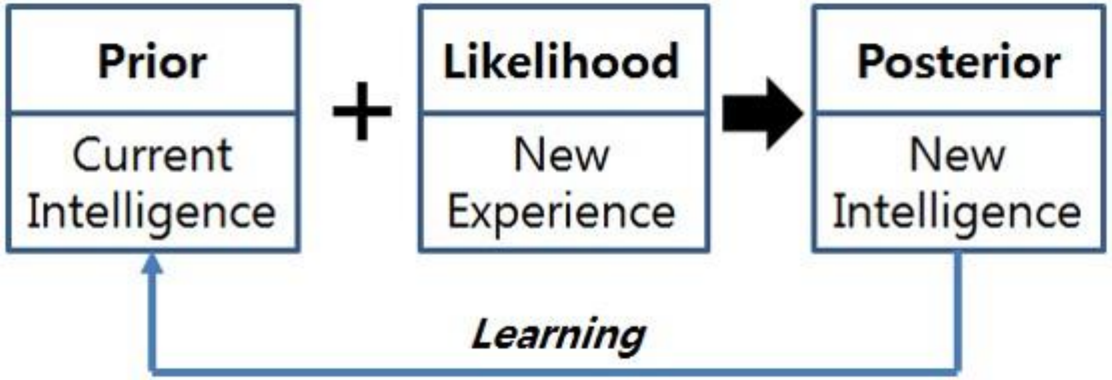
<u>Bayesian Learning</u>

- Repeated actions (complicated updating)
- Observe each other

<u>DeGroot Model</u>

- Repeated communication
- "naïve" updating

We will only touch on the Bayesian approach: a model of Bala and Goyal



Simple Bayesian Learning

The Questions

- Will society converge?
  - Converging in having the same opinion.
- Will they aggregate information properly?
  - Will they eventually converge to the right opinion?
- …

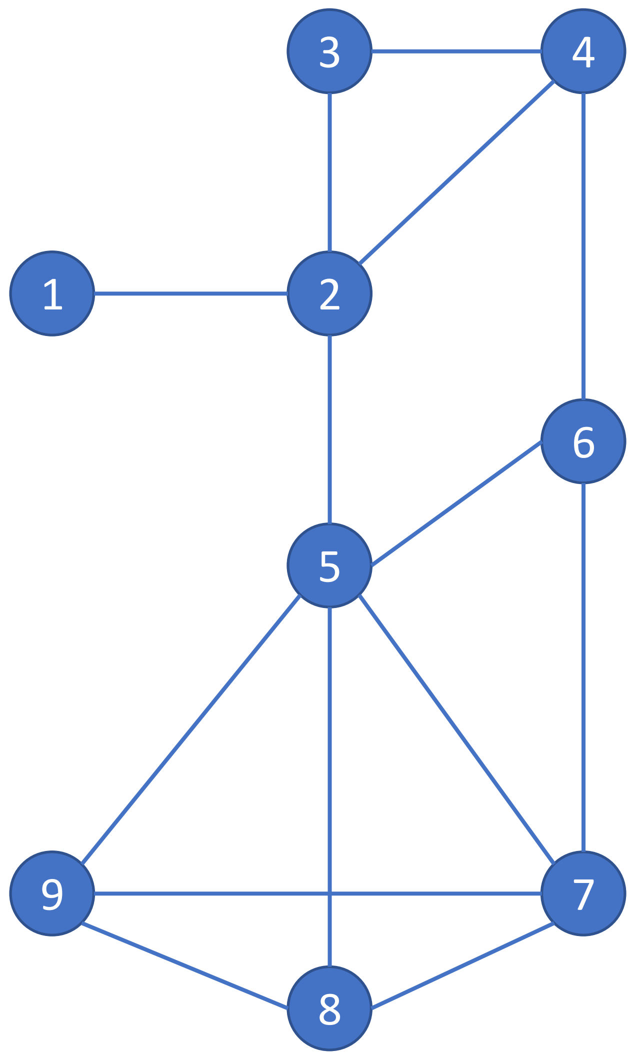
Setting

n players in an undirected connected component g

- Choose action A or B each period
- A pays 1 for sure
- B pays 2 with probability p and 0 with probability $1 - p$

Learning:

- Each period get a payoff based on choice
- Also observe neighbors' choices
- Maximize discounted stream of payoffs $E[\sum_t \delta_t \pi_{it}]$ (π_{it}: payoff at t for agent i, δ_t: belief of agent i at time t)
- p is unknown, takes on finite set of values



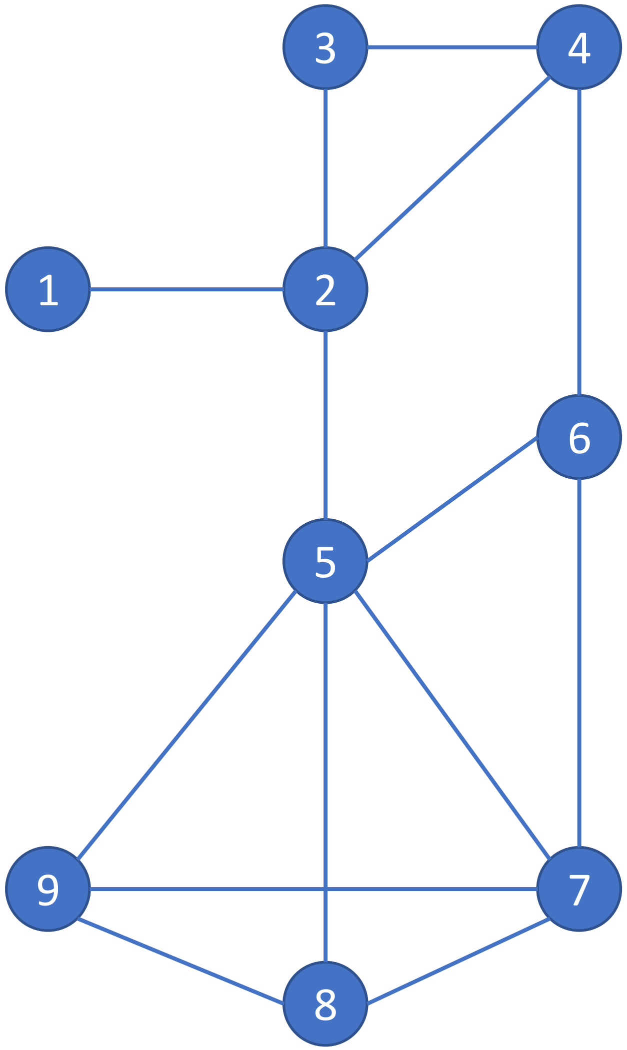
Challenges

Complicated inferences about the choice for A and B for non-neighbors.

- In this model we assume bounded rationality: just look at the history of As and Bs and do not make inferences

If my prior is $p < \frac{1}{2}$, then by experimentation I may try to look at whether this belief is right. However, I could play strategically and let the others experiment and see what happens (free rider) – complicated game

- In this model we assume that players are not strategic



Convergence to Conformism

Proposition: If p is not exactly $1/2$, then with probability 1 there is a time such that all agents in a given component play just one action (and all play the same action) from that time onward

Proof:

- Suppose contrary
- Some agent in some component plays B infinitely often
- That agent will converge to true belief by the law of large numbers
- Must be that belief converges to $p > 1/2$, or that agent would stop playing B
- With probability 1, all agents who see B played infinitely often converge to a belief that B pays 2 with prob $p > 1/2$
- Neighbors of agent must play B, after some time, and so forth
- All agents must play B from some time on

Do we Play the Right Action?

- If B is the right action, then play the right action if converge to it, but might not
- If A is the right action, then must converge to right action

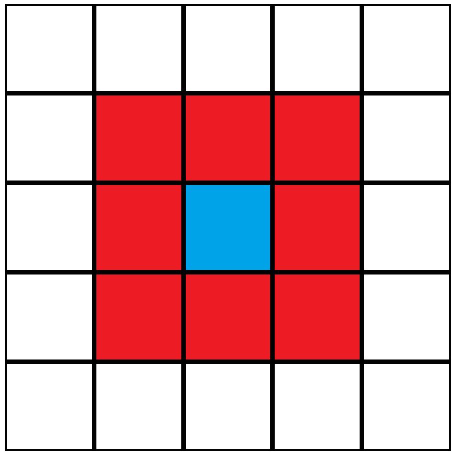
What is the probability of converging to correct action?

