ΝΑΝΟΗΛΕΚΤΡΟΝΙΚΗ & ΚΒΑΝΤΙΚΕΣ ΠΥΛΕΣ

2η Διάλεξη

Βιβλιογραφία: EXPLORATIONS IN QUANTUM COMPUTING, Colin P. Williams (2nd edition, Springer-Verlag, 2011), chapter 1.

Quantization: From Bits to Qubits

- Ket Vector Representation of a Qubit
- Superposition States of a Single Qubit
- Bloch Sphere Picture of a Qubit
- Reading the Bit Value of a Qubit

Assumptions about the properties of bit that are no longer necessarily true at the quantum scale

Assumption	Classically	Quantum mechanically
A bit always has a definite value	True	False. A bit need not have a definite value until the moment after it is read
A bit can only be 0 or 1	True	False. A bit can be in a superposition of 0 and 1 simultaneously
A bit can be copied without affecting its value	True	False. A qubit in an unknown state cannot be copied without necessarily changing its quantum state
A bit can be read without affecting its value	True	False. Reading a qubit that is initially in a superposition will change the qubit
Reading one bit in the computer memory has no affect on any other (unread) bit in the memory	True	False. If the bit being read is entangled with another qubit, reading one qubit will affect the other
To compute the result of a computation, you must run the computer	True	False

Quantization: From Bits to Qubits

Ket Vector Representation of a Qubit: $|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

A quantum system can be found to be in one of a discrete set of states: $|0\rangle$ or $|1\rangle$

Superposition

If it is not observed it may also exist in a superposition of those states simultaneously: $|\psi\rangle = a|0\rangle + b|1\rangle$ such that $|a|^2 + |b|^2 = 1$.

Dirac Notation:

For every "ket" $|\psi\rangle$ (column vector), there is a corresponding "bra" $\langle\psi|$ (row vector):

$$|\psi\rangle = a|0\rangle + b|1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\langle\psi| = a^*\langle 0| + b^*\langle 1| = (a^* \;\; b^*)$$

The ket and the bra contain *equivalent information* about the quantum state

Inner and Outer Products

For a pair of qubits in states: $|\psi\rangle = a|0\rangle + b|1\rangle$ and $|\phi\rangle = c|0\rangle + d|1\rangle$

The *inner product* $\langle\psi|\phi\rangle$ defines the **overlap** between (normalized) states: $|\psi\rangle$, $|\phi\rangle$

$$\langle\psi|\phi\rangle = \underbrace{((\langle\psi|)\cdot(|\phi\rangle))}_{\text{bra (c) ket}} = (a^* \ \ b^*)\cdot\begin{pmatrix} c \\ d \end{pmatrix} = a^*c + b^*d$$

$$\langle\psi|\psi\rangle = (a^* \ \ b^*)\cdot\binom{a}{b} = a^*a + b^*b = |a|^2 + |b|^2 = 1$$

The *outer product* $|\psi\rangle\langle\phi|$ is a matrix:

$$|\psi\rangle\langle\phi| = (|\psi\rangle)\cdot(\langle\phi|) = \begin{pmatrix} a \\ b \end{pmatrix}\cdot(c^* \ \ d^*) = \begin{pmatrix} ac^* & ad^* \\ bc^* & bd^* \end{pmatrix}$$

The outer product describes the structure of **unitary operators**, which correspond to **quantum logic gates**. For example, a NOT gate:

