ΝΑΝΟΗΛΕΚΤΡΟΝΙΚΗ & ΚΒΑΝΤΙΚΕΣ ΠΥΛΕΣ

9^{η} Διάλεξη

Βιβλιογραφία: EXPLORATIONS IN QUANTUM COMPUTING, Colin P. Williams (2nd edition, Springer-Verlag, 2011), chapter 2.

Universal Quantum Gates

- DEUTSCH gate
- BARENCO gate
- Universal set of gates
- Special 2-Qubit Gates
- Interrelationships Between Types of 2-Qubit Gates

Entangled States from Quantum Gates

- Pure vs Mixed entangled states
- Entropy of entanglement
- Concurrence
- Tangle within a pure 2-qubit state

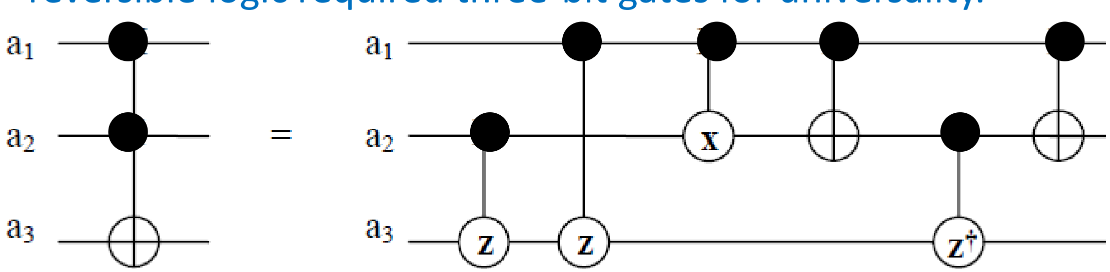
Universal Quantum Gates

The **Deutsch** gate is universal for quantum logic in the sense that any unitary transform on an arbitrary number of qubits can be simulated by repeated applications of D(θ) on three qubits at a time.

$$
\text{DEUTSCH} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & i\cos(\theta) & \sin(\theta) \\
0 & 0 & 0 & 0 & 0 & 0 & \sin(\theta) & i\cos(\theta)
\end{pmatrix}
\begin{matrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{matrix}
$$

θ is any constant angle such that 2θ/π is an irrational number.

- ✓ Circuits for an arbitrary $2^n \times 2^n$ unitary matrix built from this gate are very inefficient in gate count.
- ✓ Deutsch gate is not elementary and can be decomposed into two-qubit gates, whereas classical reversible logic required three-bit gates for universality.



phase rotation

$$
x = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{matrix} |0\rangle \\ |1\rangle \end{matrix}
$$

transforms inputs to superpositions

$$
z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{matrix} |0\rangle \\ |1\rangle \end{matrix}
$$

A special case of the Deutsch gate for θ=π/2 is the 3-qubit Toffoli gate, that is simulated using 2-qubit gates.

the BARENCO universal gate

The set of BARENCO gates is universal for quantum logic in the sense that any unitary transform, on any number of qubits, can be simulated by these gates acting on only two qubits at a time.

$$\text{BARENCO} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\alpha}\cos(\theta) & -ie^{i(\alpha-\phi)}\sin(\theta) \\ 0 & 0 & -ie^{i(\alpha+\phi)}\sin(\theta) & e^{i\alpha}\cos(\theta) \end{pmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

ϕ, α and θ are fixed irrational multiples of π, and of each other.

- ✓ In practice, it is quite hard to use the Barenco gate as a primitive gate as it requires a 2-qubit Hamiltonian having three "tunable" parameters, ϕ, α and θ.
- ✓ Since the BARENCO gate is a controlled-U gate can be further decomposed into a sequence of 1-qubit gates and a single (fixed) 2-qubit gate such as CNOT.

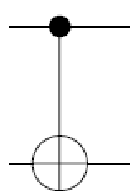
Thus, the set $\{R_X(\alpha), R_Y(\beta), R_Z(\gamma), Ph(\delta), CNOT\}$ forms a universal set of gates used in constructing practical quantum circuits:

$$R_X(\theta) \equiv \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$R_y(\theta) \equiv \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$R_Z(\theta) \equiv \begin{pmatrix} e^{-\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{pmatrix}$$

$$Ph(\theta) \equiv \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$



$$\equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

All 2-dimensional unitary matrices are universal?

Even though BARENCO is a 2-qubit gate we see that the unitary matrix in it is effectively two-dimensional, not four-dimensional, because of the identity matrix acting on the first two states, $|00\rangle$ and $|01\rangle$.

- ✓ Since any unitary transform can be built from this gate, this may imply that 2-dimensional unitary matrices are universal.
- ✓ However, it does not follow from this that one-qubit gates are universal, as indeed they are not.
- ✓ There is a distinction between the number of qubits and the number of dimensions affected by a gate transformation.

For example, a CNOT operation acts on the second qubit of a 2-qubit system with a four-dimensional unitary matrix, in the sense that all four states are affected by the transformation, even though the operation represented is a single-qubit (2-dimensional) NOT .

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix}$$

However, NOT gate is not universal for quantum computing!

In general, a one-qubit operation need not be two-dimensional, and similarly, a two-qubit operation need not be four-dimensional.

Sets of universal gates with fixed-angle

Families of gates that are universal for quantum computing but do not lead to efficient quantum circuits due to the need to repeat fixed angle rotations many times to approximate a desired 1-qubit gate to adequate precision.