- Arbitrarily high if each action has some agent who initially has arbitrarily high prior that the action is the best one

Temporal and Spatial Evolution of Learning

Simulations:

- Set of farmers in a $k \times k$ grid
- Each farmer owns a single plot of land
- Each farmer i observes the actions and payoffs of surrounding 8 neighbors

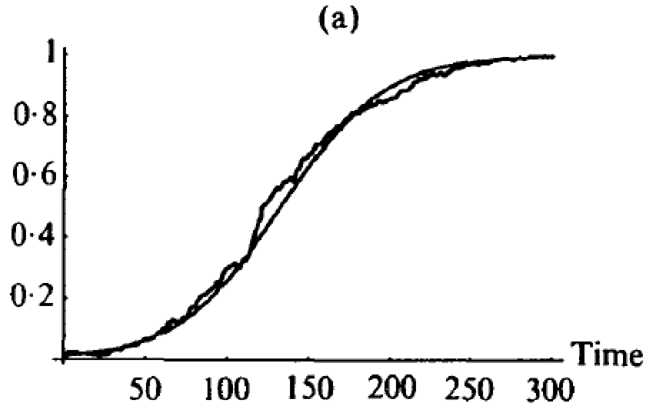


Temporal Patterns

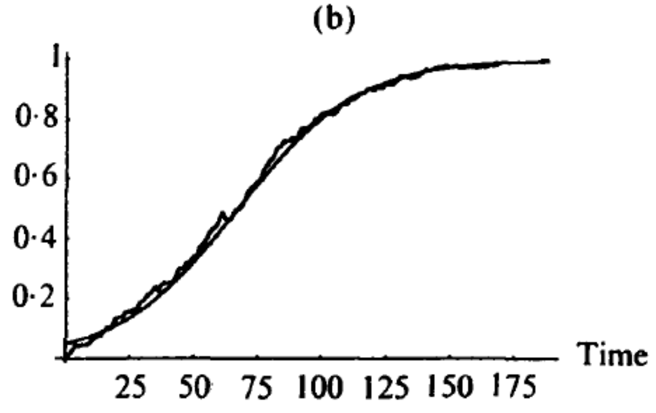
Assume 2 crops.

- Crop 1: payoff equal to $\frac{1}{2}$
- Crop 2: payoff either equal to $0{,}45$ or to $0{,}55$

Assuming the best for crop 2:

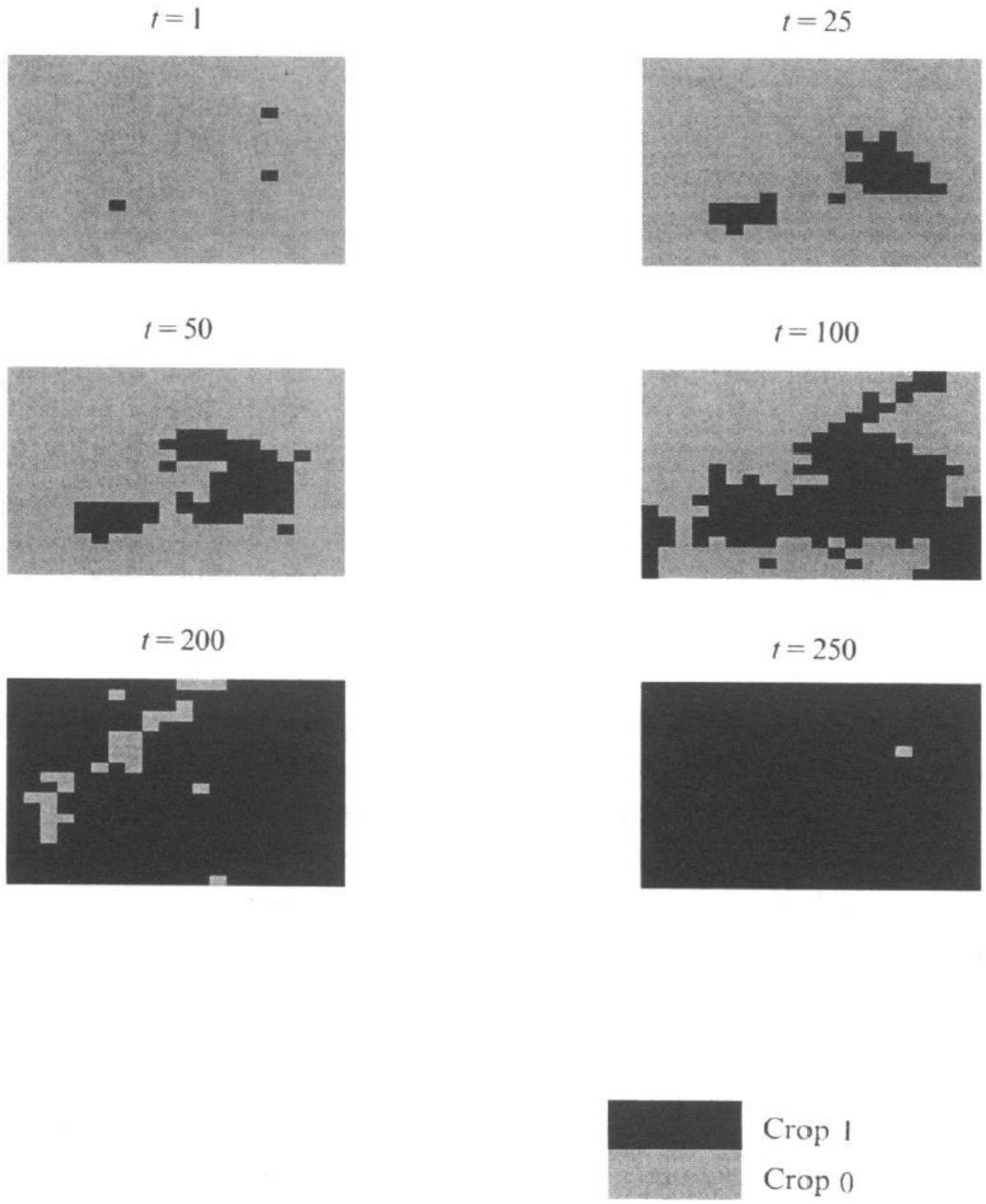


Assuming the best for crop 2 is $0{,}57$:



Spatial Patterns

We consider the previous example.



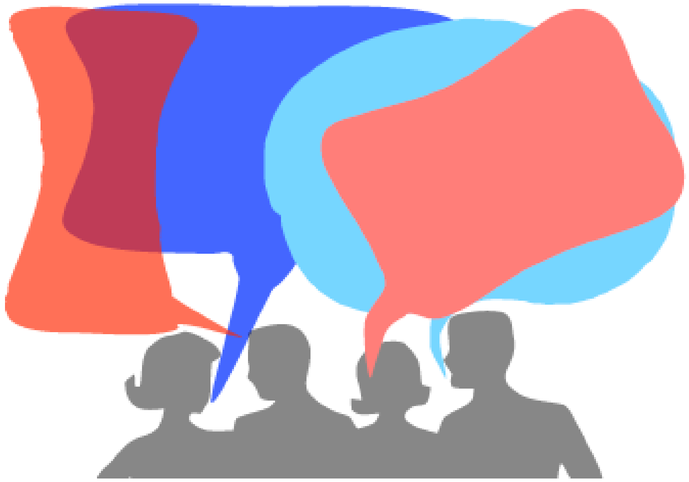
Limitations of this Model

- Homogeneity of actions and payoffs across players. What if heterogeneity?
- Repeated actions over time (experimentation is not always possible – e.g., global warming)
- Stationarity
- Networks are not playing role here!

Linear Network Systems

Focusing on Communication

Opinion Dynamics in Social Influence Networks



$$p_i^+ = \sum_{1 \le j \le n} a_{ij} p_j$$

a_{ij}:weight that i assigns to the opinion p_j of j

Questions:

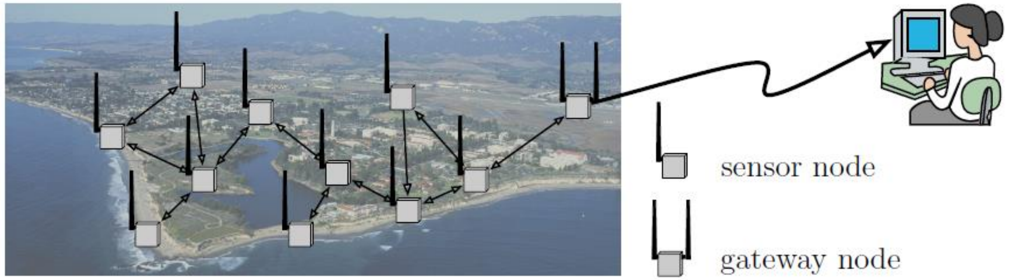
(i) Is this model of human opinion dynamics believable? Is there empirical evidence in its support?

(ii) How does one measure the coefficients a_{ij}?

(iii) Under what conditions do the pdfs converge to the same pdf? In other words, when do the agents achieve consensus? And to what final pdf?

(iv) What are more realistic, empirically-motivated models, possibly including stubborn individuals or antagonistic interactions?

Averaging Algorithms in Wireless Sensor Networks



$$x_i^+ = avg(x_i, \{x_j, for\ all\ neighbor\ nodes\ j\ or\ i\})$$

x_i: a scalar quantity (e.g., temperature)

Questions:

(i) Does each node converge to a value? Is this value the same for all nodes?

