Άσκηση 10.1

Θα λυθεί για την κατάσταση $|p\rangle = \sqrt{p}\,|1,1\rangle + \sqrt{1-p}\,|0,0\rangle$

$$\langle A\rangle = \langle p|\hat{A}|p\rangle = p\,\langle 1,1|\hat{A}|1,1\rangle + \sqrt{p(1-p)}\,\langle 1,1|\hat{A}|0,0\rangle + \sqrt{p(1-p)}\,\langle 0,0|\hat{A}|1,1\rangle + (1-p)\,\langle 0,0|\hat{A}|0,0\rangle$$

Βρίσκουμε

$$\langle 1,1|\hat{A}|1,1\rangle = \frac{i}{2}\langle 1,1|\sigma_1\otimes\sigma_2 - \sigma_2\otimes\sigma_1|1,1\rangle = \frac{i}{2}\left(\langle 1|\sigma_1|1\rangle\langle 1|\sigma_2|1\rangle - \langle 1|\sigma_2|1\rangle\langle 1|\sigma_1|1\rangle\right) = 0$$

$$\langle 0,0|\hat{A}|0,0\rangle = \frac{i}{2}\langle 0,0|\sigma_1\otimes\sigma_2 - \sigma_2\otimes\sigma_1|0,0\rangle = \frac{i}{2}\left(\langle 0|\sigma_1|0\rangle\langle 0|\sigma_2|0\rangle - \langle 0|\sigma_2|0\rangle\langle 0|\sigma_1|0\rangle\right) = 0$$

$$\langle 0,0|\hat{A}|1,1\rangle = \frac{i}{2}\langle 0,0|\sigma_1\otimes\sigma_2 - \sigma_2\otimes\sigma_1|0,0\rangle = \frac{i}{2}\left(\langle 0|\sigma_1|1\rangle\langle 0|\sigma_2|1\rangle - \langle 0|\sigma_2|1\rangle\langle 0|\sigma_1|1\rangle\right) = 0$$

άρα $\langle\hat{A}\rangle = 0$

Βρίσκουμε τον τελεστή $A^2 = -\frac{1}{4}(\sigma_1\otimes\sigma_2 - \sigma_2\otimes\sigma_1)(\sigma_1\otimes\sigma_2 - \sigma_2\otimes\sigma_1)$

$$= -\frac{1}{4}\left(\sigma_1\sigma_1\otimes\sigma_2\sigma_2 - \sigma_1\sigma_2\otimes\sigma_2\sigma_1 - \sigma_2\sigma_1\otimes\sigma_1\sigma_2 + \sigma_2\sigma_2\otimes\sigma_1\sigma_1\right)$$

$$= -\frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} - (i\sigma_3)\otimes(-i\sigma_3) - (-i\sigma_3)\otimes(i\sigma_3) + \mathbb{1}\otimes\mathbb{1}\right) = -\frac{1}{2}\left(\mathbb{1}\otimes\mathbb{1} - \sigma_3\otimes\sigma_3\right)$$

$$\langle A^2\rangle = \langle p|\hat{A}^2|p\rangle = p\,\langle 1,1|A^2|1,1\rangle + \sqrt{p(1-p)}\,\langle 1,1|A^2|0,0\rangle + \langle 0,0|A^2|1,1\rangle + (1-p)\,\langle 0,0|A^2|0,0\rangle$$

$$\langle 1,1|A^2|1,1\rangle = -\frac{1}{2}\langle 1,1|\mathbb{1}\otimes\mathbb{1} - \sigma_3\otimes\sigma_3|1,1\rangle = -\frac{1}{2}\left(\langle 1|1\rangle\langle 1|1\rangle - \langle 1|\hat{\sigma}_3|1\rangle\langle 1|\hat{\sigma}_3|1\rangle\right) = 0$$

$$\langle 0,0|A^2|0,0\rangle = -\frac{1}{2}\langle 0,0|\mathbb{1}\otimes\mathbb{1} - \sigma_3\otimes\sigma_3|0,0\rangle = \cdots = 0$$

$$\langle 0,0|A^2|1,1\rangle = -\frac{1}{2}\langle 0,0|\mathbb{1}\otimes\mathbb{1} - \sigma_3\otimes\sigma_3|1,1\rangle = \left(\langle 0|1\rangle\langle 0|1\rangle - \langle 0|\sigma_3|1\rangle\langle 0|\sigma_3|1\rangle\right) = 0$$

