Παραδείγματα: Αόριστα Ολοκληρώματα

Άσκηση 1

$$\int (5x^4 - 3x^3 + \frac{1}{x} - 2)dx$$

Λύση:

$$\int (5x^4 - 3x^3 + \frac{1}{x} - 2)dx = \int 5x^4\,dx - \int 3x^3\,dx + \int \frac{1}{x}dx - \int 2\,dx = x^5 - \frac{3}{4}x^4 + \ln|x| - 2x + C$$

Άσκηση 2

$$\int \left(\frac{1}{\sqrt[3]{x}} - \sqrt{x}\right)dx$$

Λύση:

$$\int \left(\frac{1}{\sqrt[3]{x}} - \sqrt{x}\right)dx = \int \frac{1}{\sqrt[3]{x}}dx - \int \sqrt{x}dx = \int x^{-\frac{1}{3}}dx - \int x^{\frac{1}{2}}\,dx = \frac{3}{2}x^{\frac{2}{3}} - \frac{2}{3}x^{\frac{3}{2}} + c$$
$$= \frac{3\sqrt[3]{x^2}}{2} - \frac{2\sqrt[2]{x^3}}{3} + C$$

Άσκηση 3

$$\int \frac{5x^4 + 6x^2 - 3}{x^2}dx$$

Λύση:

$$\int \frac{5x^4 + 6x^2 - 3}{x^2}dx = \int \frac{5x^4}{x^2}dx + \int \frac{6x^2}{x^2}dx - \int \frac{3}{x^2}dx = \int 5x^2dx + \int 6dx - \int 3x^{-2}dx$$
$$= \frac{5x^3}{3} + 6x + \frac{3}{x} + C$$

Άσκηση 4

$$\int 3(3x+1)^4 dx$$

Λύση:

Θέτουμε:

$$u = 3x + 1 \qquad (1)$$

Άρα

$$du = 3dx \qquad (2)$$

Από (1) και (2) ισχύει ότι:

$$\int 3(3x+1)^4 dx = \int u^4 du = \frac{u^5}{5} + C = \frac{(3x+1)^5}{5} + C$$

Άσκηση 5

$$\int 2xcosx^2 dx$$

Λύση:

Θέτουμε:

$$u = x^2 \qquad (1)$$

Άρα

$$du = 2x\,dx \qquad (2)$$

Από (1) και (2) ισχύει ότι:

$$\int 2xcosx^2 dx = \int cos\ u\ du = \sin u + C = \sin x^2 + C$$

Άσκηση 6

$$\int x^3(x^4-3)^7 dx$$

Λύση:

Θέτουμε:

$$u = x^4 - 3 \qquad (1)$$

Άρα

$$du = 4x^3 dx \Leftrightarrow x^3 dx = \frac{1}{4} du \qquad (2)$$

Από (1) και (2) ισχύει ότι:

$$\int x^3(x^4-3)^7 dx = \int \frac{1}{4} u^7 du = \frac{1}{4}\frac{u^8}{8} + C = \frac{(x^4-3)^8}{32} + C$$

Άσκηση 7

$$\int \frac{x}{\sqrt{x-1}} dx$$

Λύση:

Θέτουμε:

$$u = x - 1 \Leftrightarrow x = u + 1 \qquad (1)$$

Άρα

$$du = dx \qquad (2)$$

Από (1) και (2) ισχύει ότι:

$$\int \frac{x}{\sqrt{x-1}} dx = \int \frac{u+1}{\sqrt{u}} du = \int \left(\frac{u}{\sqrt{u}} + \frac{1}{\sqrt{u}}\right) du = \int \left(u \cdot u^{-\frac{1}{2}} + u^{-\frac{1}{2}}\right) du = \int \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right) du$$
$$= \frac{2}{3} u^{3/2} + \frac{1}{2} u^{1/2} + C = \frac{2}{3}(x-1)^{3/2} + \frac{1}{2}(x-1)^{1/2} + C$$

