Quick Study® ACADEMIC

LINEAR ALGEBRA

matrices.lines.determinates.vectors

MATRICES

MATRICES - SOME DEFINITIONS

- **Matrix:** A rectangular array of numbers (named with capital letters) called **entries** with the **size** of the matrix described by the number of rows (horizontals) and columns (verticals); for example, a 3 by 4 matrix (3 X 4) has 3 rows and 4 columns;

$$\begin{bmatrix} 3 & -2 & 0 & 6 \\ -1 & 9 & 5 & 0 \\ 0 & -3 & 6 & 3 \end{bmatrix} \text{ is a 3 by 4 matrix}$$

- **Square matrix:** Has the same number of rows and columns
- **Diagonal matrix:** A square matrix with all entries equal to zero, except the entries on the **main diagonal** (diagonal from upper left to lower right); for example, this is a **main diagonal**

$$\begin{bmatrix} 2 & -1 & 7 \\ 0 & 4 & -5 \\ 1 & -2 & -5 \end{bmatrix} \text{ and this is a diagonal matrix } \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **Identity matrix** (denoted by I): A square matrix with entries that are all zeros except entries on the main diagonal, which must all equal the number one
  - **Triangular matrix:** A square matrix with all entries below the main diagonal equal to zero (upper triangular), or with all entries above the main diagonal equal to zero (lower triangular)
- **Equal matrices:** Are the same size and have equal entries
- **Zero matrix:** Every entry is the number zero
- **Scalar:** A magnitude or a multiple
- **Row equivalent matrices:** Can be produced through a sequence of **row operations**, such as:
  - Row interchange: Interchanging any 2 rows
  - Row scaling: Multiplying a row by any nonzero number
  - Row addition: Replacing a row with the sum of itself and any other row or multiple of that other row
- **Column equivalent matrices:** Can be produced through a sequence of **column operations**, such as:
  - Column interchange: interchanging any 2 columns
  - Column scaling: multiplying a column by any nonzero number
  - Column addition: replacing a column with the sum of itself and any other column or multiple of that other column.
- **Elementary matrices:** Square matrices that can be obtained from an identity matrix, I, of the same dimensions through a single row operation
  - The **rank** of matrix **A**, denoted rank(**A**), is the common dimension of the row space and column space of matrix **A**
- The **nullity** of matrix **A**, denoted nullity(**A**), is the dimension of the nullspace of **A**

MATRIX OPERATIONS

- **Addition:** If matrices **A** and **B** are the same size, calculate **A + B** by adding the entries that are in the same positions in both matrices
- **Subtraction:** If matrices **A** and **B** are the same size, calculate **A – B** by subtracting the entries in **B** from the entries in **A** that are in the same positions
- **Multiplying by a scalar:** The product of $k\mathbf{A}$, where k is a scalar, is obtained by multiplying every entry in matrix **A** by k
- **Multiplying matrices:** If the number of columns in **A** equals the number of rows in **B**, calculate the **product AB** by multiplying the entries in row i of **A** by the entries in column j of **B**, adding these products and placing the resulting sum in the ij position of the final matrix; the final resulting product matrix will have the same number of rows as matrix **A** and the same number of columns as matrix **B**
- **Multiplicative inverses:** If **A** and **B** are square matrices and $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ (remember **I** is the identity matrix) then **A** and **B** are inverses; the inverse of a matrix **A** may be denoted as $\mathbf{A}^{-1}$; therefore, $\mathbf{B} = \mathbf{A}^{-1}$ and $\mathbf{A} = \mathbf{B}^{-1}$; to find the inverse of an invertible matrix **A**:
  - First, use a sequence of row operations to change **A** to **I**, the identity matrix; then,
  - Use these exact same row operations on **I**; this will result in the inverse matrix $\mathbf{A}^{-1}$ of matrix **A**
- The **transpose** of **A** with dimensions of $m \times n$ is the matrix $\mathbf{A}^t$ of dimensions $n \times m$ whose columns are the rows of **A** in the same order; that is, row one becomes column one, row two becomes column two, etc.
- **Orthogonal matrix:** A square, invertible matrix such that $\mathbf{A}^t = \mathbf{A}^{-1}$; that is, $\mathbf{A}^t\mathbf{A} = \mathbf{A}\mathbf{A}^t = \mathbf{I}$
- **Normal matrix:** A square matrix that satisfies $\mathbf{A}^t\mathbf{A} = \mathbf{A}\mathbf{A}^t$; that is, commutes with its transpose
- The **trace** of a square matrix **A** is the sum of all of the entries on the main diagonal, and is denoted as **tr(A)**

