ΝΑΝΟΗΛΕΚΤΡΟΝΙΚΗ & ΚΒΑΝΤΙΚΕΣ ΠΥΛΕΣ

3η Διάλεξη

Βιβλιογραφία: EXPLORATIONS IN QUANTUM COMPUTING, Colin P. Williams (2nd edition, Springer-Verlag, 2011), chapter 1.

Multi-qubit Quantum Memory Registers

- The Computational Basis
- Direct Product for Forming Multi-qubit States
- Interference Effects
- Entanglement

Multi-qubit Quantum Memory Registers

- A useful quantum computational device will need to have a multi-qubit quantum memory register.

- In general, this is assumed to consist of a collection of n-qubits, which are assumed to be ordered, indexed and addressable so that selective operations can be applied to any single qubit or any pair of qubits at will.

- A n-qubit register can be found in a superposition of all the 2^n possible ***bit strings***:

$$|00\ldots0\rangle, |00\ldots1\rangle, \ldots, |11\ldots1\rangle$$

- When we describe the state of a multi-qubit quantum memory register as a superposition of its possible bit-string configurations, we say the state is represented in the ***computational basis***.

The Computational Basis

The most general form for a pure state of a ***2-qubit*** quantum memory register can be written as:

$$|\psi\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$$

$$\text{where } |c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 = 1$$

Thus, the ***register contains many different bit string configurations at once***, each with their own amplitude.

Similarly, the general state of a 3-qubit register can be written as:

$$|\psi\rangle = c_0|000\rangle + c_1|001\rangle + c_2|010\rangle + c_3|011\rangle + c_4|100\rangle + c_5|101\rangle + c_6|110\rangle + c_7|111\rangle$$

$$\text{where } |c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 + |c_4|^2 + |c_5|^2 + |c_6|^2 + |c_7|^2 = 1$$

The most general form for a pure state of an n-qubit quantum memory register is:

$$|\psi\rangle = c_0|00\ldots0\rangle + c_1|00\ldots1\rangle + \cdots + c_{2^n-1}|11\ldots1\rangle = \sum_{i=0}^{2^n-1} c_i|i\rangle$$

where $\sum_{i=0}^{2^n-1} |c_i|^2 = 1$ and $|i\rangle$ represents the "computational basis eigenstate"

The “computational basis eigenstate” $|i\rangle$

$|i\rangle$ represents the “computational basis eigenstate” whose bit values match those of the decimal number *i* expressed in base-2 notation, padded on the left (if necessary) with “0” bits in order to make a full complement of *n* bits.

For example, the 5-qubit computational basis eigenstate corresponding to $|6\rangle$ is: $|00110\rangle$
This is because 6 in base-2 is “110” and then we pad on the left with two “0” bits to make a total of 5 bits.
As for the case of single qubits, such ***ket vectors*** can be regarded as ***column vectors*** according to the lexicographic ordering:

$$|00\cdots00\rangle = \begin{bmatrix}1\\0\\0\\\vdots\\0\\0\\0\end{bmatrix}, |00\cdots01\rangle = \begin{bmatrix}0\\1\\0\\\vdots\\0\\0\\0\end{bmatrix}, |00\cdots10\rangle = \begin{bmatrix}0\\0\\1\\\vdots\\0\\0\\0\end{bmatrix} \cdots |11\cdots10\rangle = \begin{bmatrix}0\\0\\0\\\vdots\\0\\1\\0\end{bmatrix}, |11\cdots11\rangle = \begin{bmatrix}0\\0\\0\\\vdots\\0\\0\\1\end{bmatrix}$$

Complex amplitudes in a multi-qubit quantum memory register

In a multi-qubit quantum state: $|\psi\rangle = \sum_{i=0}^{2^n-1} c_i|i\rangle$ it is not necessary for every amplitude to be non-zero.

