INTRODUCTION TO MANIFOLDS

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Preface

These are very brief lecture notes for a course on manifolds consisting of around ten 90 minute lectures. I have given similar lectures on this topic, in Japanese, several times to third or fourth year undergraduates at Tokyo Metropolitan University and Keio University. In 2006/7 I gave the course in English and wrote these notes to accompany the lectures.

The notes contain very few proofs; in the lectures I gave more details or referred to a textbook. The lectures also contained more examples. However, the philosophy of the course was to introduce the main ideas, with a minimum of technicalities.

All of the students had studied calculus, linear algebra and differential geometry of curves and surfaces. Many of the students would take further (equally brief) courses on differential forms, differential geometry of manifolds, or homology theory, and this is why the course ends the way it does. Some of the students would continue in a graduate program, where they would be expected to "know" manifold theory (and therefore have to fill in many remaining details themselves when the need arose).

It is my understanding that similar courses are given at many other Japanese universities. Despite the brevity of the course, its early appearance in the curriculum (by current international standards) serves the purpose of introducing a nontrivial topic to undergraduate geometry students.

Corrections and comments are welcome!

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Introduction

Roughly speaking, a manifold is a geometrical object which, locally, looks like $\mathbf{R}^n$.

Why do we need manifolds?

There are many reasons. For example:

— For the accurate description of "configuration spaces" of physical objects we need manifolds. For example, the positions of an aircraft can be described by five numbers: the three coordinates of a point of the aircraft, and the two Euler angles giving its orientation. This is a five-dimensional manifold.

— The (maximal) domain of a solution of a differential equation is often a manifold. For example, the domain of a solution of a complex ordinary differential equation is a Riemann surface (a two-dimensional manifold).

— Many common geometrical objects are manifolds. For example, a torus is a two-dimensional manifold.

— Many common geometrical concepts can be described as manifolds. For example, the group of all rotational transformations of $\mathbf{R}^3$ is a manifold. It is a three-dimensional manifold, because two parameters are needed to fix the axis of rotation, and one parameter is needed to fix the angle of rotation.

We cannot deal with "arbitrary" geometrical objects (as they are too complicated). In order to use the tools of calculus, we insist that each small piece of the object looks like $\mathbf{R}^n$. Such an object will be called an n-dimensional manifold. We shall give the precise definition later.

1. Review of vector calculus notation

Before giving the precise definition of manifold, we review some notation from calculus.

1.1 Derivative of a scalar function

Let $x_1, \ldots, x_n$ be the standard coordinates of $\mathbf{R}^n$. Let y be the standard coordinate of $\mathbf{R}$.

Let U be an open subset of $\mathbf{R}^n$. Let

$$f : U \to \mathbf{R}, \quad (x_1, \ldots, x_n) \mapsto y = f(x_1, \ldots, x_n)$$

be a function.

Let $p = (p_1, \ldots, p_n) \in U$.

Definition. If there exist $A_1, \ldots, A_n \in \mathbf{R}$ such that

$$f(p + q) - f(p) = \sum_{i=1}^{n} A_i q_i + o(|q|)$$

for any $q = (q_1, \ldots, q_n)$ in some open neighbourhood of $0 = (0, \ldots, 0) \in \mathbf{R}^n$, then we say that f is differentiable at p. We say that $\sum_{i=1}^{n} A_i q_i$ is a linear approximation to f at p.

It follows from this definition that

$$A_i = \frac{\partial f}{\partial x_i}(p)$$

(the partial derivative of f at p in the direction of $e_i = (\overbrace{0, \ldots, 0}^{i-1}, 1, \overbrace{0, \ldots, 0}^{n-i})$).

Proposition. If $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ are continuous on U, then

$$f(p + q) - f(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p) q_i + o(|q|)$$

for any $p \in U$ and for any q in some open neighbourhood of $0 \in \mathbf{R}^n$. In this case we say that f is a C^1 function.

Proof. [omitted]

The map

$$df : U \to \mathbf{R}^n, \quad p \mapsto df_p = df(p) = (\frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_n}(p))$$

is called the derivative of f. If f is a C^1 function, then df is a C^0 function, i.e. a continuous map.

1.2 Derivative of a vector function

Next, let $y_1, \ldots, y_m$ be the standard coordinates of $\mathbf{R}^m$, and let

$$g : U \to \mathbf{R}^m, \quad x \mapsto y = g(x)$$

be a map. Here, "$x \mapsto y = g(x)$" is an abbreviation for

$$(x_1, \ldots, x_n) \mapsto (y_1, \ldots, y_m) = (g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)).$$

Definition. If there exist $A_{ij} \in \mathbf{R}$ such that

$$g(p+q) - g(p) = \left(\sum_{i=1}^{n} A_{1i} q_i, \ldots, \sum_{i=1}^{n} A_{mi} q_i \right) + o(|q|)$$

for any $q = (q_1, \ldots, q_n)$ in some open neighbourhood of $0 = (0, \ldots, 0) \in \mathbf{R}^n$, then we say that g is differentiable at p. We say that $(\sum_{i=1}^{n} A_{1i} q_i, \ldots, \sum_{i=1}^{n} A_{mi} q_i)$ is a linear approximation to g at p.

It follows from this definition that

$$A_{ij} = \frac{\partial g_i}{\partial x_j}(p).$$

Proposition. If the functions $\frac{\partial g_i}{\partial x_j}$ are continuous on U, then

$$g(p+q) - g(p) = \left(\sum_{i=1}^{n} \frac{\partial g_1}{\partial x_i}(p) q_i, \ldots, \sum_{i=1}^{n} \frac{\partial g_m}{\partial x_i}(p) q_i \right) + o(|q|)$$

for any $p \in U$ and for any q in some open neighbourhood of $0 \in \mathbf{R}^n$. In this case we say that g is a C^1 map.

Proof. [omitted]

The map

$$dg : U \to M_{m,n}\mathbf{R}, \quad p \mapsto dg_p = dg(p) = \left(\frac{\partial g_i}{\partial x_j}(p) \right)_{1 \le i \le m, 1 \le j \le n}$$

is called the derivative of g. If g is a C^1 map, then dg is a C^0 map, i.e. a continuous map.

(The vector space $M_{m,n}\mathbf{R}$, consisting of all $m \times n$ real matrices, is isomorphic to the vector space $\mathbf{R}^{mn}$. It is also isomorphic to the vector space $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^m)$, consisting of all linear maps $\mathbf{R}^n \to \mathbf{R}^m$.)

We can write $g(p+q) - g(p) = Aq + o(|q|)$ in the above definition, where Aq is an abbreviation for the matrix product

$$\begin{pmatrix} A_{11} & \ldots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \ldots & A_{mn} \end{pmatrix} \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix}.$$

Finally, let V, W be vector spaces of dimensions n, m. Assume that V has a norm $|\ |$. Let U be an open subset of V, and let $p \in U$. Let $g : U \to W$ be a map.

Definition. If there exists a linear map $A_p \in \mathrm{Hom}(V, W)$ such that

$$g(p+q) - g(p) = A_p(q) + o(|q|)$$

for any $q = (q_1, \ldots, q_n)$ in some open neighbourhood of $0 = (0, \ldots, 0) \in V$, then we say that g is differentiable at p. We say that A_p is a linear approximation to g at p.

It follows from this definition that

$$A_p(v) = \lim_{t \to 0} \frac{g(p+tv) - g(p)}{t}$$

for any $v \in V$ (the derivative of g at p in the direction of v).

Definition. If the map

$$dg : U \to \mathrm{Hom}(V, W), \quad p \mapsto A_p$$

is continuous on U, we say that g is a C^1 map and that dg is the derivative of g.

Thus, if $g : U \to W$ is a C^1 map, we obtain a C^0 (continuous) map $dg : U \to \mathrm{Hom}(V, W)$. Similarly, if $dg : U \to \mathrm{Hom}(V, W)$ is C^1, we obtain a C^0 map $d(dg) = d^2 g : U \to \mathrm{Hom}(V, \mathrm{Hom}(V, W))$. We can define $d^r g$ for $r = 1, 2, 3, \ldots$ in the same way.

Proposition. If the functions

$$\frac{\partial^{\alpha_1 + \cdots + \alpha_n} g_i}{(\partial x_1)^{\alpha_1} \ldots (\partial x_n)^{\alpha_n}} \quad (0 \le \alpha_i, \quad \alpha_1 + \cdots + \alpha_n \le r)$$

are continuous on U, then $dg, d^2 g, \ldots, d^r g$ are continuous on U. In this case we say that g is a C^r map. If this holds for all $r \ge 0$, we say that g is a C^∞ map, or a smooth map.

Proof. [omitted]

1.4 **Definition.** Let U_1, U_2 be open subsets of $\mathbf{R}^n$. If a map $g : U_1 \to U_2$ satisfies the conditions

(1) g is bijective

(2) g is smooth

(3) g^{-1} is smooth

then we say that g is a diffeomorphism. We say that U_1, U_2 are diffeomorphic.

Theorem. (Inverse Function Theorem) Let U be an open subset of $\mathbf{R}^n$, and let $p \in U$. Let $g : U \to \mathbf{R}^n$ be a smooth map. If $dg_p \in \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ is a linear isomorphism, then there exist open neighbourhoods U_0, V_0 of $p, g(p)$ such that $g|_{U_0} : U_0 \to V_0$ is a diffeomorphism.

