Review of Selected Topics in Probability
Random Variables

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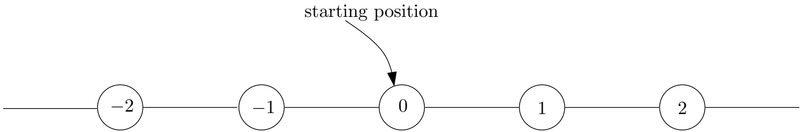
Lecture 4

Random Walk on the Line



An individual is placed at vertex 0. At each time step $t = 1, 2, \ldots$, he independently and equiprobably decides to move either one vertex to its right or one vertex to its left. What is the probability that after n steps he is back where he started?

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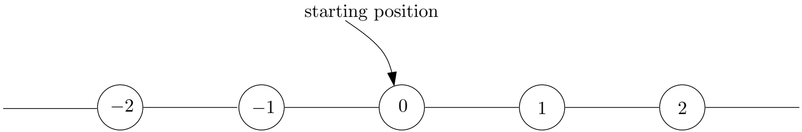

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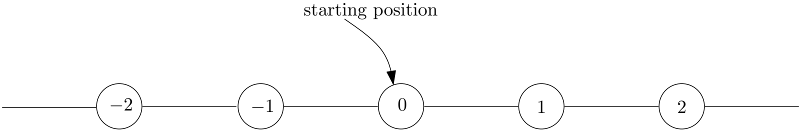

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$$\Pr\{\text{at } 0 \text{ after } n \text{ steps} | \text{started at } 0\} = \begin{cases} 0 & \text{,if } n \text{ is odd} \\ \binom{n}{n/2}\frac{1}{2^n} & \text{,if } n \text{ is even.} \end{cases}$$

Random Variables in the Random Walk

Let

$$X_t = \begin{cases} +1 & \text{,with probability } \frac{1}{2} \\ -1 & \text{,with probability } \frac{1}{2}. \end{cases}$$

- X_t is a random variable.
- So is $S_n = \sum_{t=1}^{n} X_t$, denoting the position at time n.

Definition (Random Variable)

Random Variable X is a function mapping an outcome to a real number, i.e. $X : \mathcal{S} \to \mathbb{R}$.

Note: In fact, they are smart ways to name events!

Examples and Properties

- Discrete Random Variables: X takes values in a countable set. Examples: The outcome of a die, the sum of the outcomes of 5 dice, the number of Heads when flipping a coin 732 times, the number of births in a population in the time interval $[0, \sqrt{2}]$ etc.
- Continuous Random Variables: X takes values in a non-countable set. Examples: The time between two phone calls, the sum of the waiting times between 500 consecutive births in a population, etc.

Defining Discrete Random Variables

To define a *discrete* random variable X, we need:

- The (countable) set of values $\mathcal{A}$ that X can take on, i.e. $X \in \mathcal{A}$.

- Definition (Probability Mass Function)

The probability mass function of X is the function $p : \mathcal{A} \to \mathbb{R}$

$$p(a) \overset{def}{=} \Pr(X = a) \tag{1}$$

such that

1. $p(a) \geq 0, \forall a \in \mathcal{A}$
2. $\sum_{a \in \mathcal{A}} p(a) = 1.$

Defining Continuous Random Variables

To define a *continuous* random variable Y, we need:

- The (non-countable) set of values $\mathcal{A}$ that Y can take on, i.e. $Y \in \mathcal{A}$.

- Definition (Probability Density Function)

  The probability density function of Y is the function $f : \mathcal{A} \to \mathbb{R}$ satisfying

  1. $f(y) \geq 0, \forall y \in \mathcal{A}$
  2. $\int_{-\infty}^{\infty} f(y)dy = 1$
  3. $\forall a, b \in \mathbb{R}$,

$$\Pr(a < Y < b) \stackrel{def}{=} \int_a^b f(y)dy \qquad (2)$$

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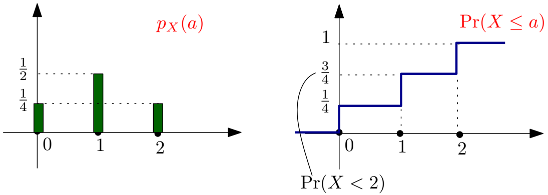
Important note: For a continuous random variable $\Pr(Y < a) = \Pr(Y \leq a)$ and $\Pr(Y = a)$ is 0 or undefined.