For example, a particular 3-qubit quantum state: $|\psi\rangle = a|001\rangle + b|010\rangle + c|100\rangle$ does not contain any contributions from the eigenstates: $|000\rangle, |011\rangle, |101\rangle, |110\rangle, |111\rangle$.

As a column vector, this 3-qubit state would actually be:

$$|\psi\rangle = a|001\rangle + b|010\rangle + c|100\rangle \equiv \begin{pmatrix} 0 \\ a \\ b \\ 0 \\ c \\ 0 \\ 0 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \text{amplitude of } |000\rangle \text{ component} \\ \text{"} \quad |001\rangle \quad \text{"} \\ \text{"} \quad |010\rangle \quad \text{"} \\ \text{"} \quad |011\rangle \quad \text{"} \\ \text{"} \quad |100\rangle \quad \text{"} \\ \text{"} \quad |101\rangle \quad \text{"} \\ \text{"} \quad |110\rangle \quad \text{"} \\ \text{"} \quad |111\rangle \quad \text{"} \end{pmatrix}$$

The **size** of these column vectors **grow exponentially** with the number of qubits, making it **computationally intractable to simulate arbitrary quantum computations** on classical computers. **A 100-qubit quantum memory register requires 2^{100} complex amplitudes!!**

In very few qubits, we run out of particle in the known Universe with which to make a classical memory large enough to represent a quantum state.

Direct Product of Quantum States

How the state of the *n*-qubit register is related to the states of the individual qubits?
The answer is provided by the **direct product** of the *n* individual quantum states.

Let $|\phi\rangle = \sum_{j=0}^{2^m-1} a_j|j\rangle$ be an *m*-qubit *pure* state, and $|\psi\rangle = \sum_{k=0}^{2^n-1} b_k|k\rangle$ be an *n*-qubit *pure* state.

The quantum state of a memory register is computed by taking their direct product: $|\phi\rangle \otimes |\psi\rangle$

$$|\phi\rangle \otimes |\psi\rangle = \sum_{j=0}^{2^m-1} a_j|j\rangle \otimes \sum_{k=0}^{2^n-1} b_k|k\rangle = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2^m-1} \end{pmatrix} \otimes \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{2^n-1} \end{pmatrix} = \begin{pmatrix} a_0 \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{2^n-1} \end{pmatrix} \\ a_1 \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{2^n-1} \end{pmatrix} \\ \vdots \\ a_{2^m-1} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{2^n-1} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_0 b_0 \\ a_0 b_1 \\ \vdots \\ a_0 b_{2^n-1} \\ \vdots \\ a_1 b_0 \\ a_1 b_1 \\ \vdots \\ a_1 b_{2^n-1} \\ \vdots \\ a_{2^m-1} b_0 \\ a_{2^m-1} b_1 \\ \vdots \\ a_{2^m-1} b_{2^n-1} \end{pmatrix}$$

Direct or “tensor” or “Kroenecker” product

The tensor product of a set of n-vectors $|z_j\rangle$, specified by the quantum numbers z_j for $j = 1, 2, \ldots, n$, is written interchangeably as:

