Non-dictatorial Judgment Aggregation
Algorithms and Characterizations

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Where to begin

Alice in the Wonderland: "Where shall I begin, please your Majesty?" The Rabbit asked

"Begin at the beginning," the King said, very gravely, "and go on till you come to the end: then stop."

The paradoxical beginning

- Nicolas de Caritat (marquis de Condorcet), 1743 – 1794.

  The truth belongs to those who seek it, not to those who claim to own it.

▶ The Marquis de Condorcet paradox

Voter 1: $x > y > z$
Voter 2: $y > z > x$
Voter 3: $z > x > y$

- Majority rule no good.

The Arrow's and the marquis Condorcet's theorems

Theorem (K. Arrow's Impossibility theorem, 1950)

The only way to resolve Condorcet's paradox is to proclaim a dictator.

Fortunately, things, ain't as bad.

Theorem (marquis de Condorcet, 1785)

If each voter is "right" with probability more than $1/2$, then the majority decision is w.h.p. "right".

- It was the first time that the individuals' positions were assumed to be restricted (Aggregation Theory rather than Voting Theory).

The jury paradox

	Valid contract	Breach	Defendant liable
	p	q	r
Judge 1	1	1	1
Judge 2	1	0	0
Judge 3	0	1	0
Majority	1	1	0

- Again, the only way out is to proclaim one of the juror's a dictator.

Still another paradox

- A suspect for pickpocketing is brought in front of a panel of three judges, together with two witnesses who fell victim to pickpocketing at the same time in different places but both claim to recognize the defendant as the thief.
  - Three issues:

    p: Witness A is right
    q: Witness B is right
    r: The suspect is not guilty

  - Domain (rational position patterns):
    $X = \{(1,1,1),(0,1,0),(1,0,0),(0,0,1)\}$.
  - The position patterns of the three judges could be any three vectors $x^1, x^2, x^3 \in X$.

The pickpocket paradox cont'ed

	Witness A is right p	Witness B is right q	The suspect is innocent r
Judge 1	1	0	0
Judge 2	0	1	0
Judge 3	0	0	1
Minority	1	1	1

- Aggregator $\bar{f}(x^1, x^2, x^3) \mapsto x^1 \oplus x^2 \oplus x^3$. Minority aggregator.

Abstract approach: Dokow & Holzman 2010

- A set of n individuals (the society) . Individuals: $i = 1, \ldots, n$.
- A set of m issues (the agenda). Issues: $j = 1, \ldots, m$.
- For each issue j, a set A_j of an individual's possible positions on j. Non-degeneracy condition $\forall j, |A_j| \geq 2$.
- Boolean framework: $\forall j, A_j = \{0, 1\}$.
- Non-Boolean framework: More than two positions are allowed for at least one issue.
- A set of permissible (rational) position patterns $X \subseteq \prod_{j=1}^{m} A_j$. We refer to X as the *domain*. Non-degeneracy condition: $\forall j$, the j-th projection of $X = A_j$.
- An *aggregator* $F : X^n \to X$.
- Elements in the domain of an aggregator X^n are $n \times m$ matrices, denoted by $[x]$. The i-th row x^i of $[x]$ is the position pattern of individual i, whereas the j-th column x_j is the column-vector of all individuals' votes on issue j. Notation: $F([x]) = F(x^1, \ldots, x^n)$.

Profile: n position patterns for m issues

$$[x] = \begin{pmatrix} x_1^1 & x_2^1 & \cdots & x_j^1 & \cdots & x_m^1 \\ x_1^2 & x_2^2 & \cdots & x_j^2 & \cdots & x_m^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^i & x_2^i & \cdots & x_j^i & \cdots & x_m^i \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^n & x_2^n & \cdots & x_j^n & \cdots & x_m^n \end{pmatrix}$$

position pattern: $x^i = (x_1^i, x_2^i, \cdots, x_m^i) \in X$

column-vector on issue j : $x_j = \begin{pmatrix} x_j^1 \\ x_j^2 \\ \vdots \\ x_j^n \end{pmatrix}$

Aggregators

Aggregators are subject to certain minimal requirements:

- Independence from Irrelevant Alternatives (a.k.a. Issue by Issue Aggregation): If individuals change their position patterns, but all retain the same value on issue j, then the j-component of the social (aggregated) position pattern does not change.
- Being conservative a.k.a. supportive: The value of the social (aggregated) position pattern on issue j is equal to at least one individual's position on j.