Άρα $\langle A^2\rangle = 0$

$$(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2 = 0$$

Άσκηση 10.3

$$\hat{A} = \hat{m}\cdot\hat{\sigma} = \begin{pmatrix} m_3 & m_1 - i m_2 \\ m_1 + i m_2 & -m_3 \end{pmatrix} \longrightarrow \begin{matrix} \langle 0|\hat{A}|0\rangle = -m_3 & \langle 0|\hat{A}|1\rangle = m_1 + i m_2 \\ \langle 1|\hat{A}|1\rangle = m_3 & \langle 1|\hat{A}|0\rangle = m_1 - i m_2 \end{matrix}$$

$$\hat{B} = \hat{n}\cdot\hat{\sigma} = \begin{pmatrix} n_3 & n_1 - i n_2 \\ n_1 + i n_2 & -n_3 \end{pmatrix} \longrightarrow \begin{matrix} \langle 0|\hat{B}|0\rangle = -n_3 & \langle 0|\hat{B}|1\rangle = n_1 + i n_2 \\ \langle 1|\hat{B}|1\rangle = n_3 & \langle 1|\hat{B}|0\rangle = n_1 - i n_2 \end{matrix}$$

Ενδεικτικά για

$$|\phi_-\rangle = \frac{1}{\sqrt{2}}\left(|1,1\rangle - |0,0\rangle\right)$$

$$Cor(m,n) = \langle \phi_-|\hat{A}\otimes\hat{B}|\phi_-\rangle = \frac{1}{2}\Big(\langle 1,1|A\otimes B|1,1\rangle + \langle 0,0|\hat{A}\otimes B|0,0\rangle$$

$$-\langle 0,0|\hat{A}\otimes B|1,1\rangle - \langle 1,1|\hat{A}\otimes B|0,0\rangle\Big)$$

$$= \frac{1}{2}\Big(\langle 1|A|1\rangle\langle 1|B|1\rangle + \langle 0|A|0\rangle\langle 0|B|0\rangle - \langle 0|\hat{A}|1\rangle\langle 0|B|1\rangle - \langle 1|\hat{A}|0\rangle\langle 1|\hat{B}|0\rangle\Big)$$

$$= \frac{1}{2}\Big(m_3 n_3 + (-m_3)(-n_3) - (m_1 + i m_2)(n_1 + i n_2) - (m_1 - i m_2)(n_1 - i n_2)\Big)$$

$$= m_3 n_3 + m_2 n_2 - m_1 n_1$$

Άσκηση 10.2

έχουμε $\langle 0,0|p\rangle = \sqrt{1-p}$, $\langle 1,1|p\rangle = \sqrt{p}$, $\langle 0,1|p\rangle = \langle 1,0|p\rangle = 0$

ρ_1

$$\langle i|\rho_1|j\rangle = \langle i,0|\hat{\rho}_1|j,0\rangle + \langle i,1|\hat{\rho}_1|j,1\rangle$$

$$= \langle i,0|p\rangle\langle p|j,0\rangle + \langle i,1|p\rangle\langle p|j,1\rangle$$