Universal gate family	Meaning	Noteworthy properties
$\{R_x, R_y, R_z, Ph, \text{CNOT}\}$	The union of the set of 1-qubit gates and CNOT is universal	The most widely used set of gates in current quantum circuits
BARENCO(ϕ, α, θ).	A single type of 2-qubit gate is universal	The surprise here is that whereas in classical reversible computing no 2-bit classical reversible gate is universal, in quantum computing almost *all* 2-qubit gates are universal
$\{H, S, T, \text{CNOT}\}$ where $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the Walsh-Hadamard gate, $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ is the "phase gate", and $T = \begin{pmatrix} 1 & 0 \\ 0 & \exp(i\pi/4) \end{pmatrix}$ is the "$\pi/8$ gate"	Three fixed-angle 1-qubit gates together with CNOT	The surprise here is that fixed angle gates can form a universal set. In fact, the Solvay-Kitaev theorem implies that any 1-qubit gate can be approximated to accuracy ϵ using $\mathcal{O}(\log^c 1/\epsilon)$ gates from the set $\{H, S, T, \text{CNOT}\}$ where c is a positive constant

Special 2-Qubit Gates

The physical interactions available within different types of quantum computer hardware can give rise to different 2-qubit gates that are more "natural", than CNOT, since they are associated with different Hamiltonians. The four most common alternatives to CNOT are:

$$\text{CSIGN} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

CSIGN arises naturally in Linear Optical quantum computing.

$$\text{SWAP}^\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(1+e^{i\pi\alpha}\right) & \frac{1}{2}\left(1-e^{i\pi\alpha}\right) & 0 \\ 0 & \frac{1}{2}\left(1-e^{i\pi\alpha}\right) & \frac{1}{2}\left(1+e^{i\pi\alpha}\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

SWAP$^\alpha$ arises naturally in spintronic quantum computing. The duration of the exchange operation determines the exponent achieved in SWAP$^\alpha$.

$$\text{iSWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

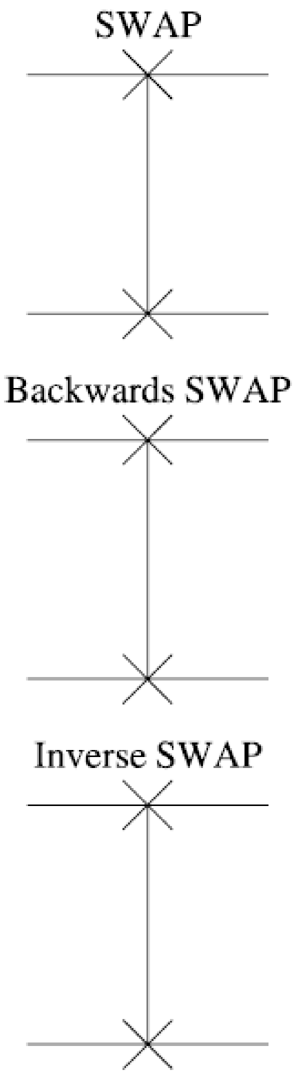
iSWAP arises naturally in superconducting quantum computing via Hamiltonians implementing the XY model.

The Berkeley **B** Gate: Hamiltonian is $\mathcal{H} = \frac{\pi}{8}(2X \otimes X + Y \otimes Y)$. Gate is $U = \exp(i\mathcal{H})$. $\mathrm{B} = e^{i\frac{\pi}{8}(2X\otimes X + Y \otimes Y)}$

$$= \begin{pmatrix} \cos\left(\frac{\pi}{8}\right) & 0 & 0 & i\sin\left(\frac{\pi}{8}\right) \\ 0 & \cos\left(\frac{3\pi}{8}\right) & i\sin\left(\frac{3\pi}{8}\right) & 0 \\ 0 & i\sin\left(\frac{3\pi}{8}\right) & \cos\left(\frac{3\pi}{8}\right) & 0 \\ i\sin\left(\frac{\pi}{8}\right) & 0 & 0 & \cos\left(\frac{\pi}{8}\right) \end{pmatrix} = \frac{\sqrt{2-\sqrt{2}}}{2} \begin{pmatrix} 1+\sqrt{2} & 0 & 0 & i \\ 0 & 1 & i(1+\sqrt{2}) & 0 \\ 0 & i(1+\sqrt{2}) & 1 & 0 \\ i & 0 & 0 & 1+\sqrt{2} \end{pmatrix}$$

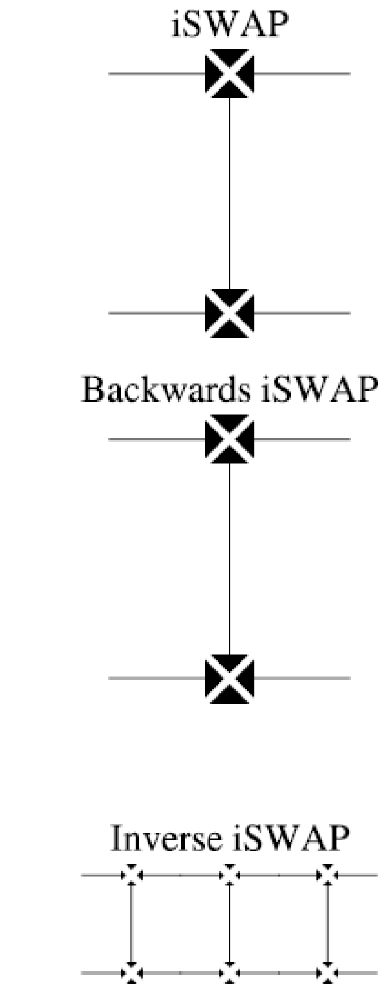
Special quantum gates: SWAP, iSWAP, and SWAP$^\alpha$

Unlike CNOT, these gates do not have a preferred “control” qubit and can be inserted “right way up” or “upside down” without affecting the operation the gate performs.