Aggregators cont'ed

$$\bar{f}([x]) = \bar{f}(x^1, \ldots, x^n) = \bar{f} \begin{pmatrix} f_1 & \cdots & f_j & \cdots & f_m \\ \frown & & \frown & & \frown \\ x_1^1 & \cdots & x_j^1 & \cdots & x_m^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^i & \cdots & x_j^i & \cdots & x_m^i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^n & \cdots & x_j^n & \cdots & x_m^n \\ \smile & & \smile & & \smile \\ \shortparallel & & \shortparallel & & \shortparallel \\ f_1(x_1) & \cdots & f_j(x_j) & \cdots & f_m(x_m) \end{pmatrix}$$

$$\bar{f}([x]) = \bar{f}(x^1, \ldots, x^n) = (f_1(x_1), \ldots, f_m(x_m)) \in X.$$

Possibility domains

Definition

An n-ary aggregator $\bar{f} = (f_1, ..., f_m)$ is called dictatorial if there is a fixed i such that the social (aggregated) position pattern is always equal to i'th individual's position pattern.

Definition

X is a possibility domain if for some $n \in \mathbb{N}$ there exists an n-ary non-dictatorial aggregator for X.

jump

Majority/minority aggregators

A ternary $\bar{f} = (f_1, ..., f_m)$ is a majority aggregator (notationally maj) for X if every f_j is a majority operation:

$$f_j(x, x, y) = f_j(x, y, x) = f_j(y, x, x) = x.$$

Minority aggregators (notationally min):

$$f_j(x, x, y) = f_j(x, y, x) = f_j(y, x, x) = y$$

$f_j(x, x, x) = x$ (if two operands are equal the minority operator returns the third).

Basic Question I: When dictators can be avoided? The characterization theorem

Theorem (Kirousis, Kolaitis and Livieratos, 2017)

X is a possibility domain if and only if X admits either:

- *a majority aggregator or*
- *a minority aggregator or*
- *a non-dictatorial binary aggregator.*

What about the proof?

Idea : Post's lattice

Let A be a binary set (e.g., $\{0, 1\}$).

- Clone: Set $\mathcal{C}$ of operations on powers of A that contains all projections and is closed under *superpositions*: If $g \in \mathcal{C}$ is n-ary and $f_1, \ldots, f_n \in \mathcal{C}$ are k-ary, then $g(f_1, \ldots, f_n) \in \mathcal{C}$.
- Post (1941) provided a complete classification of Boolean clones.

Theorem (Post, 1941)

If $\mathcal{C}$ is a Boolean clone of conservative *operations, and $\wedge, \vee, \mathrm{maj}, \min \notin \mathcal{C}$, then the projection functions are the only operators in $\mathcal{C}$.*

Basic Question II: How easy is to decide if dictators can be avoided? The complexity theorem

Assume, at a first approach, that X is extensively part of the input (i.e. X is *not* implicitly given by a means of a succinct representation, like, e.g., a formula).

Theorem (KKL, 2018)

There is a polynomial-time algorithm for solving the following problem: given a domain X, determine whether or not X is a possibility domain, and if it is, produce a non-dictatorial aggregator of arity at most three.

Rough sketch of proof

- Known algorithms that check whether X is closed under a majority or a minority aggregator.
- Notice however a binary non-dictatorial aggregator $\bar{f}$ may have different functions f_j in each coordinate. This raises exponentially the search space.
- By supportiveness, f_j is determined by specifying for pairs $(u, u') \in A_j$, with $u \neq u'$, whether $f_j(u, u') = u$ or $f_j(u, u') = u'$.