$i = j = 0$

$$\langle 0|\hat{\rho}_1|0\rangle = \langle 0,0|p\rangle\langle p|0,0\rangle + \langle 0,1|p\rangle\langle p|0,1\rangle = \sqrt{1-p}\,\sqrt{1-p} + 0\cdot 0 = 1-p$$

$$\langle 1|\hat{\rho}_1|1\rangle = \langle 1,0|p\rangle\langle p|1,0\rangle + \langle 1,1|p\rangle\langle p|1,1\rangle = 0\cdot 0 + \sqrt{p}\sqrt{p} = p$$

$$\langle 0|\hat{\rho}_1|1\rangle = \langle 0,0|p\rangle\langle p|1,0\rangle + \langle 0,1|p\rangle\langle p|1,1\rangle = \sqrt{1-p}\cdot 0 + 0\cdot\sqrt{p} = 0$$

άρα

$$\rho_1 = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}$$

Άσκηση 10.5

γράφουμε $\sigma_1 = \sigma_+ + \sigma_-$, $\sigma_2 = -i(\sigma_+ - \sigma_-)$, οπότε

$$K_\pm = i(\sigma_+ + \sigma_-)\otimes(\sigma_+ - \sigma_-) \pm (-i)(\sigma_+ - \sigma_-)\otimes(\sigma_+ + \sigma_-) \longrightarrow \begin{cases} \hat{K}_+ = -2i(\sigma_+\otimes\sigma_+ - \sigma_-\otimes\sigma_-) \\ \hat{K}_- = 2i(\sigma_+\otimes\sigma_- - \sigma_-\otimes\sigma_+) \end{cases}$$

Θεωρούμε τη βάση που αποτελείται από τα $|1,1\rangle, |1,0\rangle, |0,1\rangle, |0,0\rangle$.

$\hat{K}_+|0,0\rangle = -2i\,\sigma_+|0\rangle\otimes\sigma_+|0\rangle + 2i\,\sigma_-|0\rangle\otimes\sigma_-|0\rangle =$
$= -2i\,|1\rangle\otimes|1\rangle = -2i\,|1,1\rangle$

$K_+|1,1\rangle = -2i\,\sigma_+|1\rangle\otimes\sigma_+|1\rangle + 2i\,\sigma_-|1\rangle\otimes\sigma_-|1\rangle =$
$= 2i\,|0\rangle\otimes|0\rangle = 2i\,|0,0\rangle$

$\hat{K}_+|0,1\rangle = 0$

$\hat{K}_+|1,0\rangle = 0$

Άρα έχουμε ιδιοτιμή 0 με διπλό εκφυλισμό και

$$\hat{K}_+\begin{pmatrix}|0,0\rangle\\|1,1\rangle\end{pmatrix} = \begin{pmatrix}0 & -2i\\ 2i & 0\end{pmatrix}\begin{pmatrix}|0,0\rangle\\|1,1\rangle\end{pmatrix}$$

δηλ. ο $\hat{K}_+$ αντιστοιχεί στον πίνακα $2\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = 2\sigma_2$ στον υπόχωρο που ορίζουν τα $|0,0\rangle$ και $|1,1\rangle$.

Δεδομένου ότι οι ιδιοτιμές του $2\sigma_2$ είναι οι ± 2 με ιδιοδιανύσματα $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$ και $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$ παίρνουμε ότι τα $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|0,0\rangle \pm i\,|1,1\rangle\right)$ ικανοποιούν

$\hat{K}_+|\psi_+\rangle = 2|\psi_+\rangle$

$K_+|\psi_-\rangle = -2|\psi_-\rangle$

$\hat{K}_-|0,0\rangle = 2i\,\sigma_+|0\rangle\otimes\sigma_-|0\rangle - 2i\,\sigma_-|0\rangle\otimes\sigma_+|0\rangle = 0$

$K_-|1,1\rangle = 2i\,\sigma_+|1\rangle\otimes\sigma_-|1\rangle - 2i\,\sigma_-|1\rangle\otimes\sigma_+|1\rangle = 0$

$K_-|0,1\rangle = 2i\,\sigma_+|0\rangle\otimes\sigma_-|1\rangle - 2i\,\sigma_-|0\rangle\otimes\sigma_+|1\rangle$
$= 2i\,|1\rangle\otimes|0\rangle = 2i\,|1,0\rangle$

