Lecture 6: “The Local Lemma”

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Summary of previous lecture

1. The Variance of a random variable:

$$Var[X] = E\left[(X - E[X])^2\right]$$

2. The Chebyshev Inequality:

$$\Pr\{|X - \mu| \geq t\} \leq \frac{Var[X]}{t^2}$$

3. A direct consequence:

$$\Pr\{X = 0\} \leq \Pr\{|X - \mu| \geq \mu\} \leq \frac{Var[X]}{\mu^2}$$

4. The Second Moment method:

$$\left.\begin{array}{l} \text{if } E[X] \to \infty \text{ and} \\ Var[X] = o(E^2[X]) \end{array}\right\} \Rightarrow \Pr\{X = 0\} \to 0$$

Thus, $\Pr\{X \geq 1\} \to 1 \Rightarrow$ existence w.h.p.

Summary of previous lecture

5. Covariance of two random variables:

$$\text{Cov}(X, Y) = E[XY] - E[X] \cdot E[Y]$$

   stochastic dependence of $X, Y \uparrow \Rightarrow |Cov(X, Y)| \uparrow$
   X, Y independent $\Rightarrow Cov(X, Y) = 0$

6. Variance of a sum of r.v. $X = X_1 + X_2 + \cdots + X_n$:

$$Var[X] = \sum_{1 \leq i, j \leq n} Cov(X_i, X_j)$$

   The sum is over ordered pairs.

7. An upper bound for the variance of a sum of <u>indicator</u> r.v. $X = X_1 + X_2 + \cdots + X_n$:

$$Var[X] \leq E[X] + \sum_{1 \leq i \neq j \leq n} Cov(X_i, X_j)$$

   Hint: $Cov(X_i, X_i) = Var(X_i) \leq E[X_i]$ (since $X_i \sim$ Bernoulli).

Summary of previous lecture

8. Non-trivial dependence: $\sim$

$$i \sim j \Rightarrow \begin{cases} A_i \neq A_j \text{ and} \\ A_i, A_j \text{ dependent} \end{cases}$$

9. Δ approach:

$$\Delta = \sum_{i \sim j} \Pr\{A_i \wedge A_j\}$$

10. The upper bound using Δ:

$$\text{Since} \quad Cov(X_i, X_j) \leq \Pr\{A_i \wedge A_j\}$$

$$\text{we have:} \quad Var[X] \leq E[X] + \Delta$$

11. The Second Moment method (variation with Δ):

$$\left. \begin{array}{l} \text{if } E[X] \to \infty \text{ and} \\ \Delta = o(E^2[X]) \end{array} \right\} \Rightarrow \Pr\{X = 0\} \to 0$$

Summary of previous lecture

12. Symmetric events:

$$A_i, A_j \text{ symmetric } \Rightarrow Pr\{X_i|X_j\} = \Pr\{X_j|X_i\}$$

13. Δ^* approach:

$$\Delta^* = \sum_{j \sim i} \Pr\{A_j|A_i\}$$

14. The Second Moment method (variation with Δ^*):

$$\left.\begin{array}{l}\text{if } E[X] \to \infty \text{ and} \\ \Delta^* = o(E[X])\end{array}\right\} \Rightarrow \Pr\{X = 0\} \to 0$$

15. Threshold functions in $G_{n,p}$ $p_o = p_o(n)$ is a threshold of property A iff

$$p >> p_o \Rightarrow \Pr\{G_{n,p} \text{ has the property A }\} \to 1$$
$$p << p_o \Rightarrow \Pr\{G_{n,p} \text{ has the property A }\} \to 0$$

16. Example - K_4 in random graphs (threshold: $p_o = n^{-2/3}$)

Importance of Stochastic Independence

1) Chebyshev inequality:
   - is more powerful than Markov's inequality
   - however provides only a polynomially small upper bound on deviations:
   $$\Pr\{|X-\mu| \geq \lambda\sigma\} \leq \frac{1}{\lambda^2} = \lambda^{-2}$$
2) Exponentially small deviation probability:
   - Normal Distribution: accurate estimation of probability of large deviations using pdf:
   $$\Pr\{|X-\mu| \geq \lambda\sigma\} = \Pr\{|Z| \geq \lambda\} \sim \sqrt{\frac{2}{\pi}}\,\frac{e^{-\frac{\lambda^2}{2}}}{\lambda}$$
   - Central Limit Theorem: Any r.v. X that is a sum of many **independent** and identically distributed (i.i.d.) random variables with same mean μ and variance σ^2 has the above deviation probability:
   $$\Pr\left\{\left|\frac{X-n\mu}{\sigma\sqrt{n}}\right| \geq \lambda\right\} \sim \sqrt{\frac{2}{\pi}}\,\frac{e^{-\frac{\lambda^2}{2}}}{\lambda}$$

Importance of Stochastic Independence (II)

3) Method of positive probability when events $A_1, \cdots, A_n$ are "bad" and stochastically independent:

$$\forall i, \Pr\{A_i\} \leq p < 1 \Rightarrow \Pr\left\{ \bigwedge_i \overline{A_i} \right\} \geq (1-p)^n \sim e^{-pn} > 0$$

Remark: This is an exponentially small positive probability but it is enough for this method (to prove that the desired property holds).