$$\text{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

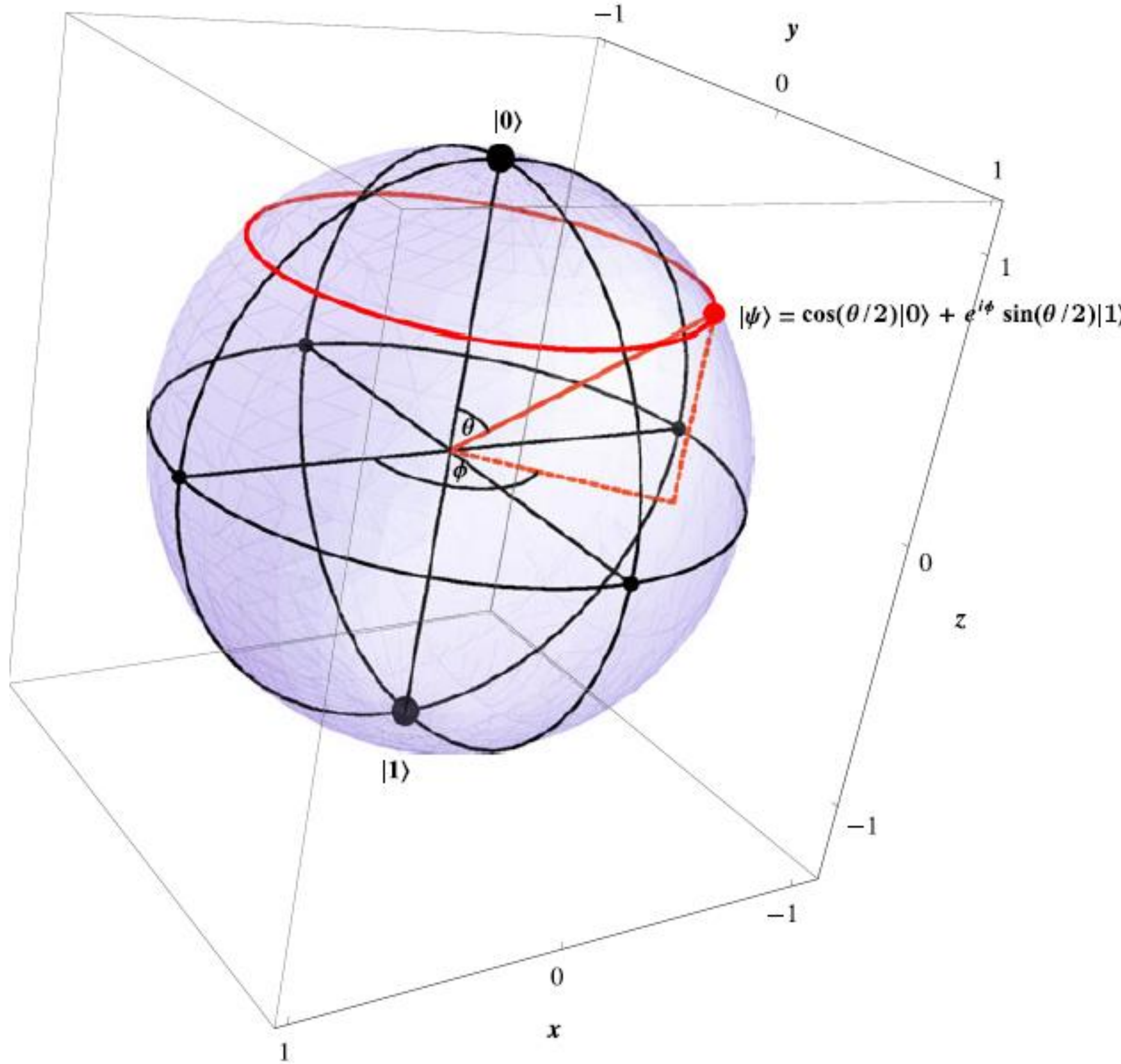
Bloch Sphere Picture of a Qubit

A **pure** quantum state of a **single qubit is a unit vector in Bloch sphere.**

A pair of elevation and azimuth angles (θ,ϕ) in the range $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ pick out a point on the Bloch sphere.

Orthogonal states, $|0\rangle$ and $|1\rangle$, are not found to be at right angles on the Bloch sphere.

Orthogonal quantum states, i.e. states $|\psi\rangle$ and $|\chi\rangle$ for which $\langle\psi|\chi\rangle$= 0, are represented by antipodal points on the Bloch sphere (rather than being drawn at right angles).



Bloch sphere showing the computational basis states $|0\rangle$ and $|1\rangle$, and a general qubit state: $|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle$, where θ, and ϕ are real numbers.