Proof cont'ed.: Basic Idea

- Consider the graph H_X whose vertices are triples (u, u', k), with $u, u' \in A_k$ and $u \neq u'$. Connect a vertex (u, u, k) with a directed edge towards another vertex (v, v', l) of H_X if (intuitively)
  - An edge from (u, u, k) to (v, v', l) means that if for some binary aggregator $\bar{f}$, $f_k(u, u')$ is forced to take the value u, then $f_l(v, v')$ is forced to take the value v.
  - So, in some sense, the situation is reminiscent of 2-SAT, where the transitive closure of a certain graph is strongly connected iff there is no satisfying truth assignment. $\square$

A stronger notion of possibility domain

Definition (Kirousis, Kolaitis & Livieratos, 2018)

An aggregator $\bar{f} = (f_1, \ldots, f_m)$ is uniform non-dictatorial if $f_j \restriction_{B_j}$ is not a projection function, for all $j = 1, \ldots, m$ and for every two-element subset $B_j \subseteq X_j$.

- Boolean framework: known as *locally non-dictatorial aggregators* (Nehring & Puppe, 2010).

Example

A ternary aggregator each component of which is either maj, or $\min$, or $\wedge^{(3)}(x, y, z) := \wedge(\wedge(x, y), z)$, or $\vee^{(3)} := \vee(\vee(x, y), z)$.

Definition

X is a uniform possibility domain if it admits a uniform non-dictatorial aggregator.

When can dictators can be *uniformly* avoided? The first characterization of uniform possibility: quantifier inversion

Theorem (Kirousis, Kolaitis & Livieratos, 2018)

The following are equivalent:

- *X is a uniform possibility domain.*
- *For every j and every two-element subset $B_j \subseteq A_j$, there is an aggregator $\bar{f} = (f_1, \ldots, f_m)$ such that $f_j\restriction_{B_j}$ is not a projection function.*

recall

Relation with Constraint Satisfaction Problems

- The above "quantifier inversion theorem" paved the way for connecting Aggregation Theory with deep dichotomy results about Constraint Satisfaction Problems (CSP).

Theorem (Second characterization of uniform possibility, KKL, 2018 informal statement)

A domain X is a uniform possibility domain if and only if an associated with X (multi-sorted) CSP is solvable in polynomial time.

jump

Multi-sorted relations

Definition

Let $A_1, \ldots, A_m$ be finite sets, each belonging to a specific "sort". A multi-sorted relation over these sets is a subset of the Cartesian product of an arbitrary collection of these sets (with repetitions allowed).
E.g: $R \subseteq A_3 \times A_1 \times A_1 \times A_7$.

Polymorphisms

Definition

Let A be a set and $R \subseteq A^m$ an m-ary (single-sorted) relation on A. A function $f : A^n \to A$ is called an (single-sorted, n-ary) polymorphism of A if for any n vectors

$$(x_1^1, \dots, x_m^1), \dots, (x_1^n, \dots, x_m^n) \in R,$$

we have that

$$(f(x_1^1, \dots, x_1^n), \dots, f(x_m^1, \dots, x_m^n)) \in R$$

- From now on all functions considered will be *conservative (supportive)*. That is for all $(x^1, \dots, x^n) \in A^n$,

$$f(x^1, \dots, x^n) \in \{x^1, \dots, x^n\}.$$

$$
\begin{array}{cccccc}
A & \cdots & A & \cdots & A & \\
\frown & & \frown & & \frown & \\
(x_1^1 & \cdots & x_j^1 & \cdots & x_m^1) & \in R \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
(x_1^i & \cdots & x_j^i & \cdots & x_m^i) & \in R \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
(x_1^n & \cdots & x_j^n & \cdots & x_m^n) & \in R \\
\smile & & \smile & & \smile & \\
\shortparallel & & \shortparallel & & \shortparallel & \\
(f(x_1) & \cdots & f(x_j) & \cdots & f(x_m)) & \in R
\end{array}
$$

- Observe, that in all columns we have the same A and the same f (single-sorted R).
- If R is multi-sorted, then the A's differ. "Each A" is one of the $A_1, \ldots, A_m$. Also the corresponding f's differ accordingly. We then talk about multi-sorted polymorphism $\bar{f} = (f_1, \ldots, f_m)$.