4) THE IMPORTANCE OF SMALL DEPENDENCE:

When the dependencies are small and/or rare then similar (to independence) results hold!

- rare dependencies $\Rightarrow$ The Lovász Local Lemma
- small dependencies $\Rightarrow$ Janson Inequality

Importance of Stochastic Independence (III)

The Local Lemma:

- Provides a more general way of proving that **rarely dependent events** hold with positive, though exponentially small, probability.

Summary of this lecture

1) Dependency Graph
2) The Lovász Local Lemma
3) Example - Diagonal Ramsey Numbers

Dependency Graph

Definition 1

*Let $A_1, \ldots, A_n$ be events in a probability space. An undirected graph $G = (V, E)$ with $|V| = n$ is called a **dependency graph** for the events $A_1, \ldots, A_n$ if for all vertices i, j such that $(i, j) \in E$, the corresponding events A_i, A_j are stochastically dependent.*

Note: The **degree** $d(i)$ of vertex i (corresponding to the event A_i) in this graph captures **the "level" of dependence** of this event (actually, the number of dependent events).

The Lovász Local Lemma

Theorem 1 (Symmetric Lovász Local Lemma)

Let $A_1, \ldots, A_n$ be a set of events with dependency graph $G = (V, E)$. We denote $d(i)$ the degree of vertex $i \in V$. Suppose that:

a. $\Pr\{A_i\} \leq p, \;\; 1 \leq i \leq n$ *and*

b. $d(i) \leq d, \;\; 1 \leq i \leq n$

Then:

$$\text{If } \; 4dp < 1 \quad \Rightarrow \Pr\left\{ \bigwedge_{i=1}^{n} \overline{A_i} \right\} > 0$$

Intuition: **Rare dependencies of ("bad" for a property) events** of small probability lead to a positive probability existence proof.

Proof of theorem 1

Step 1:
We will first show that:

$$\forall S \subseteq V : \Pr\left\{ A_i \middle| \bigwedge_{j \in S, |S|=s} \overline{A_j} \right\} \leq 2p$$

Proof (by induction on the size of S):

- **Base Case**: $|S| = 0$: $\Pr\{A_i\} \leq p < 2p$
- **Induction Hypothesis**: The condition holds for all $S : |S| < s$.
- **Induction Step:** We must prove that the condition holds for all S such that $|S| = s$.
  Consider $\Pr\left\{ A_i \middle| \overline{A_1} \wedge \overline{A_2} \wedge \cdots \wedge \overline{A_s} \right\}$ and (w.l.o.g.) reorder the s events such that the ones that A_i may be dependent on are in the beginning.

Proof of theorem 1

By definition of conditional probability we have that:

$$\Pr\left\{A_i \middle| \overline{A_1} \wedge \overline{A_2} \wedge \cdots \wedge \overline{A_s}\right\} = \frac{\Pr\left\{A_i \wedge \overline{A_1} \wedge \cdots \wedge \overline{A_s}\right\}}{\Pr\left\{\overline{A_1} \wedge \cdots \wedge \overline{A_s}\right\}}$$

$$= \frac{\Pr\left\{A_i \wedge \overline{A_1} \wedge \cdots \wedge \overline{A_d} \middle| \overline{A_{d+1}} \wedge \cdots \wedge \overline{A_s}\right\}}{\Pr\left\{\overline{A_1} \wedge \cdots \wedge \overline{A_d} \middle| \overline{A_{d+1}} \wedge \cdots \wedge \overline{A_s}\right\}} = \frac{\mathcal{A}}{\mathcal{B}}$$

Bound of $\mathcal{A}$:
Since A_i is independent of the events A_{d+1} to A_s (by the reordering we have done), we can bound the enumerator as follows:

$$\begin{aligned}
\mathcal{A} &= \Pr\left\{A_i \wedge \overline{A_1} \wedge \cdots \wedge \overline{A_d} \middle| \overline{A_{d+1}} \wedge \cdots \wedge \overline{A_s}\right\} \\
&\leq \Pr\left\{A_i \middle| \overline{A_{d+1}} \wedge \cdots \wedge \overline{A_s}\right\} \\
&= \Pr\{A_i\} \\
&\leq p
\end{aligned}$$

