Κεφ 8

① $\hat{\rho}_0 = \frac{1}{2}\begin{pmatrix} 1+n_3 & n_1 - i n_2 \\ n_1 + i n_2 & 1 - n_3 \end{pmatrix}$, $\hat{H} = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$, $e^{-i\hat{H}t} = \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix}$

$$\hat{\rho}(t) = \begin{pmatrix} e^{-i\omega t} & 0 \\ 0 & e^{i\omega t} \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1+n_3 & n_1 - i n_2 \\ n_1 + i n_2 & 1 - n_3 \end{pmatrix}\begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} (1+n_3)e^{-i\omega t} & (n_1 - i n_2)e^{-i\omega t} \\ (n_1+in_2)e^{i\omega t} & (1-n_3)e^{i\omega t} \end{pmatrix}\begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+n_3 & (n_1 - in_2)e^{-2i\omega t} \\ (n_1+in_2)e^{2i\omega t} & (1-n_3) \end{pmatrix}$$

Οι φασματικοί προβολείς του $\hat{\sigma}_1$ είναι $\hat{P}_+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\hat{P}_- = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

$$\mathrm{Prob}(+1,t) = \frac{1}{4}\,\mathrm{Tr}\begin{pmatrix} 1+n_3 & (n_1-in_2)e^{-2i\omega t} \\ (n_1+in_2)e^{2i\omega t} & 1-n_3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} =$$

$$= \frac{1}{4}\,\mathrm{Tr}\begin{pmatrix} 1+n_3+(n_1-in_2)e^{-2i\omega t} & \\ & 1-n_3+(n_1+in_2)e^{2i\omega t} \end{pmatrix} = \frac{1}{2}\left(1 + n_1\cos 2\omega t - n_2 \sin 2\omega t\right)$$

$$\mathrm{Prob}(-1,t) = 1 - \mathrm{Prob}(+1,t) = \frac{1}{2}\left(1 - n_1\cos 2\omega t + n_2 \sin 2\omega t\right)$$

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② Στη βάση που ορίζουν τα $|1\rangle, |2\rangle, |3\rangle$, $\hat{H} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 3\omega \end{pmatrix}$, $e^{-i\hat{H}t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\omega t} & 0 \\ 0 & 0 & e^{-3i\omega t} \end{pmatrix}$

$|\alpha_\pm\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \pm\sqrt{2} \\ 1 \end{pmatrix}$, $\hat{\rho}_0 = \frac{2}{3}|\alpha_+\rangle\langle\alpha_+| + \frac{1}{3}|\alpha_-\rangle\langle\alpha_-| =$

$$= \frac{2}{12}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix} + \frac{1}{12}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{2} & 1 \end{pmatrix} = \frac{2}{12}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 2 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix} + \frac{1}{12}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ -\sqrt{2} & 2 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 3 & \sqrt{2} & 3 \\ \sqrt{2} & 6 & \sqrt{2} \\ 3 & \sqrt{2} & 3 \end{pmatrix}$$

$$\hat{\rho}(t) = e^{-i\hat{H}t}\rho_0 e^{i\hat{H}t} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\omega t} & 0 \\ 0 & 0 & e^{-3i\omega t} \end{pmatrix}\frac{1}{12}\begin{pmatrix} 3 & \sqrt{2} & 3 \\ \sqrt{2} & 6 & \sqrt{2} \\ 3 & \sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\omega t} & 0 \\ 0 & 0 & e^{3i\omega t} \end{pmatrix}$$

$$= \frac{1}{12}\begin{pmatrix} 3 & \sqrt{2} & 3 \\ \sqrt{2}e^{-i\omega t} & 6e^{-i\omega t} & \sqrt{2}e^{-i\omega t} \\ 3e^{-3i\omega t} & \sqrt{2}e^{-3i\omega t} & 3e^{-3i\omega t} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\omega t} & 0 \\ 0 & 0 & e^{3i\omega t} \end{pmatrix} = \frac{1}{12}\begin{pmatrix} 3 & \sqrt{2}e^{i\omega t} & 3e^{3i\omega t} \\ \sqrt{2}e^{-i\omega t} & 6 & \sqrt{2}e^{2i\omega t} \\ 3e^{-3i\omega t} & \sqrt{2}e^{-2i\omega t} & 3 \end{pmatrix}$$