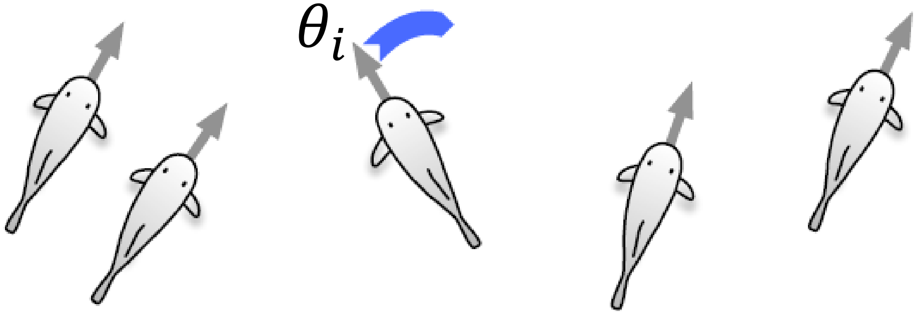
(ii) Is this value equal to the average of the initial conditions? In other words, when do the agents achieve average consensus?

(iii) What properties do the graph and the corresponding matrix need to have in order for the algorithm to converge?

(iv) How quick is the convergence?

Flocking Dynamics in Animal Behavior

$$\Delta\theta_i = average(\{\theta_j, for\ all\ neighbors\ j\}) - \theta_i$$



Questions:

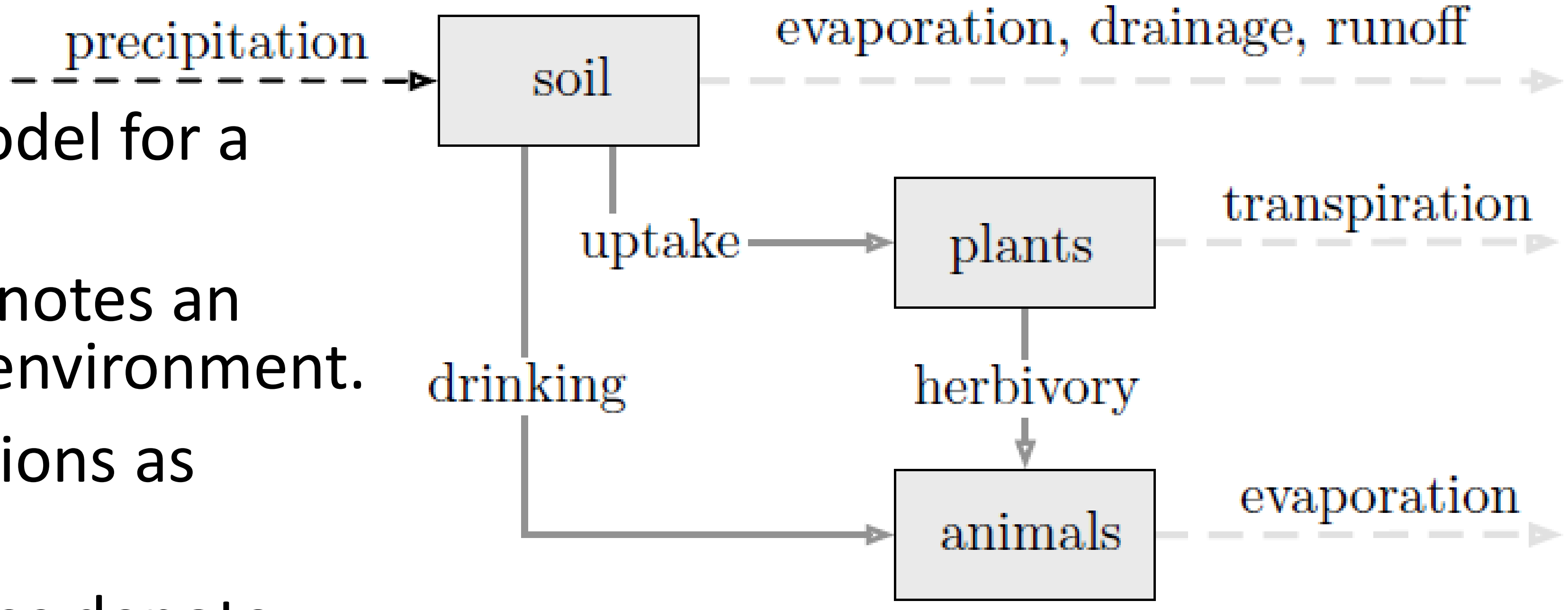
(i) how valid is this model in understanding and reproducing animal behavior?

(ii) what are equilibrium headings and when are they attractive?

(iii) what properties does the graph need to have to ensure a proper flocking behavior?

Dynamical Flow Systems in Ecosystems
(Compartmental Systems)

Water flow model for a desert ecosystem.

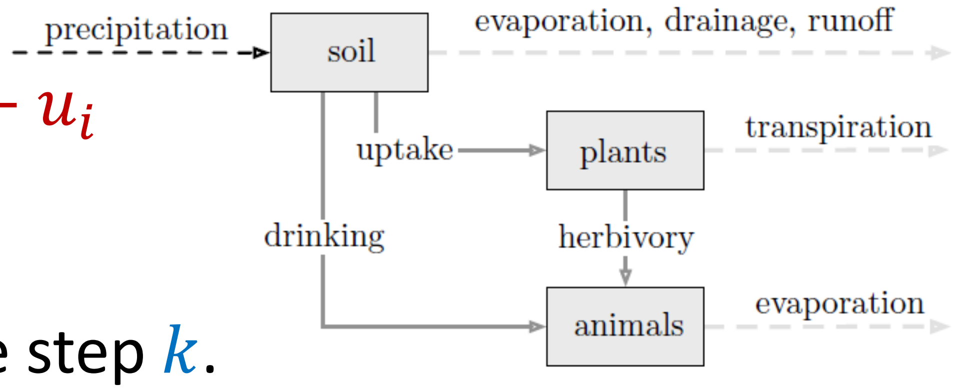


The Noy-Meir water flow model for a desert ecosystem.

1. The black dashed line denotes an inflow from the outside environment.
2. Each compartment functions as storage unit.
3. The light-gray dashed lines denote outflows into the outside environment

Discrete-Time Model

$$q_i(k+1) = \sum_{1 \le j \le n} a_{ji} q_j + u_i$$



$q_i(k)$: quantity of water at compartment i in time step k.

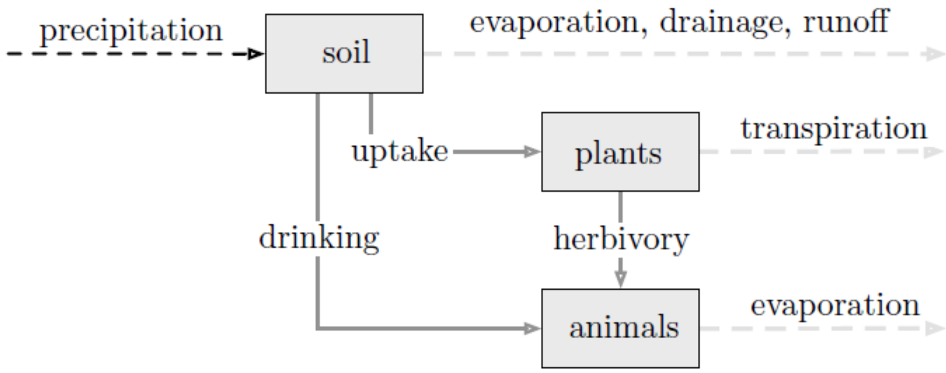
a_{ij}: (routing fractions) fraction of water from i to j in one step.

u_i: non-negative supply to compartment i.

$$A_{Noy-Meir} = \begin{bmatrix} 1 - a_{edr} - a_u - a_d & a_u & a_d \\ 0 & 1 - a_t - a_h & a_h \\ 0 & 0 & 1 - a_e \end{bmatrix}, U = \begin{bmatrix} a_p \\ 0 \\ 0 \end{bmatrix}$$

$$Q(k+1) = A^T_{Noy-Meir} Q_k + U$$

Continuous-Time Model



$$\frac{dq_i(t)}{dt} = \sum_{j=1, j\neq i}^{n} \left(f_{ji}q_j(t) - f_{ij}q_i(t)\right) - f_{0,i}q_i(t) + u_i$$

$q_i(t)$: denote the quantity of water at compartment i at time $t \in \Re^+ \cup \{0\}$

