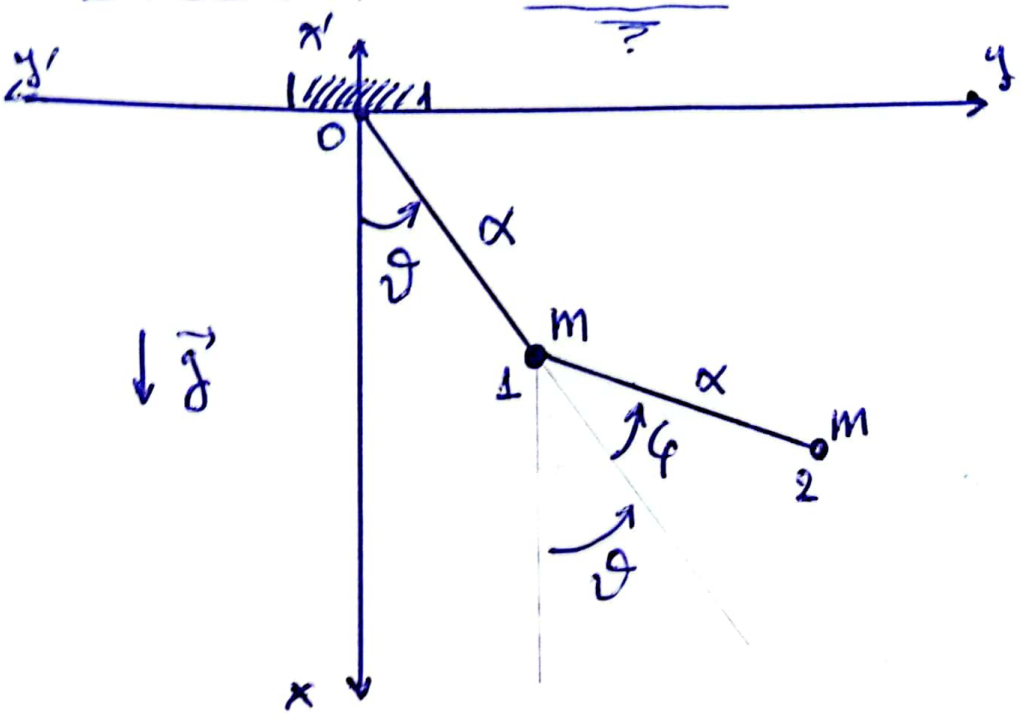
Διπλό εκκρεμές (ειδική περίπτωση)



▶ Έστω μονάδες $\alpha = m = g = 1$ (1)

▶ $x_1 = \eta\mu\vartheta \longrightarrow \dot{x}_1 = -\sigma\upsilon\nu\vartheta\,\dot{\vartheta}$ (2α)

$x_2 = \eta\mu\vartheta + \alpha\upsilon(\vartheta+\varphi) \longrightarrow \dot{x}_2 = -\sigma\upsilon\nu\vartheta\cdot\dot{\vartheta} - \sigma\upsilon\nu(\vartheta+\varphi)\cdot(\dot{\vartheta}+\dot{\varphi})$ (2β)

$y_1 = \sigma\upsilon\nu\vartheta \longrightarrow \dot{y}_1 = \alpha\upsilon\vartheta\cdot\dot{\vartheta}$ (2γ)

$y_2 = \sigma\upsilon\nu\vartheta + \sigma\upsilon\nu(\vartheta+\varphi) \longrightarrow \dot{y}_2 = \alpha\upsilon\vartheta\cdot\dot{\vartheta} + \alpha\upsilon(\vartheta+\varphi)\cdot(\dot{\vartheta}+\dot{\varphi})$ (2δ)

▶ $2K = v_1^2 + v_2^2 = \dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 =$

$= \dot{\vartheta}^2 + \dot{\vartheta}^2 + (\dot{\vartheta}+\dot{\varphi})^2 + 2\dot{\vartheta}(\dot{\vartheta}+\varphi)\cdot[\underbrace{\sigma\upsilon\nu\vartheta\cdot\sigma\upsilon\nu(\vartheta+\varphi) + \alpha\upsilon\vartheta\cdot\alpha\upsilon(\vartheta+\varphi)}_{\sigma\upsilon\nu\varphi}$

$= \dot{\varphi}^2 + 2\dot{\vartheta}\dot{\varphi} + 3\dot{\vartheta}^2 + 2\dot{\vartheta}^2\alpha\upsilon\varphi +$