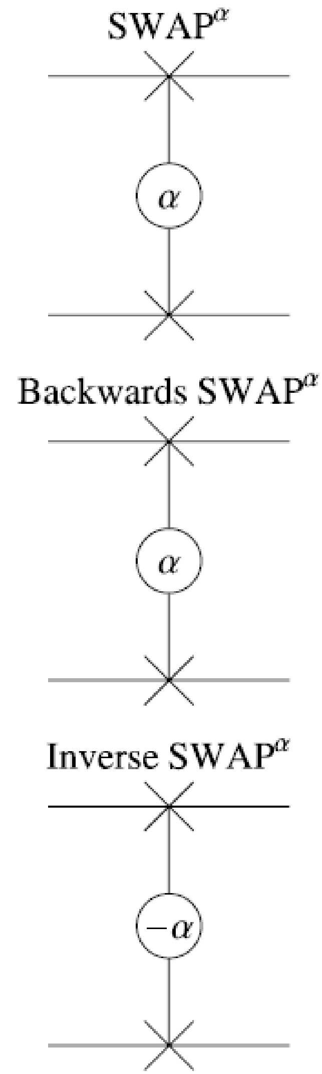
SWAP is inverse

$$\mathrm{SWAP} = \mathrm{SWAP}^{-1}$$



iSWAP is not inverse

$$\mathrm{iSWAP}^\dagger = \mathrm{iSWAP}^{-1} = \mathrm{iSWAP}^3$$



SWAP$^\alpha$ is not inverse

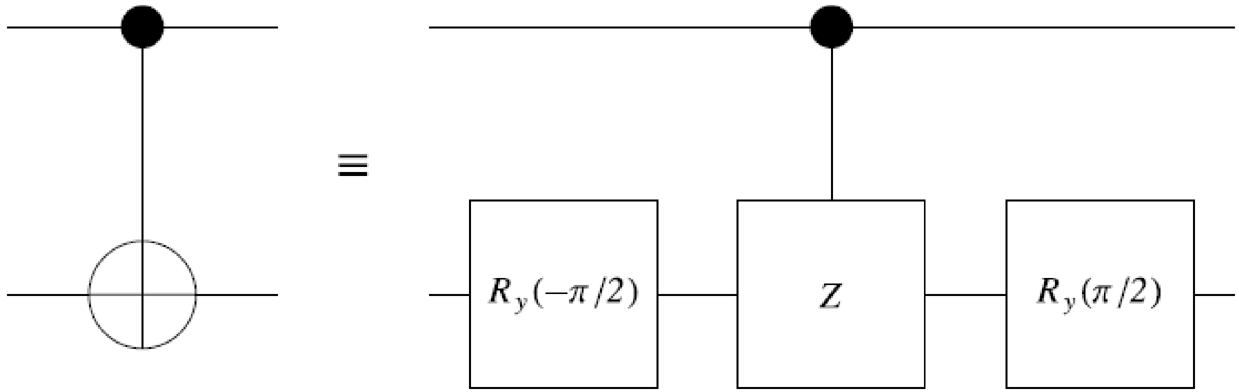
$$(\mathrm{SWAP}^\alpha)^\dagger = (\mathrm{SWAP}^\alpha)^{-1} = \mathrm{SWAP}^{-\alpha})$$

Interrelationships Between Types of 2-Qubit Gates

The Hamiltonians in different types of physical systems arise from physical interactions Nature provides. From these Hamiltonians one can always find 2-qubit gates from which we can, in conjunction with 1-qubit gates, build CNOT gates.

CNOT from CSIGN

Quantum circuit for obtaining a CNOT gate given the ability to achieve 1-qubit gates and CSIGN