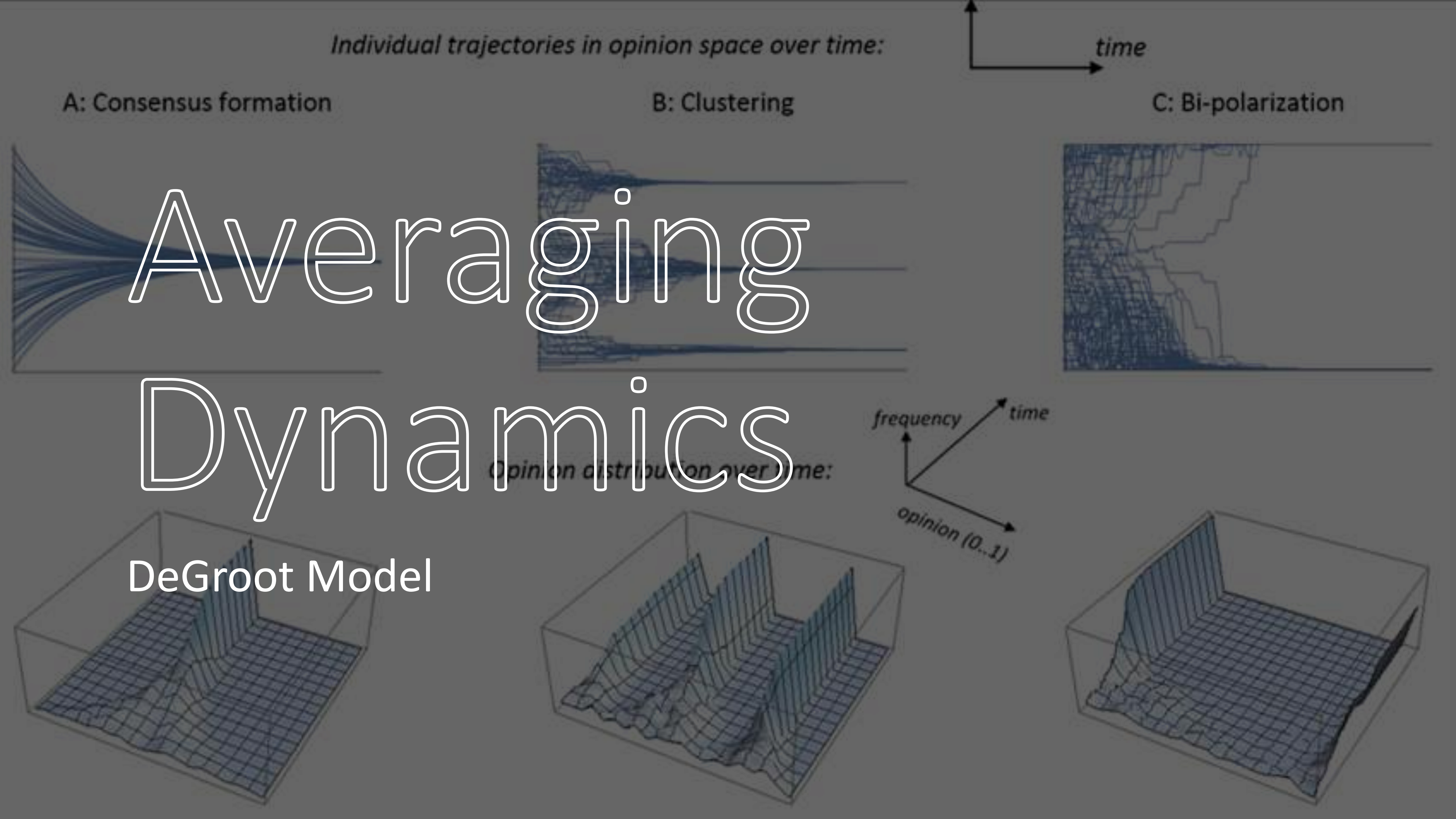
f_{ij}: denote the flow rates of commodity at compartment i flowing to compartment j (positive)

u_i: external supply to compartment i

$f_{0,i}$: outflow rate of compartment i to the environment

Averaging Dynamics

DeGroot Model



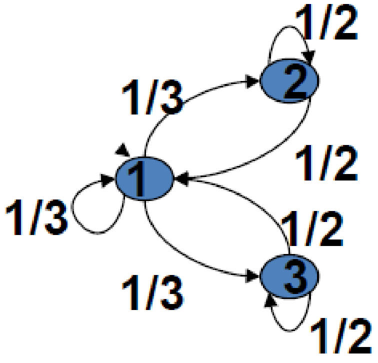
Some Basic Characteristics

- Repeated Communication
- External Information only during initialization
- Information dissemination
- Influence, convergence speed, network impact

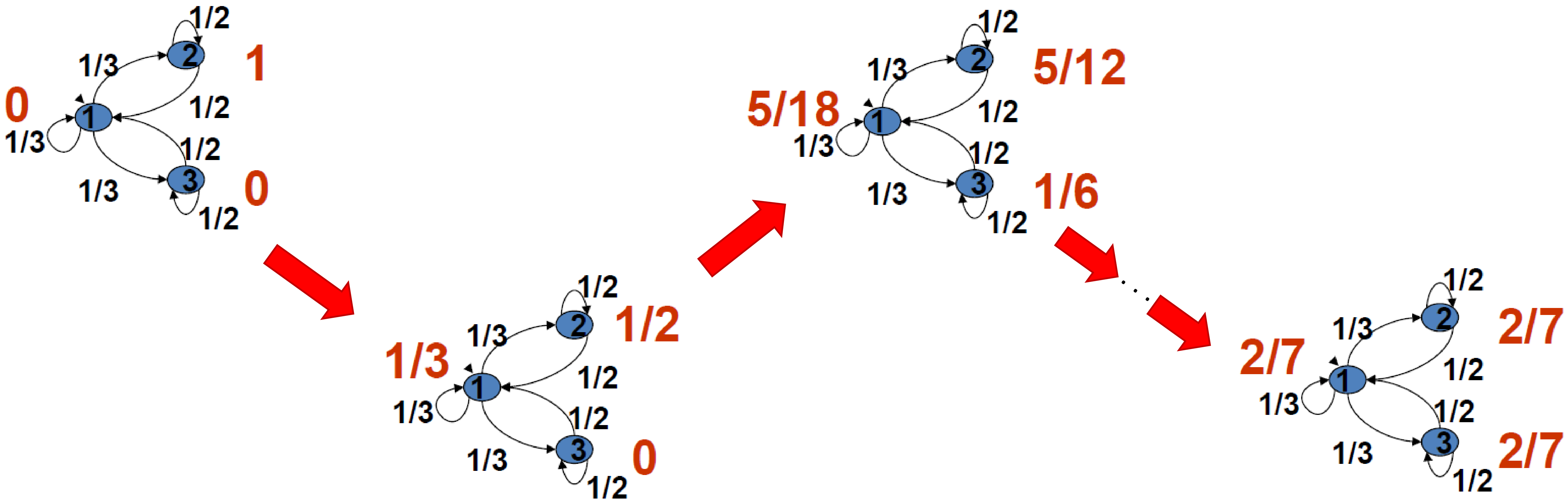
Bounded Rationality Model

- Repeatedly average (with weights) beliefs with neighbors
- Weights do not change (otherwise Bayesian)