Pure 1-qubit states on Bloch Sphere

Bloch sphere labeled with pure 1-qubit states at the extremes of the x-, y-, and z-axes:

X-axis: $|\nearrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

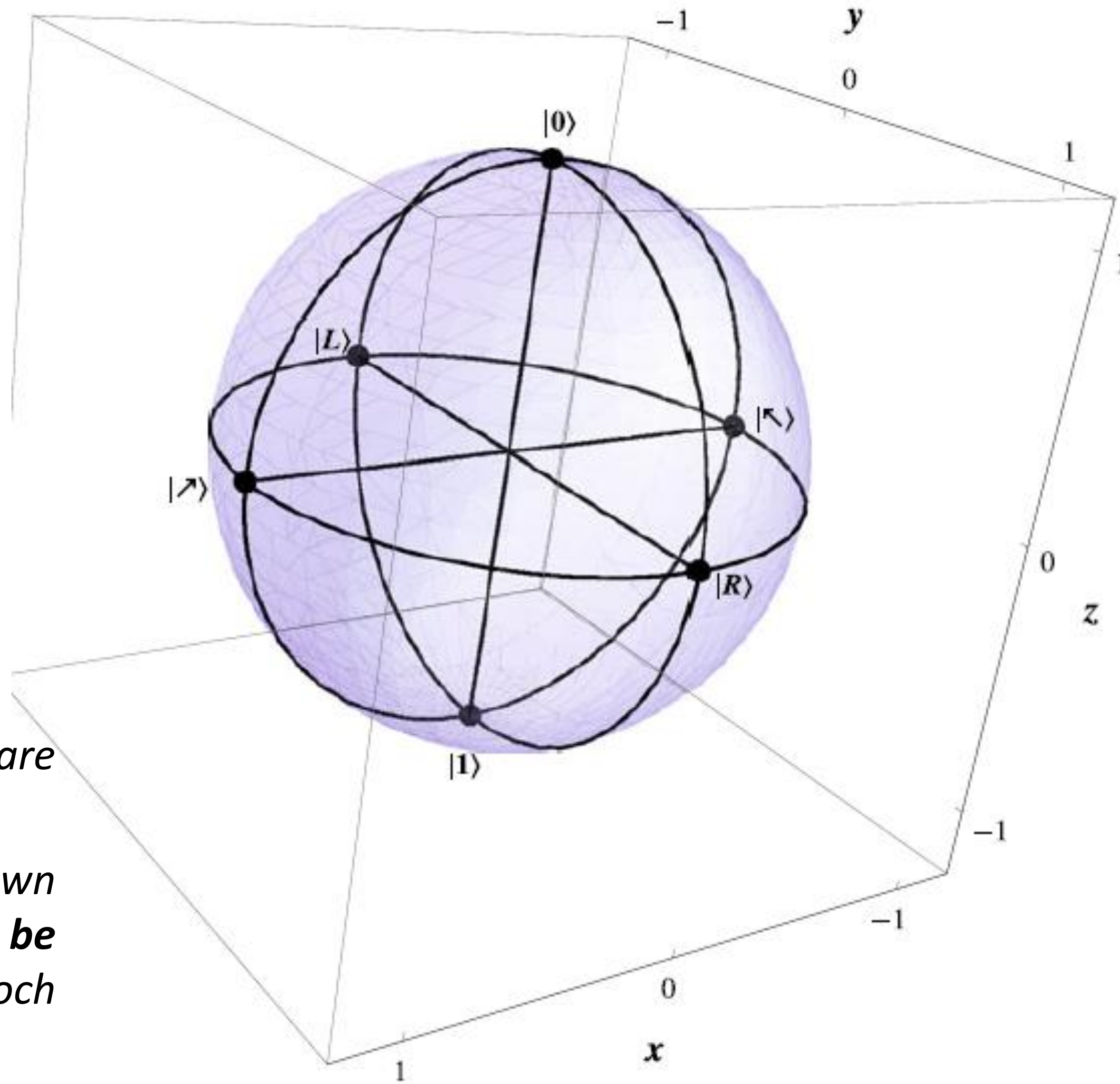
$|\nwarrow\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Y-axis:

$|R\rangle = |\circlearrowright\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

$|L\rangle = |\circlearrowleft\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

Z-axis: $|0\rangle$, and $|1\rangle$



Orthogonal quantum states are located at antipodal points.