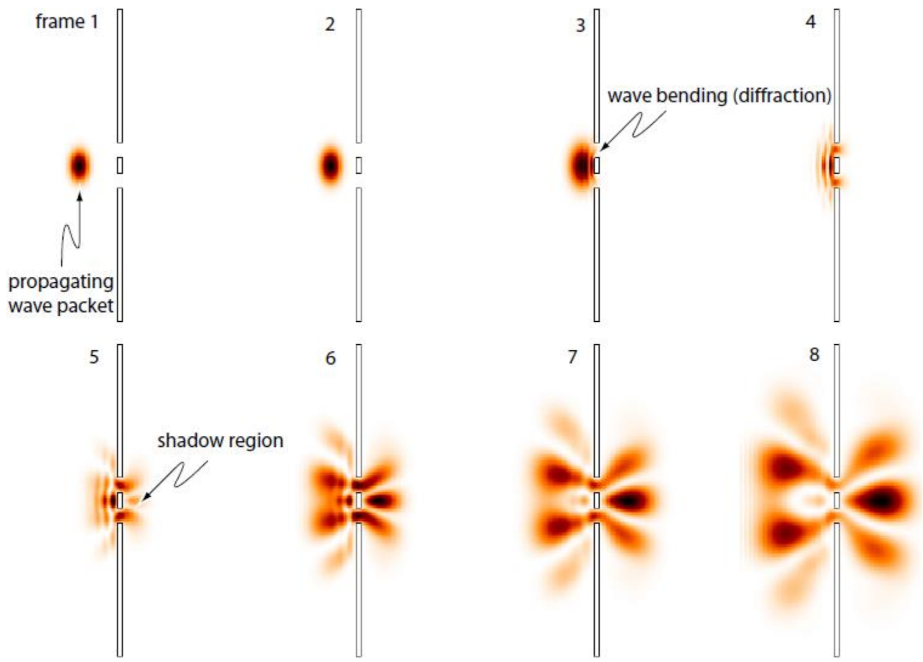
$$|z_1\rangle \otimes |z_2\rangle \otimes \cdots \otimes |z_n\rangle = |z_1\rangle|z_2\rangle \cdots |z_n\rangle = |z_1 z_2 \cdots z_n\rangle$$

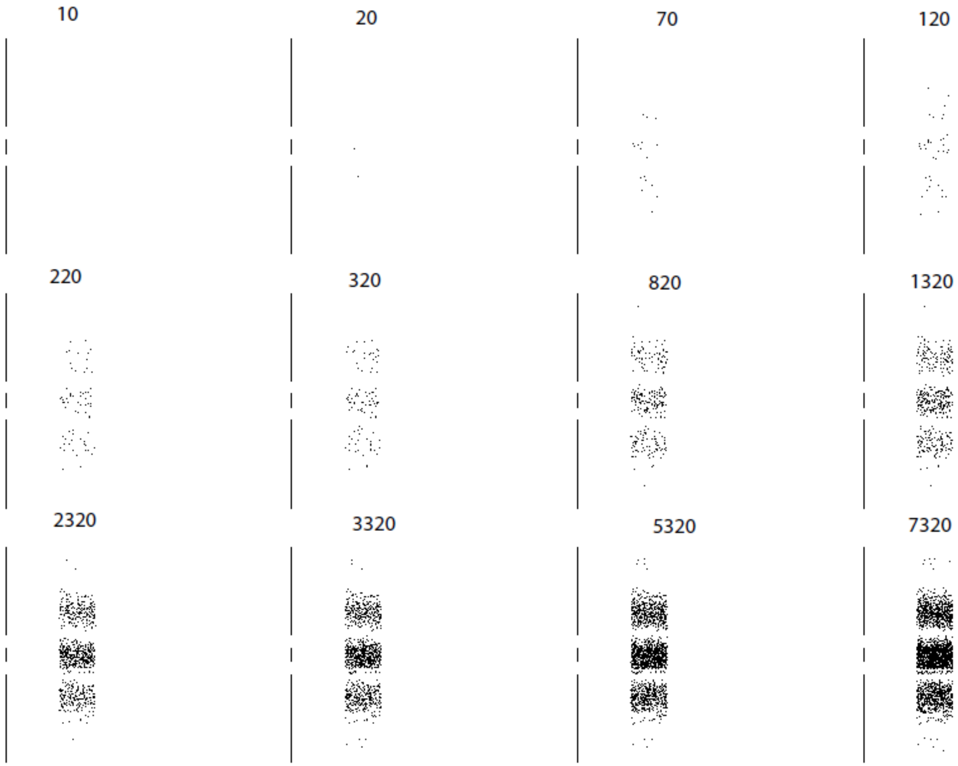
For example, let $|\phi\rangle = a|0\rangle + b|1\rangle$ and $|\psi\rangle = c|0\rangle + d|1\rangle$. Then the direct product

$$|\phi\rangle \otimes |\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} c \\ d \end{pmatrix} \\ b\begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

Interference Effects: The mystery of wave-particle duality

Really small things (atomic size) sometimes behave like particles and sometimes behave like waves, ***depending upon the way the measurement is made***. The experiment that exemplifies this peculiar behavior is the **two-slit interference** experiment that has been performed for light, electrons, neutrons, atoms and molecules.