COMPLEX MATRICES

- Entries are all **complex numbers, a + b*i***
- The **conjugate** of a complex matrix: Denoted as $\overline{A}$, entries are all conjugates of the complex matrix **A**; remember that the conjugate of **a + b*i*** is **a – b*i*** and conversely
- **Conjugate transpose:** $\mathbf{A}^{\mathbf{H}} = (\overline{A})t = (\overline{A^t})$; notice that $\mathbf{A}^{\mathbf{H}}$ means matrix **A** was both transposed and conjugated
- **Hermitian complex matrix:** **A** if $\mathbf{A}^{\mathbf{H}} = \mathbf{A}$
- **Skew-Hermitian complex matrix:** **A** if $\mathbf{A}^{\mathbf{H}} = -\mathbf{A}$
- If A is a complex matrix and $\mathbf{A}^{\mathbf{H}} = \mathbf{A}^{-1}$, then it is **unitary**
- A complex matrix is normal if $\mathbf{A}^{\mathbf{H}}\mathbf{A} = \mathbf{A}\mathbf{A}^{\mathbf{H}}$

PROPERTIES OF MATRICES

- When the sizes of the matrices are correct, allowing the indicated operations to be performed, the following properties are true.
  - Commutative:
    - $A + B = B + A$,
    - but $AB = BA$ is FALSE
  - Associative:
    - $A + (B + C) = (A + B) + C$
    - $A(BC) = (AB)C$
  - Symmetric:
    - If $A^t = A$, then matrix **A** is symmetric
    - If $A^t = -A$, the matrix **A** is skew-symmetric
  - Matrix Distribution:
    - $A(B + C) = AB + AC$
    - $A(B - C) = AB - AC$
  - Scalar Distribution:
    - $k(A + B) = kA + kB$
    - $k(A - B) = kA - kB$
    - $A(k + l) = kA + lA$
    - $A(k - l) = kA - lA$
  - Scalar Products:
    - $k(lA) = (kl)A$
    - $k(AB) = kA(B) = A(kB)$
  - Negative of a Matrix: $-1(A) = -A$
  - Addition of a Zero Matrix: $A + 0 = 0 + A = A$
  - Addition of Opposites: $A + (-A) = A - A = 0$
  - Multiplication by a Zero Matrix: $A(0) = (0)A = 0$
  - CAUTION:
    - If $AB = AC$ then **B** does NOT necessarily equal **C**
    - If $AB = 0$ then **A** and/or **B** do NOT necessarily equal zero
    - Multiplicative Inverses: $A(A^{-1}) = A^{-1}(A) = I$
    - **Product of inverses:** If **A** and **B** are invertible (if they have inverses), then AB is invertible and $A^{-1}(B^{-1}) = (BA)^{-1}$; notice that the order of the matrices must be reversed
  - Exponents: If A is a square matrix and k, m and n are nonzero integers, then
    - $A^0 = I$
    - $A^n = A\,(A)\,(A)\ldots(A)$, n times
    - $A^mA^n = A^{m+n}$; $(A^m)^n = A^{mn}$
    - $(kA)^{-1} = \frac{1}{k}A^{-1}$ if **A** is invertible;
  - Transpose:
    - $(A^t)^t = A$
    - $(A + B)^t = A^t + B^t$
    - $(kA)^t = kA^t$
    - $(AB)^t = B^tA^t$; notice the order of the matrices is reversed
  - Trace: If **A** and **B** are square matrices of the same size then
    - $tr(A + B) = tr(A) + tr(B)$
    - $tr(kA) = k\,tr(A)$
    - $tr(A^t) = tr(A)$
    - $tr(AB) = tr(BA)$
  - **Square matrices:** If **A** is a $m \times m$ square matrix and invertible then
    - $AX = 0$ has only the trivial solution for **X**
    - **A** is row equivalent to **I** of the same dimensions $m \times m$
    - $AX = B$ is consistent for every $m \times 1$ matrix **B**

LINEAR EQUATIONS

- Definition: Equations of the form $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b$, where $a_1, a_2, a_3, \ldots, a_n$ and **b** are real-number constants and the variables, $x_1, x_2, x_3, \ldots, x_n$ are all to the first degree
- **Solutions:** Numbers $s_1, s_2, s_3, \ldots, s_n$ that make the linear equation true when substituted for the variables $x_1, x_2, x_3, \ldots, x_n$
- **Linear systems:** Finite sets of linear equations with matching variables that have the same solution values for all equations in the system
- **Inconsistent** linear systems have no solutions
- **Consistent** linear systems have at least one solution
- **Coefficient matrix for a linear system:** The coefficients of the variables after the variables have all been arranged in the same order in all equations
- **Augmented matrix for a linear system:** The matrix of the linear system together with the constants for each equation; a mental record must be kept of the positions of the variables, the + signs and the = signs; example: the linear system