2. Smooth manifolds

Definition. Let M be a Hausdorff topological space. We say that M is an n-dimensional topological manifold if it satisfies the following condition: for any $p \in M$, there exists

(1) an open subset U with $p \in U \subseteq M$,

(2) an open subset $E \subseteq \mathbf{R}^n$, and

(3) a homeomorphism $\psi : U \to E$.

Such a U is called a (local) coordinate neighbourhood, and ψ is called a (local) coordinate function. We also say that (U, ψ) (or $\psi : U \to E$) is a (local) chart.

Notation. We write $x = \psi(p)$ and regard $(x_1, \ldots, x_n)$ as "local coordinates" for the manifold M.

Definition. Let M be a topological manifold. Let A be a set. We say that S is a C^0-atlas (or coordinate neighbourhood system) for M if $S = \{(U_\alpha, \psi_\alpha) \mid \alpha \in A\}$ where

(1) U_α is an open subset of M, for all $\alpha \in A$

(2) $\psi_\alpha : U_\alpha \to E_\alpha$ is a homeomorphism to an open subset E_α of $\mathbf{R}^n$, for all $\alpha \in A$

(3) $\cup_{\alpha \in A} U_\alpha = M$.

Definition. Let S be a C^0-atlas for M. If $\psi_\alpha \circ \psi_\beta^{-1}$ is a C^∞-map for all $\alpha, \beta \in A$, we say that S is a C^∞-atlas for M. We call $\psi_\alpha \circ \psi_\beta^{-1}$ a coordinate transformation or transition function.

The domain of the map $\psi_\alpha \circ \psi_\beta^{-1}$ is assumed to be $\psi_\beta(U_\alpha \cap U_\beta)$ (which could be the empty set). Thus, $\psi_\alpha \circ \psi_\beta^{-1}$ is a homeomorphism from $\psi_\beta(U_\alpha \cap U_\beta)$ to $\psi_\alpha(U_\alpha \cap U_\beta)$.

Definition. Let M be a topological manifold. If S is a C^0-atlas for M, we say that the pair (M, S) is a C^0-manifold. If S is a C^∞-atlas for M, we say that the pair (M, S) is a C^∞-manifold (or smooth manifold, or differentiable manifold).

To simplify notation, we often say "M is a C^∞-manifold" instead of "(M, S) is a C^∞-manifold", if there is little danger of ambiguity.

The concept of C^r-manifold can be defined in a similar way. However, from now on in these notes, "manifold" always means "C^∞-manifold".

The concept of complex manifold can be defined in a similar way, using coordinate charts $\psi : U \to \mathbf{C}^n$. However, the term "complex manifold" will always mean "complex manifold with holomorphic (complex analytic) transition functions".

Examples. : The following topological spaces possess natural C^∞-atlases. They are examples of manifolds.

(1) $\mathbf{R}^n$

(2) S^n

(3) Any open subset of a manifold.

(4) The product $M_1 \times M_2$ of any manifolds M_1, M_2.

(5) $\mathbf{R}P^n$

The following topological spaces possess natural holomorphic atlases. They are all examples of complex manifolds.

(6) $\mathbf{C}^n$

(7) Any open subset of a complex manifold.

(8) The product $M_1 \times M_2$ of any complex manifolds M_1, M_2.

(9) $\mathbf{C}P^n$

Let us consider some examples of (real) manifolds related to matrices.

(10) The set

$$\mathrm{M}_{n,m}\mathbf{R} = \{\text{all } n \times m \text{ real matrices}\}$$

is a vector space of dimension nm. As a topological space, it is homeomorphic to $\mathbf{R}^{nm}$. Therefore, by (1), it is a manifold.

(11) The set

$$\mathrm{GL}_n\mathbf{R} = \{\text{all invertible } n \times n \text{ real matrices}\}$$

is an open subset of $\mathrm{M}_n\mathbf{R} = \mathrm{M}_{n,n}\mathbf{R}$ (because it is the complement of $\det^{-1}(0)$, and $\det : \mathrm{M}_n\mathbf{R} \to \mathbf{R}$ is a continuous function). Therefore, by (10) and (3), it is a manifold. It is also a group (under matrix multiplication). It is called the *general linear group.*

(12) Consider the set

$$\mathrm{SO}_n = \{A \in \mathrm{M}_n\mathbf{R} \mid A^t = A^{-1},\ \det A = 1\}.$$

Is it a manifold? It is certainly a group, and it is called the *special orthogonal group.* In fact, it is a manifold, and we shall be able to prove this later. For the moment, let us consider three special cases.

(i) $n = 1$: We have $\mathrm{SO}_1 = \{1\}$. This is a topological space with a single point. It is a manifold of dimension zero.

(ii) $n = 2$: By direct calculation, we have

$$\begin{aligned}\mathrm{SO}_2 &= \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, a = d, b = -c\} \\ &= \{A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^2 + b^2 = 1\}.\end{aligned}$$

It follows that SO_2 is homeomorphic to S^1. Hence it is a manifold of dimension one.

If we write $a = \cos\theta$, $b = \sin\theta$, we see that the matrix A above represents "rotation through the angle θ".

(ii) $n = 3$: It is not easy to describe SO_3 explicitly. Geometrically, any matrix in SO_3 represents "rotation in $\mathbf{R}^3$ through the angle θ around a line through the origin". Since the line can be specified by two angles ϕ, ψ, we guess that SO_3 should be a manifold of dimension three. However, it is definitely not homeomorphic to $S^1 \times S^1 \times S^1$!

3. Example: S^2 as a real and complex manifold

Let $S^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ (the two-dimensional unit sphere).

3.1 S^2 as a real manifold

We define two charts $(U, \phi), (V, \psi)$ as follows: $U = S^2 - \{(0,0,1)\}$, $V = S^2 - \{(0,0,-1)\}$, and

$$\phi : U \to \mathbf{R}^2, \quad \phi(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}\right)$$
$$\psi : V \to \mathbf{R}^2, \quad \psi(x_1, x_2, x_3) = \left(\frac{x_1}{1 + x_3}, \frac{x_2}{1 + x_3}\right).$$

These maps are obtained by stereographic projection from the north and south poles. They are bijective and their inverses are given by

$$\phi^{-1}(u_1, u_2) = \left(\frac{2u_1}{u_1^2 + u_2^2 + 1}, \frac{2u_2}{u_1^2 + u_2^2 + 1}, \frac{u_1^2 + u_2^2 - 1}{u_1^2 + u_2^2 + 1}\right)$$
$$\psi^{-1}(v_1, v_2) = \left(\frac{2v_1}{v_1^2 + v_2^2 + 1}, \frac{2v_2}{v_1^2 + v_2^2 + 1}, \frac{1 - v_1^2 - v_2^2}{v_1^2 + v_2^2 + 1}\right).$$

(In the case of $\phi : U \to \mathbf{R}^2$ we denote the standard coordinates of $\mathbf{R}^2$ by (u_1, u_2); in the case of $\psi : V \to \mathbf{R}^2$ we denote the standard coordinates of $\mathbf{R}^2$ by (v_1, v_2).)

If S^2 has the induced topology (as a subspace of $\mathbf{R}^3$), it is easy to check that ϕ and ψ are homeomorphisms. Since $U \cup V = S^2$, it follows that S^2 is a topological manifold and $\{(U, \phi), (V, \psi)\}$ is a C^0-atlas.

We have

$$\psi \circ \phi^{-1}(u_1, u_2) = \left(\frac{u_1}{u_1^2 + u_2^2}, \frac{u_2}{u_1^2 + u_2^2}\right),$$

and the domain of $\psi \circ \phi^{-1}$ is

$$\phi(U \cap V) = \{(u_1, u_2) \in \mathbf{R}^2 \mid (u_1, u_2) \neq (0, 0)\}.$$

This is a C^∞ map. Similarly,

$$\phi \circ \psi^{-1}(v_1, v_2) = \left(\frac{v_1}{v_1^2 + v_2^2}, \frac{v_2}{v_1^2 + v_2^2}\right),$$

with domain

$$\psi(U \cap V) = \{(v_1, v_2) \in \mathbf{R}^2 \mid (v_1, v_2) \neq (0, 0)\},$$

and this is also a C^∞ map. We deduce that $\{(U, \phi), (V, \psi)\}$ is a C^∞-atlas. Hence S^2 is a (real) C^∞ manifold.

3.2 S^2 as a complex manifold

Consider $\mathbf{R}^2$ with standard coordinates (u_1, u_2), and $\mathbf{C}$ with standard coordinate z. Define $\alpha : \mathbf{R}^2 \to \mathbf{C}$ by $\alpha(u_1, u_2) = u_1 + iu_2 = z$. Define $\tilde{\phi} : U \to \mathbf{C}$ by $\tilde{\phi} = \alpha \circ \phi$.

Consider $\mathbf{R}^2$ with standard coordinates (v_1, v_2), and $\mathbf{C}$ with standard coordinate w. Define $\beta : \mathbf{R}^2 \to \mathbf{C}$ by $\beta(v_1, v_2) = v_1 - iv_2 = w$. Define $\tilde{\psi} : V \to \mathbf{C}$ by $\tilde{\psi} = \beta \circ \psi$.

The maps α and β are homeomorphisms. Therefore $\tilde{\phi}$ and $\tilde{\psi}$ are homeomorphisms.