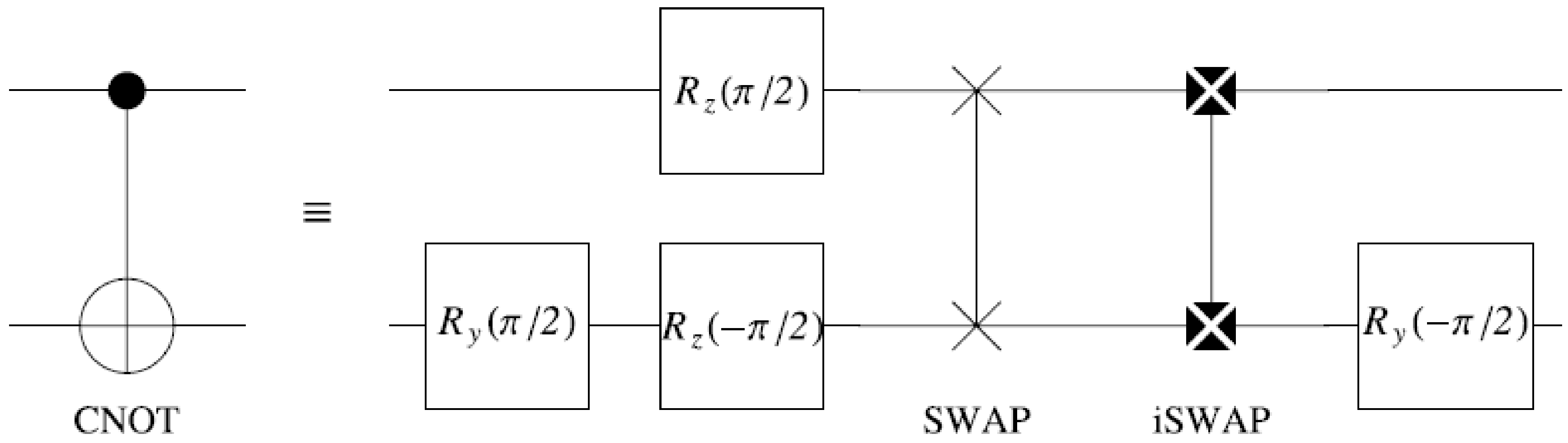


$$\underset{\text{CNOT}}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} \equiv \left(\mathbb{1}_2 \otimes R_y\left(\frac{\pi}{2}\right)\right) \cdot \underset{\text{CSIGN}}{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}} \cdot \left(\mathbb{1}_2 \otimes R_y\left(-\frac{\pi}{2}\right)\right)$$

Note that the order is reversed with respect to the left to right sequence in the circuit diagram

CNOT from iSWAP and one SWAP

Quantum circuit for obtaining a CNOT gate given the ability to achieve 1-qubit gates, iSWAP, and SWAP



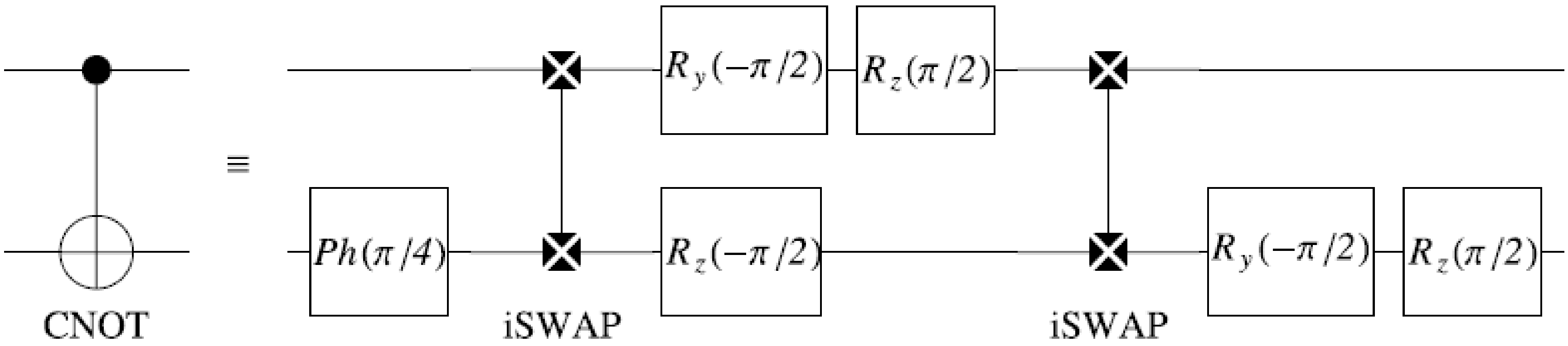
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \equiv \left(\mathbb{1}_2 \otimes R_y\left(-\frac{\pi}{2}\right)\right) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\cdot \left(R_z\left(\frac{\pi}{2}\right) \otimes \left(R_z\left(-\frac{\pi}{2}\right) \cdot R_y\left(\frac{\pi}{2}\right)\right)\right)$$

Note that the order is reversed with respect to the left to right sequence in the circuit diagram

CNOT from Two iSWAPs

Quantum circuit for obtaining a CNOT gate given the ability to achieve 1-qubit gates and iSWAP



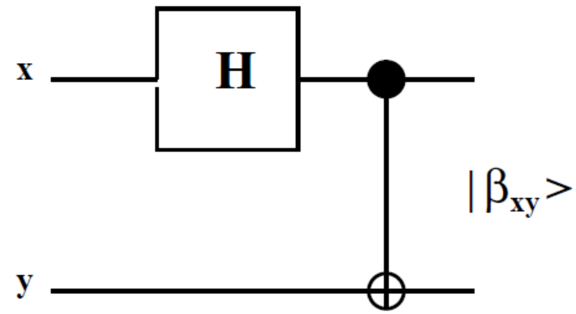
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \equiv \left(\mathbb{1}_2 \otimes \left(R_z\left(\frac{\pi}{2}\right) \cdot R_y\left(-\frac{\pi}{2}\right)\right)\right) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\cdot \left(\left(R_z\left(\frac{\pi}{2}\right) \cdot R_y\left(-\frac{\pi}{2}\right)\right) \otimes R_z\left(-\frac{\pi}{2}\right)\right) \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \left(\mathbb{1}_2 \otimes Ph\left(\frac{\pi}{4}\right)\right)$$