$$\begin{array}{c}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = b_3 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m
\end{array}$$

can be written as the **coefficient matrix**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{3n} \\ a_{m1} & a_{m2} & a_{m3} & a_{mn} \end{bmatrix} \text{ or}$$

can be written as the **augmented matrix**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & a_{3n} & b_3 \\ a_{m1} & a_{m2} & a_{m3} & a_{mn} & b_m \end{bmatrix} \text{ where}$$

the subscripts indicate the equation and the position in the equation for each coefficient or constant; that is, a_{11} is equation **1** variable **1** while a_{23} is equation **2** variable **3**; these are often referred to as either a_{mn} or a_{ij} where the m and i indicate the equation or the **row** in the matrix and the n and j indicate the variable position or the **column** in the matrix

- **Row-echelon:** A form of a matrix in which:
  - The first nonzero entry, if there is one, in a row is the number **1**, called the leading **1**;
  - Any rows that have all entries equal to zero are moved to the bottom of the matrix;
  - Any two consecutive rows that are not all zeros, the lower of the two rows has the leading **1** further to the right than the leading **1** in the higher row
- **Reduced row-echelon:** A form of a matrix with the same characteristics as a row-echelon, but also has zeros everywhere in each column that contain a lead **1** except in the position of the lead **1**

SOLVING LINEAR SYSTEMS USING MATRICES

- ***Gaussian* elimination**
  - Row operations are used to reduce the augmented matrix to a row-echelon form; then,
  - Back-substitution is used to solve the resulting corresponding system of equations
- ***Gauss-Jordan* elimination**
  - Row operations are used to reduce the augmented matrix to a reduced row-echelon form; then,
  - Each equation in the corresponding system of equations is solved for the lead **1** variable and arbitrary values are assigned for the remaining variables, yielding a general solution; or
  - If the reduced row-echelon matrix has zeros everywhere except the main diagonal of the matrix (disregarding the constants on the far right) which contains the lead **1**'s, then the variables with the lead **1**'s as coefficients have the numerical values indicated at the far right ends of their rows

MATRIX INVERSION

- When a system of linear equations has the same number of variables and equations, resulting in a square coefficient matrix size $\boldsymbol{m \times m}$, if the coefficient matrix is invertible then
  - $\mathbf{B}$ is a $\boldsymbol{m \times 1}$ matrix of the system constants;
  - $\mathbf{X}$ is the solution matrix; and,
  - $\mathbf{AX = B}$ has exactly one solution, which is $\mathbf{X = A^{-1}B}$
- When solving a sequence of n systems that have equal square coefficient matrices the solution matrices $\mathbf{X_1, X_2, X_3, \ldots X_n}$ can be found
  - with $\mathbf{X_1 = A^{-1}B_1;\ X_2 = A^{-1}B_2; \ldots X_n = A^{-1}B_n}$ if $\mathbf{A}$ is invertible
  - by reducing the systems to reduced row-echelon form and applying *Gauss-Jordan* elimination
- *Cramer's Rule* is used for solving systems of linear equations and is found under the determinates section

DETERMINATES

- Definition: A number value calculated for every square matrix; denoted by $\mathbf{det(A)}$ or $|A|$ or $\mathbf{D}$; if a matrix has two proportional rows, then its determinate equals zero
- **Determinate of order 1:** The determinate of a matrix with only one entry where the determinate is equal to that one entry; that is, if $\mathbf{A} = [a_{11}]$, then the determinate of $\mathbf{A}$ or $\mathbf{det(A)}$ or $\mathbf{D} = |a_{11}| = \mathbf{a_{11}}$

EVALUATION OF DETERMINATES

- A determinate of order 2 is the determinate of a 2 by 2 matrix and it is $\mathbf{D} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \mathbf{a_{11}a_{22} - a_{12}a_{21}}$
- A determinate of order 3 is the determinate of a 3 by 3 matrix and may be calculated as $\mathbf{D} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \mathbf{a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21}}$ $\mathbf{- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23}}$; this process can be displayed by writing the determinate matrix with the first 2 columns repeated after the matrix and then using diagonals to find the products, for example

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ becomes } \begin{vmatrix} a_{11} & a_{12} & a_{13} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \end{matrix}$$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}$$