Let us compute the coordinate transformations:

$$\begin{aligned}
\tilde{\psi} \circ \tilde{\phi}^{-1}(z) &= \beta \circ \psi \circ \phi^{-1} \circ \alpha^{-1}(z) \\
&= \beta \circ \psi \circ \phi^{-1}(u_1, u_2) \\
&= \beta \left(\frac{u_1}{u_1^2 + u_2^2}, \frac{u_2}{u_1^2 + u_2^2} \right) \\
&= \frac{u_1}{u_1^2 + u_2^2} - i \frac{u_2}{u_1^2 + u_2^2} \\
&= \frac{1}{z}
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\tilde{\phi} \circ \tilde{\psi}^{-1}(w) &= \alpha \circ \phi \circ \psi^{-1} \circ \beta^{-1}(w) \\
&= \frac{1}{w}.
\end{aligned}$$

The domain of $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is

$$\tilde{\phi}(U \cap V) = \{ z \in \mathbf{C} \mid z \neq 0 \},$$

so $\tilde{\psi} \circ \tilde{\phi}^{-1}$ is complex analytic. Similarly, the domain of $\tilde{\phi} \circ \tilde{\psi}^{-1}$ is

$$\tilde{\psi}(U \cap V) = \{ w \in \mathbf{C} \mid w \neq 0 \},$$

so $\tilde{\phi} \circ \tilde{\psi}^{-1}$ is complex analytic. It follows that $\{(U, \tilde{\phi}), (V, \tilde{\psi})\}$ is a complex analytic atlas. Hence S^2 is a complex manifold.

How about the case of S^n ? Similar arguments show that S^n is a real manifold of dimension n. If n is odd, S^n cannot be a complex manifold (as an m-dimensional complex manifold is, automatically, a $2m$-dimensional real manifold). If n is even and $n \geq 4$, it is not easy to decide whether S^n admits a complex analytic atlas. In fact, it is known that S^4 does *not* admit a complex analytic atlas. It is an open question whether S^6 admits a complex analytic atlas. It is known that S^{2m} does *not* admit a complex analytic atlas when $m \geq 4$.

4. Smooth maps

4.1 Smooth functions on a manifold

We begin with the definition of smooth function.

Definition. Let M be a smooth manifold. Let $f : M \to \mathbf{R}$ be a function on M. Let $p \in M$.

(1) We say that f is smooth (or C^∞) at p if $f \circ \psi^{-1}$ is smooth at $\psi(p)$ for some local coordinate function $\psi : U \to E \subseteq \mathbf{R}^n$ with $p \in U$.

(2) We say that f is a smooth map (or C^∞-map) if f is smooth at all points of M.

In this situation, the function

$$f \circ \psi^{-1} : (x_1, \dots, x_n) \to f(\psi^{-1}(x_1, \dots, x_n))$$

is a function on an open subset of $\mathbf{R}^n$. Therefore, the partial derivatives

$$\frac{\partial}{\partial x_1} f \circ \psi^{-1} \bigg|_{\psi(p)}, \dots, \frac{\partial}{\partial x_n} f \circ \psi^{-1} \bigg|_{\psi(p)}$$

are defined. For brevity, we can write

$$\frac{\partial f}{\partial x_1}\bigg|_p, \dots, \frac{\partial f}{\partial x_n}\bigg|_p,$$

although it is important to keep in mind that these partial derivatives depend on the choice of ψ. We do not (yet) have any concept of "derivative of f".

4.2 Smooth maps between manifolds

The previous definition can be extended to the case of maps between manifolds.

Definition. Let M, M' be smooth manifolds with $\dim M = n$, $\dim M' = n'$. Let $\phi : M \to M'$ be a map. Let $p \in M$.

(1) We say that ϕ is smooth (or C^∞) at p if $\psi' \circ \phi \circ \psi^{-1}$ is smooth at $\psi(p)$ for some local coordinate functions $\psi : U \to E \subseteq \mathbf{R}^n$, $\psi' : U' \to E' \subseteq \mathbf{R}^{n'}$ with $p \in U$, $\phi(p) \in U'$.

(2) We say that ϕ is a smooth map (or C^∞-map) if ϕ is smooth at all points of M.

The previous definition is a special case of this definition (the case $M' = \mathbf{R}$).

Definition. If $\phi : M \to M'$ satisfies the conditions

(1) ϕ is bijective

(2) ϕ is smooth

(3) ϕ^{-1} is smooth

then we say that ϕ is a diffeomorphism. We say that M, M' are diffeomorphic.

If $x_1, \dots, x_n$ are the standard coordinates of $\mathbf{R}^n$, and $y_1, \dots, y_{n'}$ are the standard coordinates of $\mathbf{R}^{n'}$, then we have n' functions of n variables:

$$y_i = \psi' \circ \phi \circ \psi^{-1}(x_1, \dots, x_n), \quad 1 \le i \le n'$$

and we have their partial derivatives

$$\frac{\partial y_i}{\partial x_j}$$

(which depend on the choice of ψ, ψ').

4.3 Some smooth functions on S^2

Let us verify that the "height function"

$$f : S^2 \to \mathbf{R}, \quad f(x, y, z) = z$$

is a smooth function. It is necessary to compute the functions $f \circ \phi^{-1}$, $f \circ \psi^{-1}$ (where ϕ, ψ are as in section 3.1). We have

$$\begin{aligned} f \circ \phi^{-1}(u_1, u_2) &= f\left(\frac{2u_1}{u_1^2 + u_2^2 + 1}, \frac{2u_2}{u_1^2 + u_2^2 + 1}, \frac{u_1^2 + u_2^2 - 1}{u_1^2 + u_2^2 + 1} \right) \\ &= \frac{u_1^2 + u_2^2 - 1}{u_1^2 + u_2^2 + 1}, \end{aligned}$$

and this is smooth on its domain $\phi(U) = \mathbf{R}^2$. Similarly

$$\begin{aligned} f \circ \psi^{-1}(v_1, v_2) &= f\left(\frac{2v_1}{v_1^2 + v_2^2 + 1}, \frac{2v_2}{v_1^2 + v_2^2 + 1}, \frac{1 - v_1^2 - v_2^2}{v_1^2 + v_2^2 + 1} \right) \\ &= \frac{1 - v_1^2 - v_2^2}{v_1^2 + v_2^2 + 1}, \end{aligned}$$

which is smooth on its domain $\psi(V) = \mathbf{R}^2$. We conclude that f is smooth on S^2.

A similar calculation shows that $f = F|_{S^2} : S^2 \to \mathbf{R}$ is smooth for any smooth function $F : \mathbf{R}^3 \to \mathbf{R}$.

The function

$$f : S^2 \to \mathbf{R}, \quad f(x, y, z) = \frac{1}{z}$$

is *not* a smooth function on S^2. (It is not even a well-defined function on S^2.)

The function

$$f : S^2 \to \mathbf{R}, \quad f(X, Y, Z) = Z$$

is *not* a complex analytic function on S^2. (We replace x, y, z by X, Y, Z in order to avoid confusion with the local coordinates $z = u_1 + iu_2$ and $w = v_1 - iv_2$.)

For example,

$$\begin{aligned}
f \circ \tilde{\phi}^{-1}(z) &= f\left(\frac{2u_1}{u_1^2+u_2^2+1}, \frac{2u_2}{u_1^2+u_2^2+1}, \frac{u_1^2+u_2^2-1}{u_1^2+u_2^2+1}\right) \\
&= \frac{u_1^2+u_2^2-1}{u_1^2+u_2^2+1} \\
&= \frac{|z|^2-1}{|z|^2+1}.
\end{aligned}$$

This is not a complex analytic function of z.

4.4 Some smooth maps from S^2 to S^2

Let us verify that the function

$$f : S^2 \to S^2, \quad f(x,y,z) = (x,y,-z)$$

is a smooth map. We shall use the same atlas $\{(U,\phi),(V,\psi)\}$ for each S^2. However, to avoid confusion, we shall write $\{(U_1,\phi_1),(V_1,\psi_1)\}$ for the atlas of the domain manifold S^2, and $\{(U_2,\phi_2),(V_2,\psi_2)\}$ for the atlas of the target manifold S^2. We have the following situation:

$$\begin{tikzcd}
\mathbf{R}^2 & & \mathbf{R}^2 \\
U_1 \cup V_1 = S^2 \arrow[u, "\phi_1"] \arrow[d, "\psi_1"] \arrow[rr, "f"] & & S^2 = U_2 \cup V_2 \arrow[u, "\phi_2"'] \arrow[d, "\psi_2"'] \\
\mathbf{R}^2 & & \mathbf{R}^2
\end{tikzcd}$$

(the domains of ϕ_i, ψ_i are U_i, V_i). It is possible to verify that all the maps

$$\phi_2 \circ f \circ \phi_1^{-1},\ \phi_2 \circ f \circ \psi_1^{-1},\ \psi_2 \circ f \circ \phi_1^{-1},\ \psi_2 \circ f \circ \psi_1^{-1}$$

are smooth on their respective domains. This shows that f is a smooth map.

More generally, let $F : \mathbf{R}^3 \to \mathbf{R}^3$ be any smooth map such that $|F(x,y,z)| = |(x,y,z)|$ for all (x,y,z). Then a similar argument shows that the restriction $f = F|_{S^2} : S^2 \to S^2$ is a smooth map.

5. Smooth maps 2

5.1 The Riemann sphere $S^2_R = \mathbf{C} \cup \{\infty\}$

Let S^2_R be the disjoint union of the set of complex numbers $\mathbf{C}$ and a set with one element. This set with one element will be denoted $\{\infty\}$, and its element will be called "infinity". Thus, an element $p \in S^2_R$ is either a complex number or ∞.