Άσκηση 8

$$\int x^2 e^x \, dx$$

Λύση:

$$\int x^2 e^x \, dx = \int x^2 (e^x)' \, dx = x^2 e^x - \int (x^2)' e^x \, dx = x^2 e^x - \int 2x e^x \, dx = x^2 e^x - 2 \int x (e^x)' \, dx$$
$$= x^2 e^x - 2x e^x + 2 \int (x)' e^x \, dx = x^2 e^x - 2x e^x + 2 \int e^x \, dx$$
$$= x^2 e^x - 2x e^x + 2e^x + C$$

Άσκηση 9

$$\int x^2 sinx \, dx$$

Λύση:

$$\int x^2 sinx \, dx = \int x^2 (-cosx)' \, dx = -x^2 cosx + \int (x^2)' cosx \, dx = -x^2 cosx + \int 2x \, cosx \, dx$$
$$= -x^2 cosx + 2 \int x \, (sinx)' \, dx = -x^2 cosx + 2x sinx - 2 \int (x)' \, sinx \, dx$$
$$= -x^2 cosx + 2x sinx - 2 \int sinx \, dx = -x^2 cosx + 2x sinx + 2cosx + C$$

Άσκηση 10

$$\int x^2 \ln |x| \, dx$$

Λύση:

$$\int x^2 \ln |x| \, dx = \int \left(\frac{x^3}{3}\right)' \ln |x| \, dx = \frac{x^3}{3} \ln |x| - \int \frac{x^3}{3} (\ln|x|)' \, dx = \frac{x^3}{3} lnx - \int \frac{x^3}{3} \frac{1}{x} \, dx$$
$$= \frac{x^3}{3} \ln |x| - \frac{1}{3} \int x^2 \, dx = \frac{x^3}{3} \ln |x| - \frac{1}{3} \frac{x^3}{3} + C = \frac{x^3}{3} \left(\ln |x| - \frac{1}{3}\right) + C$$

Άσκηση 11

$$\int \frac{x^2-3x+2}{x+1}\,dx$$

Λύση:

$$\int \frac{x^2-3x+2}{x+1}\,dx \overset{*}{\leftrightarrow} \int \frac{(x+1)(x-4)+6}{x+1}\,dx =$$

** Ο αριθμητής προκύπτει από την διαίρεση πολυωνύμων*

$$\begin{aligned}\int \frac{(x+1)(x-4)+6}{x+1}\,dx &= \int \left(\frac{(x+1)(x-4)}{x+1}+\frac{6}{x+1}\right)dx \\ &= \int \frac{(x+1)(x-4)}{x+1}dx + \int \frac{6}{x+1}dx = \int (x-4)dx + 6\int \frac{1}{x+1}dx \\ &= \frac{x^2}{2} - 4x + 6\ln|x+1| + C\end{aligned}$$

Άσκηση 12

$$\int \frac{1}{x^2-1}\,dx$$

Λύση:

$$\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A_1}{(x-1)} + \frac{A_2}{(x+1)}, \qquad \text{όπου } a_i \in \mathbb{R}$$

$$\frac{1}{(x-1)(x+1)} = \frac{A_1}{(x-1)} + \frac{A_2}{(x+1)} \Rightarrow 1 = A_1(x+1) + A_2(x-1) \Rightarrow 1 = xA_1 + A_1 + xA_2 - A_2$$
$$\Rightarrow \quad 1 = (A_1 + A_2)x + (A_1 - A_2)$$

Συνεπώς:

$$\begin{cases} A_1 + A_2 = 0 \\ A_1 - A_2 = 1 \end{cases}$$

$$\text{Άρα } A_1 = \frac{1}{2} \text{ και } A_2 = -\frac{1}{2}$$

Άρα:

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \left(\frac{\frac{1}{2}}{(x-1)} + \frac{-\frac{1}{2}}{(x+1)}\right)dx = \int \left(\frac{1}{2}\frac{1}{(x-1)} - \frac{1}{2}\frac{1}{(x+1)}\right)dx \\ &= \frac{1}{2}\int \frac{1}{(x-1)}dx - \frac{1}{2}\int \frac{1}{(x+1)}dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C\end{aligned}$$