- Cofactor or *Laplace* expansion
  - Definition: The **cofactor** of entry $\mathbf{a}_{ij}$ in matrix $\mathbf{A}$ is $(\mathbf{-1})^{i+j}(\mathbf{M}_{ij})$ where $\mathbf{M}_{ij}$ is the determinate of the submatrix that results from omitting the i^{th} row and the j^{th} column from the original matrix $\mathbf{A}$; cofactors are denoted as $\mathbf{C}_{ij}$; for example, the cofactor of the a_{12} entry in the matrix $\begin{bmatrix} 3 & 1 & -4 \\ -2 & 0 & 5 \\ 1 & -3 & 8 \end{bmatrix}$ is $(\mathbf{-1})^{1+2}(\mathbf{M_{12}})$ where $\mathbf{M_{12}}$ is the determinate of the submatrix $\begin{bmatrix} -2 & 5 \\ 1 & 8 \end{bmatrix}$ and $\mathbf{M_{12}} =$ $\mathbf{-2(8) - 5(1) = -21}$ and $\mathbf{C_{12} = -1(-21) = 21}$
  - **Cofactor expansion** or ***Laplace* expansion:** A method for calculating the determinate of a square matrix by adding $\mathbf{a}_{ij}C_{ij}$ for any full row or full column of a matrix; that is, $\mathbf{det(A)} = |A|$ $= \sum_{j=1}^{n} a_{ij}C_{ij}$ when eliminating a row and $\mathbf{det(A)} = |A| = \sum_{i=1}^{n} a_{ij}C_{ij}$ when eliminating a column
  - Example of cofactor expansion: Given matrix $\mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ the determinate can be calculated using **cofactor expansion** by eliminating any row or column, so
    $\mathbf{det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}}$ using the first row and
    $\mathbf{det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}}$ using the second row and
    $\mathbf{det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}}$ using the third row and
    $\mathbf{det(A) = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}}$ using the first column and
    $\mathbf{det(A) = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}}$ using the second column
    and $\mathbf{det(A) = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}}$ using the third column
- ***Cramer's* Rule:** If there is a system of linear equations that create a $\boldsymbol{n \times n}$ coefficient matrix $\mathbf{A}$ with $\mathbf{det(A)} = |A| \neq \mathbf{0}$, then the system has a unique solution which is $\mathbf{x_1} = \frac{|A_1|}{|A|}$; $\mathbf{x_2} = \frac{|A_2|}{|A|}$; $\mathbf{x_3} = \frac{|A_3|}{|A|}$; …; $\mathbf{x_n} = \frac{|A_n|}{|A|}$ where $\boldsymbol{j} = \mathbf{1, 2, 3}, \ldots, \boldsymbol{n}$ and $\mathbf{A}\boldsymbol{j}$ is the matrix created by replacing all entries in the $\boldsymbol{j^{th}}$ column by the matrix of equation constants, $\mathbf{B} = \begin{vmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{vmatrix}$
  - Applying *Cramer's Rule* to solving a 2 by 2 system of linear equations using a *2 X 2* coefficient matrix: create the *2 X 2* coefficient matrix $\mathbf{A}$ of the system; calculate $|A|$; replace the entries in column **1** of $\mathbf{A}$ with the constants of the system creating matrix $\mathbf{A_1}$; calculate $|A_1|$; replace the entries in column **2** of $\mathbf{A}$ with the constants of the system creating matrix $\mathbf{A_2}$; calculate $|A_2|$; then find the values of the variables by using $\mathbf{x_1} = \frac{|A_1|}{|A|} =$

$$\frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} \text{ and } \mathbf{x_2} = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} =$$

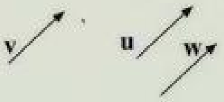
$$\frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

PROPERTIES OF DETERMINATES

- If **A** is a square matrix, then
  - **AX = 0** has only the zero solution
  - $\mathbf{det(A)} = |A| = \mathbf{det(A^t)} = |A^t|$
  - it is invertible if and only if $\mathbf{det(A)} = |A| \neq \mathbf{0}$
  - $\mathbf{det(A)} = |A| = \mathbf{0}$ when **A** has a row or a column of zeros
  - $\mathbf{det(A)} = |A| = \mathbf{0}$ when **A** has 2 identical rows or columns
  - $\mathbf{det(A)} = |A| = \mathbf{a_{11}a_{22}a_{33}\ldots a_{nn}}$ when **A** is a $n \times n$ triangular matrix
- If **A** is invertible, then $\mathbf{det(A^{-1})} = |A^{-1}| = \frac{1}{|A|} = \frac{1}{\det(A)}$
- If **A** and **B** are square matrices with the same dimensions, then $\mathbf{det(AB)} = |AB| = |A||B| = \mathbf{det(A)\ det(B)}$
- If **B** results from **A** through row or column operations, then
  - $\mathbf{det(B)} = |B| = -|A| = -\mathbf{det(A)}$ when **B** resulted from 2 row or column interchanges
  - $\mathbf{det(B)} = |B| = k|A| = k\mathbf{det(A)}$ when **B** resulted from a scalar multiplication of a row or column in **A**
  - $\mathbf{det(B)} = |B| = |A| = \mathbf{det(A)}$ when **B** resulted from the addition of 2 rows or row multiples or 2 columns or column multiples in **A**