More on multi-sortedness

- Multi-sorted constraint language Γ: any set of multi-sorted relations (over $A_1, \ldots, A_m$).
- Γ is conservative if, for any j and for any $B \subseteq A_j$, $B \in \Gamma$.
- MPol(Γ): the set of *multi-sorted* polymorphisms of Γ.
- If Γ is conservative then MPol(Γ) is equal to the set of *supportive/conservative* multi-sorted polymorphisms of Γ.

- The Galois connection.

Évariste Galois 1811–1832

Domains are multi-sorted relations

- A domain $X \subseteq \prod_{j=1}^{m} A_j$ can be seen as a multi-sorted relation.
- Γ_X is the multi-sorted *conservative* constraint language comprised from X and *all* $B_j \subseteq A_j$.
- $\mathrm{MPol}(\Gamma_X)$ is the set of X's *aggregators*.
- Thus, X is a ("plain") impossibility domain if and only if:

$$\mathrm{MPol}(\Gamma_X) = \{(pr_d^n, \dots, pr_d^n) \mid n, d \in \mathbb{N} \text{ s.t. } 1 \leq d \leq n\}.$$

The Constraint Satisfaction Problem (CSP)

CSP(Γ)

- Given, a domain set A and a set of variables V, and
- a set of relations (single-sorted constraint language) Γ and a set of constraints (referring to the relations of Γ),

is there a value assignment to the variables that satisfies *all* constraints?

Multi-sorted CSP, MCSP(Γ)

- Given domain sets $A_1, \ldots, A_m$ (one for each "sort"), a set of variables V, with specified sorts they can take values from, and
- a set of multi-sorted relations (multi-sorted language Γ) plus a set of constraints (referring to the relations of Γ)

is there a "correct" value assignment to the variables that satisfies *all* constraints?

Bulatov's Dichotomy Theorem for conservative CSP

Theorem (Bulatov, 2011)

Let X be a set of feasible voting patterns. If for any j and any two-element subset $B_j \subseteq A_j$ there is either:

- *a binary aggregator $\bar{f} = (f_1, ..., f_m)$ for X s.t. $f_j \restriction_{B_j} \in \{\wedge, \vee\}$, or*
- *a ternary aggregator $\bar{f} = (f_1, ..., f_m)$ for X s.t. $f_j \restriction_{B_j} \in \{maj, \oplus\}$,*

then $\mathrm{MCSP}(\Gamma_X)$ is tractable. Otherwise it is NP-complete.

goto

Interlude: Dichotomy for CSP

- Feder-Vardi Conjecture (1993): CSP(Γ) is in P or NP-complete.
- Schaefer (1978): The conjecture is true in the Boolean framework (languages Γ where the set of values that can be assigned to the variables are 0 or 1, SAT)
- From 1993 until 2017, the Dichotomy Conjecture was proved for successively larger classes of languages.
- Finally in 2017 was proved in the general case independently by A. Bulatov and D. Zhuk.
- In this paper, we use the 2010 dichotomy result by Bulatov for conservative languages.

The second characterization of uniform possibility again

From Bulatov's theorem and from the quantifier inversion theorem we get:

Theorem (Second characterization of uniform possibility, KKL, 2018 formal statement)

If X is a uniform possibility domain then $\mathrm{MCSP}(\Gamma_X)$ is tractable; otherwise it is NP-complete.

- In Bulatov's theorem, for each j, B_j, we have a different aggregator, however the definition of uniform possibility domains demands a single aggregator for all j, B_j; this is the reason the quantifier inversion theorem was necessary.

Consequences

Corollary

X is a uniform possibility domain if and only if there is an aggregator $\bar{f} = (f_1, \ldots f_m)$, such that for all j and all $x, y \in A_j$, we have $f_j(x, x, y) = f_j(x, y, x) = f_j(x, x, y)$.

Corollary

In the Boolean case: a domain is locally non-dictatorial if and only if it admits a ternary anonymous aggregator.

Anonymous: Its arguments can be commuted without changing its value.

Theorem (KKL, 2018)

There is a polynomial-time algorithm for solving the following problem: given a domain X, determine whether or not X is a possibility domain, and if it is, produce a non-dictatorial aggregator of arity at most three.