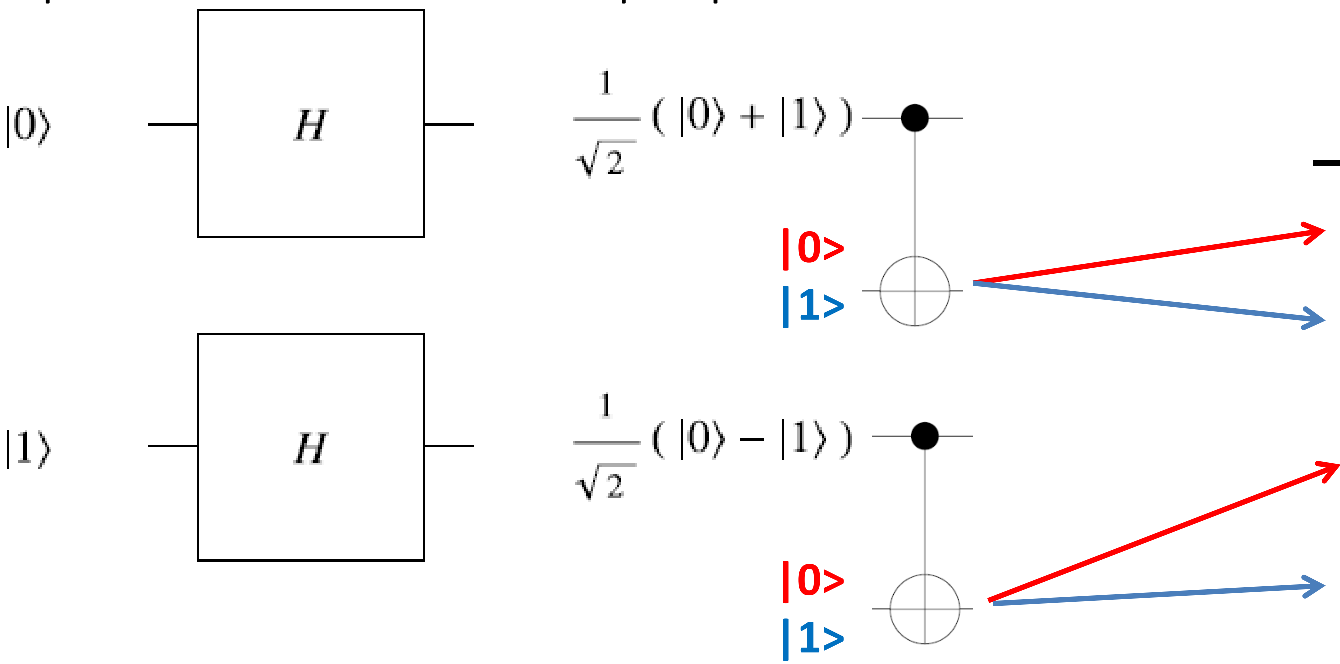
Note that the order is reversed with respect to the left to right sequence in the circuit diagram

Entangled States from Quantum Gates

A circuit that applies a Hadamard gate on the control input-bit of CNOT creates the four basic entangled states (depicted up to the normalization factor) which are called the Bell or EPR states.



Input Bits → Input qubits → Entangled input qubits



Four Bell states

in	out
$\|00>$	$(\|00> + \|11>)$
$\|01>$	$(\|01> + \|10>)$
$\|10>$	$(\|00> - \|11>)$
$\|11>$	$(\|01> - \|10>)$

A quantum network is a device consisting of a number of quantum logic gates whose computational steps are synchronized in time, and a quantum computer is a family of quantum networks. A quantum computation is a unitary evolution of the network which process entangled input qubits and send them into some output qubits.

Pure vs Mixed entangled states

Definition: A state of a bipartite system (composed of two distinct subsystems) is called entangled if it cannot be written as a direct product of two states from the two subsystem Hilbert spaces $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

Pure States: If there are no local states $|\psi_1\rangle \in \mathcal{H}_1$ and $|\psi_2\rangle \in \mathcal{H}_2$, such that the state of the system $|\Psi\rangle$ can be written as a product thereof:

$\nexists\ |\psi_1\rangle \in \mathcal{H}_1, |\psi_2\rangle \in \mathcal{H}_2$ such that $|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, then $|\Psi\rangle$ is an *entangled state.*

Reduced density matrix: For each state $|\Psi\rangle$, there are local bases $|\tilde{\varphi}_i^S\rangle$ and $|\tilde{\varphi}_i^S\rangle$ where:

$$|\Psi\rangle = \sum_i \sqrt{\lambda_i}\, |\varphi_i^S\rangle \otimes |\phi_i^S\rangle \qquad \lambda_i = \mathcal{S}_i^2 \text{ are known as } \textit{Schmidt coefficients},$$

The reduced density matrix of the first subsystem is: $\varrho_1 = \sum_i \lambda_i |\varphi_i^S\rangle\langle\varphi_i^S|$, and can define:

$$\textit{separable mixed states}\ \varrho = \sum_i p_i\, \rho_i^{(1)} \otimes \rho_i^{(2)} \text{ with } p_i > 0 \text{ and } \sum_i p_i = 1$$