VECTORS

- **Definition:** An arrow in 2- or 3-dimensional space with 2 characteristics: direction and length or magnitude; or, a list of numbers (also called a **linear array** containing **coordinates**, **entries** or **components**); the endpoints of vectors uniquely determine every vector
- **Initial point:** The tail end of the arrow
- **Terminal point:** The tip of the arrow
- **Scalars:** The numerical quantities of vectors
- Notation: Vectors are named with small case letters, and if the initial point of vector **v** is **A** and the terminal point is **B**, then $\mathbf{v} = \overrightarrow{AB}$
- Vectors are **equal** if they have the same direction and length (or entries) regardless of the position; for example, **v = u = w**



Also they are equal if **v** = (3, –1 ,5)
u = (3, –1, 5) and **w** = (3, –1, 5)
They have the same components

- The **zero vector** has a length of zero in any direction and is denoted as 0
- The **negative** of vector **v**, written –**v**, is the vector having the same magnitude or length, but the opposite direction of **v**

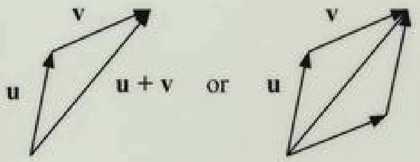
NORM OF A VECTOR

- The norm or length of a vector, $\|u\|$, in *n*-space is the square root (always nonnegative) of **u · u**; that is, if $\mathbf{u} = (\mathbf{a_1, a_2, a_3}, \ldots \mathbf{a_n})$ then $= \|u\| = \sqrt{u \cdot u} = \sqrt{a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2}$
- $\|u\| \geq 0$ and $\|u\| = \mathbf{0}$ if and only if $\mathbf{u} = \mathbf{0}$
- If $\|u\| = \mathbf{1}$ or if $\mathbf{u \cdot u = 1}$, then **u** is a **unit vector**

- A **linear combination** of vectors $\mathbf{v_1, v_2, v_3, \ldots, v_n}$ is vector **w** when $\mathbf{w} = k_1\mathbf{v_1} + k_2\mathbf{v_2} + k_3\mathbf{v_3} + \ldots + k_n\mathbf{v_n}$ and $k_1, k_2, k_3, \ldots k_n$ are scalars
- A **linear space** is the space spanned by a set of vectors $S = (\mathbf{v_1, v_2, v_3, \ldots v_n})$; denoted by **lin(S)**
- A **linearly independent** nonempty set of vectors, $S = \{\mathbf{v_1, v_2, v_3, \ldots, v_n}\}$, has only one solution to the vector equation $k_1\mathbf{v_1} + k_2\mathbf{v_2} + k_3\mathbf{v_3} + \ldots + k_n\mathbf{v_n} = \mathbf{0}$ and that is all scalars, k_n, equal zero; if there are other solutions to the equation then S is a **linearly dependent** set
- A **basis** for vector space V is a set of vectors S in V if S is linearly independent and S spans V
- The **matrix** $\mathbf{M = A} - t\mathbf{I_n}$, where $\mathbf{I_n}$ is the n-square identity matrix, t is an indeterminate, and $\mathbf{A} = |a_{ij}|$ n-square matrix has a negative matrix $t\mathbf{I_n} - \mathbf{A}$ and a determinant $\Delta(t) = \mathbf{det}(t\mathbf{I_n} - \mathbf{A}) = (-1)^n\ \mathbf{det(A} - t\mathbf{I_n})$ which is the **characteristic polynomial** of **A**

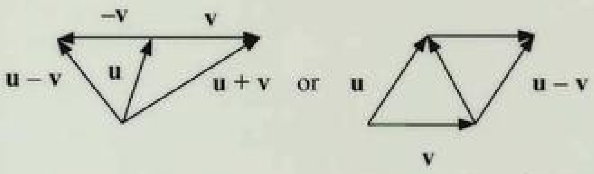
OPERATIONS OF VECTORS

- Addition: If **u** is a vector with endpoint $(\mathbf{a_1, b_1, c_1})$ and **v** is a vector with endpoint $(\mathbf{a_2, b_2, c_2})$, then **u + v** has the endpoint $(\mathbf{a_1 + a_2, b_1 + b_2, c_1 + c_2})$; this can be seen graphically using the **parallelogram law** which places the initial point of **v** at the terminal point of **u**, then the vector **u + v** is the vector that connects the initial point of **u** to the terminal point of **v**