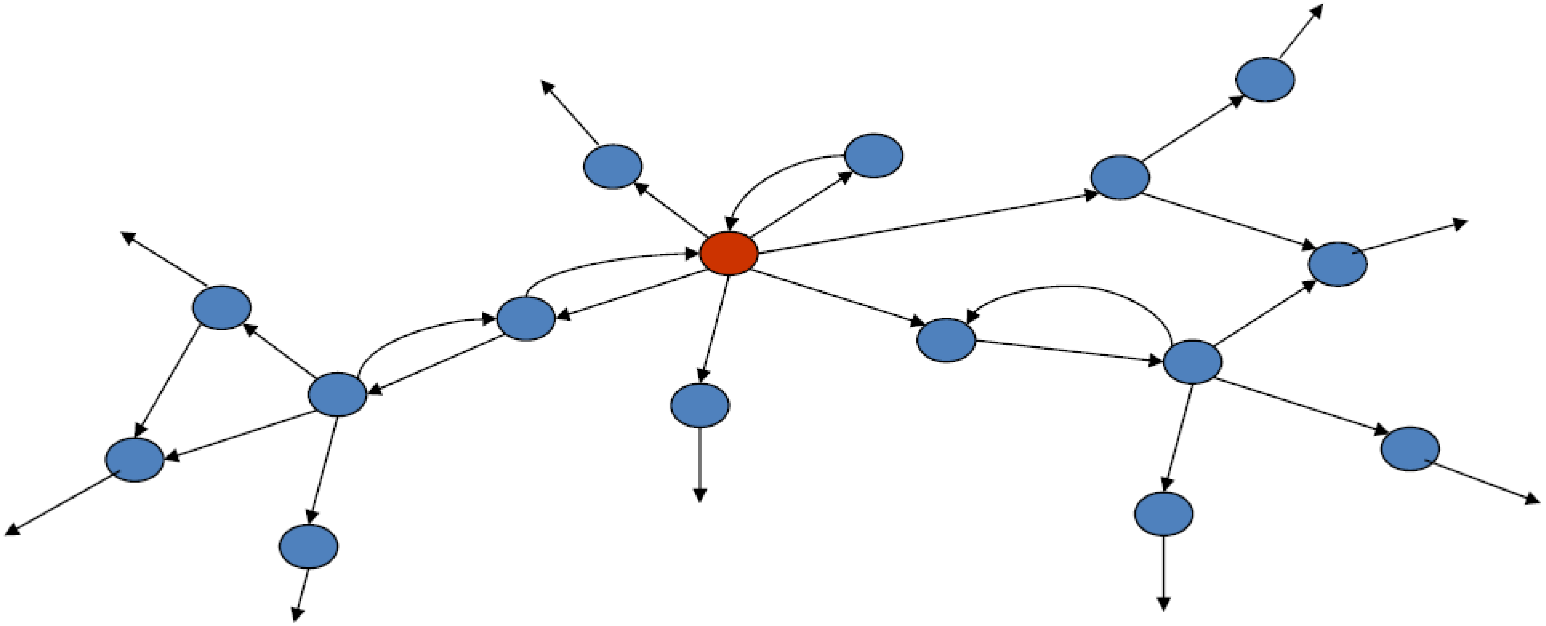
Example



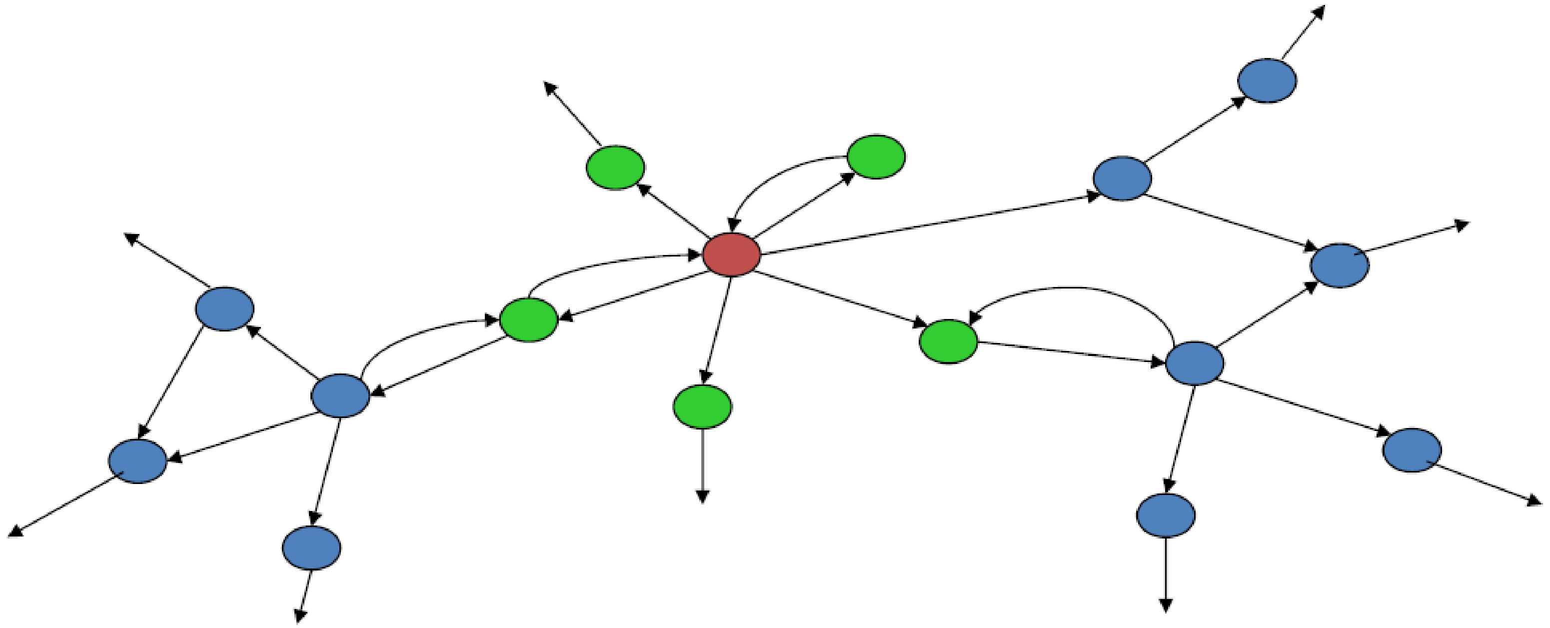
$$T = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$



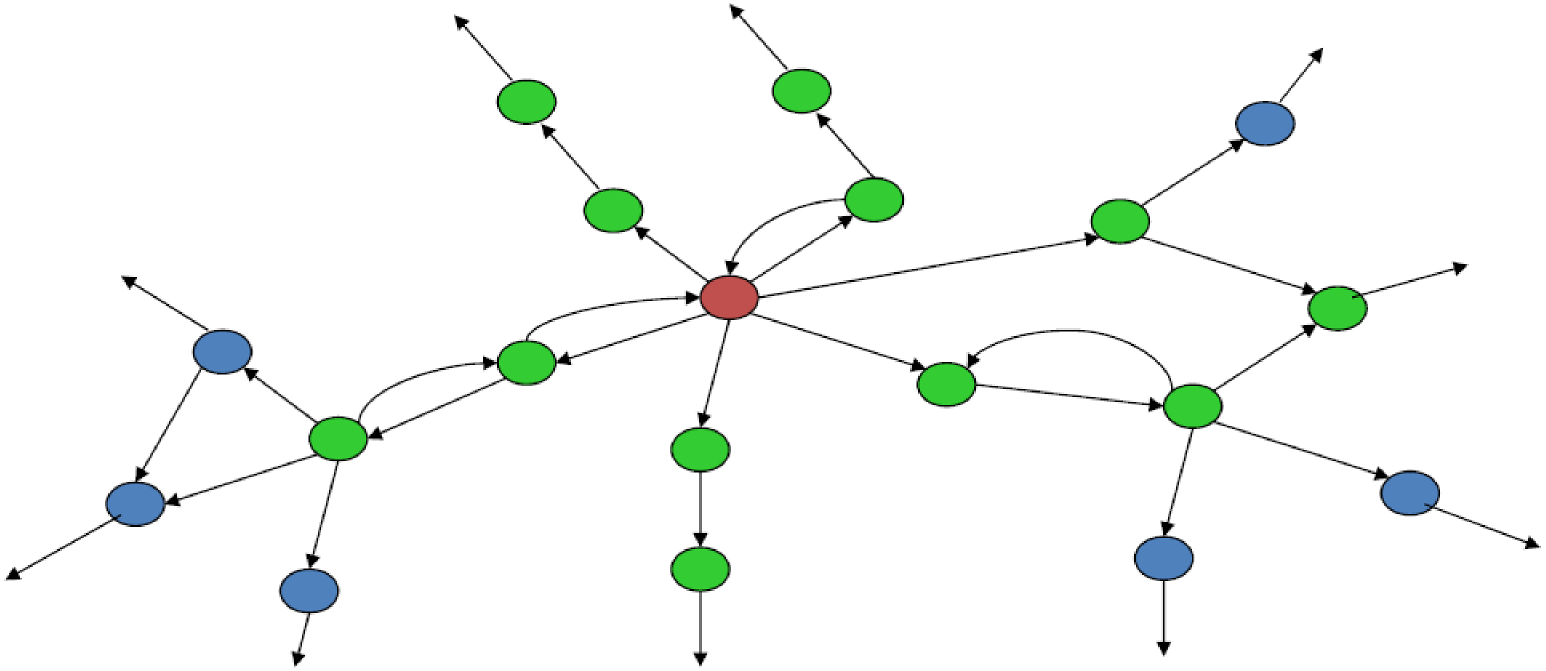
Each Updating Step we get Information from Further Away



Each Updating Step we get Information from Further Away



Each Updating Step we get Information from Further Away



De Groot Model

- Individuals: $\{1, \ldots, n\}$
- A: weighted directed network, row-stochastic matrix
- Start with beliefs, attitude, etc. $b_i(0) \in [0,1]$ – or $[0,1]^k$ (vectors)…
- Updating: $b_i^+(t) = \sum_j a_{ij} b_j(t-1)$