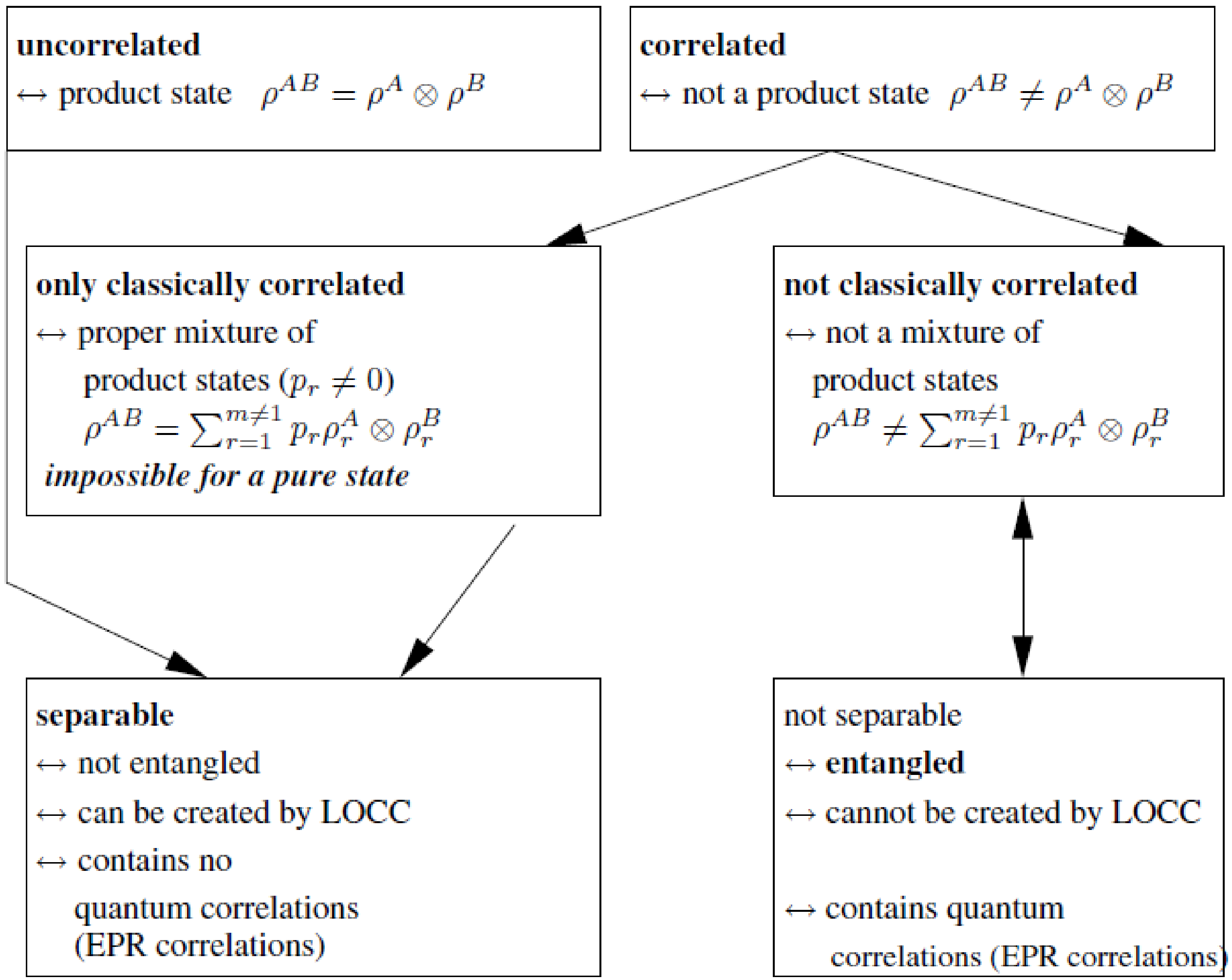
that yield correlated measurement results. These correlations can be described in terms of the classical probabilities p_i, and are therefore considered classical.

A mixed state ϱ is entangled if there are no local states $\rho_i^{(1)}$, $\rho_i^{(2)}$, and non-negative weights p_i, such that ϱ can be expressed as a convex mixture thereof:

$$\nexists\ \varrho_i^{(1)}, \varrho_i^{(2)}, p_i \geq 0 \quad \text{such that} \quad \varrho = \sum_i p_i\, \rho_i^{(1)} \otimes \rho_i^{(2)}$$

Distinction of cases for the characterization of the states of bipartite systems

A pure overall state is either not correlated or it is entangled.



LOCC=“*local operations and classical communication*”. The composite system S^{AB} is prepared from product states by means of *local operations* (*LO*) on the subsystems S^A and S^B, that exchange information via *classical communication* (*CC*) to coordinate their local operations.

Entropy of entanglement

General requirements for measures *C* of entanglement:

- $C = 0$ for product states $\rho = \rho_A \otimes \rho_B$.
- C is invariant under local unitary transformations. The measure should not depend on the choice of basis.

Pure States: A measure that fulfills these requirements is the entropy of entanglement. It uses the von Neumann entropy of a density operator: $S(\rho) = -Tr\{\rho \log_2(\rho)\}$

Separability criterion for pure states:

$$\begin{cases} \mathrm{tr}\varrho_r^2 = 1 \Rightarrow \varrho_r \text{ is pure } \Rightarrow |\Psi\rangle \text{ is separable} \\ \mathrm{tr}\varrho_r^2 < 1 \Rightarrow \varrho_r \text{ is mixed} \Rightarrow |\Psi\rangle \text{ is entangled} \end{cases}$$

with r referring to either one of the two subsystems.

The entropy of entanglement for a *single* partite state $\rho_1 = |\Psi\rangle\langle\Psi|$:

- $S(\rho) = 0$, for a pure state the information gain from such a measurement vanishes.
- $S(\rho) = \log_2 N$, is maximum for a completely mixed state. N is the dimension of the Hilbert space.