We define a topology on the set S^2_R as follows: a subset $X \subseteq S^2_R$ is said to be open if and only if

either X is an open subset of $\mathbf{C}$

or $\infty \in X \subseteq S^2_R$ and $S^2_R - X$ is compact.

It can be verified that this makes S^2_R a topological space.

We define local charts for S^2_R as follows. Let $U = \mathbf{C}$ and $V = S^2_R - \{0\}$. Define $\Phi : U \to \mathbf{C}$ by

$$\Phi(p) = p.$$

Define $\Psi : V \to \mathbf{C}$ by

$$\Psi(p) = \begin{cases} \frac{1}{p} \text{ if } p \neq \infty \\ 0 \text{ if } p = \infty. \end{cases}$$

It can be verified that $\{(U, \Phi), (V, \Psi)\}$ is a smooth atlas (in fact, a complex analytic atlas).

Proposition. The manifolds S^2 and S^2_R are diffeomorphic. In fact, the map

$$S^2_R \to S^2, \quad p \mapsto \begin{cases} \tilde{\phi}^{-1}(z) \text{ if } p = z \\ (0,0,1) \text{ if } p = \infty \end{cases}$$

is a complex analytic diffeomorphism.

Proof. [omitted]

Thus, the Riemann sphere S^2_R is diffeomorphic to the standard sphere S^2.

The Riemann sphere can be used to construct many examples of smooth maps $\hat{f} : S^2 \to S^2$, by extending maps $f : \mathbf{C} \to \mathbf{C}$. For example, if $f(z) = z^2 + 1$, the map

$$\hat{f} : S^2_R \to S^2_R, \quad p \mapsto \begin{cases} z^2 + 1 \text{ if } p = z \\ \infty \text{ if } p = \infty \end{cases}$$

is a complex analytic map. To prove this, it is necessary to verify that the maps

$$\Phi_2 \circ \hat{f} \circ \Phi_1^{-1},\ \Phi_2 \circ \hat{f} \circ \Psi_1^{-1},\ \Psi_2 \circ \hat{f} \circ \Phi_1^{-1},\ \Psi_2 \circ \hat{f} \circ \Psi_1^{-1}$$

in the diagram below are complex analytic.

$$\begin{array}{ccc}
\mathbf{C}\ (\text{with } z_1) & & \mathbf{C}\ (\text{with } z_2) \\
\uparrow \Phi_1 & & \Phi_2 \uparrow \\
U_1 \cup V_1 = S^2_R & \xrightarrow{\hat{f}} & S^2_R = U_2 \cup V_2 \\
\downarrow \Psi_1 & & \Psi_2 \downarrow \\
\mathbf{C}\ (\text{with } w_1) & & \mathbf{C}\ (\text{with } w_2)
\end{array}$$

For example, let us check $\Psi_2 \circ \hat{f} \circ \Phi_1^{-1}$. The domain of this map is

$$\{z_1 \in U_1 \mid \hat{f} \circ \Phi_1^{-1}(z_1) \in V_2\} = \{z_1 \in U_1 \mid z_1 \in \mathbf{C}, z_1^2 + 1 \neq 0\}.$$

We obtain

$$\begin{aligned}
\Psi_2 \circ \hat{f} \circ \Phi_1^{-1}(z_1) &= \Psi_2 \circ \hat{f}(z_1) \\
&= \Psi_2(z_1^2 + 1) \\
&= (z_1^2 + 1)^{-1}.
\end{aligned}$$

Since $z_1^2 + 1 \neq 0$, this is complex analytic.

Using the diffeomorphism between S^2_R and S^2, we obtain a smooth map $S^2 \to S^2$.

More generally, let $f : \mathbf{C} \to \mathbf{C}$ be a complex polynomial map. Define a new map $\hat{f} : S^2_R \to S^2_R$ by

$$\hat{f}(p) = \begin{cases} f(p) \text{ if } p = z \\ \infty \text{ if } p = \infty \end{cases}$$

Then it can be verified that $\hat{f}$ is a complex analytic map.

Even more generally, for any rational function f, it is possible to define a complex analytic extension $\hat{f} : S^2_R \to S^2_R$.

Question: Is it possible to define a complex analytic extension $\hat{f} : S^2_R \to S^2_R$ of the exponential map $f(z) = e^z$? (Answer: No.)

5.2 The circle

In this section we shall consider some smooth maps defined on the circle, S^1.

Proposition. The manifolds S^1 and $\mathbf{R}P^1$ are diffeomorphic.

Proof. It is necessary to find a smooth map $f : S^1 \to \mathbf{R}P^1$ such that f^{-1} (exists and) is a smooth map.

Question: Are the following maps diffeomorphisms?

(1) $f(\cos\theta, \sin\theta) = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2})$

(2) $f(x, y) = [x - 1, y]$.

(Answer: Yes.)

Question: Are S^2 and $\mathbf{R}P^2$ are diffeomorphic? (Answer: No.)

6. Tangent vectors

The derivative of a map $f : \mathbf{R}^n \to \mathbf{R}^m$ is given by its matrix of partial derivatives (Jacobi matrix). How should we define the derivative of a map $f : M_1 \to M_2$ between manifolds?

The following simple example illustrates the problem. Let

$$S^2 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

as usual, and let $f(x, y, z) = z$. The function f takes its maximum value at the point $(0, 0, 1)$, so we expect that the derivative of f should be zero there. Regarding f as a function on $\mathbf{R}^3$, we have

$$\left(\tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}, \tfrac{\partial f}{\partial z}\right)\Big|_{(0,0,1)} = (0, 0, 2),$$

which is not zero.

With respect to local coordinate functions ψ_1, ψ_2, the derivative of $f : M_1 \to M_2$ should be represented by the Jacobi matrix of $\psi_2 \circ f \circ \psi_1^{-1}$. In the above example, the derivative of $\psi_2 \circ f \circ \psi_1^{-1}$ is zero for any choice of ψ_1, ψ_2, which seems correct. However, this is not a satisfactory definition of "the derivative of f".

To define the derivative of a map between manifolds, we have to introduce the concept of of tangent vector, tangent space, and tangent bundle.

6.1 Motivation

The tangent vector (or velocity vector) of a smooth curve

$$\gamma : (-\epsilon, \epsilon) \to \mathbf{R}^n$$

at a point $p = \gamma(0)$ is defined to be the derivative

$$\frac{d\gamma}{dt}\bigg|_0 .$$

The tangent space to this smooth curve at p is defined to be the one-dimensional vector space spanned by the tangent vector. (Thus, it is well-defined only when the tangent vector is nonzero.) The tangent vector depends on the parameter of the curve, but the tangent space does not.

Consider a parametrization

$$\alpha : (-\epsilon, \epsilon) \times (-\delta, \delta) = D \to \mathbf{R}^n$$

of a smooth surface $\alpha(D)$ in $\mathbf{R}^n$. We denote the coordinates of $(-\epsilon, \epsilon)$, $(-\delta, \delta)$ by u, v respectively. The tangent space to this surface at a point $p = \alpha(0, 0)$ is defined to be the two-dimensional vector space spanned by

$$\frac{\partial \alpha}{\partial u}\bigg|_{(0,0)}, \quad \frac{\partial \alpha}{\partial v}\bigg|_{(0,0)} .$$

(Thus, it is well-defined only when the above two vectors are linearly independent.) This definition is independent of the parametrization of the surface. Elements of this vector space have the form

$$\lambda \frac{\partial \alpha}{\partial u}\bigg|_{(0,0)} + \mu \frac{\partial \alpha}{\partial v}\bigg|_{(0,0)},$$

where $\lambda, \mu \in \mathbf{R}$. They are called tangent vectors to the surface.

We wish to define tangent vectors for a general smooth manifold, without assuming that the manifold is a subset of Euclidean space. The key idea comes from the following "differential operator expressions" for tangent vectors of curves and surfaces:

$$\frac{d}{dt}\bigg|_0 \gamma, \quad \left(\lambda \frac{\partial}{\partial u}\bigg|_{(0,0)} + \mu \frac{\partial}{\partial v}\bigg|_{(0,0)} \right) \alpha$$

6.2 The definition

Let $S = \{(U_\alpha, \psi_\alpha) \mid \alpha \in A\}$ be a smooth atlas for a manifold M.

Definition. Let $p \in M$. Let $\mathcal{M}_p$ be the set of all smooth real-valued functions, each of which is defined on some open neighbourhoood of p. A tangent vector to M at p is a map

$$v : \mathcal{M}_p \to \mathbf{R}$$

such that:

(1) $v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$

(2) $v(fg) = v(f)g(p) + f(p)v(g)$

for all $f, g \in \mathcal{M}_p$, $\lambda, \mu \in \mathbf{R}$. The set of all tangent vectors to M at p is denoted by T_pM. It is called the tangent space to M at p.

Proposition. T_pM is a vector space (with the obvious operations).

Proof. If v_1, v_2 satisfy (1) and (2) above, it is easy to verify that $\lambda_1 v_1 + \lambda_2 v_2$ also satisfies (1) and (2).