Matrices

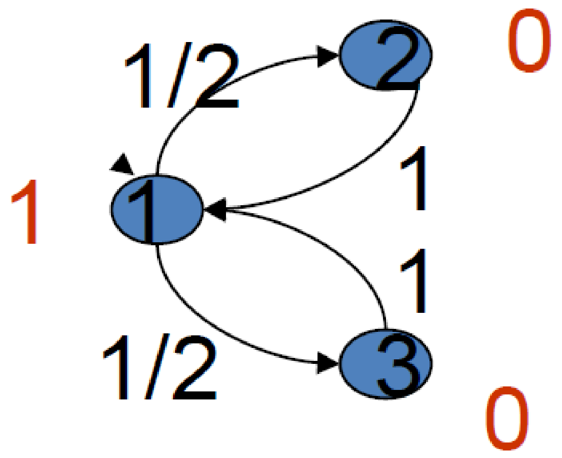
DeGroot Model:

$$b_i(t) = \sum_j a_{ij} b_j(t-1)$$

$$\begin{bmatrix} b_1(t) \\ \dots \\ b_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_1(t-1) \\ \dots \\ b_n(t-1) \end{bmatrix}$$

$$b(t) = A \cdot b(t-1) \Rightarrow$$
$$b(t) = A^t \cdot b(0)$$

Convergence – It is not Always Possible



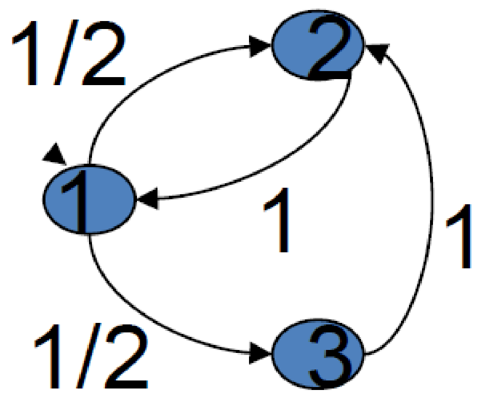
$$A = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

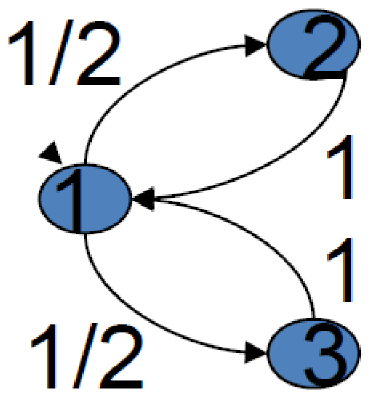
$$b(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, b(1) = A \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$b(2) = A \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, b(3) = A \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ldots (periodic)$$

Convergence

- A converges if $\lim_{t\to\infty} A^t b(0)$ exists for all $b(0)$
- A is aperiodic if the greatest common divisor of its cycle lengths is 1





Convergence Theorem

Assumption: A is strongly connected.

Theorem:

i. A is convergent if and only if A is aperiodic

ii. A is convergent if and only if $\lim_{t\to\infty} A^t = [1\,1\,\ldots 1]^T s^T$, where s is the unique left hand-side eigenvector with eigenvalue 1

Under these assumptions the DeGroot model converges to:

$$\lim_{t\to\infty} A^t b(0) = [1\,1\,\ldots 1]^T \cdot s^T \cdot b(0)$$

Proof

A is primitive if $\exists t_0 \colon A^t_{ij} > 0$, for all $t \geq t_0$

Known theorems:

- If A is strongly connected and stochastic then it is aperiodic if and only if it is primitive
- If A is strongly connected and primitive, then $\lim_{t \to \infty} A^t = [1\ 1\ \ldots 1]^T s$

where s is the unique lhs eigenvector with eigenvalue 1. All entries of s are positive (s must be rescaled so that all entries add to 1)

Proof

$\Rightarrow$ Strongly connectedness, stochasticity and aperiodicity implies convergence

$\Leftarrow$ Strongly connectedness, stochasticity and convergence implies that A is primitive

Aperiodicity

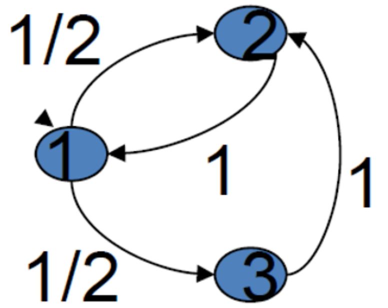
Aperiodicity is easy to achieve

- Have some agent weigh him or herself

- Or have at least one communicating dyad and a transitive triple…

Influence in DeGroot Model

Influence



The DeGroot model converges and achieves consensus.

- Converges to (normalized) eigenvector weighted sum of original beliefs.

$$A^1 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 \end{bmatrix} \quad A^5 = \begin{bmatrix} 1/2 & 3/8 & 1/8 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} \dots \quad A^\infty = \begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{bmatrix}$$

Limiting Beliefs

- When group reaches a consensus, what is it?

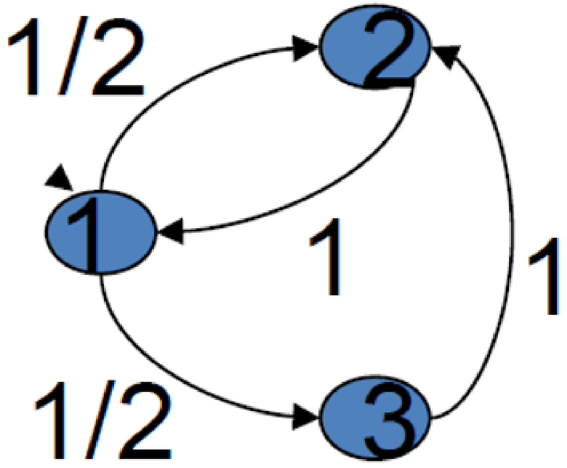
- Who are the influential agents in terms of steering the limiting belief?

Example of Influence

$$\begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 2/5 \\ 2/5 \end{bmatrix}$$

$$\begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 2/5 \\ 2/5 \end{bmatrix}$$

$$\begin{bmatrix} 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 2/5 & 1/5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1/5 \\ 1/5 \end{bmatrix}$$



Who has Influence?

- Note that $s = s \cdot A$, which is

$$s_i = \sum_j a_{ji} s_j$$

- Eigenvector Centrality for unweighted graph:

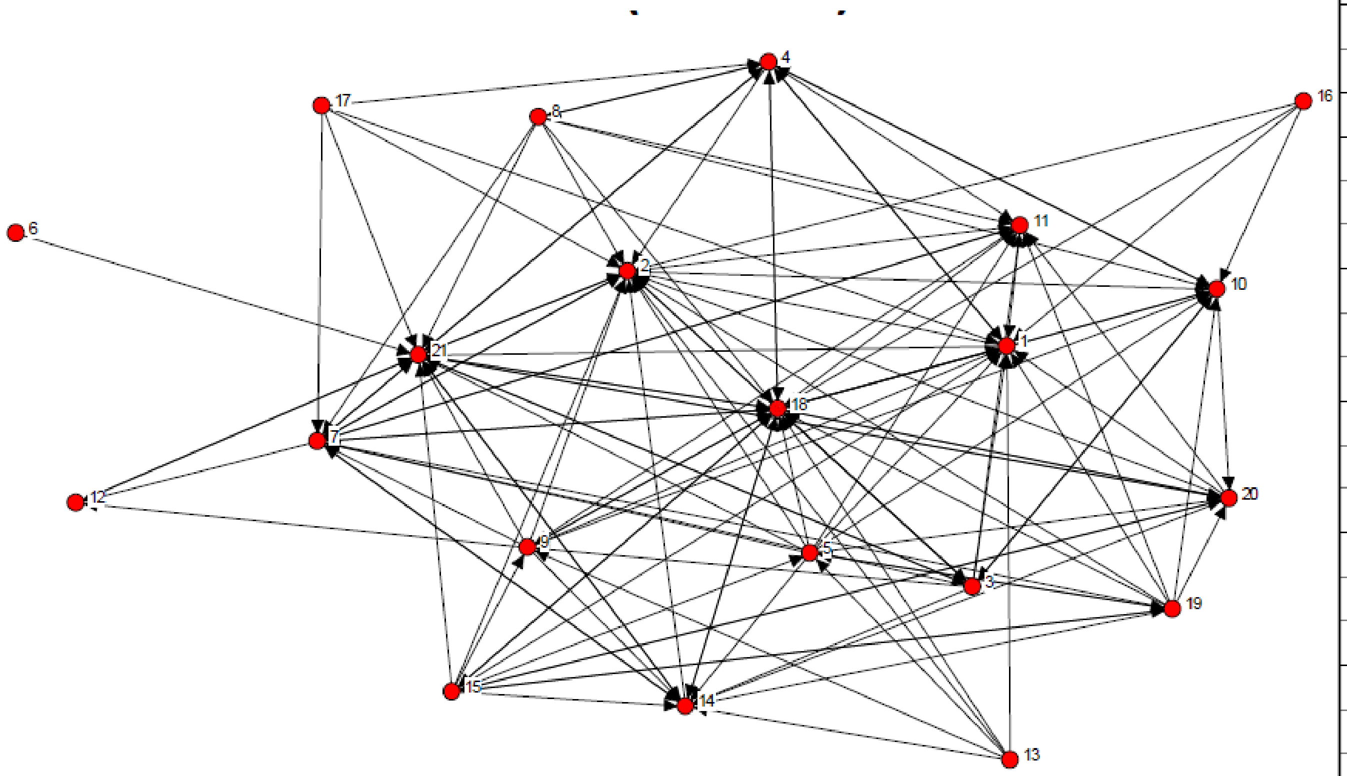
$$C_i = a \sum_j g_{ij} C_j$$

Stubborn Agents

- An agent that puts too much weight on itself will drag others to his/her belief

- Groups that are highly introspective and visible will have major influence on others