The operation that maps an unknown state to its antipodal state ***cannot be expressed as a rotation*** *on the Bloch sphere.*

Rather it is the sum of a rotation (in longitude through 180 degrees) and a reflection (in latitude with respect to the equatorial plane of the Bloch sphere).

Reading the Bit Value of a Qubit

Measuring the bit value of a qubit initially in state: $a|0\rangle + b|1\rangle$ yields the answer:

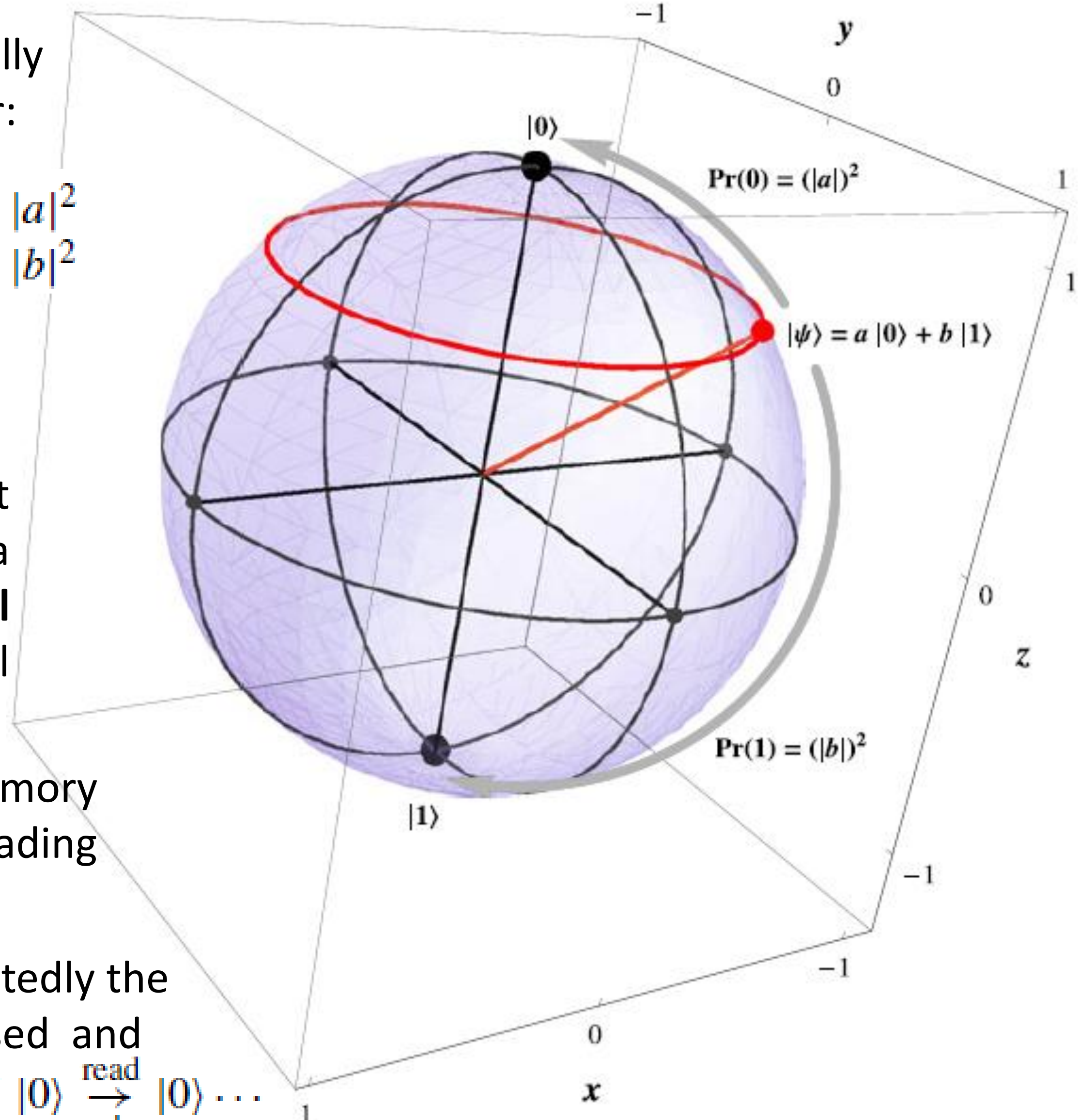
$$\text{Read}(a|0\rangle + b|1\rangle) = \begin{cases} 0 & \text{with probability } |a|^2 \\ 1 & \text{with probability } |b|^2 \end{cases}$$