For example, if $M = \mathbf{R}^n$ with its standard atlas $\{(\mathbf{R}^n, \mathrm{id})\}$, the maps

$$e_i : f \mapsto \frac{\partial f}{\partial x_i}(p), \quad 1 \le i \le n$$

are tangent vectors at $p = (p_1, \dots, p_n)$. This seems very different from the intuitive idea that a tangent vector of $\mathbf{R}^n$ is just an ordinary vector $(\lambda_1, \dots, \lambda_n)$. However, such an ordinary vector corresponds to

$$\lambda_1 \frac{\partial}{\partial x_1}\bigg|_p + \dots + \lambda_n \frac{\partial}{\partial x_n}\bigg|_p,$$

which is a tangent vector in the sense of the above definition. We shall prove that *every* tangent vector of $\mathbf{R}^n$ is of this type.

Theorem. $\dim T_p\mathbf{R}^n = n$ for any $p \in \mathbf{R}^n$. In fact,

$$\left.\frac{\partial}{\partial x_1}\right|_p , \ \cdots \ , \ \left.\frac{\partial}{\partial x_n}\right|_p$$

is a basis of $T_p\mathbf{R}^n$.

Proof. [omitted]

More generally, this result holds for any manifold:

Theorem. Let M be an n-dimensional smooth manifold. Then $\dim T_pM = n$ for any $p \in M$. In fact,

$$\left.\frac{\partial}{\partial x_1}\right|_p , \ \cdots \ , \ \left.\frac{\partial}{\partial x_n}\right|_p$$

is a basis of $T_p\mathbf{R}^n$, where $x_1, \dots, x_n$ are the standard coordinates of $\mathbf{R}^n$, and $\psi : U \to \mathbf{R}^n$ is a local chart with $p \in U$. (This basis depends on the local chart ψ.)

Proof. [omitted]

Example. Let $M = \{(x,y) \in \mathbf{R}^2 \mid 1 < x^2 + y^2 < 3\}$. Let $p = (\frac{3}{2}, \frac{3}{2}) \in M$. We shall consider two local charts at p, and two bases of T_pM.

First local chart: the inclusion map $\psi : M \to \mathbf{R}^2, \psi(x,y) = (x,y)$ is a local chart. (This M is an open subset of $\mathbf{R}^2$, so $\{(M,\psi)\}$ is a C^∞-atlas.) If we use the same standard coordinates x, y on M and $\mathbf{R}^2$, then

$$\left.\frac{\partial}{\partial x}\right|_p , \quad \left.\frac{\partial}{\partial y}\right|_p$$

is a basis of T_pM.

Second local chart: the map $\phi : U \to \mathbf{R}^2, \phi(x,y) = (\sqrt{x^2+y^2}, \tan^{-1} y/x) = (r,\theta)$ is a local chart, where $U = \{(x,y) \in M \mid x > 0, y > 0\}$. This time we use (r,θ) as standard coordinates for $\mathbf{R}^2$; thus

$$E = \phi(U) = \{(r,\theta) \in \mathbf{R}^2 \mid 1 < r < \sqrt{3}, 0 < \theta < \pi/2\}$$

is an open rectangle. Then

$$\left.\frac{\partial}{\partial r}\right|_p , \quad \left.\frac{\partial}{\partial \theta}\right|_p$$

is a basis of T_pM.

What is the relation between $\left.\frac{\partial}{\partial x}\right|_p, \left.\frac{\partial}{\partial y}\right|_p$ and $\left.\frac{\partial}{\partial r}\right|_p, \left.\frac{\partial}{\partial \theta}\right|_p$? By linear algebra, they must be related by a linear transformation. To find the linear transformation, we apply the tangent vectors to a function f:

$$\begin{aligned}
\left.\frac{\partial}{\partial r}\right|_p (f) &= \left.\frac{\partial}{\partial r} f \circ \phi^{-1}\right|_{\phi(p)} \\
&= \left.\frac{\partial}{\partial r} (f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1})\right|_{\phi(p)} \\
&= \left.\frac{\partial}{\partial x}(f \circ \psi^{-1})\right|_{\psi(p)} \left.\frac{\partial x}{\partial r}\right|_{\phi(p)} \ + \ \left.\frac{\partial}{\partial y}(f \circ \psi^{-1})\right|_{\psi(p)} \left.\frac{\partial y}{\partial r}\right|_{\phi(p)} .
\end{aligned}$$

Here we use $(r, \theta) = \phi \circ \psi^{-1}(x, y) = (\sqrt{x^2 + y^2}, \tan^{-1} y/x)$, i.e. $(x, y) = \psi \circ \phi^{-1}(r, \theta) = (r \cos\theta, r \sin\theta)$. We have $\frac{\partial x}{\partial r}\Big|_{\phi(p)} = 1/\sqrt{2}$ and $\frac{\partial y}{\partial r}\Big|_{\phi(p)} = 1/\sqrt{2}$. Therefore

$$\frac{\partial}{\partial r}\bigg|_p = \tfrac{1}{\sqrt{2}} \frac{\partial}{\partial x}\bigg|_p + \tfrac{1}{\sqrt{2}} \frac{\partial}{\partial y}\bigg|_p .$$

A similar calculation gives

$$\frac{\partial}{\partial \theta}\bigg|_p = \tfrac{-\sqrt{3}}{2} \frac{\partial}{\partial x}\bigg|_p + \tfrac{\sqrt{3}}{2} \frac{\partial}{\partial y}\bigg|_p .$$

7. The derivative of a smooth map

Definition. Let M, M' be smooth manifolds with $\dim M = n$, $\dim M' = n'$. Let $\phi : M \to M'$ be a smooth map. Let $p \in M$. We define the derivative (or differential) of ϕ at p to be the map

$$(d\phi)_p : T_pM \to T_{\phi(p)}M'$$

given by

$$(d\phi)_p(v)(f) = v(f \circ \phi)$$

for any $v \in T_pM$ and any $f \in \mathcal{M}'_{\phi(p)}$.

It follows easily that $(d\phi)_p$ is a linear transformation from the vector space T_pM to the vector space $T_{\phi(p)}M'$.

If $\phi : M \to M'$ and $\psi : M' \to M''$ are smooth maps, then the composition $\psi \circ \phi : M \to M''$ is also a smooth map. It follows easily that

$$(d(\psi \circ \phi))_p = (d\psi)_{\phi(p)} \circ (d\phi)_p.$$

Thus, the above definition is very natural. To understand it in more concrete terms, let us consider the special case $M = \mathbf{R}^n$, $M' = \mathbf{R}^{n'}$. Let us write $\phi = (\phi_1, \ldots, \phi_{n'})$ as usual. Let us use the standard coordinates $x_1, \ldots, x_n$ for $\mathbf{R}^n$, and the standard coordinates $y_1, \ldots, y_{n'}$ for $\mathbf{R}^{n'}$. Then:

(1) We have a linear isomorphism

$$T_p\mathbf{R}^n \to \mathbf{R}^n, \quad \lambda_1 \left.\frac{\partial}{\partial x_1}\right|_p + \ldots + \lambda_n \left.\frac{\partial}{\partial x_n}\right|_p \mapsto (\lambda_1, \ldots, \lambda_n).$$

(2) We have a linear isomorphism

$$T_{\phi(p)}\mathbf{R}^{n'} \to \mathbf{R}^{n'}, \quad \mu_1 \left.\frac{\partial}{\partial y_1}\right|_{\phi(p)} + \ldots + \mu_{n'} \left.\frac{\partial}{\partial y_{n'}}\right|_{\phi(p)} \mapsto (\mu_1, \ldots, \mu_{n'}).$$

(3) Using (1) and (2), the linear map $(d\phi)_p : T_pM \to T_{\phi(p)}M'$ can be identified with a linear map $A : \mathbf{R}^n \to \mathbf{R}^{n'}$. What is this linear map?

An equivalent question is: "What is the matrix A of the linear transformation $(d\phi)_p$ with respect to the basis $\left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p$ of T_pM and the basis $\left.\frac{\partial}{\partial y_1}\right|_{\phi(p)}, \ldots, \left.\frac{\partial}{\partial y_{n'}}\right|_{\phi(p)}$ of $T_{\phi(p)}M'$?".

To answer this, we apply the tangent vector $(d\phi)_p(\frac{\partial}{\partial x_1}|_p)$ to a function $g \in \mathcal{M}'_{\phi(p)}$. (In the following calculation, we omit the points p and $\phi(p)$ from the notation, for clarity.)

$$\begin{aligned}(d\phi)_p\left(\frac{\partial}{\partial x_i}\right)(g) &= \frac{\partial}{\partial x_i}(g\circ\phi)\\ &= \sum_{j=1}^{n'}\frac{\partial g}{\partial y_j}\frac{\partial \phi_j}{\partial x_i}\\ &= \sum_{j=1}^{n'}\frac{\partial \phi_j}{\partial x_i}\left(\frac{\partial}{\partial y_j}\right)(g)\end{aligned}$$

Since this holds for all g, we must have

$$(d\phi)_p\left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^{n'}\frac{\partial \phi_j}{\partial x_i}\left(\frac{\partial}{\partial y_j}\right).$$

That is, the matrix is

$$A = \begin{pmatrix} \frac{\partial\phi_1}{\partial x_1} & \cdots & \frac{\partial\phi_{n'}}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial\phi_1}{\partial x_n} & \cdots & \frac{\partial\phi_{n'}}{\partial x_n}\end{pmatrix},$$

which is the Jacobi matrix of ϕ.