Von Neumann entropy gives the minimum of bits required to store the result of a random variable

The entropy of entanglement for bipartite pure states is the von Neumann entropy of one of the reduced states: $E(\rho) = S(\rho_A) = S(\rho_B)$, where $\rho_A = Tr_B(\rho)$ and vice versa.

- If ρ is a product state $|\uparrow\uparrow\rangle$, ρ_A and ρ_B are pure states and the entropy vanishes: $E(\rho)=0$.
- If the state is maximally entangled: $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$, the subsystems become completely mixed: $\rho_A=\rho_B=\frac{1}{2}\mathbf{1}$, and the maximally entanglement entropy is $E(\rho) = S(\rho_A) = S(\rho_B) = 1$.

Concurrence

- ✓ There is some variability in the potential for 2-qubit gates to generate entanglement.
- ✓ Two-qubit gates built as the direct product of two 1-qubit gates cannot generate any entanglement.
- ✓ Other gates, such as CNOT, map unentangled inputs into maximally entangled outputs.

Entanglement measure is a way to quantify the degree of entanglement within a state, using an ensemble of *input states* over which to average this entanglement measure.

The "**entangling power**" of a quantum gate characterizes the degree, on average, of entanglement of its outputs over received unentangled states as inputs. The more the output is entangled, the greater the entangling power of the gate.

Tangle provides a quantitative measure of the degree of entanglement within a quantum state and is the square of the concurrence, which for a 2-qubit pure state, $|\psi\rangle$, is:

$$\text{Concurrence}(|\psi\rangle) = |\langle\psi|\widetilde{\psi}\rangle|$$

where $|\widetilde{\psi}\rangle$ is the spin-flipped version of $|\psi\rangle : |\widetilde{\psi}\rangle = (Y \otimes Y)|\psi^*\rangle, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the Pauli-Y matrix, and $|\psi^*\rangle$ is $|\psi\rangle$ with its amplitudes complex conjugated.

Thus, if $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ then $|\psi^*\rangle = a^*|00\rangle + b^*|01\rangle + c^*|10\rangle + d^*|11\rangle$ and $|\widetilde{\psi}\rangle = -d^*|00\rangle + c^*|01\rangle + b^*|10\rangle - a^*|11\rangle$.

Hence, the concurrence of a general 2-qubit state $|\psi\rangle$ is given by:

$$C := \text{Concurrence}(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle) = |2b^*c^* - 2a^*d^*| \geq 0.$$

Tangle within a pure 2-qubit state

Tangle is a quantitative measure for the degree of entanglement within a pure 2-qubit state.

- ✓ Concurrence defines a **"spin-flip"** transformation, which is a mathematical specification of a transformation that maps the state of each component qubit into its orthogonal state.
- ✓ There is no perfect spin-flip "gate" because the **spin-flip transformation is not unitary** (If there were it would be a universal NOT gate).
- ✓ If a 2-qubit state $|\psi\rangle$ is a direct product state (an unentangled state) its spin-flipped version, $|\tilde{\psi}\rangle$, will be orthogonal to $|\psi\rangle$ and the overlap $\langle\psi|\tilde{\psi}\rangle$ will be zero (zero concurrence).
- ✓ A spin-flip transformation over maximally entangled states, such as Bell states, is invariant up to an unimportant overall phase:

Bell states	**Spin-flip transformation**
$\lvert\beta_{00}\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$	$\lvert\beta_{00}\rangle \xrightarrow{\text{spin-flip}} -\lvert\beta_{00}\rangle$
$\lvert\beta_{01}\rangle = \frac{1}{\sqrt{2}}(\lvert 01\rangle + \lvert 10\rangle)$	$\lvert\beta_{01}\rangle \xrightarrow{\text{spin-flip}} \lvert\beta_{01}\rangle$
$\lvert\beta_{10}\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$	$\lvert\beta_{10}\rangle \xrightarrow{\text{spin-flip}} \lvert\beta_{10}\rangle$
$\lvert\beta_{11}\rangle = \frac{1}{\sqrt{2}}(\lvert 01\rangle - \lvert 10\rangle)$	$\lvert\beta_{11}\rangle \xrightarrow{\text{spin-flip}} -\lvert\beta_{11}\rangle$

The overlap between a maximally entangled state and its spin-flipped counterpart is unity (concurrence is one).

For density matrices, the concurrence is defined as: $C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$, where λ_i are the eigenvalues, in decreasing order, of the Hermitian operator: $R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$, with σ_y=Y in the spin-flipped density matrix: $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^* (\sigma_y \otimes \sigma_y)$.

Examples

Problem 1: A general pure state in $\mathcal{H}$ can be given by a superposition of pure states

$$|\Psi_\mathrm{e}\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle \otimes |\psi_2\rangle + |\phi_1\rangle \otimes |\phi_2\rangle\right)$$

where $|\psi_i\rangle \neq |\phi_i\rangle$ $(i = 1, 2)$. We may now ask what the state $|\Psi_\mathrm{e}\rangle$ looks like if one has access to only one of the subsystems?

