Lecture 7: “The Janson Inequality”

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Summary of previous lecture

1. The Lovász Local Lemma
2. Example - Diagonal Ramsey Numbers

Summary of this lecture

1) On the importance of stochastic independence
2) The Janson Inequality
3) Example - Triangle-free sparse Random Graphs
4) Example - Paths of length 3 in $G_{n,p}$

1) On the importance of stochastic independence

- The local Lemma demonstrates that **rare dependencies** yield results similar to the case of stochastic independence.
- The Janson inequality actually does the same, but for the case **when the total amount of dependencies is rather small.**

Intuition

1. Let B_i be the undesired events.
2. non-trivial dependence $\sim$:

$$i \sim j \Leftrightarrow i \neq j \text{ and } B_i, B_j \text{ dependent}$$

3. $\Delta = \sum_{i \sim j} \Pr\{B_i \wedge B_j\}$: measure of dependencies.
4. If the events were independent then the probability of the desired property is defined as follows:

$$\Pr\{\wedge \overline{B_i}\} = \prod_i \Pr\{\overline{B_i}\} = M$$

5. The Janson inequality shows that $\Pr\{\wedge \overline{B_i}\}$ remains very close to M if the dependencies are small.

2) The Janson Inequality

Theorem 1

Let B_i be undesired events. Define

1) $\Delta = \sum_{i \sim j} \Pr\{B_i \wedge B_j\}$ *and*
2) $M = \prod_i \Pr\{\overline{B_i}\}$

$$\textit{If } \Pr\{B_i\} \leq \epsilon \ \textit{ then } \ M \leq \Pr\left\{\bigwedge_i \overline{B_i}\right\} \leq M \cdot exp\left(\frac{1}{1-\epsilon} \cdot \frac{\Delta}{2}\right)$$

Remark: If

a. ϵ **is small** (e.g. ϵ is constant or smaller i.e. undesired events are not very probable) **and**
b. Δ **is small e.g. o(1)** (i.e. there are small dependencies)

Then,

$$e^{\left(\frac{1}{1-\epsilon} \cdot \frac{\Delta}{2}\right)} \to 1 \ \Rightarrow \Pr\left\{\bigwedge_i \overline{B_i}\right\} \simeq M = \prod_i \Pr\{\overline{B_i}\}$$

3) Example - Triangle-free sparse Random Graphs

Theorem 2

Consider the $G_{n,p}$ graph space. For every constant c,

$$\text{If } p = \frac{c}{n} \Rightarrow \Pr\{\nexists\ K_3\} \to e^{-c^3/6}$$

($G_{n,p}$ with $p = \frac{c}{n}$ is sparse because the connectivity threshold is $p = \frac{c \log n}{n}$.)

Proof:

- Let S be any fixed set of 3 vertices ($|S| = 3$).
- We define the event $B_S = \{\text{S is } K_3 \text{ (triangle)}\}$.
- $\Pr\{B_S\} = p^3 = \frac{c^3}{n^3}$

We want to prove that

$$\Pr\left\{ \bigwedge_{S, |S|=3} \overline{B_S} \right\} \to e^{-c^3/6}$$

Proof of theorem 2

<u>Intuition:</u> If events B_S were independent, then:

$$M = \Pr\left\{ \bigwedge_{S,|S|=3} \overline{B_S} \right\} = \prod_S \Pr\left\{\overline{B_S}\right\}$$

$$= (1-p^3)^{\binom{n}{3}} \sim e^{-\frac{c^3}{n^3}\cdot\frac{n^3}{3!}} = e^{-c^3/6}$$

Since events are dependent, we will show that dependencies in this sparse graph are small, to get a very similar result, via applying the Janson inequality:

$$\epsilon = \Pr\{B_S\} = \frac{c^3}{n^3} \to 0$$

$$M = \prod_{S,|S|=3} \Pr\{\overline{B_S}\} = (1-p^3)^{\binom{n}{3}} \sim e^{-c^3/6}$$

$$\Delta = \sum_{S\sim T} \Pr\{B_S \wedge B_T\}$$

Proof of theorem 2

- Non-trivial dependence:

$$S \sim T \;\Leftrightarrow\; \left\{ \begin{array}{l} S \neq T \\ |S \wedge T| \geq 2 \end{array} \right\} \;\Leftrightarrow\; |S \wedge T| = 2$$