For a map $\phi : M \to M'$ between general manifolds M, M' the situation is very similar. Let ψ, ψ' be local coordinate functions for M, M' near the points $p, \phi(p)$. Then the derivative

$$d\phi_p : T_pM \to T_{\phi(p)}M'$$

is represented by the matrix

$$\begin{pmatrix} \frac{\partial(\psi'_1\circ\phi\circ\psi^{-1})}{\partial x_1} & \cdots & \frac{\partial(\psi'_{n'}\circ\phi\circ\psi^{-1})}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial(\psi'_1\circ\phi\circ\psi^{-1})}{\partial x_n} & \cdots & \frac{\partial(\psi'_{n'}\circ\phi\circ\psi^{-1})}{\partial x_n}\end{pmatrix}$$

with respect to the bases $\left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p$ of T_pM and $\left.\frac{\partial}{\partial y_1}\right|_{\phi(p)}, \ldots, \left.\frac{\partial}{\partial y_{n'}}\right|_{\phi(p)}$ of $T_{\phi(p)}M'$.

Example. : Let us consider the case of

$$S^2 = \{(x,y,z) \in \mathbf{R}^3 \mid x^2+y^2+z^2 = 1\}$$

and $f(x,y,z) = z$, from the previous section. With respect to a local coordinate function ψ for S^2, the linear transformation

$$df|_{(a,b,c)} : T_{(a,b,c)}S^2 \to T_c\mathbf{R} \cong \mathbf{R}$$

is represented by the Jacobi matrix

$$\left(\frac{\partial f\circ\psi^{-1}}{\partial u}, \frac{\partial f\circ\psi^{-1}}{\partial v}\right)$$

What is the effect of changing the local coordinate function ψ?

8. Immersions and embeddings

8.1 Definitions

Let $\phi : M \to M'$ be a smooth map between two manifolds M, M'.

Definition. We say that ϕ is an *immersion* if the linear map $d\phi_p$ is injective (i.e. $\mathrm{Ker}\, d\phi_p = \{0\}$ or $\mathrm{rank}\, d\phi_p = \dim M$) for all $p \in M$. We say that ϕ is an *submersion* if the linear map $d\phi_p$ is surjective (i.e. $\mathrm{rank}\, d\phi_p = \dim M'$) for all $p \in M$

Definition. We say that ϕ is an *embedding* if (1) ϕ is an immersion and (2) the map ϕ itself is injective.

In some textbooks there is an additional condition in the definition of embedding: (3) the map ϕ is a homeomorphism from M to $\phi(M)$, where $\phi(M)$ has the subspace topology induced from M'. We shall say that ϕ is a *closed embedding* if it satisfies conditions (1), (2), (3).

Examples. : In the examples below, $\phi : \mathbf{R} \to \mathbf{R}^2$.

(1) $\phi(t) = (t, t^2)$ is an embedding (and a closed embedding).

(2) $\phi(t) = (t^2, t^3)$ is not an immersion (and, therefore, not an embedding).

(3) $\phi(t) = (t^2 - 4, t^3 - 4t)$ is an immersion but not an embedding.

8.2 Some questions

Many manifolds admit natural immersions and embeddings in $\mathbf{R}^n$, if n is sufficiently large. Usually, this happens when M is defined as a subset of $\mathbf{R}^n$.

For example, S^n admits an embedding $\phi : S^n \to \mathbf{R}^{n+1}$. It follows that S^n admits an embedding in $\mathbf{R}^m$ for any $m > n$. (There is no immersion $\phi : S^n \to \mathbf{R}^m$ if $m \leq n$. Why?)

If M is not defined as a subset of $\mathbf{R}^n$, it may be difficult to find an embedding $\phi : M \to \mathbf{R}^n$ (or to prove that no such embedding exists).

For example, does $\mathbf{R}P^n$ admit an immersion or embedding in $\mathbf{R}^m$ for some m? If so, then what is the smallest such m?

Even if M is defined as a subset of $\mathbf{R}^n$, it may be possible to find an immersion or embedding in $\mathbf{R}^m$ for some m which is less than n.

For example, the torus $S^1 \times S^1$ is naturally a subset of $\mathbf{R}^2 \times \mathbf{R}^2 = \mathbf{R}^4$. It is easy to show that the inclusion map $\phi : S^1 \times S^1 \to \mathbf{R}^4$ is an embedding. However, there is also a well known embedding $\phi : S^1 \times S^1 \to \mathbf{R}^3$. (There is no immersion $\phi : S^1 \times S^1 \to \mathbf{R}^2$. Why?)

We shall return to these questions later.

8.3 Tangent vectors revisited

Let $\phi : M \to \mathbf{R}^N$ be an immersion of an n-dimensional manifold M in $\mathbf{R}^N$.

Let $p \in U \subseteq M$, with $\psi : U \to \mathbf{R}^n$ a local chart. We shall use coordinates $x_1, \ldots, x_n$ for $\mathbf{R}^n$, and coordinates $X_1, \ldots, X_N$ for $\mathbf{R}^N$.

The derivative $d\phi_p : T_pM \to T_{\phi(p)}\mathbf{R}^N$ is expressed locally by

$$d\phi_p(\frac{\partial}{\partial x_i}\Big|_p) = \sum_{j=1}^{N} a_{ji} \, \frac{\partial}{\partial X_j}\Big|_{\phi(p)}, \quad \text{where} \quad a_{ji} = \frac{\partial \, \phi_j \circ \psi^{-1}}{\partial x_i} ((\psi(p)).$$

Now, if we identify $T_{\phi(p)}\mathbf{R}^N$ with $\mathbf{R}^N$ in the obvious way, we can identify $d\phi_p(\frac{\partial}{\partial x_i}\Big|_p)$ with the "geometrical vector" $\frac{\partial \phi \circ \psi^{-1}}{\partial x_i}((\psi(p)) = (a_{1i}, \ldots, a_{Ni})$ in $\mathbf{R}^N$.

Since $d\phi_p$ is injective, we can identify the "abstract tangent vector" $\frac{\partial}{\partial x_i}\Big|_p$ with the "geometrical vector" $\frac{\partial \phi \circ \psi^{-1}}{\partial x_i}((\psi(p))$ in $\mathbf{R}^N$. In other words, an immersion of M in $\mathbf{R}^N$ allows us to regard tangent vectors of M as ordinary vectors.

For example, let M be the 2-dimensional manifold

$$S^2 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

The inclusion map

$$i : S^2 \to \mathbf{R}^3, \quad i(x, y, z) = (x, y, z)$$

is an immersion (in fact, an embedding). Let $\psi : U \to \mathbf{R}^2$ be a local chart for S^2. If we use coordinates u, v on $\mathbf{R}^2$, then the abstract tangent vectors

$$\frac{\partial}{\partial u}, \ \frac{\partial}{\partial v}$$

correspond to the geometrical tangent vectors

$$(i \circ \psi^{-1})_u, \ (i \circ \psi^{-1})_v.$$

This is the usual way to describe tangent vectors in elementary surface theory.

9. Submanifolds

Manifolds are often defined by equations, for example the sphere is given by the equation $\sum_{i=1}^{n} x_i^2 = 1$. However, not every equation defines a manifold. For example the subspace of $\mathbf{R}^2$ defined by $x_1^2 - x_2^2 = 0$ is not a manifold.

More generally, we can consider a system of equations of the form

$$f_1(x_1, \ldots, x_n) = 0, \ \ldots \ , f_k(x_1, \ldots, x_n) = 0.$$

When does this system define a manifold?

9.1 Definition of submanifold

It is useful to introduce the concept of submanifold. Roughly speaking L is a submanifold of M if it is a manifold *and* its manifold structure is compatible with the manifold structure of M.

Definition. Let M be an n-dimensional manifold. Let $L \subseteq M$ (L is a subset of M, with the induced topology). We say that L is an l-dimensional submanifold of M if

(1) (when $l = n$) L is an open subset of M.

(2) (when $0 \leq l \leq n-1$) For any point $p \in L$, there exists a chart (U, ψ) of M such that $p \in U$ and

$$L \cap U = \{m \in U \mid \psi_{l+1}(m) = \cdots = \psi_n(m) = 0\}.$$

This definition immediately implies:

Proposition. If L is an l-dimensional submanifold of M, then

(1) L is itself an l-dimensional manifold, and

(2) the inclusion map $i : L \to M$, $i(p) = p$, is an embedding.

Example. The sphere S^{n-1} is an $(n-1)$-dimensional submanifold of $\mathbf{R}^n$.

9.2 When is $f^{-1}(c)$ a submanifold?

The proof of the following theorem is based on the Implicit Function Theorem (which is based on the Inverse Function Theorem):

Theorem. Let M, N be n, k-dimensional manifolds, respectively. Let $c \in N$. Let $f : M \to N$ be a (smooth) map. Let $L = f^{-1}(c)$. If, for every $p \in L$, we have $\operatorname{rank} df_p = k$, then L is an $(n-k)$-dimensional submanifold of M.

Proof. [omitted]

Examples.

(1) $M = \mathbf{R}^n$, $N = \mathbf{R}$, $c = 1$, $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$. The matrix of df_p is

$$\left(\tfrac{\partial f}{\partial x_1}(p), \ldots, \tfrac{\partial f}{\partial x_n}(p) \right) = 2p.$$

For any $p \in L$, we have $p \neq 0$, so $\operatorname{rank} df_p = 1$. We conclude that $L = S^{n-1}$ is a submanifold of $\mathbf{R}^n$. (This argument is very simple, because it does not involve charts!)

(2) $M = \mathbf{R}^n$, $N = \mathbf{R}$, $c = 0$, $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$. The theorem does not apply to this example (and we can conclude nothing). However, it is obvious that $L = \{(0, \ldots, 0)\}$, hence L is a 0-dimensional manifold.