The diagonal of this parallelogram is u + v

- **Subtraction:** For vectors **u** and **v**, their difference is defined to be **u – v = u + (–v)**; graphically:



u – v is the other diagonal of the parallelogram

Also, if $\mathbf{u} = (\mathbf{a_1, b_1})$ and $\mathbf{v} = (\mathbf{a_2, b_2})$, then $\mathbf{u - v} = (\mathbf{a_1 - a_2, b_1 - b_2})$; this relationship is also true in 3-space

- **Scalar multiplication:** The product of any nonzero real number scalar k and nonzero vector **v** is the product of the magnitude of **v** and scalar k; retaining the same direction if $k > 0$ and changing to the opposite direction if $k < 0$
  - If **v** has the endpoint $(\mathbf{a_1, b_1, c_1})$, then $k\mathbf{v}$, called the **scalar multiple of v**, has the endpoint $(k\mathbf{a_1}, k\mathbf{b_1}, k\mathbf{c_1})$; if either $k = 0$ or $\mathbf{v} = \mathbf{0}$ then $k\mathbf{v} = \mathbf{0}$
- **Translation equations**
  - When the initial point is not at the origin; that is, $\mathrm{v} = \overrightarrow{PQ}$ and $\mathbf{P} = (\mathbf{a_1, b_1})$ when $\mathbf{Q} = (\mathbf{a_2, b_2})$, then $\mathbf{v} = (\mathbf{a_2 - a_1, b_2 - b_1})$
  - In translation of axes, if **v** has a terminal point of $(\mathbf{a_1, b_1})$ and the origin is translated to **(x, y)** then the translated components $(\mathbf{a_2, b_2})$ are found by using the equations $\mathbf{a_2 = a_1 - x}$ and $\mathbf{b_2 = b_1 - y}$; this can also be done in 3-space: **v** has the terminal point of $(\mathbf{a_1, b_1, c_1})$ and the origin is translated to (x, y, z) so the translated components of **v** are $(\mathbf{a_1 - x, b_1 - y, c_1 - z})$

OPERATIONS OF VECTORS CONT.