Proof of theorem 1

Bound of $\mathcal{B}$:

$$\begin{aligned}
\mathcal{B} &= \Pr\left\{\overline{A_1} \wedge \cdots \wedge \overline{A_d} \middle| \overline{A_{d+1}} \wedge \cdots \wedge \overline{A_s}\right\} \\
&= 1 - \Pr\left\{A_1 \vee \cdots \vee A_d \middle| \overline{A_{d+1}} \wedge \cdots \wedge \overline{A_s}\right\} \\
&\geq 1 - \sum_{i=1}^{d} \Pr\left\{A_i \middle| \overline{A_{d+1}} \wedge \cdots \wedge \overline{A_s}\right\} \\
&\geq 1 - \sum_{i=1}^{d} 2p \quad \text{(by the induction hypothesis)} \\
&= 1 - 2pd \\
&> \frac{1}{2}
\end{aligned}$$

Thus,

$$\Pr\left\{A_i \middle| \overline{A_1} \wedge \overline{A_2} \wedge \cdots \wedge \overline{A_s}\right\} = \frac{\mathcal{A}}{\mathcal{B}} < \frac{p}{1/2} = 2p$$

□

Proof of theorem 1

Step 2:

$$
\begin{aligned}
\Pr\left\{\bigwedge_{i=1}^{n} \overline{A_i}\right\} &= \Pr\left\{\overline{A_1}\right\} \cdot \Pr\left\{\overline{A_2}\middle|\overline{A_1}\right\} \cdots \Pr\left\{\overline{A_n}\middle|\overline{A_1} \wedge \cdots \wedge \overline{A_{n-1}}\right\} \\
&= \prod_{i=1}^{n} \Pr\left\{\overline{A_i}\middle|\overline{A_1} \wedge \cdots \wedge \overline{A_{i-1}}\right\} \\
&= \prod_{i=1}^{n} \left(1 - \Pr\left\{A_i\middle|\overline{A_1} \wedge \cdots \wedge \overline{A_{i-1}}\right\}\right) \\
&\geq \prod_{i=1}^{n} (1 - 2p) \\
&= (1 - 2p)^n > 0
\end{aligned}
$$

Since $\left.\begin{array}{l} 4dp < 1 \\ d \geq 1 \end{array}\right\} \Rightarrow p < \frac{1}{4d} < \frac{1}{4} \Rightarrow 2p < \frac{1}{2} \Rightarrow 1 - 2p > \frac{1}{2} > 0$ □

Example - Diagonal Ramsey Numbers

Definition 2

The diagonal Ramsey number $R(k,k)$ is the smallest integer n such that in any two-coloring of the edges of the complete graph on n vertices K_n there is a monochromatic K_k.

Previous bounds:

- Method of positive probability: $R(k,k) \geq \frac{k}{e\sqrt{2}} \cdot 2^{\frac{k}{2}}$
- Deletion method: $R(k,k) \geq \frac{k}{e} \cdot 2^{\frac{k}{2}}$

Theorem 2

$$R(k,k) \geq \frac{k\sqrt{2}}{e} \cdot 2^{\frac{k}{2}}$$

Proof of theorem 2

- Construct a probability sample space by two-coloring at random, equiprobably (for the two colors) and independently (for the various edges) every edge of K_n.
- Let S be any fixed set of k vertices.
- Define the event $A_S = \{\text{S is monochromatic}\}$.

$$\Pr\{A_S\} = \left(\frac{1}{2}\right)^{\binom{k}{2}} + \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{1-\binom{k}{2}} = p$$

- Define a dependency graph G with vertices corresponding to k-sets such that for two sets S and T, the edge $(S,T) \in G$ iff $|S \cap T| \geq 2$ (i.e. S, T are dependent)
- Upper bound on degree d:

$$d = \Big|\{T : |S \cap T| \geq 2\}\Big| = \binom{k}{2} \cdot \binom{n-2}{k-2} < \binom{k}{2}\binom{n}{k-2}$$

Remark: We need at least 2 common vertices. The rest $(k-2)$ may be common or not so we have to choose them from a $(n-2)$-set and not from a $(n-k)$-set.

Proof of theorem 2

Since we want to prove that there exists a 2-coloring in which there is no monochromatic K_k we want to show that

$$\Pr\left\{ \bigwedge_{|S|=k} \overline{A_S} \right\} > 0$$

So, to employ the Local Lemma we have to have $4dp < 1 \Rightarrow$

$$4\binom{n}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} < 1$$

Working out the asymptotics we get:

$$R(k,k) \geq \frac{k\sqrt{2}}{e} \cdot 2^{\frac{k}{2}}(1+o(1))$$

□