Implicitly described domains

In the sequel, we work exclusively in the Boolean domain.

There are two approaches on how to implicitly define the domain X:

- *The Classical approach (List & Pettit, 2002).* First, given a propositional formula ϕ and $x \in \{0,1\}$, let

$$\phi^x := \begin{cases} \phi & \text{if } x = 1, \\ \neg\phi & \text{if } x = 0. \end{cases}.$$

  Now consider a sequence $\bar{\phi} = (\phi_1, , \ldots, \phi_m)$ of propositional formulae, each of an arbitrary number of variables. Let

$$X_{\bar{\phi}} := \left\{ (x_1, \ldots, x_m) \in \{0,1\}^m \mid \bigwedge_{j=1}^{m} \phi_j^{x_j} \text{ is satisfiable } \right\}.$$

  As a non-degeneracy condition, assume that the projection of $X_{\bar{\phi}}$ on all coordinates is $\{0,1\}$.

recall

Implicitly described domains continued

- *The integrity constraint approach (Grandi & Endriss, 2013).* Let ϕ be a (single) propositional formula (integrity constraint) on m variables. Let

  $$X_\phi \subseteq \{0,1\}^m$$

  be the set of truth assignments that satisfy ϕ. As a non-degeneracy condition we assume that the projection of X on each coordinate is $\{0,1\}$.

The two approaches are "equivalent" Dokow & Holzman, 2009, in the sense that from one we can get to the other, but in a non-unique way.

Complexity-wise, the two models "probably" differ

Theorem

(KKL, 2018)

1. *In the classical framework, where the input given as an m-sequence of formulae $\bar{\phi}$ of arbitrary numbers of variables, the problem of deciding whether $X_{\bar{\phi}}$ is a (uniform) possibility domain is in Δ_3^{P} (resp. Σ_3^{P}).*
2. *In the integrity constraint approach, where the input is given as a single m-variable formula ϕ, the problem of deciding whether X_ϕ is a (uniform) possibility domain is in $\Sigma_2^{\mathrm{P}} \cap \Pi_2^{\mathrm{P}}$ (resp. Σ_2^{P}).*

- Open question: We do not know if matching lower bounds exist (we have a non-trivial lower bound result for one case).

Several similar results —but not for the question of existence of a dictatorial aggregator— in Endriss, Grandi & Porello 2012 and Endriss & de Haan, 2015 (for the agenda approach).

Another approach to characterizations

Basic Problem: Find an "efficiently decidable" class $\mathcal{C}$ of propositional formulae so that a domain X is a possibility domain if and only if there is $\phi \in \mathcal{C}$ such that $X = \mathrm{Mod}(\phi)$.

Similar problem examined in Grandi & Endriss 2013. Our motivation the following classical result:

Theorem (Dechter & Pearl, 1992, Scheafer, 1978)

A subset $X \subseteq \{0,1\}^m$ is component-wise closed under the operators $\wedge, \vee$, $\oplus$, or maj if and only if it is the conjunction of generalized clauses that are Horn, dual Horn, affine, or bijunctive, respectively.

- Horn (dual Horn) clauses: disjunction of literals at most one of which is positive (resp. negative).
- Affine clauses: exclusive disjunction (logical direct sum) of literals.
- Bijunctive clauses: disjunction of at most two literals.

Characterizations of integrity constraints

Theorem (Díaz, Kirousis, Kokonezi & Livieratos)

A domain X is a possibility domain if and only if $X = \mathrm{Mod}(\phi)$, where ϕ is such that

- *either its variables can be partitioned into two non-empty subsets so that no clause contains variables from both sets, or*
- *its clauses are exclusive disjunctions of their literals (affine clauses), or*
- *if we change the logical sign of some of its a variables, we get formula with some Horn clauses whose variables appear positively in all the remaining clauses.*

Moreover, it can be efficiently checked whether a formula is one of the above types.

Similar result for uniformly possibility domains.

Future work

Do not dwell in the past, do not dream of the future, concentrate the mind on the present moment.

- By

Siddhartha Gautama
a.k.a. Buddha

Thank You for Your Attention