A quantum mechanical particle impinging on a double slit it will pass through both slits and self-interfere beyond the slit, resulting in an oscillatory pattern of probability amplitude for where the particle will be found.

Interference Effects in quantum memory registers

One of the most striking differences between quantum memory registers and classical memory registers is the possibility of encountering "quantum interference" effects in the quantum case that are absent in the classical case.

Quantum interference can occur whenever ***there is more than one way*** to obtain a particular computational result.

The different pathways can interfere constructively to increase the net probability of that result, or they can interfere destructively to reduce the probability of that result.

Let ***A*** be some observable that can act on an *n*-qubit register, with eigenvalue "a" when the corresponding state of the memory register is $|\psi_a\rangle$: $A|\psi_a\rangle = a|\psi_a\rangle$

The question is: *what is the probability to obtain the value "a" when the observable* ***A*** *is measured and the quantum memory register is in state*: $|\psi\rangle = c_j|j\rangle + c_k|k\rangle$

Classical probabilistic memory register*:*

$$P^{\text{CLASSICAL}}(a) = P_j(a)p_j + P_k(a)p_k = |c_j|^2 P_j(a) + |c_k|^2 P_k(a) = |c_j|^2|\langle\psi_a|j\rangle|^2 + |c_k|^2|\langle\psi_a|k\rangle|^2$$

Quantum memory register*:*

$$\begin{aligned} P^{\text{QUANTUM}}(a) &= |\langle\psi_a|\psi\rangle|^2 = |c_j\langle\psi_a|j\rangle + c_k\langle\psi_a|k\rangle|^2 \\ &= |c_j|^2|\langle\psi_a|j\rangle|^2 + |c_k|^2|\langle\psi_a|k\rangle|^2 + 2\,\text{Re}(c_j c_k^*\langle\psi_a|j\rangle\langle\psi_a|k\rangle^*) \end{aligned}$$

Entanglement

Entanglement is crucial to obtain the exponential speedups in quantum algorithms.

Definition: Entangled Pure State. A multi-qubit pure state is entangled if and only if it *cannot be factored into the direct product* of a definite state for each qubit individually:

$$|\psi\rangle_{AB} \neq |\psi\rangle_A \otimes |\psi\rangle_B \text{ for any choice of states } |\psi\rangle_A \text{ and } |\psi\rangle_B.$$

In a multi-qubit memory register if qubits are entangled then actions performed on one subset of qubits can have an impact on another, "untouched", subset of qubits.

For example, consider a 2-qubit memory register comprised of qubits A and B, in state

$$\frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B) \longleftrightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

If qubit ***A*** is measured in the computational basis and found to be in state $|1\rangle$ then even though qubit ***B*** has not yet been touched, its quantum state is now determined to be $|1\rangle$ too.

For example, we can prepare two entangled quantum registers ***A*** and ***B*** where register ***A*** contains a set of indices running from 0 to $2^n - 1$ and register ***B*** contains a set of values of a function that ***depends upon the value of the index in register A***. The joint state is:

$$\sum_{i=0}^{2^n-1} |i\rangle_A |f(i)\rangle_B$$

By measuring the value of the function in register ***B*** to be "c", we can project out the set of indices in register ***A*** consistent with the observed function value, giving a superposition:

$$\sum_{\{i' : f(i')=c\}} |i'\rangle_A |c\rangle$$

*In one shot we get all index values in register **A** that give the same value for the function in **B!!**!*