For a local operator $a \otimes \mathbb{1}$ on the first subsystem, the expectation value observed in an experiment

$$\langle a\rangle = \langle\Psi_\mathrm{e}|a \otimes \mathbb{1}|\Psi_\mathrm{e}\rangle = \mathrm{tr}(a \otimes \mathbb{1}\ |\Psi_\mathrm{e}\rangle\langle\Psi_\mathrm{e}|) = \mathrm{tr}_1(a\,\mathrm{tr}_2|\Psi_\mathrm{e}\rangle\langle\Psi_\mathrm{e}|) = \mathrm{tr}_1(a\varrho_1)\ ,$$

where $\mathrm{tr}_{1,2}$ denotes the partial trace over the first/second subsystem, and $\varrho_1 = \mathrm{tr}_2|\Psi_\mathrm{e}\rangle\langle\Psi_\mathrm{e}|$ is the reduced density matrix of the first subsystem, and the state of the first subsystem alone is given by ϱ_1.

Then, the state of the second subsystem is described by its reduced density matrix $\varrho_2 = \mathrm{tr}_1|\Psi_\mathrm{e}\rangle\langle\Psi_\mathrm{e}|$.
The state of the composite system is not equal to the product of both subsystem states,

$$\rho = |\Psi_\mathrm{e}\rangle\langle\Psi_\mathrm{e}| \neq \rho_1 \otimes \rho_2$$

Moreover, if one performs a local measurement on one subsystem, this leads to a state reduction of the entire system state, not only of the subsystem on which the measurement had been performed. Therefore, the probabilities for an outcome of a measurement on one subsystem are influenced by prior measurements on the other subsystem. Thus, measurement results on – possibly distant and non-interacting – subsystems are correlated.

Examples

Problem 2: If we consider a typical product state, $\Psi_1 = |\uparrow\uparrow\rangle = (1,0,0,0)$ and the maximally entangled state $\Psi_2 = \frac{1}{2}(|\uparrow\rangle + |\downarrow\rangle)\otimes(|\uparrow\rangle + |\downarrow\rangle) = \frac{1}{2}(1,1,1,1)$, find the concurrence $C(\Psi_1)$ and $C(\Psi_2)$

The concurrence for pure 2-qubit states $|\Psi\rangle = \alpha|\uparrow\uparrow\rangle + \beta|\uparrow\downarrow\rangle + \gamma|\downarrow\uparrow\rangle + \delta|\downarrow\downarrow\rangle$ is

$$C := 2|\alpha\delta - \beta\gamma| \geq 0$$

we find $C(\Psi_1) = 0$, the state is not entangled.

we again find $C(\Psi_2) = 0$ the state is not entangled.

Problem 3: We now consider the effect of an "entangling gate": $\mathrm{CN} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\frac{\varphi}{2} & -\sin\frac{\varphi}{2} \\ & & \sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}$ which is close to the CNOT gate if φ = π, but is experimentally realizable.

Find: i) $\Psi_3 = \mathrm{CN}\cdot\Psi_2$ ii) $C(\Psi_3)$ iii) the entanglement entropy for this state, iv) plot the entanglement entropy *S* and concurrence *C* of ρ_3 as a function of the rotation angle φ.

i) $\Psi_3 = \mathrm{CN}\cdot\Psi_2 = \frac{1}{2}\left(1, 1, \cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}, \cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right).$

This corresponds to the *'pre-measurement'* in theory of quantum measurement process, which entangles the system with the apparatus.

ii) For this state, the concurrence is $C(\Psi_3) = \sin\frac{\varphi}{2}.$ This state is entangled for any finite angle φ.

The entanglement reaches its maximum of ½ for φ = π, where CN ≈ CNOT, apart from the − sign, and returns to 0 for φ = 2π.

Examples

Problem 3:

iii) To calculate the entanglement entropy for this state we need the full density operator:

$$\rho_3 = \frac{1}{4}\begin{pmatrix} 1 & 1 & c_- & c_+ \\ 1 & 1 & c_- & c_+ \\ c_- & c_- & 1-\sin\varphi & \cos\frac{\varphi}{4} \\ c_+ & c_+ & \cos\frac{\varphi}{4} & 1+\sin\varphi \end{pmatrix}, \quad \text{where} \quad c_\pm = \cos\frac{\varphi}{2} \pm \sin\frac{\varphi}{2}.$$

For the subsystems, this yields

$$\rho_A = Tr_B(\rho) = \frac{1}{2}\begin{pmatrix} 1 & \cos\frac{\varphi}{2} \\ \cos\frac{\varphi}{2} & 1 \end{pmatrix} \quad \text{and} \quad \rho_B = Tr_A(\rho) = \frac{1}{2}\begin{pmatrix} 1-\frac{1}{2}\sin\varphi & \cos^2\frac{\varphi}{2} \\ \cos^2\frac{\varphi}{2} & 1+\frac{1}{2}\sin\varphi \end{pmatrix},$$

where we used the trigonometric identity: $1+\cos(\varphi/4) = \cos^2(\varphi/2)$. The difference between ρ_A and ρ_B reflects the asymmetric role that control and target bit play in the CNOT gate. $S(\rho) = -Tr\{\rho \log_2(\rho)\}$

iv)

- ✓ This figure shows the resulting entanglement entropy $E(\rho_3)=S(\rho_A)=S(\rho_B)$ as a function of the rotation angle φ.
- ✓ Clearly, the dependence is different from that of the concurrence $C(\Psi_3)$ for the same state, which starts linearly with φ and reaches a maximum value of 0.5.
- ✓ However, both entanglement measures reach maxima for the same state and vanish when the state is separable.