UNIVERSITY OF PATRAS

DEPARTMENT OF BUSINESS ADMINISTRATION

FURTHER OPERATIONAL RESEARCH TECHNIQUES

Lecture 3: The Minimum Cost Flow Problem

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Example

• Two factories E1 and E2 supply two warehouses A1 and A2. The quantity produced at each factory, the quantity demanded at each warehouse and the transportation network are shown below:



- Problem:
 - Transport the produce quantity form the production facilities (factories) to the demand sites (warehouses) to cover the demand at minimum cost.

Formulation as LP

- Variables
 - x_{ij} the flow along edge (i, j)
- Objective Function (minimization of total cost)

 $Min \ z = 900 \cdot x_{E1A1} + 200 \cdot x_{E1E2} + 400 \cdot x_{E1K\Delta} + 300 \cdot x_{E2K\Delta} + 100 \cdot x_{K\SigmaA2} + 300 \cdot x_{A1A2} + 200 \cdot x_{A2A1}$

• Constraints

 $x_{E1A1} + x_{E1E2} + x_{E1K\Delta} \le 50$ $x_{E2K\Delta} \le 40 + x_{E1E2}$ $x_{E1A1} + x_{A2A1} - x_{A1A2} \ge 30$ $x_{K\Delta A2} + x_{A1A2} - x_{A2A1} \ge 60$ $x_{E1K\Delta} + x_{E2K\Delta} = x_{K\Delta A2}$

(Quantity produced at E_1) (Quantity produced at E_2) (Cover demand at A_1) (Cover demand at A_2) (Balance of flow at $K\Delta$)

• Constraints (continued)

 $x_{E1E2} \le 10$ (Capacity along edge $E_1 - E_2$)

 $x_{K\Delta A2} \le 80$ (Capacity alomng edge $K\Delta - A_2$)

 $x_{E1A1} \ge 0, x_{E1E2} \ge 0, \dots x_{A2A1} \ge 0$

- Observation: since total production (50+40=90) is equal to total demand (30+60=90), all constraints must be satisfied as <u>equalities</u> for the solution to be feasible!
- The problem can be formulated as follows:

Formulation as LP/3

 $Min \ z = 900 \cdot x_{E1A1} + 200 \cdot x_{E1E2} + 400 \cdot x_{E1K\Delta} + 300 \cdot x_{E2K\Delta} + 100 \cdot x_{K\SigmaA2} + 300 \cdot x_{A1A2} + 200 \cdot x_{A2A1}$ $\mu.\tau.\pi.$

$x_{E1A1} + x_{E1E2} + x_{E1K\Delta} = 50$	(Node E ₁)
$x_{E2K\Delta} - x_{E1E2} = 40$	(Node E ₂)
$x_{\rm A1A2} - (x_{E1A1} + x_{\rm A2A1}) = -30$	(Node A ₁)
$x_{\rm A2A1} - (x_{\rm K\Delta A2} + x_{\rm A1A2}) = -60$	(Node A ₂)
$x_{\mathrm{K}\Delta\mathrm{A}2} - (x_{\mathrm{E}1\mathrm{K}\Delta} + x_{\mathrm{E}2\mathrm{K}\Delta}) = 0$	(Node $K\Delta$)
$x_{E1E2} \leq 10$	(Capacity of edge $E_1 - E_2$)
$x_{\mathrm{K}\Delta\mathrm{A}2} \leq 80$	(Capacity of edge $K\Delta - A_2$)

 $x_{\text{E1A1}} \geq 0, x_{\text{E1E2}} \geq 0, \dots x_{\text{A2A1}} \geq 0$

The Minimum Cost Flow Problem: General Statement

• Data

- c_{ij} the unit cost along edge (i, j)
- u_{ij} the capacity (maximum flow) along edge (i, j)
- b_i the net flow created at node i
- Observation:
 - b_i>0 if node i is a production node
 - b_i<0 if node i is a demand node</p>
 - b_i=0 if node i is an intermediate node
- Problem: Transport the total production quantity through the network at minimum cost

- Variables
 - x_{ij} the flow along edge (i, j)
- Objective function (minimize cost)

$$Min \quad \mathbf{z} = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{c}_{ij} \cdot \mathbf{x}_{ij}$$

• Constraints

$$\sum_{j=1}^{n} \mathbf{x}_{jj} - \sum_{j=1}^{n} \mathbf{x}_{ji} = \mathbf{b}_{i} \qquad \text{for every node i}$$

(flow out of node i- flow into node i= net flow)

$$\mathbf{0} \leq \mathbf{x}_{ij} \leq \mathbf{u}_{ij}$$
 for every edge (i, j)

• The problem can be solved as an LP, using a special for of the Simplex method (network Simplex)

Observations

- The problem is feasible if $\sum_{i=1}^{n} b_i = 0$
- If the produced and the demanded quantities (b_i) as well as the capacities (u_{ij}) are integer numbers, then in any basic feasible solution (including the optimal solution), all variables are integer.
- The following problems can be formulated as Minimum Cost Flow Problems
 - The transportation problem
 - The shortest path problem
 - The maximum flow problem

Solution with Solver

• A formulation of the problem and the optimal solution by Solver:

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26	kD	1000000	1000000	1000000	1000000	80														
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28	A2	1000000	1000000	1000000	1000000	1000000														
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Shortest Path as MCFP

- Transformation:
 - $-c_{ij}=d_{ij}$ the distance of edge (i, j)
 - $u_{ij} = 1$ for every edge (i, j)
 - b_o=1 for the Origin
 - $b_D = -1$ for the Destination
 - b_i=0 for all other nodes

Maximum Flow as MCFP

- Transformation:
 - We add an artificial (virtual) edge (D,O) from the sink back to the source
 - c_{ij}=0 for all edges (i, j) except the artificial edge (D,O)
 - $-c_{DO}=-1$ for the artificial edge (D, O)
 - u_{ij} : capacity of edge (i, j)
 - $u_{ij} = +\infty$ for the artificial edge (D, O)
 - b_i=0 for all nodes