(3) $M = \mathbf{R}^2$, $N = \mathbf{R}$, $c = 0$, $f_t(x, y) = y^2 - x^2(x + t)$. Here t is a fixed real number. For which values of t is L a submanifold of M?

(4) $M = M_{3,3}\mathbf{R} = M_3\mathbf{R}$ (see section 1), $N = \{A \in M_3(\mathbf{R}) \mid A^t = A\}$, $c = I$ (identity matrix), $f(A) = A^t A$. We have $L = \{A \in M_3\mathbf{R} \mid A^t = A^{-1}\} = O_3$ (the orthogonal group). The theorem applies here and we conclude that O_3 is a 3-dimensional submanifold of $M_3\mathbf{R}$.

10. The tangent bundle

The tangent bundle of a manifold M is defined, roughly speaking, to be the disjoint union of all tangent spaces to points of M:

$$\text{“ } TM = \bigcup_{p \in M} T_pM \text{ ”}.$$

“Disjoint union” means that the subsets T_pM, T_qM have empty intersection if $p \neq q$.

There is a natural manifold structure on TM. Before we define it, we consider a special case.

Definition. Let M be a submanifold of $\mathbf{R}^N$ of dimension n, and let $\phi : M \to \mathbf{R}^N$ be the inclusion map. We define $T^\phi M = \{(p, v) \in M \times \mathbf{R}^N \mid v \in d\phi_p(T_pM)\}$.

It is straightforward to prove that $T^\phi M$ is a submanifold of $M \times \mathbf{R}^N$ of dimension $2n$.

Example. Let $M = S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\}$, with $\phi(x) = x$. Then $T^\phi S^{n-1}$ can be identified with

$$\{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n \mid |x| = 1, \langle x, y \rangle = 0\}.$$

This is $F^{-1}(1, 0)$ where

$$F : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^2$$

where $F(x, y) = (|x|^2, \langle x, y \rangle)$. By the method of the previous section, it is possible to prove directly that $F^{-1}(1, 0)$ is a submanifold of $\mathbf{R}^n \times \mathbf{R}^n$ of dimension $2n - 2$. (Why do we not use $F(x, y) = (|x|, \langle x, y \rangle)$?)

For a general manifold (not necessarily a submanifold of $\mathbf{R}^N$), the definition of TM is as follows:

Definition. Let $S = \{(U_\alpha, \psi_\alpha) \mid \alpha \in A\}$ be a C^∞ atlas for a manifold M. Then

$$TM = \left(\bigcup_{\alpha \in A} E_\alpha \times \mathbf{R}^n \right) / \sim$$

where the union is a disjoint union, and where the equivalence relation $\sim$ is defined by:

$$(x, v) \sim (y, w) \iff y = \psi_\beta \circ \psi_\alpha^{-1}(x), w = d(\psi_\beta \circ \psi_\alpha^{-1})_x(v).$$

We denote the equivalence class of (x, v) by $[x, v]_p$, where $p = \psi_\alpha^{-1}(x)$. We define the *projection map* $\pi_M : TM \to M$ by $\pi_M([x, v]_p) = p$.

Theorem. (1) TM is a smooth manifold of dimension $2n$, with C^∞ atlas $\tilde{S} = \{(\tilde{U}_\alpha, \tilde{\psi}_\alpha) \mid \alpha \in A\}$, where $\tilde{U}_\alpha = \{[x, v]_p \mid p \in U_\alpha\}$ and $\tilde{\psi}_\alpha : \tilde{U}_\alpha \to E_\alpha \times \mathbf{R}^n$, $\tilde{\psi}_\alpha([x, v]_p) = (x, v)$. (2) The projection map π_M is a smooth map, and a submersion.

Proof. [omitted]

From the definition of TM, we have a linear isomorphism

$$\pi_M{}^{-1}(p) \to T_pM, \quad [x,v]_p \mapsto \sum_{i=1}^n v_i \tfrac{\partial}{\partial x_i}|_p.$$

Thus, since $TM = \cup_{p\in M}\pi_M^{-1}(p)$ and $\pi_M^{-1}(p) \cong T_pM$, we can regard TM as the disjoint union of all the tangent spaces T_pM, as stated above.

If M is a submanifold of $\mathbf{R}^N$, and $\phi : M \to \mathbf{R}^N$ is the inclusion map, we can define a map

$$TM \to T^\phi M, \quad [x,v]_p \mapsto (p, d\phi_p(\sum_{i=1}^n v_i \tfrac{\partial}{\partial x_i}|_p).$$

Using the above theorem, it is possible to prove that this map is a diffeomorphism. Therefore, the abstract tangent bundle TM can be identified with the geometrical tangent bundle $T^\phi M$.

If $\phi : M \to N$ is a smooth map, then each linear map $d\phi_p : T_pM \to T_{\phi(p)}N$ can be regarded as a map $\pi_M^{-1}(p) \to \pi_N^{-1}(\phi(p))$. We obtain a map $TM \to TN$. This map can be defined directly as follows:

Definition. Let $\phi : M \to N$ be a smooth map. We define a map $d\phi : TM \to TN$ by

$$d\phi([x,v]_p) = [\theta_\beta \circ \phi \circ \psi_\alpha^{-1}(x), d(\theta_\beta \circ \phi \circ \psi_\alpha^{-1})_x(v)]_{\phi(p)}$$

where θ_β is a local coordinate function at $\phi(p)$. We call $d\phi$ the *derivative* of ϕ.

It follows immediately that $d\phi : TM \to TN$ is smooth.

Thus, we have finally arrived at the definition of "the derivative of a smooth map".

Example. Let M be an open subset $U \subseteq \mathbf{R}^n$, with its standard atlas. Since there is only one chart, we have

$$TU = T^\phi U = \{(p,v) \mid p \in U, v \in \mathbf{R}^n\} = U \times \mathbf{R}^n$$

(where $\phi : U \to \mathbf{R}^n$ is the inclusion map). In section 1 we defined the derivative of $g : U \to \mathbf{R}^m$ to be the map

$$dg : U \to \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^m), \quad p \mapsto dg_p = \left(\frac{\partial g_i}{\partial x_j}(p)\right)_{1\le i\le m, 1\le j\le n}$$

The new definition of the derivative of $g : U \to \mathbf{R}^m$ is:

$$dg : TU \ (= U \times \mathbf{R}^n) \to T\mathbf{R}^m \ (= \mathbf{R}^m \times \mathbf{R}^m), \quad dg(p,v) = (g(p), dg_p(v)).$$

Thus, the "new dg" is equivalent to "g and the old dg".

For a general n-dimensional manifold M, the relation between TM and $M \times \mathbf{R}^n$ is not easy to judge. For example, is it true that TS^n "is" $S^n \times \mathbf{R}^n$?

Definition. Let M be a manifold of dimension n. We say that the manifold M is *parallelizable* if there exists a diffeomorphism $\Phi : TM \to M \times \mathbf{R}^n$ such that every restricted map $\Phi|_{T_pM} : T_pM \to \{p\} \times \mathbf{R}^n$ is a linear isomorphism of vector spaces.

Remark. TM and $M \times \mathbf{R}^n$ are examples of *vector bundles.* The definition says that a manifold is parallelizable if and only if TM and $M \times \mathbf{R}^n$ are *isomorphic* vector bundles. Similarly, if M is a submanifold of $\mathbf{R}^N$, with inclusion map $\phi : M \to \mathbf{R}^N$, then the map $d\phi : TM \to T\mathbf{R}^N$ is a *homomorphism* of vector bundles. The image of $d\phi$ is the vector bundle $T^\phi M$, and $d\phi$ is an isomorphism from the vector bundle TM to the vector bundle $T^\phi M$.

Example. We have seen that any open subset of $\mathbf{R}^n$ is parallelizable.

Example. The circle S^1 is parallelizable. To prove this, it is convenient to use $T^\phi S^1$ instead of TS^1, where $\phi : S^1 \to \mathbf{R}^2$ is the standard embedding of S^1 in $\mathbf{R}^2$. Let us define

$$\Phi : T^\phi S^1 \to S^1 \times T_{(1,0)}S^1, \quad \Phi(p, v) = (p, R_p^{-1}(v))$$

where $R_p : \mathbf{R}^2 \to \mathbf{R}^2$ denotes the linear transformation which rotates $(1, 0)$ (in the positive direction) to p. It is easy to see that Φ is a diffeomorphism, and that $\Phi|_{T_p^\phi S^1} : T_p^\phi M \to \{p\} \times T_{(1,0)}S^1$ is a linear isomorphism.

We cannot use the same method to prove that S^2 is parallelizable. (Why?) In fact, S^2 is *not* parallelizable, but it is necessary to use topological methods to prove this.

The definition of TM is rather abstract, and we have not checked its properties carefully. However, we shall use it mainly as a conceptual guide: it is "most natural" to think of $d\phi$ as a map $TM \to TN$. Of course, TM is essential in discussions of global properties such as parallelizability. However, we shall usually be concerned with local properties, which can be discussed using local coordinates, as in the case where M is an open subset of $\mathbf{R}^n$. In this case, it is not important whether we use the old or the new definition of dg; the basic ingredient is the Jacobi matrix dg_p. The same comment applies to the definitions of vector fields, differential forms, and Riemannian metrics which we shall give in the next section. In each case, it is conceptually helpful to use TM, but in practice we shall not use TM for calculations.

