UNIVERSITY OF PATRAS

DEPARTMENT OF BUSINESS ADMINISTRATION

FURTHER OPERATIONAL RESEARCH TECHNIQUES

Lecture 3: The Minimum Cost Flow Problem

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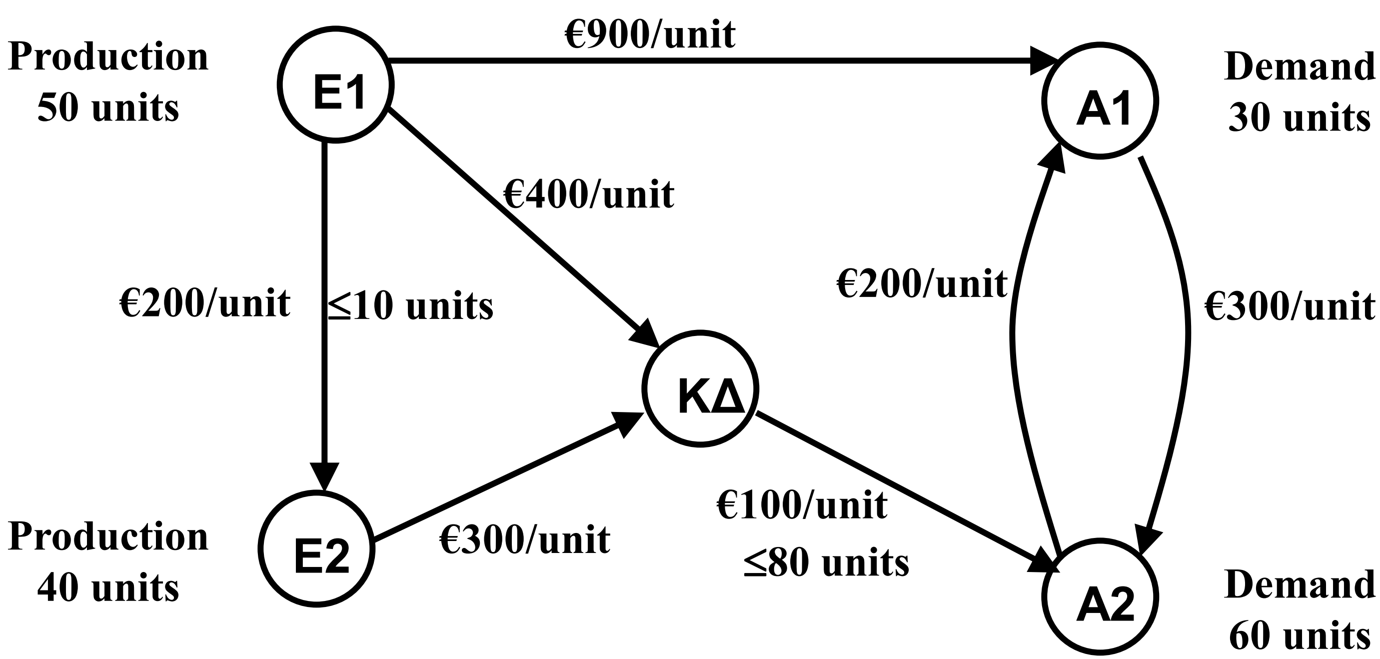
Example

- Two factories E1 and E2 supply two warehouses A1 and A2. The quantity produced at each factory, the quantity demanded at each warehouse and the transportation network are shown below:



Example /2

- **Problem:**
  - **Transport the produce quantity form the production facilities (factories) to the demand sites (warehouses) to cover the demand at minimum cost.**

Formulation as LP

- **Variables**
  - **x_{ij} the flow along edge (i, j)**
- **Objective Function (minimization of total cost)**

$$Min\ z = 900 \cdot x_{E1A1} + 200 \cdot x_{E1E2} + 400 \cdot x_{E1\mathrm{K\Delta}} + 300 \cdot x_{E2\mathrm{K\Delta}} + 100 \cdot x_{\mathrm{K\Sigma A2}} + 300 \cdot x_{\mathrm{A1}A2} + 200 \cdot x_{\mathrm{A2}A1}$$

- **Constraints**

$x_{E1A1} + x_{E1E2} + x_{E1\mathrm{K\Delta}} \leq 50$ (Quantity produced at E_1)

$x_{E2\mathrm{K\Delta}} \leq 40 + x_{E1E2}$ (Quantity produced at E_2)

$x_{E1A1} + x_{\mathrm{A2A1}} - x_{\mathrm{A1A2}} \geq 30$ (Cover demand at A_1)

$x_{\mathrm{K\Delta A2}} + x_{\mathrm{A1A2}} - x_{\mathrm{A2A1}} \geq 60$ (Cover demand at A_2)

$x_{\mathrm{E1K\Delta}} + x_{\mathrm{E2K\Delta}} = x_{\mathrm{K\Delta A2}}$ (Balance of flow at KΔ)

Formulation as LP /2

- **Constraints (continued)**

$x_{E1E2} \leq 10$ (Capacity along edge $E_1 - E_2$)

$x_{K\Delta A2} \leq 80$ (Capacity alomng edge $K\Delta - A_2$)

$x_{E1A1} \geq 0, x_{E1E2} \geq 0, \ldots x_{A2A1} \geq 0$

- **Observation: since total production (50+40=90) is equal to total demand (30+60=90), all constraints must be satisfied as <u>equalities</u> for the solution to be feasible!**

- **The problem can be formulated as follows:**

Formulation as LP/3

$$Min\ z = 900 \cdot x_{E1A1} + 200 \cdot x_{E1E2} + 400 \cdot x_{E1\text{K}\Delta} + 300 \cdot x_{E2\text{K}\Delta} + 100 \cdot x_{\text{K}\Sigma\text{A}2} + 300 \cdot x_{\text{A}1A2} + 200 \cdot x_{\text{A}2A1}$$