Τα $\hat{A}$ έχει δυνατές τιμές K με ιδιοδιάνυσμα $|\alpha_+\rangle$
$-K$ με ιδιοδιάνυσμα $|\alpha_-\rangle$
0 με ιδιοδιάνυσμα $|\alpha_0\rangle$ κάθετο στα $|\alpha_+\rangle, |\alpha_-\rangle$

$$\cancel{Prob(K,t) = \langle\alpha_+|\hat\rho(t)|\alpha_+\rangle = \tfrac{1}{12}\ldots}$$

$$Prob(K,t) = \langle\alpha_+|\hat\rho(t)|\alpha_+\rangle = \frac{1}{48}\begin{pmatrix}1 & \sqrt2 & 1\end{pmatrix}\begin{pmatrix}3 & \sqrt2\, e^{i\omega t} & 3e^{3i\omega t}\\ \sqrt2\, e^{-i\omega t} & 6 & \sqrt2\, e^{2i\omega t}\\ 3e^{-3i\omega t} & \sqrt2\, e^{-2i\omega t} & 3\end{pmatrix}\begin{pmatrix}1\\ \sqrt2\\ 1\end{pmatrix}$$

$$= \frac{1}{48}\begin{pmatrix}1 & \sqrt2 & 1\end{pmatrix}\begin{pmatrix}3 + 2e^{i\omega t} + 3e^{3i\omega t}\\ \sqrt2\, e^{-i\omega t} + 6\sqrt2 + \sqrt2\, e^{i\omega t}\\ 3e^{-3i\omega t} + 2e^{-2i\omega t} + 3\end{pmatrix} =$$

$$= \frac{1}{48}\left(3 + 2e^{i\omega t} + 3e^{3i\omega t} + 2e^{-i\omega t} + 12 + 2e^{i\omega t} + 3e^{-3i\omega t} + 2e^{-2i\omega t} + 3\right)$$

$$= \frac{1}{48}\left(18 + 4\cos\omega t + 6\cos3\omega t + 4\cos2\omega t\right) = \frac{3}{8} + \frac{1}{12}\cos\omega t + \frac{1}{8}\cos3\omega t + \frac{1}{12}\cos2\omega t$$

$$Prob(-K,t) = \langle\alpha_-|\hat\rho(t)|\alpha_-\rangle = \frac{1}{48}\begin{pmatrix}1 & -\sqrt2 & 1\end{pmatrix}\begin{pmatrix}3 & \sqrt2\, e^{i\omega t} & 3e^{3i\omega t}\\ \sqrt2\, e^{-i\omega t} & 6 & \sqrt2\, e^{2i\omega t}\\ 3e^{-3i\omega t} & \sqrt2\, e^{-2i\omega t} & 3\end{pmatrix}\begin{pmatrix}1\\ -\sqrt2\\ 1\end{pmatrix}$$

$$= \frac{1}{48}\begin{pmatrix}1 & -\sqrt2 & 1\end{pmatrix}\begin{pmatrix}3 - 2e^{i\omega t} + 3e^{3i\omega t}\\ \sqrt2\, e^{-i\omega t} - 6\sqrt2 + \sqrt2\, e^{i\omega t}\\ 3e^{-3i\omega t} - 2e^{-2i\omega t} + 3\end{pmatrix} =$$

$$= \frac{1}{48}\left(3 - 2e^{i\omega t} + 3e^{3i\omega t} - 2e^{-i\omega t} + 12 - 2e^{i\omega t} + 3e^{-3i\omega t} - 2e^{-2i\omega t} + 3\right) =$$

$$= \frac{1}{48}\left(18 - 4\cos\omega t - 6\cos3\omega t - 4\cos2\omega t\right) = \frac{3}{8} - \frac{1}{12}\cos\omega t + \frac{1}{8}\cos3\omega t - \frac{1}{12}\cos2\omega t$$

$$Prob(0,t) = 1 - Prob(K_+,t) - Prob(K_-,t) = \frac{1}{4}\left(1 - \cos3\omega t\right)$$