- $\Pr\{B_S \wedge B_T\} = p^5$
-

$$\begin{aligned} \Delta &= \binom{n}{3}\binom{3}{2}\binom{n-3}{1}p^5 \\ &= O(n^4 p^5) = O\left(n^4 \cdot \frac{c^5}{n^5}\right) = \Theta\left(\frac{1}{n}\right) = o(1) \end{aligned}$$

Since $\epsilon = o(1)$ and $\Delta = o(1)$ we apply the basic case of Janson inequality that gives us:

$$\lim_{n\to\infty} \Pr\left\{ \bigwedge_S \overline{B_s} \right\} = \lim_{n\to\infty} M = e^{-c^3/6}$$

4) Example - Paths of length 3 in $G_{n,p}$

Theorem 3

Define the event B=$\{\exists$ a path of length 3 between <u>any pair</u> of vertices in $G_{n,p}\}$. For every constant $c \geq 2$:

$$\textit{If } p = \left(\frac{c \cdot \ln n}{n^2}\right)^{1/3} \quad \Rightarrow \Pr\{B\} \to 1$$

Proof:

We define the event:

- $B_{u,v} = \{ \nexists$ a path of length 3 between vertices u and $v\}$.

$$\Pr\{\overline{B}\} = \Pr\left\{\bigcup_{u,v} B_{u,v}\right\}$$
$$\leq \sum_{u,v} \Pr\{B_{u,v}\} = O(n^2)\Pr\{B_{u,v}\}$$

Proof of theorem 3

In order to prove that $\Pr\{\overline{B}\} \to 0$ or, equivalently $\Pr\{B\} \to 1$, we must prove that

$$\Pr\{B_{u,v}\} = o(n^{-2})$$

We apply Janson Inequality to prove that $\Pr\{B_{u,v}\} = o(n^{-2})$.

We define the undesired event:

- $A_{w_1,w_2} = \{\text{the edges } (u,w_1),(w_1,w_2),(w_2,v) \text{ exist}\}$

and express the event $B_{u,v}$ as follows:

$$\Rightarrow B_{u,v} = \left\{ \bigwedge_{w_1,w_2} \overline{A_{w_1,w_2}} \right\}$$

Proof of theorem 3

$$\epsilon = \Pr\left\{A_{w_1,w_2}\right\} = p^3 = \left(\left(\frac{c \cdot \ln n}{n^2}\right)^{1/3}\right)^3 \to 0$$

$$M = \prod_{w_1,w_2} \Pr\left\{\overline{A_{w_1,w_2}}\right\} = \left(1 - p^3\right)^{(n-2)(n-3)}$$

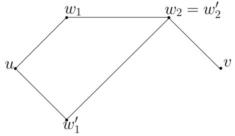
$$\leq e^{-p^3 n^2} = e^{-\frac{c \cdot \ln n}{n^2} \cdot n^2} = n^{-c} = o\left(n^{-2}\right)$$

So, the result would hold if there were no dependencies.

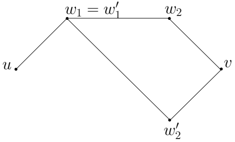
$$\Delta = \sum_{(w_1,w_2)\sim(w_1',w_2')} \Pr\left\{A_{w_1,w_2} \wedge A_{w_1',w_2'}\right\}$$

Proof of theorem 3

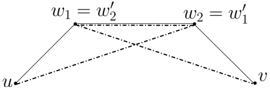
case 1:



case 2:



case 3:



Contribution to Δ:

- case 1:

$$\binom{n-2}{3} \cdot p^5$$

- case 2:

$$\binom{n-2}{3} \cdot p^5$$

- case 3:

$$\binom{n-2}{2} \cdot p^5$$

Proof of theorem 3

$$
\begin{aligned}
\Rightarrow \Delta &= O(n^3)p^5 + O(n^3)p^5 + O(n^2)p^5 \\
&= O(n^3)p^5 = O(n^3)\left(\frac{c \cdot \ln n}{n^2}\right)^{5/3} \\
&= O\left(\frac{(c \cdot \ln n)^{5/3}}{n^{1/3}}\right) \\
&= o(1)
\end{aligned}
$$

Since $\epsilon = o(1)$ and $\Delta = o(1)$ we apply the basic case of Janson inequality that gives us:

$$
\Rightarrow \lim_{n \to \infty} \Pr\left\{ \bigwedge_{w_1, w_2} \overline{A_{w_1, w_2}} \right\} = \lim_{n \to \infty} M = o(n^{-2})
$$

□