$+ 2\dot{\vartheta}\dot{\varphi}\alpha\upsilon\varphi$ (3)

▶ $U = -x_1 - x_2 = -2\alpha\upsilon\vartheta - \alpha\upsilon(\vartheta+\varphi)$ (4)

▶ $L \equiv K - U \overset{(3)}{\underset{(4)}{=}} \frac{1}{2}\dot{\varphi}^2 + \dot{\vartheta}\dot{\varphi} + \frac{3}{2}\dot{\vartheta}^2 + \dot{\vartheta}\alpha\upsilon\varphi + \dot{\vartheta}\dot{\varphi}\alpha\upsilon\varphi$

$-2\alpha\upsilon\vartheta - \alpha\upsilon(\vartheta+\varphi)$ (5)

▶ Αρχή Χαμίλτον: $0 = \delta I \overset{*1}{=} \delta \int L = \int \delta L \overset{*2}{=}$

$$\overset{⑤}{=} \int \dot{\varphi}\delta\dot{\varphi} + \dot{\vartheta}\delta\dot{\varphi} + \dot{\varphi}\delta\dot{\vartheta} + 3\dot{\vartheta}\delta\dot{\vartheta} + \omega\cos\varphi\,\delta\dot{\vartheta} - \dot{\vartheta}\sin\varphi\,\delta\varphi$$
$$+ \dot{\varphi}\omega\cos\varphi\,\delta\dot{\vartheta} + \dot{\vartheta}\omega\cos\varphi\,\delta\dot{\varphi} - \dot{\vartheta}\sin\varphi\,\delta\varphi + 2\sin\vartheta\cdot\delta\vartheta$$
$$+ \sin(\vartheta+\varphi)(\delta\vartheta+\delta\varphi) =$$

$$\overset{*3}{\overset{*4}{=}} \Big[(\dots)\delta\vartheta + (-\!\!-)\,\delta\varphi\Big]_{t_A}^{t_B} +$$

$$+ \int \delta\varphi\cdot(-\ddot{\varphi} - \ddot{\vartheta} - \dot{\vartheta}\sin\varphi - \ddot{\vartheta}\omega\cos\varphi + \sin\varphi\,\dot{\vartheta}\dot{\varphi} - \dot{\vartheta}\sin\varphi) +$$
$$+ \delta\vartheta\cdot(-\ddot{\varphi} - 3\ddot{\vartheta} + \sin\varphi\cdot\dot{\varphi} - \ddot{\varphi}\omega\cos\varphi + (\dot{\varphi})^2\sin\varphi + 2\sin\vartheta) \Rightarrow$$

$$\overset{*5}{\Rightarrow} \begin{cases} \ddot{\varphi} + \ddot{\vartheta}(1+\omega\cos\varphi) + \sin\varphi\,\dot{\vartheta}(\dot{\varphi}-2) = 0 & (6\alpha) \\ \ddot{\varphi}(1+\omega\cos\varphi) + 3\ddot{\vartheta} + \sin\varphi\cdot\dot{\varphi}\cdot(\dot{\varphi}+1) + 2\sin\vartheta = 0 & (6\beta) \end{cases}$$

▶ Δοκιμάστε να επαληθεύσετε τις εξισώσεις (6α) κ' (6β).

▶ Υποσημειώσεις

*1 $\int \dots \equiv \int_{t=t_A}^{t=t_B} dt\,\{\dots\}$ όπου $t_A, t_B > t_A$ τυχαίες σταθερές.

*2 $\delta L(\varphi, \vartheta, \dot{\varphi}, \dot{\vartheta}) = \frac{\partial L}{\partial \vartheta}\delta\vartheta + \frac{\partial L}{\partial \varphi}\delta\varphi + \frac{\partial L}{\partial \dot{\vartheta}}\delta\dot{\vartheta} + \frac{\partial L}{\partial \dot{\varphi}}\delta\dot{\varphi}$

*3 $\delta\varphi = \delta\vartheta = 0$ για $t = t_A$ κ' $t = t_B$.

*4 $\omega\cos\varphi\,\delta\dot{\vartheta} = \omega\cos\varphi\,(\delta\vartheta)^{\cdot} = (\omega\cos\varphi\,\delta\vartheta)^{\cdot} - (\omega\cos\varphi)^{\cdot}\delta\vartheta$, κ.ο.κ.

*5 Εφόσον $\delta I = 0$ για τυχαία $\delta\varphi$ κ' $\delta\vartheta$.