③ $\psi_1(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} \frac{1}{\sqrt{1+\frac{it}{2m\sigma^2}}} e^{\frac{imx^2}{2t} - \frac{\sigma^2(p_0 - \frac{mx}{t})^2}{1 - \frac{im\sigma^2}{t}}}$

στο βιβλίο υπολογίσαμε $\Delta x(t) = \sigma\sqrt{1+\frac{t^2}{4m^2\sigma^4}}$

για ελεύθερο σωμάτιο: $\langle \hat{\dot{x}} \rangle = \langle \hat{P}/m \rangle = \frac{P_0}{m}$

άρα $\delta t = \frac{\Delta x}{|\langle \dot{x} \rangle|} = \frac{m\sigma}{P_0}\sqrt{1+\frac{t^2}{4m^2\sigma^4}}$

Οι πιθανότητες για την ενέργεια δεν εξαρτώνται από το χρόνο. Άρα υπολογίζονται για την αρχική κατάσταση $\psi_0(x) = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{x^2}{4\sigma^2} + ip_0 x}$

$\langle \psi_0 | \hat{H} | \psi_0 \rangle = -\frac{1}{2m}\int dx\, \psi_0(x)\psi_0''(x) = -\frac{1}{2m}\psi_0(x)\psi_0'(x)\Big|_{-\infty}^{\infty} + \frac{1}{2m}\int dx\, |\psi'(x)|^2 = \frac{1}{2m}\int dx\, |\psi'(x)|^2$

Ομοίως $\langle \psi_0 | \hat{H}^2 | \psi_0 \rangle = \frac{1}{4m^2}\int dx\, |\psi''(x)|^2$

βρίσκουμε $\psi_0'(x) = \left(-\frac{x}{2\sigma^2} + ip_0\right)\psi_0(x)$, $\psi_0''(x) = \left[-\frac{1}{2\sigma^2} + \left(-\frac{x}{2\sigma^2} + ip_0\right)^2\right]\psi_0(x)$

Άρα $\int dx\, |\psi_0'(x)|^2 = \int dx\, |\psi_0(x)|^2 \left(p_0^2 + \frac{x^2}{4\sigma^4}\right) = p_0^2 + \frac{\langle x^2 \rangle}{4\sigma^4} = p_0^2 + \frac{1}{4\sigma^2}$

αφού $\langle x^2 \rangle = \sigma^2$

$\int dx\, |\psi''(x)|^2 = \int dx\, |\psi_0(x)|^2 \left[\left(\frac{1}{2\sigma^2} + p_0^2 - \frac{x^2}{4\sigma^4}\right)^2 + \frac{p_0^2 x^2}{\sigma^4}\right] =$

$= \int dx\, |\psi_0(x)|^2 \left[\left(p_0^2 + \frac{1}{2\sigma^2}\right)^2 - \left(\frac{1}{2\sigma^2} + p_0^2\right)\frac{x^2}{2\sigma^4} + \frac{x^4}{16\sigma^8} + \frac{p_0^2 x^2}{\sigma^4}\right] =$

$= \left(p_0^2 + \frac{1}{2\sigma^2}\right)^2 + \frac{\langle x^2 \rangle}{2\sigma^4}\left(p_0^2 - \frac{1}{2\sigma^2}\right) + \frac{\langle x^4 \rangle}{16\sigma^8} = p_0^4 + \frac{3}{16}\sigma^4 + \frac{3p_0^2}{2\sigma^2}$