11. Vector fields, differential 1-forms, Riemannian metrics

We shall give a brief preview of these three important concepts.

11.1 Vector fields

Definition. A vector field on M is a C^∞ map $X : M \to TM$ such that $\pi_M \circ X = \mathrm{id}$ (the identity map of M).

The last condition means that $X(p) \in T_pM$ for all $p \in M$. Thus, a vector field is "a smooth assignment of a tangent vector at each point of the manifold". From now on we shall write X_p instead of $X(p)$.

Example. If $M = \mathbf{R}^n$ we have $T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$. Therefore, a vector field on $\mathbf{R}^n$ is a map of the form

$$X : \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n, \quad p \mapsto (p, x(p))$$

where $x : \mathbf{R}^n \to \mathbf{R}^n$ is smooth.

Example. If (U, ψ) is a chart for a manifold M, where $\psi : U \to \mathbf{R}^n$, then we can define a vector field

$$X_i : U \to TU, \quad (X_i)_p = \tfrac{\partial}{\partial x_i}|_p.$$

Of course this is a vector field on U, not on M. (We can regard it as a "locally defined vector field on M"). In terms of the abstract definition of TU, we have $(X_i)_p = [\psi(p), (0, \dots, 0, 1, 0, \dots, 0)]_p$, where the 1 appears in the i-th position. We write $X_i = \frac{\partial}{\partial x_i}$.

Example. Let M be a submanifold of $\mathbf{R}^N$, and let $\phi : M \to \mathbf{R}^N$ be the inclusion map. We can define a "geometrical vector field" on M as a smooth map $x : M \to \mathbf{R}^N$ such that $x(p)$ belongs to the geometrical tangent space $d\psi_p(T_pM)$ for all $p \in M$. An abstract vector field X on M can be converted to a geometrical vector field x by considering the following composition of maps:

$$M \to TM \to T^\phi M \subseteq T\mathbf{R}^N = \mathbf{R}^N \times \mathbf{R}^N.$$

We have

$$p \mapsto X_p \ (= [x, v]_p) \mapsto d\phi(X_p) = (\phi(p), x(p))$$

where $x(p)$ is the element of $\mathbf{R}^N$ which corresponds to $d\phi_p(X_p)$ under the natural identification $T_p\mathbf{R}^N = \mathbf{R}^N$. For example, if $X_i = \frac{\partial}{\partial x_i}$ (from the previous example), we obtain $x = \partial(\phi \circ \psi^{-1})/\partial x_i$.

Example. Let M be a 2-dimensional submanifold of $\mathbf{R}^3$ (a surface), and let $\phi : M \to \mathbf{R}^3$ be the inclusion map. Let (U, ψ) be a chart for M, with $\psi : U \to \mathbf{R}^2$. If we use standard coordinates (u, v) on $\mathbf{R}^2$, we have standard vector fields $\partial/\partial u, \partial/\partial v$ on U (which we regard as locally defined vector fields on the surface M. These abstract vector fields correspond to the geometrical vector fields α_u, α_v, where $\alpha = \phi \circ \psi^{-1}$. This example is just a special case of the previous example, but we mention it separately as it is important in elementary surface theory. (Of course it is possible to generalize this discussion to the case where $\phi : M \to \mathbf{R}^3$ is any embedding or immersion, not necessarily the inclusion map of a submanifold.)

Theorem. Let X be a vector field on an n-dimensional manifold M. Let (U, ψ) be a chart for M. Then there exist C^∞ functions $a_1, \dots, a_n : U \to \mathbf{R}$ such that

$$X|_U = \sum_{i=1}^{n} a_i \tfrac{\partial}{\partial x_i}$$

where $\partial/\partial x_i$ denotes the i-th standard vector field on U.

The proof of this theorem follows from the definition of TM: the functions a_i exist because $\partial/\partial x_1|_p, \dots, \partial/\partial x_n|_p$ is a basis of T_pM, and they are smooth because of the definition of the manifold structure of TM. (It is possible to *define* a vector field as a collection of maps $X|_U$ of the above type. This would avoid introducing the tangent bundle TM.)

Example. Any vector field on $M = \mathbf{R}^2$ can be written in the form $X_{(x_1,x_2)} = f(x_1, x_2)\partial/\partial x_1 + g(x_1, x_2)\partial/\partial x_2$. The corresponding geometrical vector field is the map $x = (f, g)$.

Example. Consider the geometrical vector field on S^1 (a submanifold of $\mathbf{R}^2$) given by $x(x_1, x_2) = (-x_2, x_1)$. If this corresponds to the abstract vector field $X = f\partial/\partial x_1 + g\partial/\partial x_2$, what are the functions f, g?

Definition. The set of all vector fields on a manifold M is denoted by $\mathcal{X}(M)$.

The set $\mathcal{X}(M)$ is a vector space (in fact, it is a module over the ring $C^\infty(M)$ of smooth functions on M).

11.2 Differential 1-forms

Let M be a manifold of dimension n. The cotangent bundle T^*M is defined in a similar way to TM, using the dual vector spaces $(T_pM)^*$ instead of T_pM. There is a natural projection map $\pi_M^* : T^*M \to M$. It can be proved that T^*M is a manifold of dimension $2n$, and that π_M^* is a smooth submersion.

Definition. A differential 1-form (or "1-form") on M is a C^∞ map $\omega : M \to T^*M, p \mapsto \omega_p$ such that $\pi_M^* \circ \omega = \text{id}$ (the identity map of M).

Example. If $f : M \to \mathbf{R}$ is smooth, then the map

$$p\ (\in M) \mapsto df_p\ (\in \text{Hom}(T_pM, T_{f(p)}\mathbf{R}) \cong (T_pM)^*$$

is a 1-form on M. This 1-form is always denoted by df. (Thus, from now on, when we use the notation df, we must specify whether it means $df : TM \to T\mathbf{R}$ or $df : M \to T^*M$. This is not as confusing as it seems, because these two maps contain essentially the same information.)

In particular, the coordinate functions $x_i : \mathbf{R}^n \to \mathbf{R}$ give rise to the 1-forms dx_i. The 1-form dx_i means the function

$$\omega_i : p \mapsto (\omega_i)_p,$$

where

$$(\omega_i)_p(v) = (dx_i)_p(v)$$

for any tangent vector $v \in T_p\mathbf{R}^n$. On the right hand side of this formula, $(dx_i)_p$ is the derivative of the map x_i at p. In particular we have

$$dx_i(\tfrac{\partial}{\partial x_j}) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise.} \end{cases}$$

Similarly, we can define the standard 1-forms $dx_1, \ldots, dx_n$ for any chart of an n-dimensional manifold.

Theorem. Let ω be a 1-form on an n-dimensional manifold M. Let (U, ψ) be a chart for M. Then there exist C^∞ functions $a_1, \ldots, a_n : U \to \mathbf{R}$ such that

$$\omega|_U = \sum_{i=1}^{n} a_i dx_i$$

where dx_i denotes the i-th standard 1-form on U.

Example. Any 1-form on $M = \mathbf{R}^2$ can be written in the form $\omega_{(x_1,x_2)} = f(x_1,x_2)dx_1 + g(x_1,x_2)dx_2$.

Example. If $f : \mathbf{R}^2 \to \mathbf{R}$ is a smooth function, then df is a 1-form. Hence there exist smooth functions a_1, a_2 such that $df = a_1dx_1 + a_2dx_2$. What are these functions a_1, a_2? (Answer: $\partial f/\partial x_1, \partial f/\partial x_2$.)

Definition. The vector space consisting of all 1-forms on a manifold M is denoted by $\Omega^1(M)$.

11.3 Riemannian metrics

Let M be a manifold of dimension n. We can define $B(TM)$ in a similar way to TM, using the vector spaces

$$B(T_pM) = \{f : T_pM \times T_pM \to \mathbf{R} \mid f \text{ is bilinear}\}$$

instead of T_pM. There is a natural projection map $\pi_M^B : B(TM) \to M$. It can be proved that $B(TM)$ is a manifold of dimension $n + n^2$, and that π_M^B is a smooth submersion.

Definition. A Riemannian metric on M is a C^∞ map $g : M \to B(TM), g \mapsto g_p$ such that (1) $\pi_M^B \circ g = \text{id}$ (the identity map of M), and (2) for every $p \in M$, g_p is an inner product on T_pM.

If α, β are 1-forms on M, we can define a map $\alpha\beta : M \to B(TM)$ by

$$\alpha\beta(v, w) = \frac{1}{2}(\alpha(v)\beta(w) + \alpha(w)\beta(v)).$$

Theorem. Let g be a Riemannian metric on an n-dimensional manifold M. Let (U, ψ) be a chart for M. Then there exist C^∞ functions $a_{ij} : U \to \mathbf{R}, 1 \leq i \leq j \leq n$, such that

$$g|_U = \sum_{1 \leq i \leq j \leq n} a_{ij} dx_i dx_j.$$

Example. Let g be a Riemannian metric on a 2-dimensional manifold M. Let (U, ψ) be a chart for M, with $\psi : U \to \mathbf{R}^2$. If we use standard coordinates (u, v) on $\mathbf{R}^2$, we have standard 1-forms du, dv on U (which we regard as locally defined 1-forms on M). Then there exist smooth functions $E, F, G : U \to \mathbf{R}$ such that $g|_U = Edu^2 + 2Fdudv + Gdv^2$. This is usually called the "first fundamental form".