μ.τ.π.

$$x_{E1A1} + x_{E1E2} + x_{E1\text{K}\Delta} = 50 \qquad (\text{Node E}_1)$$

$$x_{E2\text{K}\Delta} - x_{E1E2} = 40 \qquad (\text{Node E}_2)$$

$$x_{\text{A1A2}} - (x_{E1A1} + x_{\text{A2A1}}) = -30 \qquad (\text{Node A}_1)$$

$$x_{\text{A2A1}} - (x_{\text{K}\Delta\text{A2}} + x_{\text{A1A2}}) = -60 \qquad (\text{Node A}_2)$$

$$x_{\text{K}\Delta\text{A2}} - (x_{\text{E1K}\Delta} + x_{\text{E2K}\Delta}) = 0 \qquad (\text{Node K}\Delta)$$

$$x_{E1\text{E2}} \leq 10 \qquad (\text{Capacity of edge E}_1 - \text{E}_2)$$

$$x_{\text{K}\Delta\text{A2}} \leq 80 \qquad (\text{Capacity of edge K}\Delta - \text{A}_2)$$

$$x_{\text{E1A1}} \geq 0, x_{\text{E1E2}} \geq 0, \ldots x_{\text{A2A1}} \geq 0$$

The Minimum Cost Flow Problem: General Statement

- **Data**
  - c_{ij} the unit cost along edge (i, j)
  - u_{ij} the capacity (maximum flow) along edge (i, j)
  - b_i the net flow created at node i
- **Observation:**
  - $b_i > 0$ if node i is a production node
  - $b_i < 0$ if node i is a demand node
  - $b_i = 0$ if node i is an intermediate node
- **Problem: Transport the total production quantity through the network at minimum cost**

Formulation as LP

- **Variables**
  - x_{ij} the flow along edge (i, j)
- **Objective function (minimize cost)**

$$\text{Min} \quad z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \cdot x_{ij}$$

- **Constraints**

$$\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n} x_{ji} = b_i \qquad \text{for every node } i$$

(flow out of node i- flow into node i= net flow)

$$0 \leq x_{ij} \leq u_{ij} \qquad \text{for every edge } (i, j)$$

- **The problem can be solved as an LP, using a special for of the Simplex method (network Simplex)**

Observations

- The problem is feasible if $\sum_{i=1}^{n} b_i = 0$

- If the produced and the demanded quantities (b_i) as well as the capacities (u_{ij}) are integer numbers, then in any basic feasible solution (including the optimal solution), all variables are integer.

- The following problems can be formulated as Minimum Cost Flow Problems
  - The transportation problem
  - The shortest path problem
  - The maximum flow problem

Solution with Solver

- **A formulation of the problem and the optimal solution by Solver:**

	E1	E2	kD	A1	A2		Differences	bi
E1	0	0	40	10	0	50	50	50
E2	0	0	40	0	0	40	40	40
kD	0	0	0	0	80	80	0	0
A1	0	0	0	0	0	0	-30	-30
A2	0	0	0	20	0	20	-60	-60
	0	0	80	30	80			

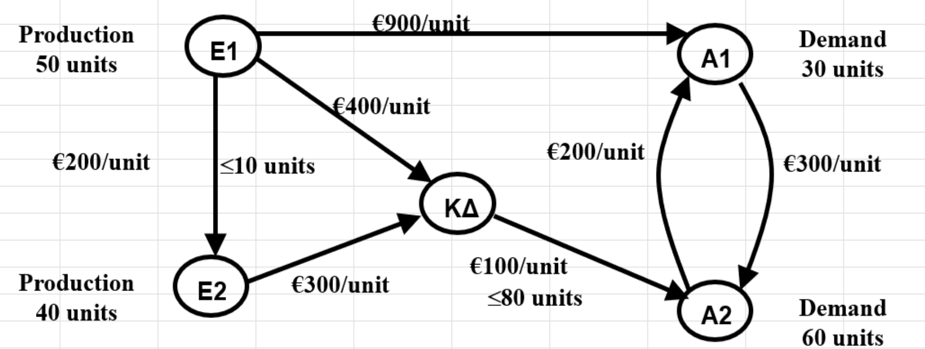
Cij

	E1	E2	kD	A1	A2
E1	1E+08	200	400	900	1E+08
E2	1E+08	1E+08	300	1E+08	1E+08
kD	1E+08	1E+08	1E+08	1E+08	100
A1	1E+08	1E+08	1E+08	1E+08	300
A2	1E+08	1E+08	1E+08	200	1E+08

Total Cost
49000

Uij

	E1	E2	kD	A1	A2
E1	1000000	10	1000000	1000000	1000000
E2	1000000	1000000	1000000	1000000	1000000
kD	1000000	1000000	1000000	1000000	80
A1	1000000	1000000	1000000	1000000	1E+07
A2	1000000	1000000	1000000	1000000	1000000



Shortest Path as MCFP

- **Transformation:**
  - $c_{ij}=d_{ij}$ the distance of edge (i, j)
  - $u_{ij}=1$ for every edge (i, j)
  - $b_O=1$ for the Origin
  - $b_D=-1$ for the Destination
  - $b_i=0$ for all other nodes

Maximum Flow as MCFP

- **Transformation:**
  - **We add an artificial (virtual) edge (D,O) from the sink back to the source**
  - **c_{ij}=0 for all edges (i, j) except the artificial edge (D,O)**
  - **c_{DO}=–1 for the artificial edge (D, O)**
  - **u_{ij} : capacity of edge (i, j)**
  - **u_{ij} =+∞ for the artificial edge (D, O)**
  - **b_i=0 for all nodes**