αφού $\langle x^2 \rangle = \sigma^2$, $\langle x^4 \rangle = 3\sigma^2$ για Γκαουσιανή

Άρα $(\Delta H)^2 = \frac{1}{4m^2}\left(p_0^4 + \frac{3}{16}\sigma^4 + \frac{3p_0^2}{2\sigma^2} - \left(p_0^2 + \frac{1}{4\sigma^2}\right)^2\right) = \frac{1}{4m^2}\left(\frac{p_0^2}{\sigma^2} + \frac{1}{8\sigma^4}\right)$

$(\Delta H) = \frac{1}{2m\sigma}\sqrt{p_0^2 + \frac{1}{8\sigma^2}}$

$(\Delta H)\,\delta t = \frac{1}{2}\sqrt{1 + \frac{1}{8\sigma^2 p_0^2}}\sqrt{1+\frac{t^2}{4m^2\sigma^4}}$

$\boxed{6}$ στο κεφ. 6.6.1 δείξαμε ότι τα γενικευμένα ιδιοδιανύσματα του $\hat{p}^2$ στο $L^2(\mathbb{R}^+)$ είναι $f_k(x) = \sqrt{\frac{2}{\pi}} \sin kx$

Άρα $\langle x| e^{-i\hat{H}t} |x'\rangle = \int_0^\infty dk \, \langle x|k\rangle\langle k|x'\rangle \, e^{-i\frac{k^2}{2m}t} =$

$$= \frac{2}{\pi} \int_0^\infty dk \sin kx \, \sin kx' \, e^{-i\frac{k^2}{2m}t} = \frac{2}{\pi} \frac{1}{(2i)^2} \int_0^\infty \left(e^{ikx} - e^{-ikx}\right)\left(e^{ikx'} - e^{-ikx'}\right) e^{-i\frac{k^2}{2m}t}$$

$$= -\frac{1}{2\pi} \int_0^\infty dk \left(e^{ik(x+x')} + e^{-ik(x+x')}\right) e^{-i\frac{k^2}{2m}t} + \frac{1}{2\pi} \int_0^\infty \left(e^{-ik(x-x')} + e^{ik(x-x')}\right) e^{-i\frac{k^2}{2m}t} dk$$

$$= -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x+x') - i\frac{k^2}{2m}t} dk + \frac{1}{2m} \int_{-\infty}^{\infty} e^{ik(x-x') - i\frac{k^2}{2m}t} dk$$

αυτός είναι ο διαδότης ελεύθερου σωματιδίου εξ (8.33) $G_t(x,-x')$

$G_t(x,x')$

Άρα $\langle x| e^{-iHt} |x'\rangle = G_t(x,x') - G_t(x,-x')$

$\boxed{7}$ $\langle x| e^{-i\hat{H}t} |x'\rangle = \int_{-\infty}^{\infty} dp \, \langle x|p\rangle \langle p|x'\rangle \, e^{-iv|p|t} = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \, e^{ip(x-x') - iv|p|t} =$

$$= \int_0^{\infty} \frac{dp}{2\pi} \, e^{ip(x-x') - ivpt} + \int_0^{\infty} \frac{dp}{2\pi} \, e^{-ip(x-x') - ivpt} = \frac{1}{\pi} \int_0^{\infty} \frac{dp}{2\pi} \cos p(x-x') \, e^{-ivpt}$$

Υπολογίζουμε το ολοκλήρωμα για $t \to t - i\varepsilon$

$$G_{t-i\varepsilon}(x,x') = \frac{1}{\pi} \int_0^{\infty} dp \, e^{-p(\varepsilon v - ivt)} \cos p(x-x') = \frac{1}{\pi} \frac{\varepsilon v - ivt}{(x-x')^2 + v^2(\varepsilon - it)^2}$$

στο όριο $\varepsilon \to 0$

$$G_t(x,x') = \frac{-ivt}{\pi} \frac{1}{(x-x')^2 - v^2 t^2}$$