Krackardt's (1987) advice network



label	s	level	dept.	age	tenure
1	0.048	3	4	33	9.3
2	0.132	2	4	42	19.6
3	0.039	3	2	40	12.8
4	0.052	3	4	33	7.5
5	0.002	3	2	32	3.3
6	0.000	3	1	59	28
7	0.143	1	0	55	30
8	0.007	3	1	34	11.3
9	0.015	3	2	62	5.4
10	0.024	3	3	37	9.3
11	0.053	3	3	46	27
12	0.051	3	1	34	8.9
13	0.000	3	2	48	0.3
14	0.071	2	2	43	10.4
15	0.015	3	2	40	8.4
16	0.000	3	4	27	4.7
17	0.000	3	1	30	12.4
18	0.106	2	3	33	9.1
19	0.002	3	2	32	4.8
20	0.041	3	2	38	11.7
21	0.201	2	1	36	12.5

Averaging

Wireless Sensors Networks

Let’s Look at Sensor Networks

Wireless sensor network:

- Spatially distributed devices
- Measurements of physical variables (e.g., temperature)
- Local computations
- Transmitting info to neighbors

Assume the following averaging distributed algorithm:

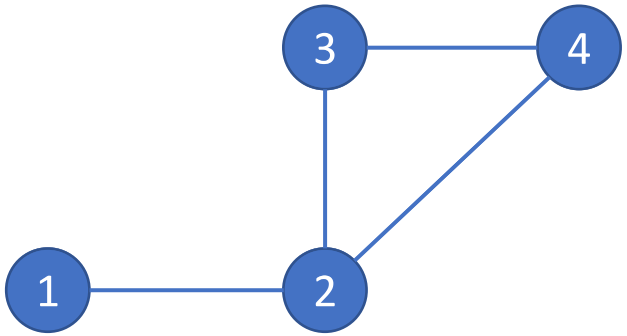
$$x_i(t+1) = average\left(x_i(t), \{x_j(t): j \in N_i\}\right)$$

The Equal-Neighbor Model

G: Connected undirected graph

$$A_{equal\_neigh} = D^{-1}A$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



The Equal-Neighbor Model

Let G an undirected graph with adjacency matrix A and let the degree matrix $D = diag(d_1, d_2, \ldots, d_n)$, where $d_1, d_2, \ldots, d_n$ are the degree of the n nodes. We define the following matrix:

$$A_{en} = D^{-1}A$$

Theorem: Let G be a connected undirected graph (possibly with self loops) with adjacency matrix A and degrees $d_1, d_2, \ldots, d_n$. Then:

1. A_{en} is well defined, row-stochastic and irreducible
2. The lhs principal eigenvector of A_{en}, normalized to have unit sum is:

$$v_{en} = \frac{1}{\sum_{1 \le i \le n} d_i} \begin{bmatrix} d_1 \\ \ldots \\ d_n \end{bmatrix}$$

3. A_{en} is double stochastic iff G is regular (i.e., all nodes have the same degree)

Proof

1. G is connected. Thus, each degree is strictly positive. The degree matrix is invertible and thus A_{en} is well-defined. Since G is connected, the corresponding directed graph to A_{en} is also strongly connected ant thus A_{en} is irreducible. Indeed, it is row stochastic since:

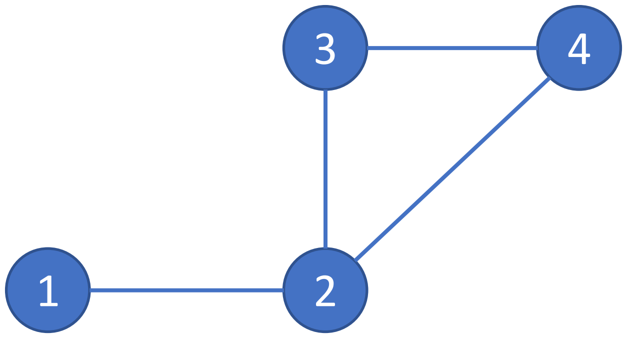
$$A_{en} \cdot \mathbf{1}_n = D^{-1} A \cdot \mathbf{1}_n = D^{-1} d = \mathbf{1}_n$$

2. $v_{en}^T A_{en} = \frac{1}{\sum_{1 \le i \le n} d_i} \begin{bmatrix} d_1 \\ \dots \\ d_n \end{bmatrix}^T A_{en} = \frac{1}{D} d^T D^{-1} A = \frac{1}{D} \mathbf{1}^T A = \frac{1}{D} d^T = v_{en}^T$

A_{en} is irreducible and aperiodic (assumption): from previous theorem it converges to $[1\ 1\ \dots 1]^T \cdot v_{en} \cdot b(0)$

Linear System – Example

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ x_4(t+1) \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$



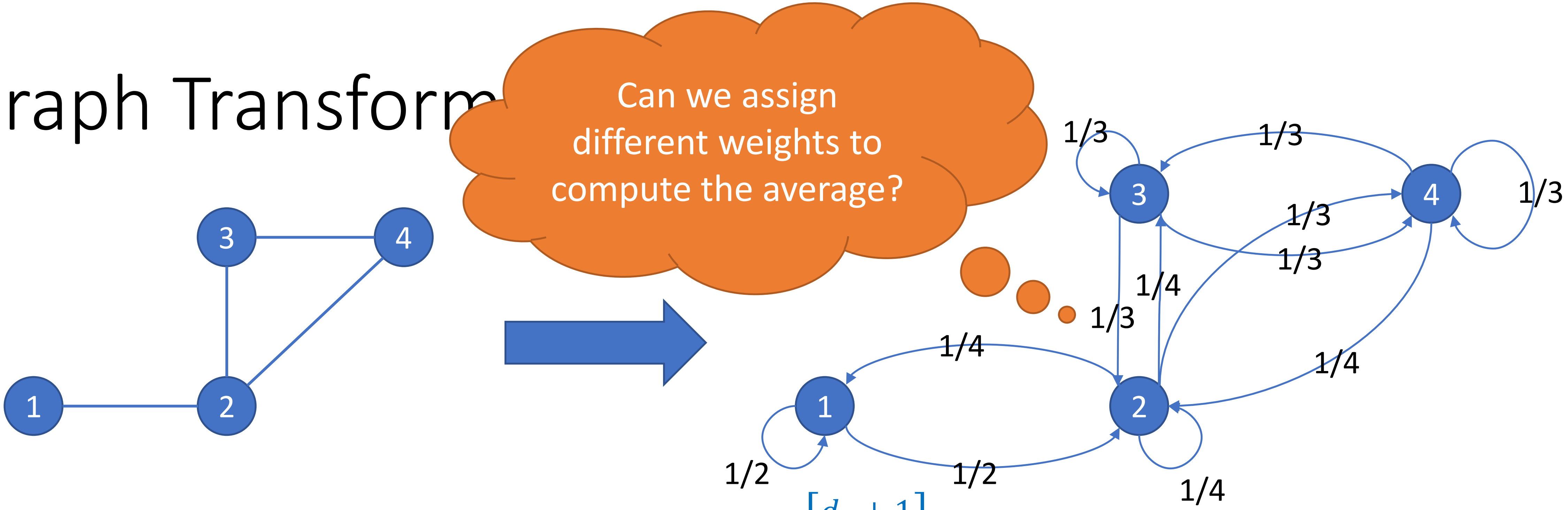
This means that for general wireless sensor networks we get a linear system for averaging:

$$x_i(t+1) = average\left(x_i(t), \{x_j(t): j \in N_i\}\right)$$

Questions:

1. Does each node converge to a value? Is the value the same for all nodes?
   Yes, it converges to the same value.
2. Is the values equal to the average of the initial conditions? No. Degree centrality
3. Properties for convergence? Aperiodic – stochastic – strongly connected
4. Speed of convergence? ☺

Graph Transform



The lhs eigenvector of this matrix is $w_{equal_{neigh}+self\_loops} = \frac{1}{n+\sum_{i=1}^{n} d_i} \begin{bmatrix} d_1 + 1 \\ \dots \\ d_n + 1 \end{bmatrix}$

The lhs eigenvector corresponding to eigenvalue 1 of A is: $\begin{bmatrix} 2/12 \\ 4/12 \\ 3/12 \\ 3/12 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix}$

If we start with initial values $\begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$, then the nodes converge to value: $\begin{bmatrix} 1/6 \\ 1/3 \\ 1/4 \\ 1/4 \end{bmatrix}^T \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \frac{11}{6} \neq 2$