$K_-|1,0\rangle = 2i\,\sigma_+|1\rangle\otimes\sigma_-|0\rangle - 2i\,\sigma_-|1\rangle\otimes\sigma_+|0\rangle$
$= -2i\,|0\rangle\otimes|1\rangle = -2i\,|0,1\rangle$

Άρα έχουμε ιδιοτιμή 0 με διπλό εκφυλισμό και

$$K_-\begin{pmatrix}|1,0\rangle\\|0,1\rangle\end{pmatrix} = \begin{pmatrix}0 & -2i\\ 2i & 0\end{pmatrix}\begin{pmatrix}|1,0\rangle\\|0,1\rangle\end{pmatrix}$$

δηλ ο K_- αντιστοιχεί στον πίνακα $2\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = 2\sigma_2$ στον υπόχωρο που ορίζουν τα $|1,0\rangle$ και $|0,1\rangle$.

Όπως και στην περίπτωση του K_+ ορίζουμε $|\phi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle \pm i\,|0,1\rangle\right)$ οπότε

$\hat{K}_-|\phi_+\rangle = 2|\phi_+\rangle$

$K_-|\phi_-\rangle = -2|\phi_-\rangle$

Άσκηση 10.5 συνέχεια

Θεωρούμε τη βάση $|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle$ στο $\mathbb{C}^2 \otimes \mathbb{C}^2$

Ισχύει ότι $\hat{\sigma}_1|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |1\rangle$, $\hat{\sigma}_1|1\rangle = |0\rangle$

$\hat{\sigma}_2|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} i \\ 0 \end{pmatrix} = -i|1\rangle$ $\hat{\sigma}_2|1\rangle = i|0\rangle$

$\sigma_3|0\rangle = -|0\rangle$ $\sigma_3|1\rangle = |1\rangle$

οπότε

$(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)|0,0\rangle = \sigma_1|0\rangle \otimes \sigma_1|0\rangle + \sigma_2|0\rangle \otimes \sigma_2|0\rangle + \sigma_3|0\rangle \otimes \sigma_3|0\rangle$

$= |1\rangle \otimes |1\rangle - |1\rangle \otimes |1\rangle + |0\rangle \otimes |0\rangle = |0,0\rangle$, άρα το $|0,0\rangle$ είναι ιδιοδιάνυσμα του $\hat{\Lambda}$ με ιδιοτιμή 1.

$(\hat{\sigma}_1 \otimes \hat{\sigma}_1 + \hat{\sigma}_2 \otimes \hat{\sigma}_2 + \hat{\sigma}_3 \otimes \hat{\sigma}_3)|1,1\rangle = \sigma_1|1\rangle \otimes \sigma_1|1\rangle + \sigma_2|1\rangle \otimes \sigma_2|1\rangle + \sigma_3|1\rangle \otimes \sigma_3|1\rangle$

$= |0\rangle \otimes |0\rangle - |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle = |1,1\rangle$

άρα το $|1,1\rangle$ είναι ιδιοδιάνυσμα του $\hat{\Lambda}$ με ιδιοτιμή 1.

$(\hat{\sigma}_1 \otimes \hat{\sigma}_1 + \hat{\sigma}_2 \otimes \sigma_2 + \hat{\sigma}_3 \otimes \sigma_3)|0,1\rangle = \sigma_1|0\rangle \otimes \sigma_1|1\rangle + \sigma_2|0\rangle \otimes \sigma_2|1\rangle + \sigma_3|0\rangle \otimes \sigma_3|1\rangle$

$= |1\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle = 2|1,0\rangle - |0,1\rangle$

ομοίως

$(\hat{\sigma}_1 \otimes \hat{\sigma}_1 + \hat{\sigma}_2 \otimes \hat{\sigma}_2 + \hat{\sigma}_3 \otimes \hat{\sigma}_3)|1,0\rangle = 2|0,1\rangle - |1,0\rangle$