and projects the qubit into either state $|0\rangle$ or state $|1\rangle$ respectively.

A measurement of a qubit with respect to North and South poles axis is called a **measurement "in the computational basis"** because the answer we get will be one of the bit values $|0\rangle$ or $|1\rangle$.

Thus, for a single qubit quantum memory register the outcome we obtain from reading it is non-deterministic.

Quantum Zeno Effect: measuring repeatedly the same state its evolution can be suppressed and fixed in a quantum state, $|\psi\rangle \overset{\text{read}}{\rightarrow} |0\rangle \overset{\text{read}}{\rightarrow} |0\rangle \overset{\text{read}}{\rightarrow} |0\rangle \cdots$ or $|\psi\rangle \overset{\text{read}}{\rightarrow} |1\rangle \overset{\text{read}}{\rightarrow} |1\rangle \overset{\text{read}}{\rightarrow} |1\rangle \cdots$



Problems with Solutions

Problem 1 Consider the three cases:

$$\text{(i)}\ |0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\text{(ii)}\ |0\rangle := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |1\rangle := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{(iii)}\ |0\rangle := \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad |1\rangle := \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}$$

Find the matrix representation of **A** in these bases. $A := |0\rangle\langle 0| + |1\rangle\langle 1|$

For all three cases:

A = I$_2$

where **I$_2$** is the 2 x 2 unit matrix.

The third case contains the first two as special cases.

Solution 1. We find:

$$\text{(i)} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{(ii)} \quad A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{(iii)} \quad A = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} + \begin{pmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problems with Solutions

Problem 2 The NOT operation (unitary operator) is defined as: $|0\rangle \to |1\rangle, \qquad |1\rangle \to |0\rangle$

(i) Find the unitary operator U_{NOT} which implements the NOT operation with respect to the basis $\{ |0\rangle, |1\rangle \}$.

(ii) Let

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Find the matrix representation of U_{NOT} for this basis.

(iii) Let

$$|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Find the matrix representation of U_{NOT} for this basis.

Solution 2.

(i) Obviously,

$$U_{NOT} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

since $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$.

(ii) For the standard basis we find

$$U_{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(iii) For the Hadamard basis we find

$$U_{NOT} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Thus, the respective matrix representations for the two bases are different.

Problems with Solutions

Problem 3 The qubit *trine* is defined by the following states:

$$|\psi_0\rangle = |0\rangle, \quad |\psi_1\rangle = -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle, \quad |\psi_2\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

where $\{\,|0\rangle,\ |1\rangle\,\}$ is an orthonormal basis set. Find

$$|\langle\psi_0|\psi_1\rangle|^2, \quad |\langle\psi_1|\psi_2\rangle|^2, \quad |\langle\psi_2|\psi_0\rangle|^2\,.$$

Solution 3.

Using $\langle 0|0\rangle = 1$, $\langle 1|1\rangle = 1$ and $\langle 0|1\rangle = 0$ we find

$$|\langle\psi_0|\psi_1\rangle|^2 = \frac{1}{4}, \quad |\langle\psi_1|\psi_2\rangle|^2 = \frac{1}{4}, \quad |\langle\psi_2|\psi_0\rangle|^2 = \frac{1}{4}\,.$$