Lecture 2: "The Method of Positive Probability (II)"

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The Basic Method (method of "positive probability")

- Construct (by using abstract random experiments) an appropriate probability sample space of combinatorial structures (thus, the sample points correspond to the combinatorial structures whose existence we try to prove).
- Prove that the probability of the desired property in this space is positive (i.e. non-zero).

$$\Downarrow$$

There is at least one point in the space with the desired property.

$$\Downarrow$$

There is at least one combinatorial structure with the desired property.

Examples in this lecture

(i) Coloring Hypergraphs

(ii) Tournaments with property S_k

(I) Coloring Hypergraphs

Definition 1

*A **Hypergraph** $H = (V, E)$ consists of:*

- V *: a finite set of vertices*
- E *: a set of subsets of V (the "edges")*

Definition 2

*A Hypergraph $H = (V, E)$ is called **n-uniform** iff all edges contain exactly n vertices.*

Property B

Definition 3

*A Hypergraph $H = (V, E)$ has **property B** (it is two-colorable) iff* $\exists$ *a two-coloring of V such that no edge is monochromatic.*

Definition 4

$\boldsymbol{m(n)}$ *is the minimum number of edges on a n-uniform hypergraph that **does not** have property B.*

Theorem 1 (Erdös, 1963)

$$m(n) \geq 2^{n-1}$$

Proof of Theorem 1 (1/2)

- Construct a probability sample space by two-coloring the vertices of H at random, equiprobably for the two colors and independently for every vertex.
- Let e be any fixed edge.
- Define the event $M_e := \{e \text{ is monochromatic}\}$.
  - i.e. all vertices of edge e must have the same color.
- Compute the probability $\Pr[M_e]$.

$$\Pr[M_e] = 2 \cdot \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2}}_{n \text{ times}} = 2^{1-n}$$

Proof of Theorem 1 (2/2)

- Define the event $M := \{\exists$ at least one monochromatic edge$\}$.
- Hence, $M = \bigcup_e M_e$
- Using Boole's inequality we can compute $\Pr[M]$

$$\Pr[M] \leq \sum_e \Pr[M_e] = |e| 2^{1-n}$$

- If $|e| \cdot 2^{1-n} < 1$ (i.e., $|e| < 2^{n-1}$) then $\Pr[M] < 1 \Rightarrow \Pr[\overline{M}] > 0$.
- Hence, there is a two-coloring without a monochromatic edge when $m(n) < 2^{n-1} \Rightarrow$ property B.
- Hence, $m(n) \geq 2^{n-1}$ is necessary for avoiding property B.

□

(II) Tournaments

Definition 5

*A **tournament** T_n is a complete directed graph on n vertices i.e., for every pair (i, j), there is either an edge from i to j or from j to i, but not both.*

Why do we call these graphs tournaments?

- Each vertex corresponds to a team playing at some tournament.
- The directed edge (i, j) means that team i wins team j.
- all teams play against each other.

The S_k Property

Definition 6

A tournament T_n is said to have ***property*** S_k *if for any set of* k *vertices in the tournament, there is some vertex that has a directed edge to each of those* k *vertices.*

Theorem 2 (Erdös, 1963)

$\forall k, \exists$ *a tournament* T_n *that has the property* S_k*.*

Proof of Theorem 2 (1/2)

- Construct a probability sample space with points random tournaments by choosing the direction of each edge at random, equiprobably for the two directions and independently for every edge.
- Let S be any fixed set of k teams and define the event $M_S := \{\nexists$ a team that wins all teams in $S\}$.
- For any team, the probability to win all teams in S is $(\frac{1}{2})^k$.
- Hence, the probability of not winning at least one of them is $1 - (\frac{1}{2})^k$.
- The probability that this is happening for all $n-k$ teams that don't belong in S is:

$$\Pr[M_S] = \left(1 - \left(\frac{1}{2}\right)^k\right)^{n-k}$$

Proof of Theorem 2 (2/2)

- Define the event $M := \{\exists$ a set S of k teams such that $\nexists$ a team $u : u \notin S$ that wins all teams in $S\}$.
- $M = \bigcup_S M_S$
- Using Boole's inequality we can compute $\Pr[M]$

$$\Pr[M] \leq \sum_{S,|S|=k} \Pr[M_S] = \binom{n}{k}\left(1 - \left(\frac{1}{2}\right)^k\right)^{n-k}$$

- If $\binom{n}{k}\left(1 - \left(\frac{1}{2}\right)^k\right)^{n-k} < 1$ then $\Pr[M] < 1 \Rightarrow \Pr[\overline{M}] > 0$.
- Hence, there is a tournament with property S_k.

□