Άρα

$$\hat{\Lambda}\begin{pmatrix} |0,1\rangle \\ |1,0\rangle \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} |0,1\rangle \\ |1,0\rangle \end{pmatrix}$$

Δηλαδή στον υπόχωρο που αντιστοιχεί στο $|0,1\rangle$ και $|1,0\rangle$ ο $\hat{\Lambda}$ δρα σαν τον τελεστή $\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$ που έχει ιδιοτιμή -3 με ιδιοδιάνυσμα $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$+1$ με ιδιοδιάνυσμα $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

δηλαδή το διάνυσμα $\frac{1}{\sqrt{2}}(|0,1\rangle - |1,0\rangle)$ είναι ιδιοδιάνυσμα του $\hat{\Lambda}$ με ιδιοτιμή -3

το διάνυσμα $\frac{1}{\sqrt{2}}(|0,1\rangle + |1,0\rangle)$ είναι ιδιοδιάνυσμα του Λ με ιδιοτιμή 1

Άσκηση 10.11

οι προβολικοί τελεστές που αντιστοιχούν στο σ_1 είναι οι $P_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$P_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

" " " σ_2 είναι οι $Q_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$

" " " $Q_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$

σ_3 είναι οι $R_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$R_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$Prob(1,1,1) = \langle\psi| P_+ \otimes Q_+ \otimes R_+ |\psi\rangle = \frac{1}{2}\Big(\langle 0,0,0| P_+ \otimes Q_+ \otimes R_+ |0,0,0\rangle$$

$$+ \langle 1,1,1| P_+ \otimes Q_+ \otimes R_+ |1,1,1\rangle + \langle 0,0,0| \hat{P}_+ \otimes \hat{Q}_+ \otimes \hat{R}_+ |1,1,1\rangle + \langle 1,1,1| \hat{P}_+ \otimes Q_+ \otimes R_+ |0,0,0\rangle\Big)$$

$$= \frac{1}{2}\Big(\langle 0|\hat{P}_+|0\rangle \langle 0|\hat{Q}_+|0\rangle \langle 0|\hat{R}_+|0\rangle + \langle 1|\hat{P}_+|1\rangle \langle 1|\hat{Q}_+|1\rangle \langle 1|\hat{R}_+|1\rangle$$

$$+ \langle 1|\hat{P}_+|0\rangle \langle 1|\hat{Q}_+|0\rangle \langle 1|\hat{R}_+|0\rangle + \langle 0|\hat{P}_+|1\rangle \langle 0|\hat{Q}_+|1\rangle \langle 0|\hat{R}_+|1\rangle\Big)$$

$$= \frac{1}{2}\Big(\frac{1}{2}\cdot\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{1}{2}\cdot 1 + \frac{1}{2}\,\frac{-i}{2}\cdot 0 + \frac{1}{2}\,\frac{i}{2}\cdot 0\Big) = \frac{1}{8}$$

$Prob(1,1,-1) =$ ίδιες εκφράσεις αλλά με $R_+ \rightarrow R_-$ =

$$= \frac{1}{2}\Big(\langle 0|P_+|0\rangle \langle 0|Q_+|0\rangle \langle 0|R_-|0\rangle + \langle 1|P_+|0\rangle \langle 1|Q_+|1\rangle \langle 1|R_-|1\rangle$$

$$+ \langle 1|\hat{P}_+|0\rangle \langle 1|\hat{Q}_+|0\rangle \langle 1|\hat{R}_-|0\rangle + \langle 0|\hat{P}_+|1\rangle \langle 0|\hat{Q}_+|1\rangle \langle 0|\hat{R}_-|1\rangle\Big)$$

$$= \frac{1}{2}\Big(\frac{1}{2}\cdot\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{2}\cdot 0 + \frac{1}{2}\,\frac{-i}{2}\cdot 0 + \frac{1}{2}\,\frac{i}{2}\,0\Big) = \frac{1}{8}$$

κ.ο.κ.