A Discrete Example

Example: Suppose we flip a coin two times and let X denote the number of heads. Then X takes on values in $\{0, 1, 2\}$, $p_X(2) = \Pr(X = 2) = \frac{1}{4}$ and $\Pr(X < 2) = p_X(0) + p_X(1) = \frac{3}{4}$.

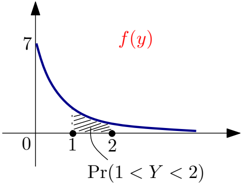


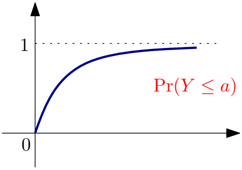
A Continuous Example: Exponential distribution with rate $\lambda = 7$

Example: Let Y be a continuous random variable, $Y \in \mathbb{R}$ and

$$f_Y(y) = \begin{cases} 7e^{-7y} & \text{,for any } y \geq 0 \\ 0 & \text{,elsewhere.} \end{cases}$$

Then $Pr(Y = 0) = 0$, $\Pr(Y \leq a) = 1 - e^{-7a}$ and $\Pr(1 < Y < 2) = \frac{1}{e^7} - \frac{1}{e^{14}} = \Pr(Y < 2) - \Pr(Y < 1)$.





Distribution Function

Definition (Distribution function)

The distribution function of a random variable X is the function $F : \mathcal{A} \to [0, 1]$

$$\begin{aligned} F(a) &= \Pr(X \leq a) \\ &= \int_{-\infty}^{a} f_X(x)dx \qquad \text{(if } X \text{ is continuous)} \\ &= \sum_{x \leq a} p_X(x) \qquad \text{(if } X \text{ is discrete)} \end{aligned} \tag{3}$$

Properties of the Distribution Function

- $F(x)$ is increasing on x.
- $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) =$

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- $F(x)$ is increasing on x.
- $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$.
- The distribution function uniquely characterizes the density function (or mass probability function in the discrete case), since $f(x) = \frac{dF(x)}{dx}$.

Multidimensional Random Variables

Definition (Joint mass probability function)

The joint mass probability function of two discrete random variables X and Y is the two dimensional function $f : \mathcal{A}_X \times \mathcal{A}_Y \to \mathbb{R}$

$$f(x, y) \stackrel{def}{=} \Pr(X = x, Y = y) \tag{4}$$

such that

1. $f(x, y) \geq 0, \forall x, y \in \mathcal{A}$
2. $\sum_{x \in \mathcal{A}_X, y \in \mathcal{A}_Y} f(x, y) = 1$

Note: This generalizes to n random variables and also to the continuous case.

Multidimensional Random Variables

Marginal Distributions

Given the joint mass probability function $f(x, y)$, the marginal distributions of f are

$$
\begin{aligned}
f_1(x) &= \Pr(X = x) \quad (5) \\
&= \Pr(X = x, \cup_{y \in \mathcal{A}_Y} \{Y = y\}) \\
&= \sum_{y \in \mathcal{A}_Y} f(x, y)
\end{aligned}
$$

and

$$f_2(y) = \Pr(Y = y). \quad (6)$$

Multidimensional Random Variables (cntd.)

Some more (easy) definitions:

- Joint Distribution Function is the function $F : \mathcal{A}_X \times \mathcal{A}_Y \to [0,1]$

$$F(x,y) \stackrel{def}{=} \Pr(X \leq x, Y \leq y) = \sum_{x' \leq x, y' \leq y} f(x', y'). \qquad (7)$$

- We write

$$\begin{aligned} f(y|x) &\stackrel{def}{=} \Pr(Y = y | X = x) \qquad (8) \\ &= \frac{f(x,y)}{f_1(x)}. \end{aligned}$$

Further reading

S. Ross. A first course in probability:
Chapter 4, “Random Variables”
Chapter 5, “Continuous Random Variables”
Chapter 6, “Jointly Distributed Random Variables”