- Dot or Inner Product
  - If **u** and **v** are vectors in n-space and $\mathbf{u} = (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \ldots \mathbf{a_n})$ and $\mathbf{v} = (\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \ldots \mathbf{b_n})$, then the **dot or inner product** $\mathbf{u} \cdot \mathbf{v} = \langle u, v\rangle = (\mathbf{a_1b_1} + \mathbf{a_2b_2} + \mathbf{a_3b_3} + \ldots + \mathbf{a_nb_n})$
  - Orthagonal or perpendicular vectors: Vectors whose dot or inner product is zero; that is, $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$
- **Distance between vectors**
  - If $\mathbf{u} = (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}, \ldots \mathbf{a_n})$ and $\mathbf{v} = (\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}, \ldots, \mathbf{b_n})$ then the distance between the vectors is $\mathbf{d(u,v)} = \|u - v\| = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 + \ldots + (a_n - b_n)^2}$
- **Projection**
  - The **projection** of a vector **u** onto a nonzero vector **v** is $\text{proj}(\mathbf{u},\mathbf{v}) = \frac{u \cdot v}{\|v\|^2} v$
- **Angle between vectors**
  - If **u** and **v** are nonzero vectors, then the angle θ between them is found using the formula $\cos\theta = \frac{u \cdot v}{\|u\|\|v\|}$ where $-1 \le \frac{u \cdot v}{\|u\|\|v\|} \le 1$
  - Angle θ, between vectors u and v, is
    1. Acute if and only if $\mathbf{u} \cdot \mathbf{v} > \mathbf{0}$
    2. Obtuse if and only if $\mathbf{u} \cdot \mathbf{v} < \mathbf{0}$
    3. Right if and only if $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$
- **Euclidean inner product** or dot product for 2- or 3-space vectors **u** and **v** with the included angle θ is defined by $u \cdot v = \begin{cases} \|u\|\|v\|\cos\theta; \ if\ u \,\&\, v \neq 0 \\ 0; \ if\ u = 0 \ or\ v = 0 \end{cases}$
- **Cross product** for 3-space vectors $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3})$ and $\mathbf{v} = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$ is $\mathbf{u \times v} = (\mathbf{u_2v_3} - \mathbf{u_3v_2}, \mathbf{u_3v_1} - \mathbf{u_1v_3}, \mathbf{u_1v_2} - \mathbf{u_2v_1})$
- **Gram-Schmidt Orthogonalization Process** for constructing an orthogonal basis $(\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \ldots, \mathbf{w_n})$ of an inner product space *V* given $(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \ldots, \mathbf{v_n})$ as a basis of *V* is $\mathbf{w_k} = \mathbf{v_k} - \mathbf{c_{k1}w_1} - \mathbf{c_{k2}w_2} - \ldots - \mathbf{c_{k,k-1}w_{k-1}}$ where $\mathbf{k} = \mathbf{2}, \mathbf{3}, \mathbf{4}, \ldots, \mathbf{n}$ and $\mathbf{c_{kj}} = \frac{\langle v_k, w_j\rangle}{\langle w_j, w_j\rangle}$ is the component of $\mathbf{v_k}$ along $\mathbf{w_j}$
- **Eigenvector:** If **A** is any n-square matrix, then a real number scalar λ is called an **eigenvalue** of **A** and **x** is **eigenvector** of **A corresponding** to λ if $\mathbf{Ax} = \lambda\mathbf{x}$, where **x** is a nonzero vector in $\mathbf{R^n}$
  - The following are equivalent:
    1. λ is an eigenvalue of **A**
    2. The system of equations $(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{o}$ has nontrivial solutions
    3. There is a nonzero vector **x** in $\mathbf{R^n}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$
    4. λ is a solution of the characteristic equation $\det(\lambda I - A) = \mathbf{0}$
  - Computing eigenvalues and eigenvectors for an n-square matrix **A**
    1. Find the characteristic polynomial $\Delta(t)$ of **A**
    2. Find the roots of $\Delta(t)$ to obtain the eigenvalues of **A**
    3. Repeat these 2 steps for each eigenvalue λ of A:
       a. Form the matrix $\mathbf{M} = \mathbf{A} - \lambda\mathbf{I}$
       b. Find a basis for the solution space of the homogeneous system $\mathbf{MX} = \mathbf{0}$
    4. For the $\mathbf{S} = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \ldots, \mathbf{v_m})$ of all eigenvectors obtained in step 3 above
       a. If $\mathbf{m} \neq \mathbf{n}$, then **A** is not diagonalizable
       b. If $\mathbf{m} = \mathbf{n}$, then **A** is diagonalizable and **P** is the matrix whose columns are the eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \ldots, \mathbf{v_n}$ with $\mathbf{D} = \mathbf{P^{-1}AP} = \mathbf{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$ where λ_j is the eigenvalue corresponding to the eigenvector $\mathbf{v_j}$
- **Bilinear forms:** A **bilinear form** on *V*, a vector space of finite dimension over field *K*, is a mapping $\mathbf{f}: \mathbf{V} \times \mathbf{V} \to K$ such that for all $\mathbf{a}, \mathbf{b} \in K$ and all $\mathbf{u_j}, \mathbf{v_j} \in V$
  - $f(\mathbf{au_1} + \mathbf{bu_2}, \mathbf{v}) = \mathbf{a}f(\mathbf{u_1}, \mathbf{v}) + \mathbf{b}f(\mathbf{u_2}, \mathbf{v})$ that is, *f* is linear in the first variable, and
  - $f(\mathbf{u}, \mathbf{av_1} + \mathbf{bv_2}) = \mathbf{a}f(\mathbf{u}, \mathbf{v_1}) + \mathbf{b}f(\mathbf{u}, \mathbf{v_2})$; *f* is linear in the second variable
- **Hermitian forms:** A Hermitian form on *V*, a vector space of finite dimension over the complex field *C*, is a mapping $f: \mathbf{V} \times \mathbf{V} \to \mathbf{C}$ such that for all $\mathbf{a}, \mathbf{b} \in C$ and all $\mathbf{u_j}, \mathbf{v_j} \in V$
  - $f(\mathbf{au_1} + \mathbf{bu_2}, v) = \mathbf{a}f(\mathbf{u_1}, v) + \mathbf{b}f(\mathbf{u_2}, v)$; that is, *f* is linear in the first variable
  - $f(\mathbf{u}, \mathbf{v}) = \overline{f(v, u)}$; that is, $f(v,v)$ is real for every $\mathbf{v} \in \mathbf{V}$
  - $f(\mathbf{u}, \mathbf{av_1} + \mathbf{bv_2}) = \bar{a}f(u, v_1) + \bar{b}f(u, v_2)$; that is, f is conjugate linear in the second variable

PROPERTIES AND THEOREMS OF VECTORS

If **u**, **v**, and **w** are nonzero vectors in *Euclidean n*-space and *k* and *s* are nonzero scalars, then

- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
- $(k + s)\mathbf{u} = k\mathbf{u} + s\mathbf{u}$
- $(ks)\mathbf{u} = k(s\mathbf{u})$
- $1\mathbf{u} = \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v})$
- $\mathbf{u} \cdot \mathbf{u} \ge \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{u} = \mathbf{0}$ if and only if $\mathbf{u} = \mathbf{0}$
- $\|u\| \ge 0$ and $\|u\| = \mathbf{0}$ if and only if $\mathbf{u} = \mathbf{0}$
- $\|ku\| = |k|\|u\|$
- $|u \cdot v| \le \|u\|\|v\|$
- $\|u + v\| \le \|u\| + \|v\|$
- $\hat{v} = \frac{1}{\|v\|} v = \frac{v}{\|v\|}$
- $\mathbf{u} \cdot (\mathbf{u \times v}) = \mathbf{0}$ when $\mathbf{u \times v}$ is orthogonal to **u**
- $\mathbf{v} \cdot (\mathbf{u \times v}) = \mathbf{0}$ when $\mathbf{u \times v}$ is orthogonal to **v**
- $\|u \times v\|^2 = \|u\|^2\|v\|^2 - (u \cdot v)^2$; Lagrange's identity
- $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ if **u** and **v** are orthogonal vectors in an inner product space; generalized *Pythagorean Theorem*
- $\langle u, v\rangle^2 \le \langle u, v\rangle\langle v, v\rangle$ if u and v are vectors in a real inner product space; ***Cauchy-Schwarz*** Inequality

■ **FIND THE SOLUTION TO A SYSTEM OF EQUATIONS USING ONE OF THESE METHODS:**

- **Graph Method** - Graph the equations and locate the point of intersection, if there is one. The point can be checked by substituting the **x** value and the **y** value into all of the equations. If it is the correct point it should make all of the equations true. *This method is weak, since an approximation of the coordinates of the point is often all that is possible.*
- **Substitution Method** for solving consistent systems of linear equations includes following steps;
  1. **Solve** one of the equations for one of the variables. It is easiest to solve for a variable which has a coefficient of one (if such a variable coefficient is in the system) because fractions can be avoided until the very end.
  2. **Substitute** the resulting expression for the variable into the **other equation**, not the same equation which was just used.
  3. **Solve** the resulting equation for the remaining variable. This should result in a numerical value for the variable, either **x** or **y**, if the system was originally only two equations.
  4. **Substitute** this numerical value back into one of the original equations and solve for the other variable.
  5. **The solution** is the point containing these **x** and **y** -values, **(x,y)**.
  6. **Check the solution** in all of the original equations.
- **Elimination Method** or the **Add/Subtract Method** or the **Linear Combination Method** - eliminate either the **x** or the **y** variable through either addition or subtraction of the two equations. These are the steps for consistent systems of two linear equations;
  1. **Write both equations in the same order**, usually **ax + by = c**, where **a**, **b**, and **c** are real numbers.
  2. **Observe the coefficients** of the **x** and **y** variables in both equations to determine:
     a. If the **x** coefficients or the **y** coefficients are the same, **subtract** the equations.
     b. If they are additive inverses (opposite signs: such as **3** and **-3**), **add** the equations.
     c. If the coefficients of the **x** variables are not the same and are not additive inverses, and the same is true of the coefficients of the **y** variables, then multiply the equations to make one of these conditions true so the equations can be either added or subtracted to eliminate one of the variables.
  3. The above steps should result in **one equation with only one variable, either x or y, but not both**. If the resulting equation has both **x** and **y**, an error was made in following the steps indicated in substitution method at left. Correct the error.
  4. **Solve** the resulting equation for the one variable (**x** or **y**).
  5. **Substitute** this numerical value back into either of the original equations and solve for the one remaining variable.
  6. The solution is the point **(x,y)** with the resulting **x** and **y**-values.
  7. **Check** the solution in all of the original equations.
- **Matrix method** - involves substantial matrix theory for a system of more than two equations and will not be covered here. Systems of two linear equations can be solved using **Cramer's Rule** which is based on determinants.
  1. For the system of equations: $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, where all of the **a**, **b**, and **c** values are real numbers, the point of intersection is **(x,y)** where $x = (D_x)/D$ and $y = (D_y)/D$.
  2. The determinant **D** in these equations is a numerical value found in this manner:
  $$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$
  3. The determinant D_x in these equations is a numerical value found in this manner:
  $$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2 - c_2b_1$$
  4. The determinant D_y in these equations is a numerical value found in this manner:
  $$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1$$
  5. Substitute the numerical values found from applying the formulas in steps 2 through 4 into the formulas for **x** and *y* in step 1 above.

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