Metropolis-Hastings Model

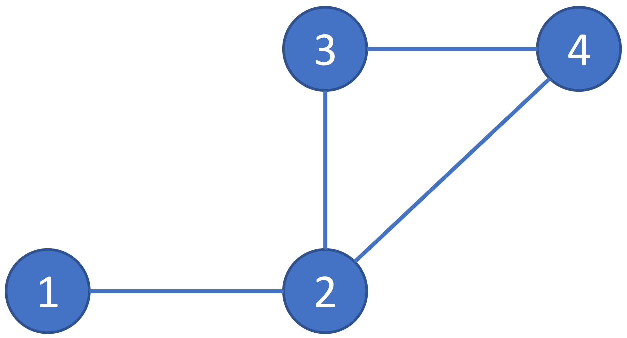
We define the weighted adjacency matrix A_{MH} as follows:

$$(A_{MH})_{ij} = \begin{cases} \dfrac{1}{1 + \max\{d_i, d_j\}}, if\ \{i,j\} \in E\ and\ i \neq j \\ 1 - \sum_{\{i,h\} \in E, i \neq h} (A_{MH})_{ih}, if\ i = j \\ 0, otherwise \end{cases}$$

Properties:

1. $(A_{MH})_{ii} > 0$ for all nodes i, $(A_{MH})_{ij} > 0$, for all pairs $\{i,j\} \in E$, and $(A_{MH})_{ij} = 0$, otherwise
2. A_{MH} is symmetric and double stochastic
3. A_{MH} is primitive iff G is connected
4. The averaging model $x(t+1) = A_{MH}x(t)$ achieves average consensus

Back in the Example



$$A_{MH} = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 5/12 & 1/3 \\ 0 & 1/4 & 1/3 & 5/12 \end{bmatrix}$$

The lhs eigenvector corresponding to eigenvalue 1 of A is: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, since A is double stochastic.

If we start with initial values $\begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$, then the nodes converge to value (we scale): $\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}^T \cdot \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} = 2$

Averaging Dynamics

Averaging Dynamics have been studied in many forms and used in many other applications:

- Control: coordination of network of robots
- Distributed: in network estimation and tracking
- Optimization and learning: distributed multi-agent optimization over network, distributed resource allocation, distributed learning
- Social/Economic Behavior: spread of influence, polarization, emergent behavior

Friedkin-Johnsen Model

Or: “what happens when stubbornness is introduced in DeGroot model”.

FJ Model

- Consensus is not realistic in most cases.
- Social experiments have shown influence among agents but settling to opinions within a convex hull *ch* of the initial opinions (no consensus)
- The FJ model is meant to represent this contraction towards *ch*(*x*(0)) which is however not necessarily a consensus point.

$$x(t+1) = (I - \Theta)Ax(t) + \Theta x(0)$$

Θ is a diagonal matrix such that: $\Theta_{ii} = \theta_i, \theta_i \in [0,1]$, where θ_i is the stubbornness of agent i. In components:

$$x_i(t+1) = (1 - \theta_i) \sum_{1 \leq j \leq n} A_{ij} x_j(t) + \theta_i x_i(0)$$

Convergence

Definition: The FJ model is said θ-connected if all nodes of the graph $G(A)$ either have $\theta_i > 0$ or are connected via directed paths to some nodes i for which $\theta_i > 0$.

Theorem: Consider the FJ model with A row stochastic, $\theta_i > 0$ for some i, and θ-connected. Then:

1. $\rho\big((I - \Theta)A\big) < 1$ (stability)
2. the state converges to $x^* = \lim_{t\to\infty} x(t) = Vx(0)$, with $V = (I - (I - \Theta)A)^{-1}\Theta$, a row-stochastic matrix
3. $x^* \in ch\big(x(0)\big)$

Speed of Convergence

Just a glimpse

Symmetric Row-Stochastic Primitive Matrix A

$$x(t+1) = Ax(t)$$

Define the essential spectral radius:

$$\rho_{ess}(A) = \max\{|\lambda|: \lambda \in spec(A)\backslash\{1\}\}$$
$$\rho_{ess}(A) = 0, if\ spec(A) = \{1, \dots, 1\}$$

We can compute the error as follows and then bound it by an ε to find the number of steps:

$$\left\| x(t) - avg\big(x(0)\big)\mathbb{I}_n \right\|_2 \leq \rho_{ess}^t(A) \left\| x(0) - avg\big(x(0)\big)\mathbb{I}_n \right\|_2$$

Time-Varying Graphs

Just to see what happens…

For Doubly Stochastic Symmetric Matrices

Let $\{A(t)\}_{t\in\mathbb{N}}$ be a sequence of symmetric and doubly stochastic matrices with associated digraphs $\{G(t)\}_{t\in\mathbb{N}}$ so that:

1. Each non-zero edge weight $a_{ij}(t)$, including the self loops $a_{ii}(t)$, is larger than a constant ε and
2. Each graph $G(t)$ is strongly connected and aperiodic

Then the solution to $x(t+1) = A(t)x(t)$ converges exponentially fast to $avg\big(x(0)\big)\mathbf{1}_n$

HK-Models (Bounded Confidence Models)

non-linear system – looking at homophily

Time-varying but the variation is state-dependent

Averaging Dynamic Model

- Set of agents $\{1,2,\dots,n\}$
- Discrete time

The dynamics are specified by:

- Initial opinion profile: $\{x_i(0) \in \Re^n, i \in [n]\}$ and the bounded confidence ε that limits the interactions of agents
  - At time t the opinion of i is given by vector $x_i(t)$
  - The neighbors of agent i are:

$$N_i(t) = \left\{ j \in [n] : \left\| x_j(t) - x_i(t) \right\|_2 \leq \varepsilon \right\}$$

  - Each agent updates its opinion by averaging the opinions of its neighbors:

$$x_i(t+1) = \frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} x_j(t)$$

HK Properties (1d)

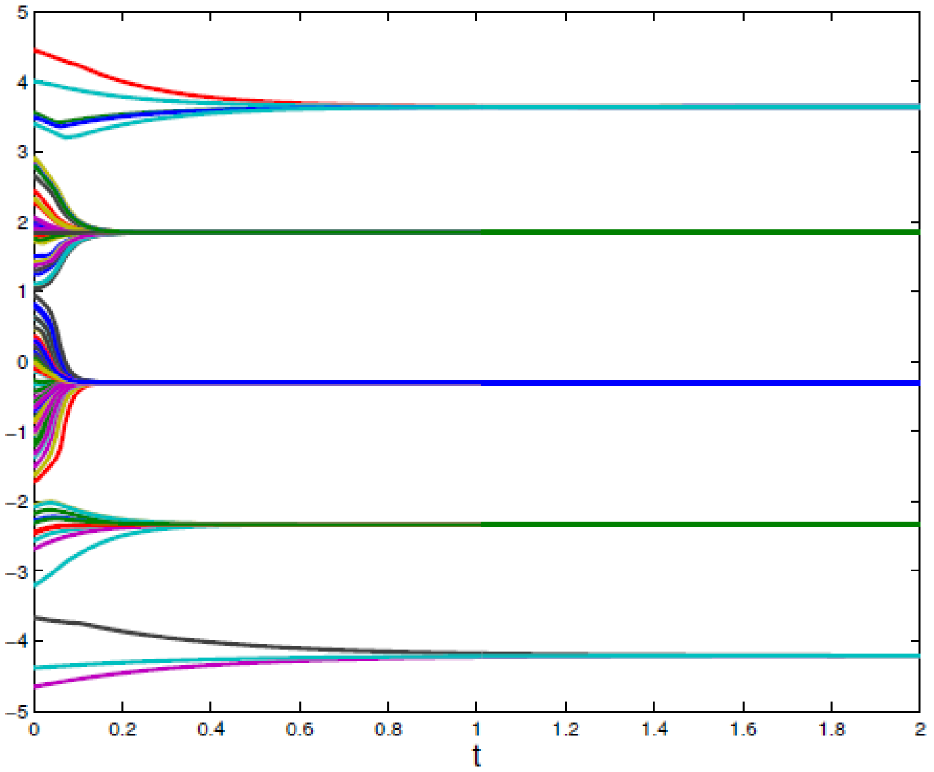
1. order preserving: $\forall t, x_i(0) \leq x_j(0) \Rightarrow x_i(t) \leq x_j(t)$

Assume $x_1 \leq x_2 \leq \cdots \leq x_n$. Then

2. $x_1(t)$ non-decreasing: $\forall t: x_1(t+1) \geq x_1(t)$
3. $x_n(t)$ non-increasing: $\forall t: x_n(t+1) \leq x_n(t)$
4. If $(i, i+1) \notin E(t) \Rightarrow (i, i+1) \notin E(t+k), \forall k > 0$ (i.e., adjacent agents that are disconnected stay disconnected)
5. If $G(t)$ is disconnected, then $G(t+k)$ is also disconnected $\forall k > 0$
6. The connected components of $G(t)$ can split as t grows, but cannot merge
7. Properties 2 and 3 are valid for any connected component of G
8. $x_i(t)$ converges to x_i^* in finite time. $\forall i, j$ it is either that $x_i^* = x_j^*$ or $\left|x_i^* - x_j^*\right| > \varepsilon$

An Example

HK model for $n = 100$ agents. A simulation with opinions $x(0)$ uniformly distributed in $[-5,5]$ and $d = 1$. The resulting clusters have consensus values that differ by >1.



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