1.1.9 Problems

1. Consider the complex numbers $z_1 = (1, 2)$, $z_2 = (-2, 3)$, and $z_3 = (1, -1)$. Compute the following:

(a) $z_1 + z_2 + z_3$; (b) $z_1z_2 + z_2z_3 + z_3z_1$; (c) $z_1z_2z_3$;
(d) $z_1^2 + z_2^2 + z_3^2$; (e) $\dfrac{z_1}{z_2} + \dfrac{z_2}{z_3} + \dfrac{z_3}{z_1}$; (f) $\dfrac{z_1^2 + z_2^2}{z_2^2 + z_3^2}$.

2. Solve the following equations:

(a) $z + (-5, 7) = (2, -1)$; (b) $(2, 3) + z = (-5, -1)$;
(c) $z \cdot (2, 3) = (4, 5)$; (d) $\dfrac{z}{(-1, 3)} = (3, 2)$.

3. Solve in $\mathbb{C}$ the equations:

(a) $z^2 + z + 1 = 0$; (b) $z^3 + 1 = 0$.

4. Let $z = (0, 1) \in \mathbb{C}$. Express $\sum\limits_{k=0}^{n} z^k$ in terms of the positive integer n.

5. Solve the following equations:

(a) $z \cdot (1, 2) = (-1, 3)$; (b) $(1, 1) \cdot z^2 = (-1, 7)$.

6. Let $z = (a, b) \in \mathbb{C}$. Compute z^2, z^3, and z^4.

7. Let $z_0 = (a, b) \in \mathbb{C}$. Find $z \in \mathbb{C}$ such that $z^2 = z_0$.

8. Let $z = (1, -1)$. Compute z^n, where n is a positive integer.

9. Find real numbers x and y in each of the following cases:

(a) $(1 - 2i)x + (1 + 2i)y = 1 + i$; (b) $\dfrac{x - 3}{3 + i} + \dfrac{y - 3}{3 - i} = i$;
(c) $(4 - 3i)x^2 + (3 + 2i)xy = 4y^2 - \frac{1}{2}x^2 + (3xy - 2y^2)i$.

10. Compute the following:

(a) $(2 - i)(-3 + 2i)(5 - 4i)$; (b) $(2 - 4i)(5 + 2i) + (3 + 4i)(-6 - i)$;
(c) $\left(\dfrac{1 + i}{1 - i}\right)^{16} + \left(\dfrac{1 - i}{1 + i}\right)^{8}$; (d) $\left(\dfrac{-1 + i\sqrt{3}}{2}\right)^{6} + \left(\dfrac{1 - i\sqrt{7}}{2}\right)^{6}$;
(e) $\dfrac{3 + 7i}{2 + 3i} + \dfrac{5 - 8i}{2 - 3i}$.

11. Compute the following:

(a) $i^{2000} + i^{1999} + i^{201} + i^{82} + i^{47}$;
(b) $E_n = 1 + i + i^2 + i^3 + \cdots + i^n$ for $n \geq 1$;
(c) $i^1 \cdot i^2 \cdot i^3 \cdots i^{2000}$;
(d) $i^{-5} + (-i)^{-7} + (-i)^{13} + i^{-100} + (-i)^{94}$.

12. Solve in $\mathbb{C}$ the following equations:

(a) $z^2 = i$; (b) $z^2 = -i$; (c) $z^2 = \frac{1}{2} - i\frac{\sqrt{2}}{2}$.

13. Find all complex numbers $z \neq 0$ such that $z + \dfrac{1}{z} \in \mathbb{R}$.

14. Prove the following:

(a) $E_1 = (2 + i\sqrt{5})^7 + (2 - i\sqrt{5})^7 \in \mathbb{R}$;
(b) $E_2 = \left(\dfrac{19 + 7i}{9 - i}\right)^{n} + \left(\dfrac{20 + 5i}{7 + 6i}\right)^{n} \in \mathbb{R}$.

15. Prove the following identities:

(a) $|z_1+z_2|^2+|z_2+z_3|^2+|z_3+z_1|^2 = |z_1|^2+|z_2|^2+|z_3|^2+|z_1+z_2+z_3|^2$;
(b) $|1+z_1\overline{z}_2|^2+|z_1-z_2|^2 = (1+|z_1|^2)(1+|z_2|^2)$;
(c) $|1-z_1\overline{z}_2|^2-|z_1-z_2|^2 = (1-|z_1|^2)(1-|z_2|^2)$;
(d) $|z_1+z_2+z_3|^2+|-z_1+z_2+z_3|^2+|z_1-z_2+z_3|^2+|z_1+z_2-z_3|^2$

$$= 4(|z_1|^2+|z_2|^2+|z_3|^2).$$

16. Let $z \in \mathbb{C}^*$ be such that $\left|z^3+\frac{1}{z^3}\right| \leq 2$. Prove that $\left|z+\frac{1}{z}\right| \leq 2$.

17. Find all complex numbers z such that

$$|z| = 1 \text{ and } |z^2+\overline{z}^2| = 1.$$

18. Find all complex numbers z such that

$$4z^2+8|z|^2 = 8.$$

19. Find all complex numbers z such that $z^3 = \overline{z}$.

20. Consider $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 1$. Prove that

$$\left|\frac{1}{z}-\frac{1}{2}\right| < \frac{1}{2}.$$

21. Let $a,\ b,\ c$ be real numbers and $\omega = -\frac{1}{2}+i\frac{\sqrt{3}}{2}$. Compute

$$(a+b\omega\ +c\omega^2)(a+b\omega^2+c\omega).$$

22. Solve the following equations:

(a) $|z|-2z = 3-4i$;